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Comparison theorems for support functions of hypersurfaces

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COMPARISON THEOREMS FOR SUPPORT FUNCTIONS OF HYPERSURFACES

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ABSTRACT. For a convex domain D that is enclosed by the hypersurface ∂D of bounded normal curvature, we prove an angle comparison theorem for angles between ∂D and geodesic rays starting from some fixed point in D, and the corresponding angles for hypersurfaces of constant normal curvature. Also, we obtain a comparison theorem for support functions of such surfaces. As a corollary, we present a proof of Blaschke's Rolling Theorem.

1. PRELIMINARIES AND THE MAIN RESULTS

Is it known the following theorem due to W. Blaschke:

Blaschke's Rolling Theorem. *Let $\mathbb{M}^m(c)$ be an m-dimensional space of constant curvature equal to c, $D \subset \mathbb{M}^m(c)$ be a convex body with the C^r-smooth boundary ∂D ($r \geqslant 2$), and $P \in \partial D$ be an arbitrary point. Let $\partial D_\lambda \subset \mathbb{M}^m(c)$ be a complete hypersurface of constant normal curvature equal to some $\lambda > 0$, and suppose that ∂D_λ touches ∂D at P so that their inner unit normals coincide.*

A. If normal curvatures k_{n} of the hypersurface ∂D at all points and in all directions satisfy the inequality $k_{\mathrm{n}} \geqslant \lambda$, then ∂D lies entirely in the closed convex domain bounded by ∂D_λ.

B. If normal curvatures of the hypersurface ∂D at all points and in all directions satisfy the inequality $\lambda \geqslant k_{\mathrm{n}}$, then the hypersurface ∂D_λ lies in D.

Moreover, the hypersurfaces ∂D and ∂D_λ can intersect only by a domain that contains the point P.

For the Euclidean space this theorem was first proved in [1]; for the general case of constant curvature spaces see [2, 3, 4].

It appears that Blaschke's Rolling Theorem can be obtained as a corollary from the following comparison theorems for angles between the radius-vector of a hypersurface and its normals. In order to give exact statements, we need to agree on some notations.

Everywhere below let $\mathbb{M}^m$ be a complete simply-connected m-dimensional Riemannian manifold such that its sectional curvatures K_σ in a direction of a 2-plane $\sigma \subset T\mathbb{M}^m$ satisfy the inequality $c_2 \geqslant K_\sigma \geqslant c_1$ with some constants c_1 and c_2.

Furthermore, let $D \subset \mathbb{M}^m$ be a closed domain with the boundary ∂D being a C^r-smooth hypersurface ($r \geqslant 2$). For $c_2 > 0$ we will additionally assume that the domain D lies inside a geodesic sphere of radius $\pi/(2\sqrt{c_2})$.

By $t_Q(\cdot) = \mathrm{dist}(Q, \cdot)$ denote a distance function from some point $Q \in D$ defined on $\mathbb{M}^m \backslash \{Q\}$, and let ∂_{t_Q} be a gradient vector field of the function t_Q, and ρ_Q be a restriction of t_Q on ∂D: $\rho_Q(\cdot) = t_Q(\cdot)|_{\partial D}$.

Theorem 1. *Suppose $D \subset \mathbb{M}^m$ and $D_{k_1} \subset \mathbb{M}^m(c_1)$ are closed domains such that normal curvatures k_n of the hypersurface ∂D at any point and in any direction with respect to the inner unit normal field N satisfy the inequality*

$$k_\mathrm{n} \geqslant k_1 > 0,$$

and normal curvatures of ∂D_{k_1} are constant and equal to k_1 with respect to the inner unit normal field N_1. Let $O \in D$ and $O_1 \in D_{k_1}$ be points with $\mathrm{dist}(O, \partial D) = \mathrm{dist}(O_1, \partial D_{k_1})$; then at all points $P \in \partial D$ and $P_1 \in \partial D_{k_1}$ such that

$$\rho_O(P) = \rho_{O_1}(P_1),$$

the inequality

(1.1) $$|\langle N, \partial_{t_O} \rangle|\,(P) \geqslant \left|\langle N_1, \partial_{t_{O_1}} \rangle\right|\,(P_1)$$

holds.

Recall that a function $h_Q \colon \partial D \to (0, +\infty)$ defined as

$$h_Q = \rho_Q \cdot |\langle N, \partial_{t_Q} \rangle|$$

is called a *support function* of the hypersurface $\partial D \subset \mathbb{M}^m$ with respect to a point $Q \in D$ (see [5, chapter 6, §5]).

Using Theorem 1 we can obtain the following comparison theorem for support functions.

Theorem 2. *Let $D \subset \mathbb{M}^m$ and $D_{k_1} \subset \mathbb{M}^m(c_1)$ be closed domains such that normal curvatures k_n of the hypersurface ∂D satisfy the inequality*

$$k_\mathrm{n} \geqslant k_1 > 0,$$

and normal curvatures of ∂D_{k_1} are constant and equal to k_1. Let $O \in D$ and $O_1 \in D_{k_1}$ be points with $\mathrm{dist}(O, \partial D) = \mathrm{dist}(O_1, \partial D_{k_1})$; then at all points $P \in \partial D$ and $P_1 \in \partial D_{k_1}$ such that $\rho_O(P) = \rho_{O_1}(P_1)$, the inequality

$$h_O(P) \geqslant h_{O_1}(P_1).$$

holds.

For Theorems 1 and 2 also holds the following dual result.

Theorem 3. *Suppose $D \subset \mathbb{M}^m$ and $D_{k_2} \subset \mathbb{M}^m(c_2)$ are closed domains such that normal curvatures k_n of the hypersurface ∂D with respect to the inner unit normal field N satisfy the inequality*

$$k_2 \geqslant k_\mathrm{n} > 0,$$

and normal curvatures of ∂D_{k_2} are constant and equal to k_2 with respect to the inner unit normal field N_2. Let $O \in D$ and $O_2 \in D_{k_2}$ be points with $\mathrm{dist}(O, \partial D) = \mathrm{dist}(O_2, \partial D_{k_2})$; then at all points $P \in \partial D$ and $P_2 \in \partial D_{k_2}$ for which the distances $\rho_O(P)$ and $\rho_{O_2}(P_2)$ are equal, the inequalities

$$\left| \langle N_2, \partial_{t_{O_2}} \rangle \right| \geqslant \left| \langle N, \partial_t \rangle \right|,$$
$$h_{O_2} \geqslant h_O$$

hold.

Remark 1. Actually, in Theorem 3 we need only the weaker restriction $c_2 \geqslant K_\sigma$ on sectional curvatures of the manifold $\mathbb{M}^m$.

Remark 2. Theorems 1 – 3 will remain true if we replace the convex domain D with a star-shaped domain of normal curvatures bounded above or below by a non-zero number λ.

2. PROOF OF THEOREM 1

In this section we will prove Theorem 1 using the similar technique as in [6], but our proof will be shorter.

Let $Q \in \partial D$ and $Q_1 \in \partial D_{k_1}$ be points such that $\mathrm{dist}(O, \partial D) = t_O(Q)$ and $\mathrm{dist}(O_1, \partial D_{k_1}) = t_{O_1}(Q_1)$. By d denote the distance $t_O(Q) = t_{O_1}(Q_1)$. Observe that inequality (1.1) holds at Q and Q_1.

In the manifolds $\mathbb{M}^m$ and $\mathbb{M}^m(c_1)$ let us introduce polar coordinate systems with origins, respectively, at O and O_1. By hypothesis of the theorem, both hypersurfaces lie in the regularity regions of these systems of coordinates. Moreover, since the second fundamental forms of ∂D and ∂D_{k_1} are positively defined, the hypersurfaces bound the convex regions. Thus they both can be explicitly defined in the introduced coordinate systems.

Suppose $\gamma(t)$ and $\gamma_1(t)$ are integral trajectories of the gradient vector fields for the functions ρ_O and ρ_{O_1} passing through the points P and P_1, and parametrized by a parameter t measuring the distance from the corresponding origin. We note that Q and Q_1 are limit points of, respectively, γ and γ_1, and $\gamma(d) = Q$, $\gamma_1(d) = Q_1$. It appears that along these integral trajectories the following equalities hold (see [7] for details)

$$(2.1) \qquad k_{\mathrm{n}}(t) = \left| \langle N, \partial_{t_O} \rangle \right|(t) \cdot \mu_{\mathrm{n}}(t) + \frac{d}{dt} \left| \langle N, \partial_{t_O} \rangle \right|,$$

$$(2.2) \qquad k_1 = \left| \langle N_1, \partial_{t_{O_1}} \rangle \right|(t) \cdot \mu_{\mathrm{n}}^{c_1}(t) + \frac{d}{dt} \left| \langle N_1, \partial_{t_{O_1}} \rangle \right|,$$

where $\mu_{\mathrm{n}}^{c_1}(t)$ is the normal curvature of a sphere of radius t in $\mathbb{M}^m(c_1)$; $k_{\mathrm{n}}(t)$ is the normal curvature of ∂D taken at the point $\gamma(t)$ in the direction of the vector $\dot{\gamma}(t)$; $\mu_{\mathrm{n}}(t)$ is the normal curvature of the geodesic sphere $S^{m-1} \subset \mathbb{M}^m$ of radius t and center O taken at the point $\gamma(t)$ in the directions of the projection of $\dot{\gamma}(t)$ on

the tangent space $T_{\gamma(t)}S^{m-1}$. All normal curvatures are calculated with respect to the corresponding inner normal vector fields.

It is known that $\mu_{\mathrm{n}}^{c_1}(t) = \mathrm{sn}'_{c_1}(t)/\mathrm{sn}_{c_1}(t)$, where

$$
\mathrm{sn}_{c_1}(t) = \begin{cases} \dfrac{1}{\sqrt{c_1}} \sin \sqrt{c_1}t, \ \text{for } c_1 > 0 \\ t, \ \text{for } c_1 = 0 \\ \dfrac{1}{\sqrt{-c_1}} \sinh \sqrt{-c_1}t, \ \text{for } c_1 < 0. \end{cases}
$$

By the comparison theorem for normal curvatures of spheres (see [8, chapter 6, §5]), we have

$$(2.3) \qquad\qquad\qquad \mu_{\mathrm{n}}^{c_1}(t) \geqslant \mu_{\mathrm{n}}(t).$$

Let us subtract (2.2) from equality (2.1); then using (2.3) and the assumption $k_{\mathrm{n}} \geqslant k_1$ of the theorem, we obtain

$$
\begin{aligned}
(2.4) \quad & 0 \leqslant k_{\mathrm{n}}(t) - k_1 \\
& \leqslant \frac{d}{dt}\left(|\langle N, \partial_{t_O}\rangle| - \left|\langle N_1, \partial_{t_{O_1}}\rangle\right| \right) + \mu_{\mathrm{n}}^{c_1}(t)\left(|\langle N, \partial_{t_O}\rangle| - \left|\langle N_1, \partial_{t_{O_1}}\rangle\right| \right).
\end{aligned}
$$

If we set $f(t) = |\langle N, \partial_{t_O}\rangle|\,(t) - \left|\langle N_1, \partial_{t_{O_1}}\rangle\right|(t)$, then it follows from (2.4) that this function satisfies the following differential inequality

$$(2.5) \qquad\qquad\qquad f'(t) + \frac{\mathrm{sn}'_{c_1}(t)}{\mathrm{sn}_{c_1}(t)} f(t) \geqslant 0.$$

Since $\mathrm{sn}_{c_1}(t) > 0$ for all positive t, inequality (2.5) is equivalent to

$$
\left(f(t) \cdot \mathrm{sn}_{c_1}(t) \right)' \geqslant 0.
$$

Therefore, the function $f \cdot \mathrm{sn}_{c_1}$ is monotonically increasing. Moreover, $f(d) \cdot \mathrm{sn}_{c_1}(d) = 0$. Thus for all t greater then d, we have $f(t) \geqslant 0$. Particularly, if $\rho_O(P) = \rho_{O_1}(P_1) = l\ (l > d)$, then $f(l) = |\langle N, \partial_{t_O}\rangle|\,(P) - \left|\langle N_1, \partial_{t_{O_1}}\rangle\right|(P_1) \geqslant 0$, as desired.

Remark 3. Theorem 1 also holds when $\mathbb{M}^m$ is a de Sitter space $\mathbb{S}_1^m(c)$ of constant positive sectional curvature equal to c, $\partial D \subset \mathbb{S}_1^m(c)$ is a connected spacelike hypersurface that is a graph over a standard unit sphere S^{m-1}. Such surfaces are called *achronal* (see [9]).

The assertion above follows from the fact that formula (2.1) can be transferred in the form as it is stated from the Riemannian case to the Lorentzian case almost directly following [7]. After that one can repeat the calculations from the proof of Theorem 1.

3. Blaschke's Rolling Theorem as a corollary

In this section we will show that Blaschke's Rolling Theorem is a corollary of Theorems 1 and 3.

We start from the part A. Let us introduce in $\mathbb{M}^m(c)$ a polar coordinate system with origin at a point $O \in D$ such that the length of the geodesic segment OP is equal to $\mathrm{dist}(O, \partial D)$. Suppose $(t; \theta^1; \dots; \theta^{m-1})$ are corresponding coordinates, and assume that the point P has the coordinates $(\mathrm{dist}(O, \partial D); 0; \dots; 0)$.

Since the domains D and D_λ are convex, the hypersurfaces ∂D and ∂D_λ that enclose these domains can be given in the introduces coordinate system explicitly by the following equations

$$(3.1) \qquad \partial D : t = p(\theta^1, \dots, \theta^{m-1}), \ \partial D_\lambda : t = q(\theta^1, \dots, \theta^{m-1}),$$

where p and q are some smooth functions, and $p(0, \dots, 0) = q(0, \dots, 0)$.

Using (3.1), we obtain

$$(3.2) \qquad |\langle N, \partial_t \rangle| = \frac{1}{\sqrt{1 + |\mathrm{grad}_{\mathbb{M}}\, p|^2}}, \ |\langle N_1, \partial_t \rangle| = \frac{1}{\sqrt{1 + |\mathrm{grad}_{\mathbb{M}}\, q|^2}},$$

where N and N_1 are inner unit normal fields for, respectively, ∂D and ∂D_λ; ∂_t is a coordinate vector field tangent to geodesic rays starting from O; $\mathrm{grad}_{\mathbb{M}}$ is a gradient operator defined in $\mathbb{M}^m(c)$.

If points $Q \in \partial D$, $Q_1 \in \partial D_\lambda$ are such that $\mathrm{dist}(O, Q) = \mathrm{dist}(O, Q_1)$, then by Theorem 1 in a view of (3.2) at these points the inequality

$$|\mathrm{grad}_{\mathbb{M}}\, p|\, (Q) \leqslant |\mathrm{grad}_{\mathbb{M}}\, q|\, (Q_1)$$

holds.

From this point the remaining arguments coincide with those in [6, section 4.4]. And from them it follows that $p \geqslant q$ for all angular parameters θ^i. The last proves the part A of Blaschke's Rolling Theorem.

Let us consider the part B of the theorem. It is easy to see that for a two-dimensional case ($m = 2$) of the part B arguments from [6] still hold. At the same time, for $m > 2$ they fail to be true. Thus for such a case we need an another approach.

If $\mathbb{M}^m(c)$ is a Euclidean space $\mathbb{E}^m$, then the part B of Blacshke's Rolling Theorem for $m > 2$ follows from the two-dimensional case with a help of projecting. More precisely, if $\pi \subset \mathbb{E}^m$ is an arbitrary two-dimensional plane parallel to a normal vector for ∂D at the point P, then an orthogonal projection $Pr_\pi\,(\partial D)$ of the hypersurface ∂D on π is a curve of curvature not greater than λ (see [1] for details).

If $c \neq 0$, then let us consider a *polar map* of the hypersurface ∂D (see [10, Theorem 2.4] and [11, Theorem 4.9]). The image of ∂D under this map is a C^r-smooth hypersurface ∂D^* that lies in a sphere (for $c > 0$), or in a de Sitter space (for $c < 0$). Moreover, normal curvatures k_{n} of ∂D^* at all points and in

every direction satisfy the inequality $k_\mathrm{n} \geqslant 1/\lambda$. Therefore, the hypersurface ∂D^* satisfies the part A of Blaschke's Rolling Theorem (here we note that, in a view of Remark 3, for a de Sitter space all arguments from the proof of the part A can be carried out directly). Thus ∂D^* lies in a closed convex domain bounded by the hypersurface $\partial D_{1/\lambda}$ of constant normal curvature equal to $1/\lambda$ that touches ∂D^* at any given point. Making the polar map of ∂D^* and $\partial D_{1/\lambda}$ once more, we will obtain that the complete hypersurface $\partial D_\lambda = \left(\partial D_{1/\lambda}\right)^*$ of constant normal curvature equal to λ that touches ∂D at the point P at the same time lies in D, as desired. The part B is proved.

Remark 4. Blaschke's Rolling Theorem also holds for non-smooth surfaces, namely, when ∂D is a λ-convex, or λ-concave hypersurface (for definitions see, for example, [6]). This generalized version of Blaschke's Rolling Theorem can be obtained from the smooth version using an approximation result in [12, Proposition 6].

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