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B. Alarcón, R. Rabanal



# HOPF BIFURCATION AT INFINITY AND DISSIPATIVE VECTOR FIELDS OF THE PLANE

BEGOÑA ALARCÓN AND ROLAND RABANAL

*Dedicated to the memory of Carlos Gutiérrez,  
on occasion of the 05th anniversary of his death.*

ABSTRACT. We describe some families of differentiable vector fields with the Hopf bifurcation at infinity, without assuming the continuous differentiability. These vector fields have isolated singular points on the plane, and the initial families are obtained by special perturbations at infinity of a vector field with some spectral property, for instance the dissipativity. The strong domination imposed by the spectral condition in the differentiable vector field is used, and then we do not apply the standard Poincaré compactification. Moreover, the perturbation of planar systems with a global period annulus is also considered.

## 1. INTRODUCTION

The purpose of this paper is to research the qualitative theory of differential equations induced by differentiable vector fields, and defined in the complement set of some compact disk on the euclidian plane. Accordingly, special attention is given to the one parameter families, for which the singular point reverses its stability as the parameter varies. For instance, in the case of a family of polynomial vector fields on the euclidian plane, denoted by  $\{\widehat{Z}_\mu : -\epsilon_0 < \mu < \epsilon_0\}$ , it is said that a Hopf bifurcation at infinity occurs for  $\mu$  crossing 0 if ‘the infinity changes its stability’ as defined, for example, in [SP87]. This phenomena on the qualitative behavior of the systems has been studied by using the standard Poincaré compactification, and it consists in compactify the respective polynomial vector field defined on the euclidian plane to an analytic vector field on the sphere, [LBK88, GSS93, SVB95]. Moreover, in this research of a Hopf bifurcation at infinity, a different method uses the strong domination imposed by the linear part of the vector field on the continuous nonlinear term, which can have sub-linear growth [Glo89, He91, KKY95, KKY97, DRY00]. Notice that, the methodologies described in the former case are related to those connected with the existence of limit cycles of arbitrarily large amplitudes in polynomial systems and then the sixteenth Hilbert problem can be considered, see for instance [Sab87, SP87, LX04]. In contrast, the present paper considers a different point of view, which has been

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useful in the study of planar systems induced by continuous differentiable vector fields, not necessarily polynomial [GT95, GS03, GPR06, AGG07, Rab13, PR14]. Thus, the one parameter families are not necessarily perturbations of linear systems, and so it complements the planar results presented in [KKY97].

Throughout this paper, given  $C \subset \mathbb{R}^2$ , a closed (compact, no boundary) curve (1-manifold),  $\overline{D}(C)$  (respectively  $D(C)$ ) denotes the compact disk (respectively open disk) bounded by  $C$ . Thus, the boundaries  $\partial\overline{D}(C)$  and  $\partial D(C)$  are equal to  $C$  besides homeomorphic to the sphere  $\partial D_1 = \{z \in \mathbb{R}^2 : \|z\| = 1\}$ .

**1.1. Statement of the results.** Let  $\overline{D}_\sigma = \{z \in \mathbb{R}^2 : \|z\| \leq \sigma\}$  be the compact disk bounded by the sphere  $\partial\overline{D}_\sigma = \{z \in \mathbb{R}^2 : \|z\| = \sigma\}$  with  $\sigma > 0$ , and suppose that  $X: \mathbb{R}^2 \setminus \overline{D}_\sigma \rightarrow \mathbb{R}^2$  is a differentiable vector field, defined on the usual complement set of  $\overline{D}_\sigma$  on  $\mathbb{R}^2$ . Here the term differentiable means *Frechet differentiable* at each point  $z \in \mathbb{R}^2 \setminus \overline{D}_\sigma$  and  $X'(z): \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is the respective derivative. This derivative is defined as the bounded linear operator induced by the usual Jacobian matrix at  $z$ . This matrix is denoted by  $DX(z)$ , so  $\det(DX)$  and  $\text{Trace}(DX)$  are the typical Jacobian determinant and divergent, respectively. On the other hand, if the point  $q \in \mathbb{R}^2 \setminus \overline{D}_\sigma$  is fixed, then a *trajectory* of  $X$  starting at  $q$  is defined as the integral curve  $I_q \ni t \mapsto \gamma_q(t) \in \mathbb{R}^2 \setminus \overline{D}_\sigma$ , determined by a maximal solution of the Initial Value Problem  $\dot{z} = X(z)$ ,  $z(0) = q$ . Of course, it means that  $\frac{d}{dt}\gamma_q(t) = X(\gamma_q(t))$ ,  $\forall t \in I_q$  and  $\gamma_q(0) = q$ . In this context, it is useful to have a term that refers to the image of the solution. Hence, we define the *orbit* of  $\gamma_q$  to be the set  $\{\gamma_q(t) : t \in I_q\}$ , and we identify the trajectory with its orbit. Similarly,  $\gamma_q^+ = \{\gamma_q(t) : t \in I_q \cap [0, +\infty)\}$  and  $\gamma_q^- = \{\gamma_q(t) : t \in (-\infty, 0] \cap I_q\}$  are the *semi-trajectories* of  $X$ , and they are called *positive* and *negative*, respectively. In consequence, each trajectory has its two limit sets,  $\alpha(\gamma_q^-)$  and  $\omega(\gamma_q^+)$ . These limit sets are well defined in the sense that they only depend on the respective solution. And lastly, such a vector field induces a well defined *positive semi-flow* (respectively *negative semi-flow*), if the condition  $q \in \mathbb{R}^2 \setminus \overline{D}_\sigma$  implies the existence and uniqueness of  $\gamma_q^+$  (respectively  $\gamma_q^-$ ).

The trajectories of a vector field may be unbounded. One way to obtain some information about the behavior of such solutions is to compactify the plane, so that the vector field is extended to a new manifold that contains the ‘points at infinity’. In this way, the so called Alexandroff compactification has been most successful in the study of planar systems induced by continuous differentiable vector fields, not necessarily polynomial (see for example [AGG07, GPR06, Rab13]). To describe the results,  $\mathbb{R}^2$  is embedded in the Riemann sphere  $\mathbb{R}^2 \cup \{\infty\}$ . Consequently,  $(\mathbb{R}^2 \setminus \overline{D}_\sigma) \cup \{\infty\}$  is the subspace of  $\mathbb{R}^2 \cup \{\infty\}$  with the induced topology, and ‘infinity’ refers to the point  $\infty$  of  $\mathbb{R}^2 \cup \{\infty\}$ . Moreover, a vector field  $X: \mathbb{R}^2 \setminus \overline{D}_\sigma \rightarrow \mathbb{R}^2 \setminus \{0\}$  (without singularities) can be extended to a map

$$\hat{X}: ((\mathbb{R}^2 \setminus \overline{D}_\sigma) \cup \{\infty\}, \infty) \longrightarrow (\mathbb{R}^2, 0)$$

(which takes  $\infty$  to 0). In this manner, all questions concerning the local theory of isolated singularities of planar vector fields can be formulated and examined in the case of the extended vector field  $\hat{X}$ , which coincides with  $X$  on  $\mathbb{R}^2 \setminus \overline{D}_\sigma$ . For instance, if  $\gamma_p^+ \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$  is an unbounded semi-trajectory of  $X: \mathbb{R}^2 \setminus \overline{D}_\sigma \rightarrow \mathbb{R}^2$  with empty  $\omega$ -limit set, then we declare that  $\gamma_p^+$  goes to infinity, and we write  $\omega(\gamma_p^+) = \infty$ . Similarly,  $\alpha(\gamma_p^-) = \infty$  denotes that  $\gamma_p^-$  comes from infinity, and it means that  $\gamma_p^- \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$  is an unbounded semi-trajectory whose  $\alpha$ -limit set is empty. Therefore, it is also possible to talk about the phase portrait of  $X$  in a neighborhood of  $\infty$ , as shown [GPR06, Proposition 29].

The point at infinity  $\infty$  of the Riemann Sphere  $\mathbb{R}^2 \cup \{\infty\}$  is an *attractor* (respectively a *repellor*) for the vector field  $X: \mathbb{R}^2 \setminus \overline{D}_\sigma \rightarrow \mathbb{R}^2$  if:

- There is a sequence of closed curves, transversal to  $X$  and tending to infinity. It means that for every  $r \geq \sigma$  there exists a closed curve  $C_r$  such that  $D(C_r)$  contains  $D_r$  and  $C_r$  has transversal contact to each small local integral curve of  $X$  at any  $p \in C_r$ .
- For some  $C_s$  with  $s \geq \sigma$ , all the trajectories  $\gamma_p$  starting at a point  $p \in \mathbb{R}^2 \setminus \overline{D}(C_s)$  satisfy  $\omega(\gamma_p^+) = \infty$ , that is  $\gamma_p^+$  goes to infinity (respectively  $\alpha(\gamma_p^-) = \infty$ , that is  $\gamma_p^-$  comes from infinity).

These definitions were previously presented in the paper [GPR06], and both can be used in the context of a one parameter family of vector fields  $\{X_\mu\} = \{X_\mu: \mathbb{R}^2 \setminus \overline{D}_\sigma \rightarrow \mathbb{R}^2; -\varepsilon < \mu < \varepsilon\}$ . Therefore, as in [AGG07] we will say that  $\{X_\mu\}$  has at  $\mu = 0$  a *Hopf bifurcation at  $\infty$*  if the following two conditions are satisfied:

- For  $\mu < 0$  (resp.  $\mu > 0$ ), the vector field  $X_\mu$  has a repellor at  $\infty$ , and for  $\mu > 0$  (resp.  $\mu < 0$ ), the vector field field  $X_\mu$  has an attractor at  $\infty$ .
- The vector field field  $X_\mu$  has no rest points in  $\mathbb{R}^2 \setminus \overline{D}_s$ , for  $-\varepsilon < \mu < \varepsilon$  and some  $s \geq \sigma$ .

**Definition 1.1.** The differentiable vector field  $X: \mathbb{R}^2 \setminus \overline{D}_\sigma \rightarrow \mathbb{R}^2$  has a *weak* (respectively *strong*) extension at  $\mu \in \mathbb{R}$  as long as there exists a constant  $s_\mu \geq \sigma$  joint to a local homeomorphism (respectively diffeomorphisms)  $\tilde{X}_\mu: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  with  $\tilde{X}_\mu(0) = 0$  such that

$$\|z\| \geq s_\mu \Rightarrow \tilde{X}_\mu(z) = X(z) + \mu z.$$

*Remark 1.2.* Let  $X: \mathbb{R}^2 \setminus \overline{D}_\sigma \rightarrow \mathbb{R}^2$  be a differentiable vector field such that its spectrum  $\text{Spc}(X) = \{\text{eigenvalues of } DX(z): z \in \mathbb{R}^2 \setminus \overline{D}_\sigma\}$  satisfies

$$(1.1) \quad \text{Spc}(X) \subset \left\{ z \in \mathbb{C} : \Re(z) = 0 \right\} \setminus \{(0, 0)\}.$$

For every  $\mu \in \mathbb{R}$ , consider the differentiable vector field  $X_\mu: \mathbb{R}^2 \setminus \overline{D}_\sigma \rightarrow \mathbb{R}^2$  given by  $X_\mu(z) = X(z) + \mu z$ . Therefore, when  $X$  has some singularity, this  $X$  has a

weak extension at  $\mu \in \mathbb{R}$  (see Corollary 3.3). In particular, for every  $\varepsilon_0 > 0$  the set  $\{s_\mu \geq \sigma : \mu \in [-\varepsilon_0, \varepsilon_0]\}$  is bounded when  $\mu \mapsto s_\mu$  is continuous.

By assuming some strong extensions, we describe the mentioned bifurcation at infinity on differentiable vector fields. Let us start with the easiest case.

**Proposition 1.3.** *Consider (1.1), and  $X_\mu: \mathbb{R}^2 \setminus \overline{D}_\sigma \rightarrow \mathbb{R}^2$  as in Remark 1.2. Suppose that  $X$  has some rest point, then the following statements hold:*

- (a)  $X_\mu$  induces a well defined positive (respectively negative) semi-flow as long as  $\mu < 0$  (respectively  $\mu > 0$ ). Moreover, the trajectories of  $X$  are unique in the sense that only depend of the initial condition.
- (b) If there is  $\varepsilon_0 > 0$  such that  $\{s_\mu \geq \sigma : \mu \in [-\varepsilon_0, \varepsilon_0]\}$ , induced by strong extensions, is a bounded set. Then there are  $s > \sigma$  and  $\varepsilon > 0$  such that

$$\{X_\mu: \mathbb{R}^2 \setminus \overline{D}_s \rightarrow \mathbb{R}^2; -\varepsilon < \mu < \varepsilon\}$$

has at  $\mu = 0$  a Hopf bifurcation at  $\infty$ .

Observe that the vector fields  $X_\mu(z) = X(z) + \mu z$  satisfy

$$X_\mu(z) - X_{\tilde{\mu}}(z) = (\mu - \tilde{\mu})z, \quad \forall z \in \mathbb{R}^2 \setminus \overline{D}_\sigma.$$

Therefore,

$$\mathbb{R} \ni \mu \mapsto X_\mu \in \mathfrak{X}(\mathbb{R}^2 \setminus \overline{D}_\sigma)$$

is continuous, if  $\mathfrak{X}(\mathbb{R}^2 \setminus \overline{D}_\sigma)$  is the space of the continuous vector fields of  $\mathbb{R}^2 \setminus \overline{D}_\sigma$ , endowed with the topology induced by the uniform convergence on compact sets.

*Remark 1.4.* If the functions  $h_\mu: \mathbb{R}^2 \setminus \overline{D}_\sigma \rightarrow (0, +\infty)$  and  $\tilde{h}_\mu: \mathbb{R}^2 \setminus \overline{D}_\sigma \rightarrow (-\infty, 0)$  are differentiable and the family  $\{X_\mu: \mathbb{R}^2 \setminus \overline{D}_s \rightarrow \mathbb{R}^2; -\varepsilon < \mu < \varepsilon\}$  is given by Proposition 1.3, the following families

$$h_\mu(z)X_\mu(z) \quad \text{and} \quad \tilde{h}_\mu(z)X_\mu(z)$$

have at  $\mu = 0$  a Hopf bifurcation at  $\infty$ . In particular,

$$h_\mu(z) = \begin{cases} \frac{1}{\mu}, & \text{if } \mu \neq 0; \\ 1, & \text{if } \mu = 0 \end{cases}$$

induces the well-defined map  $\mu \mapsto h_\mu X_\mu$ , discontinuous in zero.

**1.2. The index at infinity.** We also recall that  $\mathcal{I}(X)$ , the *index of  $X$  at infinity* is the number of the extended line  $[-\infty, +\infty]$  given by

$$\mathcal{I}(X) = \int_{\mathbb{R}^2} \text{Trace}(D\widehat{X})dx \wedge dy,$$

where  $\widehat{X}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a global differentiable vector field such that:

- In some  $\mathbb{R}^2 \setminus \overline{D}_s$  with  $s \geq \sigma$ , both  $X$  and  $\widehat{X}$  coincide.

- $z \mapsto \text{Trace}(D\widehat{X}_z)$  is Lebesgue almost-integrable on  $\mathbb{R}^2$ , in the sense of [GPR06].

This index is a well-defined number in  $[-\infty, +\infty]$ , and it does not depend on the pair  $(\widehat{X}, s)$  as shown [GPR06, Lemma 12] (see also [AGG07]).

*Remark 1.5.* If  $X$  satisfies Proposition 1.3. Then Lemma 3.1 gives the existence of some global differentiable vector field  $\widehat{X}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ , satisfying the conditions in the definition of the index at infinity. Furthermore, this extension property remains on every element of the whole family  $X_\mu: \mathbb{R}^2 \setminus \overline{D}_\sigma \rightarrow \mathbb{R}^2$  defined in Proposition 1.3. Consequently,  $\mathcal{I}(X_\mu) \in [-\infty, +\infty]$ .

This description of the index at infinity is motivated by the definitions, methods, arguments and results given in [AGG07]. In this paper the authors research the families of  $C^1$ -vector fields of the form  $\{Z_\mu\} = \{Z_\mu: \mathbb{R}^2 \setminus \overline{D}_\sigma \rightarrow \mathbb{R}^2; -\varepsilon < \mu < \varepsilon\}$ . More precisely, B. Alarcón, V. Guíñez and C. Gutiérrez prove that  $\{Z_\mu\}$  has at  $\mu = 0$  a Hopf bifurcation at  $\infty$  as long as for each  $\mu \in (-\varepsilon, \varepsilon)$  the following conditions are satisfied.

- (1\*)  $z \mapsto \text{Trace}(DZ_\mu(z))$  is Lebesgue almost-integrable on  $\mathbb{R}^2 \setminus \overline{D}_\sigma$ , as in [AGG07].
- (2\*)  $\mathcal{I}(Z_\mu)$  is well defined, this index belongs to  $[-\infty, +\infty]$ , and

$$\int_\sigma^{+\infty} \Upsilon_\mu(r) dr = +\infty,$$

where  $\Upsilon_\mu(r) = \inf \{ \|Z_\mu(z)\| : \|z\| = r \}$ .

- (3\*)  $\mu \neq 0$  implies that the product  $\mu \cdot \mathcal{I}(X_\mu) > 0$ .
- (4\*)  $Z_\mu$  has no singularities, and the *Poincaré index* at  $\infty$  of the vector field

$$\widehat{Z}_\mu : ((\mathbb{R}^2 \setminus \overline{D}_\sigma) \cup \{\infty\}, \infty) \longrightarrow (\mathbb{R}^2, 0)$$

(which extends  $Z_\mu$  at  $\infty$ ) is less than or equal 1.

In the notations of [AGG07], both conditions (1\*) and (2\*) are expressed by assuming that  $\mathcal{A}^1(2, \sigma) \supset \mathcal{A}^1(2, \sigma, \infty) \ni Z_\mu$ .

**1.3. The dissipative vector fields.** Let  $\mathcal{H}(2, \sigma)$  be the set consisting of orientation preserving local diffeomorphisms  $X: \mathbb{R}^2 \setminus \overline{D}_\sigma \rightarrow \mathbb{R}^2$  such that  $X$  is *dissipative* [GS90]. This means that

$$(1.2) \quad \text{Spc}(X) \subset \left\{ z \in \mathbb{C} : \Re(z) \leq 0 \right\} \setminus \{(0, 0)\}.$$

Moreover, the  $C^k$ -vector fields with  $k \geq 1$  define the spaces

$$\mathcal{H}^k(2, \sigma) = \{X \in \mathcal{H}(2, \sigma) : X \text{ is of class } C^k\},$$

which naturally satisfy  $\mathcal{H}^{k+1}(2, \sigma) \subset \mathcal{H}^k(2, \sigma) \subset \dots \subset \mathcal{H}^1(2, \sigma) \subset \mathcal{H}(2, \sigma)$ .

A vector field  $X \in \mathcal{H}(2, \sigma)$  induces the family  $X_\mu(z) = X(z) + \mu z$  with  $\mu \in \mathbb{R}$ . In the case that  $X$  has some singularity, Lemma 3.1 implies that  $X$  has a weak extension at  $\mu \leq 0$ . Moreover, by Remark 3.4 the condition

$$(1.3) \quad \det(DX_\mu) > 0, \text{ for all } \mu \text{ in some interval } (0, \epsilon_0)$$

implies the existence of some  $\epsilon > 0$  such that  $X$  satisfies some weak extensions, for every  $\mu \in [-\epsilon, \epsilon]$ . Thus,  $\{s_\mu \geq \sigma : \mu \in [-\epsilon, \epsilon]\}$  is a well defined set.

**Theorem 1.6.** *Consider  $X \in \mathcal{H}(2, \sigma)$  with some singular point and the family  $X_\mu(z) = X(z) + \mu z$  such that  $\mu \in \mathbb{R}$  and (1.3) holds. Suppose that*

- (1) *There is  $\varepsilon_0 > 0$  such that the set  $\{s_\mu \geq \sigma : \mu \in [-\varepsilon_0, \varepsilon_0]\}$ , induced by strong extensions is bounded.*
- (2)  *$X_\mu$  induces a well defined negative semi-flow as long as  $\mu > 0$ , and*
- (3) *The index  $\mathcal{I}(X_\mu) \in [-\infty, +\infty]$  satisfies  $\mu \cdot \mathcal{I}(X_\mu) > 0$ , if  $\mu > 0$ .*

*Then there are  $\varepsilon > 0$  and  $s > \sigma$  such that the family of vector fields*

$$\left\{ X_\mu : \mathbb{R}^2 \setminus \overline{D}_s \rightarrow \mathbb{R}^2; -\varepsilon < \mu < \varepsilon \right\}$$

*has at  $\mu = 0$  a Hopf bifurcation at  $\infty$ .*

Clearly, a similar result as described in Remark 1.4 is obtained in the general case of dissipative vector fields and families given by Theorem 1.6. In some sense, it includes the results where the strong domination imposed by the linear part is used.

Observe that we do not assume the existence of some open neighborhood of infinity free of singularities, as in (4\*). Similarly, the existence of the index is not assumed, as in (2\*); these results are obtained from the weak hypothesis on the Jacobian determinant. This condition, (1) is natural in order to work with isolated singular points. The assumption (2) is necessary since the theorem considers the vector fields not necessarily of class  $C^1$ .

In the particular case of continuous differentiable vector fields, Theorem 1.6 directly gives the next illustrative result.

**Corollary 1.7.** *If  $Z \in \mathcal{H}^1(2, \sigma)$  has a singular point, and the maps  $\mathbb{R}^2 \setminus \overline{D}_\sigma \ni z \mapsto Z_\mu(z) = Z(z) + \mu z$  are orientation preserving local diffeomorphisms, for all  $\mu$  in some interval  $(0, \epsilon_0)$ , where (1) in Theorem 1.6 holds. Then the condition*

$$\mu \neq 0 \Rightarrow \mu \cdot \mathcal{I}(Z_\mu) > 0,$$

*implies the existence of  $\varepsilon > 0$  and  $s > \sigma$  such that  $\left\{ Z_\mu : \mathbb{R}^2 \setminus \overline{D}_s \rightarrow \mathbb{R}^2; -\varepsilon < \mu < \varepsilon \right\}$  has at  $\mu = 0$  a Hopf bifurcation at  $\infty$ .*

Section 2 is devoted to present some preliminary results, on planar vector fields defined on the whole plane. Section 3 includes the proofs of Proposition 1.3 and Theorem 1.6.



## 2. GLOBAL VECTOR FIELDS

In this section we describe some families of global vector fields, and we present sufficient conditions in order to have a Hopf bifurcation at infinity. In the next proposition,  $Y(0) = 0$  means that the origin  $0 = (0, 0)$  is a singular point of  $Y$ .

**Proposition 2.1.** *Let  $Y : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a differentiable vector field whose Jacobian determinant is always positive with  $Y(0) = 0$ . Suppose that:*

- (1) *There is  $\sigma > 0$  such that the restriction  $Y| : \mathbb{R}^2 \setminus \overline{D}_\sigma \rightarrow \mathbb{R}^2$  is dissipative.*
- (2) *The maps  $Y_\mu(z) = Y(z) + \mu z$  are locally diffeomorphisms on  $\mathbb{R}^2$ , for all  $\mu \in (-\varepsilon_0, \varepsilon_0)$  and some  $\varepsilon_0 > 0$ .*

*Then there are  $s > \sigma$ , and  $\varepsilon \in (0, \varepsilon_0)$  such that for every  $\mu \in [-\varepsilon, \varepsilon]$ :*

- (a) *The restriction  $Y_\mu| : \mathbb{R}^2 \setminus \overline{D}_s \rightarrow \mathbb{R}^2$  has no singularities in  $\mathbb{R}^2 \setminus \overline{D}_s$ .*
- (b) *If  $Y_\mu$  induces a well-defined negative semi-flow on  $\mathbb{R}^2 \setminus \overline{D}_s$  for each  $\mu > 0$ , and  $\mathcal{I}(Y_u) > 0$  (respectively  $\mathcal{I}(Y_u) < 0$ ). Then  $\infty$  is an attractor (respectively a repeller) for the vector field  $Y_\mu| : \mathbb{R}^2 \setminus \overline{D}_s \rightarrow \mathbb{R}^2$ .*

*Proof.* A direct computation of the eigenvalues shows that

$$(2.1) \quad \text{Spc}(Y_\mu) = \mu + \text{Spc}(Y).$$

Thus, the following assumption  $0 \notin \text{Spc}(Y_\mu)$ , for all  $\mu \in [0, \varepsilon_0)$  – is equivalent to say that the intersection  $\text{Spc}(Y) \cap (-\varepsilon_0, 0] = \emptyset$ . Therefore, (2.1) and the definition of  $Y_\mu$  show that for each  $-\frac{\varepsilon_0}{2} \leq \mu \leq \frac{\varepsilon_0}{2}$  the following holds:

$$\{\text{Eigenvalues of } DY_\mu(z) : \|z\| \geq \sigma\} \cap \left(-\frac{\varepsilon_0}{4}, \frac{\varepsilon_0}{4}\right) = \emptyset,$$

$$\det(DY_\mu(z)) > 0, \quad \forall z \in \mathbb{R}^2.$$

Under these conditions, [GR06, Theorem 2.1] let us to obtain that

$$(2.2) \quad Y_\mu : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad \text{is globally injective, for every } \mu \in \left[-\frac{\varepsilon_0}{2}, \frac{\varepsilon_0}{2}\right]$$

(see also [Rab10, Theorem 3]). Since  $Y_\mu(0) = 0$ , (2.2) implies that

- (a.1) *There is  $\varepsilon = \frac{\varepsilon_0}{2} > 0$  such that  $Y_\mu| : \mathbb{R}^2 \setminus \overline{D}_\sigma \rightarrow \mathbb{R}^2$  has no singularities in  $\mathbb{R}^2 \setminus \overline{D}_\sigma$ , for every  $\mu \in [-\varepsilon, \varepsilon]$ .*

Thus, (a) holds.

- (b.1) *We claim that there is  $s > \sigma$  such that  $\mathbb{R}^2 \setminus \overline{D}_s$  has no periodic trajectories of  $Y_\mu$  as long as  $\mu \in (-\varepsilon, \varepsilon)$  and  $\mathcal{I}(Y_u) \neq 0$ .*

To obtain (b.1) we proceed by contradiction. Since  $Y_\mu(0) = 0$  and  $Y_\mu$  is injective:

- (\*) *We suppose that  $\{\Gamma_1, \Gamma_2, \dots, \Gamma_n, \dots\}$  is an unbounded set of periodic trajectories of  $Y_\mu$  such that*

$$\overline{D}(\Gamma_1) \subset \overline{D}(\Gamma_2) \subset \dots \subset \overline{D}(\Gamma_n) \subset \dots$$

Under these conditions, by using the Green Theorem in  $\overline{D}(\Gamma_n)$  and the arc length element  $ds$ , we obtain that

$$\int_{\overline{D}(\Gamma_n)} \text{Trace}(DY_\mu) dx \wedge dy = \oint_{\Gamma_n} \langle Y_\mu(s), \eta_n^e(s) \rangle ds,$$

where  $\eta_n^e(p)$  the unitary outer normal vector to  $\Gamma_n$  and  $\langle Y_\mu(s), \eta_n^e(s) \rangle$  is the inner product of  $Y_\mu(s)$  with  $\eta_n^e(s)$ . Thus

$$\mathcal{I}(Y_\mu) = \lim_{n \rightarrow \infty} \int_{\overline{D}(\Gamma_n)} \text{Trace}(DY_\mu) dx \wedge dy = 0,$$

because  $\langle Y_\mu(s), \eta_n^e(s) \rangle = 0$  for al  $n$ . This contradiction with  $\mathcal{I}(Y_u) \neq 0$  shows that (\*) never happens, and gives the proof of (b.1).

For  $s > \sigma$  as in (b.1) and  $\varepsilon > 0$  satisfying (a.1), we have that  $\mu \in (-\varepsilon, \varepsilon)$  implies that:

- (b.2)  $z \mapsto \text{Trace}(DY_\mu(z))$  is Lebesgue almost-integrable in  $\mathbb{R}^2$  ([GPR06, Lemma 7]).
- (b.3)  $\mathcal{I}(Y_\mu)$  is a well-defined number in  $[-\infty, +\infty)$  ([GPR06, Corollary 13]).
- (b.4)  $Y_\mu$  generates a positive semi-flow on  $\mathbb{R}^2 \setminus \overline{D}_s$ , when  $\mu < 0$  ([FGR04, Lemma 3.1]).

Under these properties: (2.2) and (b.1) to (b.4), the vector field  $Y_\mu$  satisfies all the conditions of [GPR06, Theorem 26]. Consequently:

- (b.5) For every  $r \geq s$  there exist a closed curve  $C_r$  transversal to  $Y_\mu$  contained in the regular set  $\mathbb{R}^2 \setminus \overline{D}_r$ . In particular,  $D(C_r)$  contains  $D_r$  and  $C_r$  has transversal contact to each small local integral curve of  $Y$  at any  $p \in C_r$ .

Moreover, for  $r \geq s$  large enough the closed curve  $C_r \subset \mathbb{R}^2$  is transversal to  $Y_\mu$ , near  $\infty$  and it is such that  $\mu \in (-\varepsilon, \varepsilon)$ , the index  $\mathcal{I}(Y_\mu)$  and

$$\int_{\overline{D}(C_r)} \text{Trace}(DY_\mu) dx \wedge dy = \oint_{C_r} \langle Y_\mu(s), \eta_n^e(s) \rangle ds$$

have the same sign. Thus, the point at infinity of the Riemann Sphere  $\mathbb{R}^2 \cup \{\infty\}$  is either an attractor or a repellor of  $Y_\mu: \mathbb{R}^2 \setminus \overline{D}_s \rightarrow \mathbb{R}^2$ . Therefore, (b.5) and [GPR06, Theorem 28] shown that:

- (b.6) If  $\mathcal{I}(Y_\mu) < 0$  (respectively  $\mathcal{I}(Y_\mu) > 0$ ), then  $\infty$  is a repellor (respectively an attractor) of the vector field  $Y_\mu: \mathbb{R}^2 \setminus \overline{D}_s \rightarrow \mathbb{R}^2$ .

It concludes the proof of (b), and the proposition holds.  $\square$

### 3. EXTENSIONS AND HOPF BIFURCATION AT INFINITY

This section includes the proofs of the main results. To this end, we describe the existence of globally injective extensions obtained from spectral conditions.

**3.1. On the extension of dissipative vector fields.** Recall that the dissipative vector fields define

$$\mathcal{H}(2, \sigma) = \left\{ X : \mathbb{R}^2 \setminus \overline{D}_\sigma \rightarrow \mathbb{R}^2 : \text{Spc}(X) \subset \{z \in \mathbb{C} : \Re(z) \leq 0\} \setminus \{(0, 0)\} \right\},$$

where  $\text{Spc}(X) = \{\text{Eigenvalues of } DX(z) : z \in \mathbb{R}^2 \setminus \overline{D}_\sigma\}$ . Thus,  $\det(DX) > 0$ .

**Lemma 3.1.** *Suppose that  $X \in \mathcal{H}(2, \sigma)$  has some singularity, then there are  $s_0 \geq \sigma$  and a globally injective local homeomorphism  $\tilde{X} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  with  $\tilde{X}(0) = 0$  such that*

- (a)  $\tilde{X}$  and  $X$  coincide on  $\mathbb{R}^2 \setminus \overline{D}_{s_0}$ .
- (b) The restriction  $X| : \mathbb{R}^2 \setminus \overline{D}_{s_0} \rightarrow \mathbb{R}^2$ 
  - is injective, and
  - admits a global differentiable extension  $\hat{X}$  with  $\hat{X}(0) = 0$  such that the pair  $(\hat{X}, s_0)$  satisfies the definition of the index of  $X$  at infinity, and this index  $\mathcal{I}(X) \in [-\infty, +\infty)$ .

In particular,  $X : \mathbb{R}^2 \setminus \overline{D}_\sigma \rightarrow \mathbb{R}^2$  has a weak extension at  $\mu = 0$ .

*Proof.*  $X = (f, g) \in \mathcal{H}(2, \sigma)$  satisfies  $\text{Spc}(X) \cap [0, +\infty) = \emptyset$  and  $\text{Spc}(X) \subset \{z \in \mathbb{C} : \Re(z) \leq 0\}$ . Thus, the methods of [Rab13, Proposition 2] give us that

- (a.1) Any half-Reeb component of either  $\mathcal{F}(f)$  or  $\mathcal{F}(g)$  is bounded.

Here,  $\mathcal{F}(h)$  with  $h \in \{f, g, \tilde{f}, \tilde{g}\}$  denotes the continuous foliation given by the level sets  $\{h = \text{constant}\}$  and then the leaves of the foliations are differentiable. The concept of half-Reeb component is a natural generalization of the description of the planar foliations given in [HR57]. More precisely,  $\mathcal{A}$  is a half-Reeb component of  $\mathcal{F}(h)$  if there is a homeomorphism

$$H : B = \left\{ (x, y) \in [0, 2] \times [0, 2] : 0 < x + y \leq 2 \right\} \rightarrow \mathcal{A},$$

which is a topological equivalence between  $\mathcal{F}(h)|_{\mathcal{A}}$  and  $\mathcal{F}(h_0)|_B$  with  $h_0(x, y) = xy$  such that:

- The segment  $\{(x, y) \in B : x + y = 2\}$  is sent by  $H$  onto a transversal section for the foliation  $\mathcal{F}(h)$  in the complement of the point  $H(1, 1)$ .
- Both segments  $\{(x, y) \in B : x = 0\}$  and  $\{(x, y) \in B : y = 0\}$  are sent by  $H$  onto full half-leaves of  $\mathcal{F}(h)$ .

Under the conditions (a.1) and  $\text{Spc}(X) \cap [0, +\infty) = \emptyset$ , we can apply the last section of [GR06] (see Proposition 5.1). Consequently, the proof of [Rab13, Theorem 3] gives the existence of a closed curve  $C$ , surrounding the singularity of  $X$  joint to the origin so that  $C$  is embedded in the plane, and it admits an exterior collar neighborhood  $U \subset \mathbb{R}^2 \setminus D(C)$  such that:

- (a.2)  $X(C)$  is a non-trivial closed curve,  $X(U)$  is an exterior collar neighborhood of  $X(C)$  and the restriction  $X| : U \rightarrow X(U)$  is a homeomorphism.
- (a.3) The foliation  $\mathcal{F}(f)$ , restricted to  $\mathbb{R}^2 \setminus D(C)$ , is topologically equivalent to the foliation made up by all the vertical straight lines on  $\mathbb{R}^2 \setminus D_1$ .

By Schoenflies Theorem [Jac68, Bin83, Zib05] the map  $X|: C \rightarrow X(C)$  can be extended to a homeomorphism  $X_1: \overline{D}(C) \rightarrow \overline{D}(X(C))$  with  $X_1(0) = 0$ . We extend  $X: \mathbb{R}^2 \setminus D(C) \rightarrow \mathbb{R}^2$  to  $\tilde{X} = (\tilde{f}, \tilde{g}): \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by defining  $\tilde{X}|_{\overline{D}(C)} = X_1$ . Thus  $\tilde{X}|: U \rightarrow X(U)$  is a homeomorphism and  $U$  (resp.  $X(U)$ ) is a exterior collar neighborhood of  $C$  (resp.  $X(C)$ ). Moreover, (a.3) implies that  $\tilde{X}$  is a local homeomorphism whose foliation  $\mathcal{F}(\tilde{f})$  is trivial. Therefore,  $\tilde{X}$  is injective [FGR04, Proposition 1.4], and we obtain that

(a.4) There are  $s_1 \geq \sigma$  and a globally injective local homeomorphism  $\tilde{X} = (\tilde{f}, \tilde{g}): \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $\tilde{X}(0) = 0$  and  $\tilde{X}$  and  $X$  coincide on  $\mathbb{R}^2 \setminus \overline{D}_{s_1}$ .

Under these conditions [GPR06, Theorem 11] gives the existence of some  $s_0 > s_1$  such that the restriction  $X|: \mathbb{R}^2 \setminus \overline{D}_{s_0} \rightarrow \mathbb{R}^2$  admits a global differentiable extension  $\hat{X}$  with  $\hat{X}(0) = 0$  such that the pair  $(\hat{X}, s_0)$  satisfies the definition of the index of  $X$  at infinity, and this index  $\mathcal{I}(X) \in [-\infty, +\infty)$ . This concludes the proof.  $\square$

**Corollary 3.2.** *If  $X \in \mathcal{H}(2, \sigma)$  has some singularity, and  $\mu \leq 0$ , then there are  $s_\mu \geq \sigma$ , and an injective local homeomorphims  $\tilde{X}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  with  $\tilde{X}(0) = 0$  such that*

$$\tilde{X}(z) = X(z), \quad \forall z \in \mathbb{R}^2 \setminus \overline{D}_{s_\sigma}$$

and  $X|: \mathbb{R}^2 \setminus \overline{D}_{s_\sigma} \rightarrow \mathbb{R}^2$  satisfies the statement (b) in Lemma 3.1. In particular,  $X|: \mathbb{R}^2 \setminus \overline{D}_{s_\sigma} \rightarrow \mathbb{R}^2$  has a weak extension at  $\mu \leq 0$ . Moreover,

$$\mu < 0 \Rightarrow \mathcal{I}(X_\mu) \leq \mu < 0.$$

*Proof.* Recall that  $X_\mu(z) = X(z) + \mu z$ . Since  $\text{Spc}(X_\mu) = \mu + \text{Spc}(X)$  we have

$$(3.1) \quad \text{Spc}(X_\mu) \subset \{z \in \mathbb{C} : \Re(z) \leq \mu\}.$$

Therefore, for every  $\mu \leq 0$ , keep fixed the weak extension is directly obtained by Lemma 3.1 joint to Definition 1.1.

The last part is obtained by (3.1), and the definition of index at infinity of  $X_\mu$ , because

$$\text{Trace}(DX_\mu) \leq \mu < 0.$$

Therefore, this corollary holds.  $\square$

**Corollary 3.3.** *If  $X: \mathbb{R}^2 \setminus \overline{D}_\sigma \rightarrow \mathbb{R}^2$  has some singularity, and it satisfies (1.1), then there are  $s_\mu \geq \sigma$  and an homeomorphims  $\tilde{X}: \mathbb{R}^2 \rightarrow \tilde{X}(\mathbb{R}^2)$  with  $\tilde{X}(0) = 0$  such that  $X|_{\mathbb{R}^2 \setminus \overline{D}_{s_\sigma}} = \tilde{X}|_{\mathbb{R}^2 \setminus \overline{D}_{s_\sigma}}$  satisfies Corollary 3.2. Thus  $X|: \mathbb{R}^2 \setminus \overline{D}_\sigma \rightarrow \mathbb{R}^2$  has a weak extension at  $\mu \in \mathbb{R}$ .*

*Proof.* Since  $X_\mu(z) = X(z) + \mu z$  implies that  $\text{Spc}(X_\mu) \cap \mathbb{R} = \emptyset$  for all  $\mu \in \mathbb{R}$ . Proposition 2.7 in [GR06], implies that  $X_\mu = (f_\mu, g_\mu)$  has the following property:

(a.1) Any half-Reeb component of either  $\mathcal{F}(f_\mu)$  or  $\mathcal{F}(g_\mu)$  is a bounded set.

Therefore, the proof of Lemma 3.1 holds verbatim.  $\square$

*Remark 3.4.* The vector fields  $X \in \mathcal{H}(2, \sigma)$ , as in Lemma 3.1 naturally define  $X_\mu(z) = X(z) + \mu z$ . Moreover, when the condition (1.3) holds, it is not difficult to see that (2.1) implies that

$$\text{Spc}(X_\mu) \cap \left(-\frac{\epsilon_0}{4}, +\infty\right) = \emptyset, \quad \text{when } \mu \in \left[-\frac{\epsilon_0}{4}, \frac{\epsilon_0}{4}\right].$$

Consequently, the last proof holds verbatim, and gives the existence of some  $s_\mu \geq \sigma$  such that  $X: \mathbb{R}^2 \setminus \overline{D}_{s_\mu} \rightarrow \mathbb{R}^2$  satisfies the conclusions of Lemma 3.1. In particular, there is  $\epsilon = \frac{\epsilon_0}{4} > 0$  such that  $X$  satisfies the conditions on weak extensions of Definition 1.1, for every  $\mu \in [-\epsilon, \epsilon]$ .

**3.2. Vector fields free of real eigenvalues.** This subsection is devoted to present the proof of Proposition 1.3. This proposition deals with vector fields whose Jacobian matrix is free of real eigenvalues. These vector fields induce local diffeomorphisms with zero divergence. Let us recall the mentioned proposition.

**Proposition 1.3.** *Let  $X: \mathbb{R}^2 \setminus \overline{D}_\sigma \rightarrow \mathbb{R}^2$  be a differentiable vector field such that*

$$\text{Spc}(X) \subset \left\{z \in \mathbb{C} : \Re(z) = 0\right\} \setminus \{(0, 0)\}.$$

*Consider  $X_\mu: \mathbb{R}^2 \setminus \overline{D}_\sigma \rightarrow \mathbb{R}^2$  given by  $X_\mu(z) = X(z) + \mu z$ , and suppose that  $X$  has some singularity, then the following statements hold:*

- (a)  $X_\mu$  induces a well defined positive (resp. negative) semi-flow as long as  $\mu < 0$  (resp.  $\mu > 0$ ). Moreover, the trajectories of  $X$  are unique in the sense that only depend of the initial condition.
- (b) If there is  $\epsilon_0 > 0$  such that the set  $\{s_\mu \geq \sigma : \mu \in [-\epsilon_0, \epsilon_0]\}$ , induced by strong extensions, is bounded. Then there are  $s > \sigma$  and  $\epsilon > 0$  such that

$$\{X_\mu: \mathbb{R}^2 \setminus \overline{D}_s \rightarrow \mathbb{R}^2; -\epsilon < \mu < \epsilon\}$$

*has at  $\mu = 0$  a Hopf bifurcation at  $\infty$ .*

*Proof.* By (2.1), it is not difficult to see that:

$$\begin{aligned} \mu < 0 &\Rightarrow \text{Spc}(X_\mu) \subset \{z \in \mathbb{C} : \Re(z) \leq \mu < 0\}, \\ \mu > 0 &\Rightarrow \text{Spc}(-X_\mu) \subset \{z \in \mathbb{C} : \Re(z) \leq -\mu < 0\}. \end{aligned}$$

In this context, [FGR04, Lemma 3.3] implies that the vector fields  $X_\mu$  with  $\mu < 0$ , and  $-X_\mu$  with  $\mu > 0$  induce a well defined positive semi-flow. Similarly, [Rab09, Lemma 2.1] gives the uniqueness of the trajectories induced by  $X$ . Therefore, statement (a) is true.

In order to prove the second statement, we consider

$$\tilde{s} = \sup\{s_\mu \geq \sigma : -\epsilon_0 \leq \mu \leq \epsilon_0\}.$$

By Definition 1.1 and the assumptions in (b), there exists a local diffeomorphism  $\widehat{X}_0: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  with  $\widehat{X}_0(0) = 0$  such that

$$(3.2) \quad \widehat{X}_0(z) + \mu z = X_\mu(z), \quad \text{for all } z \in \mathbb{R}^2 \setminus \overline{D}_{\tilde{s}},$$

and

$$(3.3) \quad \mathbb{R}^2 \ni z \mapsto \widehat{X}_0(z) + \mu z \in \mathbb{R}^2 \quad \text{is a local diffeomorphism,}$$

for every  $\mu \in (-\epsilon_0, \epsilon_0)$ . Under these conditions, Proposition 2.1 gives the existence of  $s \geq \tilde{s} \geq \sigma$ , and  $\varepsilon \in (0, \epsilon_0)$  such that for every  $\mu \in [-\varepsilon, \varepsilon]$ :

- The restriction  $X_\mu|: \mathbb{R}^2 \setminus \overline{D}_s \rightarrow \mathbb{R}^2$  has no rest points in  $\mathbb{R}^2 \setminus \overline{D}_s$ .
- If  $\mathcal{I}(X_u) > 0$  (resp.  $\mathcal{I}(X_u) < 0$ ). Then  $\infty$  is an attractor (resp. a repeller) for the vector field  $X_\mu|: \mathbb{R}^2 \setminus \overline{D}_s \rightarrow \mathbb{R}^2$ .

Therefore,  $\{X_\mu: \mathbb{R}^2 \setminus \overline{D}_s \rightarrow \mathbb{R}^2; -\varepsilon < \mu < \varepsilon\}$  has at  $\mu = 0$  a Hopf bifurcation at  $\infty$ , and this proposition holds.  $\square$

*Remark 3.5.* In the special case of continuous differentiable vector fields on the plane  $Y: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $Y(0) = 0$  and

$$\text{Spc}(Y) \subset \left\{ z \in \mathbb{C} : \Re(z) = 0 \right\} \setminus \{(0, 0)\}.$$

The trajectories in  $\mathbb{R}^2 \setminus \{0\}$  induced by  $Y$  are periodic orbits surrounding the origin [Rab09, PR14]. Moreover, a direct application of the last proposition gives the existence of some  $\varepsilon > 0$ , for which the family  $\{Y_\mu: \mathbb{R}^2 \setminus \overline{D}_s \rightarrow \mathbb{R}^2; -\varepsilon < \mu < \varepsilon\}$  has at  $\mu = 0$  a Hopf bifurcation at  $\infty$ .

By applying the methods of the last proof give us the next:

**3.3. Proof of Theorem 1.6.** By the assumption (1) there exists  $\tilde{s} = \sup\{s_\mu \geq \sigma : -\epsilon_0 \leq \mu \leq \epsilon_0\}$ , and there exists a local diffeomorphism  $\widehat{X}_0: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  with  $\widehat{X}_0(0) = 0$  such that (3.2) and (3.3) hold, for every  $\mu \in (-\epsilon_0, \epsilon_0)$ . Under these results, Proposition 2.1 and Corollary 3.2 imply the existence of  $s \geq \tilde{s} \geq \sigma$ , and  $\varepsilon \in (0, \epsilon_0)$  such that for every  $\mu \in [-\varepsilon, \varepsilon]$ :

- The restriction  $X_\mu|: \mathbb{R}^2 \setminus \overline{D}_s \rightarrow \mathbb{R}^2$  has no rest points in  $\mathbb{R}^2 \setminus \overline{D}_s$ .
- If  $\mathcal{I}(X_u) > 0$  (resp.  $\mathcal{I}(X_u) < 0$ ). Then  $\infty$  is an attractor (resp. a repeller) for the vector field  $X_\mu|: \mathbb{R}^2 \setminus \overline{D}_s \rightarrow \mathbb{R}^2$ .

Therefore,  $\{X_\mu: \mathbb{R}^2 \setminus \overline{D}_s \rightarrow \mathbb{R}^2; -\varepsilon < \mu < \varepsilon\}$  has at  $\mu = 0$  a Hopf bifurcation at  $\infty$ . Theorem 1.6 holds.

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B. ALARCÓN

UNIVERSIDADE FEDERAL FLUMINENSE  
 INSTITUTO DE MATEMATICA E ESTATÍSTICA  
 RUA MARIO SANTOS BRAGA S/N  
 24020-140 NITEROI, RJ, BRASIL  
*E-mail address:* balarcon@id.uff.br

R. RABANAL

DEPARTAMENTO DE CIENCIAS  
 PONTIFICIA UNIVERSIDAD CATÓLICA DEL PERÚ  
 AV. UNIVERSITARIA 1801  
 LIMA 32; PERÚ  
*E-mail address:* rrabanal@pucp.edu.pe





