CENTRE DE RECERCA MATEMÀTICA

Preprint núm. 1183
January 2014

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K. Drach

SOME SHARP ESTIMATES FOR CONVEX HYPERSURFACES OF PINCHED NORMAL CURVATURES

KOSTIANTYN DRACH

Abstract. For a convex domain D bounded by the hypersurface ∂D in a space of constant curvature we give sharp bounds on the width $R-r$ of a spherical shell with radii R and r that can enclose ∂D, provided that normal curvatures of ∂D are pinched by two positive constants. Furthermore, in the Euclidean case we also present sharp estimates for the quotient R/r.

1. Preliminaries and the main results

In [1] A. Borisenko and V. Miquel proved that a closed hypersurface with normal curvatures $k_{\rm n}$ satisfying the inequality $k_{\rm n} \geqslant 1$ in the Lobachevsky space $\mathbb{H}^m(-1)$ can be put into a spherical shell between two concentric spheres of radii R and r such that the *width* $R-r$ of the shell satisfies $R-r \leqslant \ln 2$. A similar estimate holds in Hadamard manifolds (see [2]). In [3] these results were extended for Riemannian manifolds of constant-signed sectional curvatures and hypersurfaces with normal curvatures bounded below.

In the present paper we refine some results from [3]. For this purpose we consider hypersurfaces with normal curvatures at any point and in any direction pinched by two positive constants. Such restriction allows us to obtain sharper estimates for the width $R-r$ than in [3]. Furthermore, for such surfaces we are able to derive an upper bound on the quotient R/r, which can be arbitrarily large for a hypersurface with normal curvatures just bounded below.

Let us denote by $\mathbb{M}^m(c)$ a complete simply connected m-dimensional Riemannian manifold of constant sectional curvatures equal to c. In order to state the main results, we need the following definition.

Definition 1.1. *A hypersurface $F \subset \mathbb{M}^m(c)$ is said to be κ_1, κ_2-*convex *(with $\kappa_2 \geqslant \kappa_1$, and for $c = 0$ we assume that $\kappa_1 > 0$, for $c > 0$ we assume that $\kappa_1 \geqslant 0$ and for $c < 0$ we assume that $\kappa_1 > \sqrt{-c}$), if for any point $P \in F$ there exist two nested geodesic spheres $S_2 \subset S_1 \subset \mathbb{M}^m(c)$ of constant normal curvatures equal to, respectively, κ_1 and κ_2, and passing through P such that locally near P the hypersurface F lies inside S_1 and outside S_2.*

2010 *Mathematics Subject Classification.* 53C40.

Key words and phrases. convex hypersurface, spaces of constant curvature, pinched normal curvature, λ-convexity, spherical shell.

Observe that κ_1, κ_2-convex hypersurfaces are, in particular, κ_1-convex (see [2] and [3]).

We should note that for C^r-smooth hypersurfaces with $r \geqslant 2$ the property of being κ_1, κ_2-convex is equivalent to that all its normal curvatures $k_{\rm n}$ are κ_1, κ_2-pinched, that is $\kappa_1 \leqslant k_{\rm n} \leqslant \kappa_2$. In general, since some small neighborhood of any point P on a κ_1, κ_2-convex hypersurface F lies between two tangent at P geodesic spheres, we have that F is $C^{1,1}$-smooth. Therefore, by Rademacher's theorem at almost all points a κ_1, κ_2-convex hypersurfaces has well-defined normal curvature satisfying the inequality shown above.

A closed domain $D \subset \mathbb{M}^m(c)$ is called *κ_1, κ_2-convex domain* if its boundary ∂D is a κ_1, κ_2-convex hypersurface. Such domains are homeomorphic to geodesic balls of the corresponding spaces.

We recall that for κ_1, κ_2-convex domains well-known *Blaschke's rolling theorem* holds (see [4], [5], and [6] for a smooth case, and [7], [8] for a general case). More precisely, it states the following. Suppose $D \subset \mathbb{M}^m(c)$ is a κ_1, κ_2-convex domain; for any point $P \in \partial D$ let S_1 and S_2 be two spheres of normal curvature equal to, respectively, κ_1 and κ_2, and that are tangent to ∂D at P; then $B_2 \subseteq D \subseteq B_1$, where B_i is the closed geodesic ball bounded by S_i, $i \in \{1, 2\}$.

The main result of the present paper consists of the following three theorems.

Theorem 1. *If $D \subset \mathbb{M}^m(c)$ is a κ_1, κ_2-convex domain, then the hypersurface ∂D can be put into a spherical shell between two concentric spheres of radii R and r (with $R \geqslant r$) such that*

(1) *for $c = 0$,*

$$R - r \leqslant \left(\sqrt{2} - 1\right)\left(\frac{1}{\kappa_1} - \frac{1}{\kappa_2}\right); \tag{1.1}$$

(2) *for $c = k^2$ with $k > 0$,*

$$R - r \leqslant \frac{2}{k} \arccos \sqrt{\cos\left(k(R_1 - R_2)\right)} - (R_1 - R_2); \tag{1.2}$$

(3) *for $c = -k^2$ with $k > 0$,*

$$R - r \leqslant \frac{2}{k} \operatorname{arccosh} \sqrt{\cosh\left(k(R_1 - R_2)\right)} - (R_1 - R_2), \tag{1.3}$$

where R_1 and R_2 are the radii of circles with geodesic curvatures equal to, respectively, κ_1 and κ_2, and lying in the corresponding 2-planes $\mathbb{M}^2(c)$.

Moreover, these estimates are sharp.

Remark 1.1. As $\kappa_2 \to \infty$, the estimates (1.1) – (1.3) tend to the corresponding estimates in spaces of constant curvature from [3].

Remark 1.2. By sharpness of the inequalities above and below we mean that the shown bounds are attended by so-called *rounded κ_1, κ_2-convex spindle-shaped surfaces* (see Fig. 1), which we describe in details in the next section.

In the Euclidean case we can give even more interesting estimate for the quotient R/r. This estimate is related to the result of A.V. Pogorelov about convex three-dimensional surfaces with almost umbilical points (see [9], p. 493).

Theorem 2. *If $D \subset \mathbb{E}^m$ is a κ_1, κ_2-convex domain in the Euclidean space, then the hypersurface ∂D can be put into a spherical shell between two concentric spheres of radii R and r (with $R \geqslant r$) such that*

$$\frac{R}{r} \leqslant \frac{\sqrt{\frac{\kappa_2}{\kappa_1}} + \sqrt{2}}{\sqrt{\frac{\kappa_1}{\kappa_2}} + \sqrt{2}}. \tag{1.4}$$

Moreover, this estimate is sharp.

Corollary 1.1. *In the condition of Theorem 2, it is true that*

$$\frac{R}{r} \leqslant \frac{\kappa_2}{\kappa_1}.$$

Remark 1.3. We note that if in the above $\kappa_1 = \kappa_2$, then from all estimates (1.1) – (1.4) it follows that $R = r$, and thus the domain D is a geodesic ball of the corresponding space.

Theorems 1 and 2 are based on the following result, which is useful by itself. It gives the sharp upper bound on the outer radius R of the spherical shell in terms of the inner radius r of that shell.

Theorem 3. *If $D \subset \mathbb{M}^m(c)$ is a κ_1, κ_2-convex domain, then the hypersurface ∂D can be put into a spherical shell between two concentric spheres of radii R and r (with $R \geqslant r$) such that*

(1) *for $c = 0$,*

$$R \leqslant \sqrt{(R_1 - R_2)^2 - (R_1 - r)^2} + R_2; \tag{1.5}$$

(2) *for $c = k^2$ with $k > 0$,*

$$R \leqslant \frac{1}{k} \arccos \frac{\cos\left(k(R_1 - R_2)\right)}{\cos\left(k(R_1 - r)\right)} + R_2; \tag{1.6}$$

(3) *for $c = -k^2$ with $k > 0$,*

$$R \leqslant \frac{1}{k} \operatorname{arccosh} \frac{\cosh\left(k(R_1 - R_2)\right)}{\cosh\left(k(R_1 - r)\right)} + R_2, \tag{1.7}$$

where R_1 and R_2 are the radii of circles with geodesic curvature equal to, respectively, κ_1 and κ_2, and lying in the corresponding 2-planes $\mathbb{M}^2(c)$.

Moreover, the estimates (1.5) – (1.7) are sharp.

2. Proofs of the main results

We are going to prove the above theorems by using a comparison argument. Let us introduce an object to compare with.

In $\mathbb{M}^m(c)$ let us consider a spindle-shaped κ_1-convex hypersurface that is obtained by rotating a circular arc of geodesic curvature equal to κ_1 (see [3]). Such surface have two vertexes where its normal curvatures blow up. After smoothing these vertexes using two spherical caps of normal curvature equal to κ_2 we obtain a convex $C^{1,1}$-smooth hypersurface (see Fig. 1). We will call such surfaces *rounded κ_1, κ_2-convex spindle-shaped hypersurfaces.*

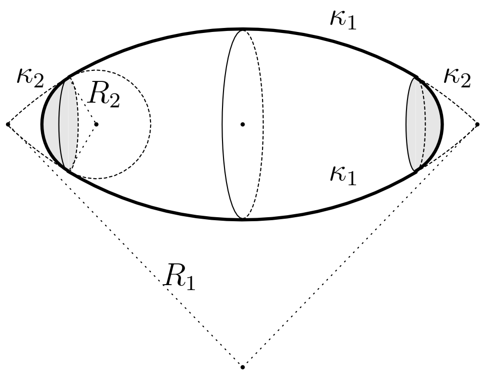

Figure 1. Rounded κ_1, κ_2-convex spindle-shaped hypersurface

It appears that the following comparison lemma holds.

Lemma 2.1. *Let $D \subset \mathbb{M}^m(c)$ be a closed κ_1, κ_2-convex domain, r be the radius of the inscribe sphere for D with center at a point O. Let $\tilde{F} \subset \mathbb{M}^m(c)$ be a rounded κ_1, κ_2-convex spindle-shaped hypersurface, and let $\tilde{r}$, $\tilde{R}$ be the radii of its inscribe and circumscribe spheres. If*

$$\tilde{r} = r,$$

then

$$\max \operatorname{dist}(O, \partial D) \leqslant \tilde{R}. \tag{2.1}$$

Moreover, this bound is sharp.

Proof. We will argue by contradiction. Suppose that (2.1) is not true, and the inverse inequality

$$\max \operatorname{dist}(O, \partial D) > \tilde{R} \tag{2.2}$$

holds. For simplicity, we denote the hypersurface ∂D by F.

Let $M \in F$ be a point such that $\max \operatorname{dist}(O, F) = |OM|$ (here and below $|\cdot|$ denotes the distance between two points); then from (2.2) it follows that on the geodesic segment OM there exists a point A such that

$$|OA| = \tilde{R} < |OM|. \tag{2.3}$$

Assume that the rounded κ_1, κ_2-convex spindle-shaped hypersurface $\tilde{F}$ is centered at O, and its rotational axis coincides with the geodesic line OA. Then $A \in \tilde{F}$.

Since the point M is a point for which the maximal distance from O is attained, we have that a totally geodesic hyperplane which touches F at M is perpendicular to the geodesic line OM. Therefore, if $\omega_2 \subseteq D$ is a sphere with the center at a point O_2, and with normal curvature equal to κ_2 that touches from inside the hypersurface F at M (such a sphere exists by Blaschke's rolling theorem), then O_2 lies on the segment OM. We note that from (2.3) it follows

$$|OO_2| > \tilde{R} - R_2, \tag{2.4}$$

where R_2 is the radius of ω_2.

Let us denote the inscribe sphere for F by ω. Since the hypersurface F is κ_1, κ_2-convex, then by Blaschke's rolling theorem F lies in a ball of radius R_1 (we remind that R_1 is the radius of a sphere of normal curvature equal to κ_1). Hence, both ω and ω_2 lies in this ball too. Now we show that there is a sphere of radius R_1 that touches externally both ω and ω_2 simultaneously.

Denote $|OO_2|$ by d. It is clear that the sphere mentioned above exists if and only if the three numbers d, $R_1 - r$, and $R_1 - R_2$ satisfy the triangle inequality. Let us check this:

(1) $(R_1 - r) + (R_1 - R_2) = 2R_1 - (r + R_2) > d$, since ω and ω_2 lie in a ball of radius R_1;
(2) $(R_1 - R_2) + d > R_1 - r$, because ω is the inscribe sphere and thus either $\omega \equiv \omega_2$ and $r = R_1 = R_2$ that is a trivial case when F is a sphere, or ω_2 touches ω from inside, which is again a trivial case when F is a sphere, or ω_2 cannot lie entirely inside ω, hence $r + d < R_2$.
(3) $(R_1 - r) + d > R_1 - R_2$, which is obviously true.

By rotational symmetry, along with a single sphere of radius R_1 there exists a family of spheres of the same radius that touches simultaneously ω and ω_2 along small $(m-2)$-dimensional spheres σ and σ_2. To continue we need the following lemma.

Lemma 2.2. [3] *If $D \subset \mathbb{M}^m(c)$ is a closed κ_1-convex domain (where for $c = 0$ we assume that $\kappa_1 > 0$, for $c > 0$ we assume $\kappa_1 \geqslant 0$ and for $c < 0$ we assume that $\kappa_1 > \sqrt{-c}$), then for any two points A, B from D every smaller circular arc of geodesic curvature equal to κ_1 that joins A and B lies in the domain D.*

Since D is, in particular, a κ_1-convex domain, then by Lemma 2.2 the part Θ of the envelope of this family lying between two hyperplanes π and π_2 (corresponding to the spheres $\sigma = \pi \cap \omega$ and $\sigma_2 = \pi_2 \cap \omega_2$), lies inside D (see Fig. 2).

Let ω^- and ω_2^+ be the spherical caps cut from ω and ω_2 by the planes π and π_2 in a way that Θ and ω^-, Θ and ω_2^+ lie on the different sides with respect to π and π_2, correspondingly.

Consider a $C^{1,1}$ smooth hypersurface $\omega^- \cup \Theta \cup \omega_2^+$ denoted by Ω. By the arguments above, Ω lie in D.

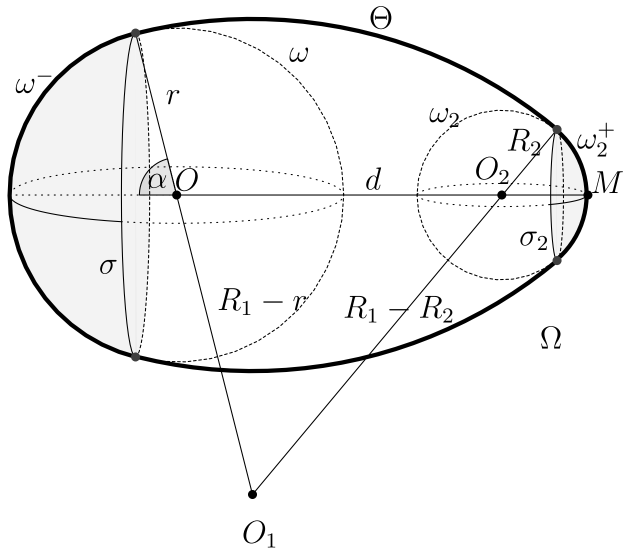

FIGURE 2. The hypersurface $\Omega = \omega^- \cup \Theta \cup \omega_2^+$.

Now, using (2.4) we are going to show that it is possible to inscribe inside Ω a sphere of radius r' strictly greater then the radius r of the inscribed sphere. This give us a desired contradiction, since by definition the inscribed sphere is a sphere of maximal radius lying in a domain .

By symmetry, the center C of the inscribed in Ω sphere lie on the geodesic OO_2. Let us consider a section of Ω by a totally geodesic two-dimensional plane Π passing trough OO_2. Then the curve Ω' defined as $\Omega \cap \Pi$ consists of four parts: an arc ω' of a circle $\omega \cap \Pi$ of radius r, an arc ω_2' of a circle $\omega_2 \cap \Pi$ of radius R_2, and two circular arcs of radius R_1. Denote by O_1 the center of one of the corresponding circles (see Fig. 2). Then $|OO_1| = R_1 - r$, $|O_2O_1| = R_1 - R_2$. Let α be the angle between the geodesic lines OO_2 and OO_1. Assume $\alpha \leqslant \pi/2$.

From the construction of $\tilde{F}$ it follows that a triangle with the side lengths $R_1 - r$, $R_1 - R_2$, and $\tilde{R} - R_2$ is a right triangle. Denote $\tilde{R} - R_2$ by $\tilde{d}$. From (2.4) we have that $d > \tilde{d}$. By the law of cosines from the geodesic triangle $\triangle OO_2O_1$, we deduce:

(1) For $c = 0$,

$$(R_1 - R_2)^2 = (R_1 - r)^2 + d^2 - 2d\,(R_1 - r)\cos\alpha$$
$$> (R_1 - r)^2 + \tilde{d}^2 - 2d\,(R_1 - r)\cos\alpha.$$

Since $(R_1 - R_2)^2 = (R_1 - r)^2 + \tilde{d}^2$, then from the above computations it follows that $\cos\alpha > 0$, thus $\alpha < \pi/2$.

(2) For $c = 1$,

$$\cos(R_1 - R_2) = \cos(R_1 - r)\cos d + \sin(R_1 - r)\sin d\cos\alpha$$
$$< \cos(R_1 - r)\cos\tilde{d} + \sin(R_1 - r)\sin d\cos\alpha.$$

Recalling that $\cos(R_1 - R_2) = \cos(R_1 - r)\cos\tilde{d}$, we obtain $\cos\alpha > 0$, $\alpha < \pi/2$.

(3) For $c = -1$,

$$\cosh(R_1 - R_2) = \cosh(R_1 - r)\cosh d - \sinh(R_1 - r)\sinh d\cos\alpha$$
$$> \cosh(R_1 - r)\cosh\tilde{d} + \sinh(R_1 - r)\sinh d\cos\alpha.$$

And since $\cosh(R_1 - R_2) = \cosh(R_1 - r)\cosh\tilde{d}$, we get $\alpha < \pi/2$.

Therefore, in all the cases the angle α is strictly less then $\pi/2$. Hence, from the right triangle $\triangle OO_1C$ we have $|OO_1| > |O_1C|$. Thus, if r' is the radius of the inscribed in Ω' circle, then $r' = R_1 - |O_1C|$ and $r' > R_1 - |OO_1| = r$. And since it is true for any plane Π, we come to the contradiction which proves (2.1).

The inequality (2.1) is sharp since the equality is obviously attained for $\tilde{F}$. Lemma 2.1 is proved. □

Theorem 3 is a direct consequence of Lemma 2.1 because the right sides of the inequalities (1.5) – (1.7) are the values of $\tilde{R}$ in terms of the inscribed sphere's radius $\tilde{r} = r$.

In order to prove Theorems 1 and 2, we should derive additional estimates for the spherical shell's width and the quotient of its radii in the case of rounded κ_1, κ_2-convex spindle-shaped hypersurfaces. These estimates are summarized in the following lemma.

Lemma 2.3. *Suppose $\tilde{F} \subset \mathbb{M}^m(c)$ is a rounded κ_1, κ_2-convex spindle-shaped hypersurface, $\tilde{r}$ and $\tilde{R}$ are the radii of the inscribe and circumscribe spheres for $\tilde{F}$; then for the width $\tilde{R} - \tilde{r}$ the estimates* (1.1) – (1.3) *hold. Moreover, when $c = 0$ for the quotient $\tilde{R}/\tilde{r}$ the estimate* (1.4) *holds.*

Proof. The estimates (1.1) – (1.3) are obtained similarly to [3]. Let us show (1.1). In the rest of the cases computations are similar.

If $R_1 = 1/\kappa_1$ and $R_2 = 1/\kappa_2$ are, as usual, the radii of the spheres of the curvatures equal to κ_1 and κ_2, then it is easy to see that

$$\tilde{R} = \sqrt{(R_1 - R_2)^2 - (R_1 - \tilde{r})^2} + R_2.$$

Let us introduce a function

$$w(\tilde r) = \tilde R - \tilde r = \sqrt{(R_1 - R_2)^2 - (R_1 - \tilde r)^2} + R_2 - \tilde r$$

defined for $\tilde r \in [R_1, R_2]$. By construction, $w \geqslant 0$, and $w(R_1) = w(R_2) = 0$. Hence, this function attains the global maximum on (R_1, R_2). Solving the equation for the derivative $dw/d\tilde r = 0$, which is a linear equation with respect to $\tilde r$, and substituting its solution in $w(\tilde r)$, we will get (1.1).

Now let us prove the estimate (1.4).

Similarly to the above, we introduce the function

$$q(\tilde r) = \frac{\tilde R}{\tilde r} = \frac{1}{\tilde r}\left(\sqrt{(R_1 - R_2)^2 - (R_1 - \tilde r)^2} + R_2\right) \tag{2.5}$$

defined for $\tilde r \in [R_1, R_2]$. Moreover, by construction, $q \geqslant 1$, and $q(R_1) = q(R_2) = 1$. Hence, this function attains the global maximum on (R_1, R_2) unless the trivial case $R_1 = R_2$. Let us find this maximal value.

Solving the equation $dq/d\tilde r = 0$ for $\tilde r$, which simplifies to a quadratic equation, we get the root

$$\tilde r_0 = \frac{1}{R_1^2 + R_2^2}\left(2R_1^2 R_2 - R_2\left(R_1 - R_2\right)\sqrt{2R_1R_2}\right).$$

The function q attains its maximum at $\tilde r_0$. Therefore

$$q(\tilde r) \leqslant q(\tilde r_0) \text{ for all } \tilde r \text{ from } [R_1, R_2]. \tag{2.6}$$

Now we proceed with the computing $q(\tilde r_0)$. It is straightforward to check that

$$(R_1 - R_2)^2 - (R_1 - \tilde r_0)^2 = \frac{(R_1 - R_2)^2}{(R_1^2 + R_2^2)^2}\left(R_2\left(R_1 - R_2\right) - R_1\sqrt{2R_1R_2}\right)^2.$$

Thus, since $R_1\sqrt{2R_1R_2} > R_2\left(R_1 - R_2\right)$,

$$\begin{aligned} &\sqrt{(R_1 - R_2)^2 - (R_1 - \tilde r_0)^2} + R_2 = \\ &= \frac{\sqrt{2R_1R_2}}{R_1^2 + R_2^2}\left(R_1 + \sqrt{2R_1R_2}\right)\left(R_1 + R_2 - \sqrt{2R_1R_2}\right). \end{aligned} \tag{2.7}$$

We can also rewrite $\tilde r_0$ in the same manner:

$$\tilde r_0 = \frac{\sqrt{2R_1R_2}}{R_1^2 + R_2^2}\left(R_2 + \sqrt{2R_1R_2}\right)\left(R_1 + R_2 - \sqrt{2R_1R_2}\right). \tag{2.8}$$

Combining (2.7) and (2.8), and recalling that $R_i = 1/\kappa_i$, $i \in \{1, 2\}$, we get

$$q(\tilde r_0) = \frac{R_1 + \sqrt{2R_1R_2}}{R_2 + \sqrt{2R_1R_2}} = \frac{\sqrt{\frac{\kappa_2}{\kappa_1}} + \sqrt{2}}{\sqrt{\frac{\kappa_1}{\kappa_2}} + \sqrt{2}}. \tag{2.9}$$

Thereby, from (2.5), (2.6), and (2.9), we finally obtain

$$\frac{\tilde{R}}{\tilde{r}} \leqslant \frac{\sqrt{\frac{\kappa_2}{\kappa_1}} + \sqrt{2}}{\sqrt{\frac{\kappa_1}{\kappa_2}} + \sqrt{2}},$$

as desired.

The above bound is sharp and is attained for the rounded κ_1, κ_2-convex hypersurface with the radius of the inscribe sphere equal to $\tilde{r}_0$. Lemma 2.3 is proved. □

Now, Theorems 1 and 2 are the direct consequences of Lemma 2.1 and Lemma 2.3.

Acknowledgments. This work was partially done while the author was visiting the Centre de Recerca Matemàtica as a participant of the Conformal Geometry and Geometric PDE's Program supported by a grant of the Clay Mathematics Institute. He would like to acknowledge both institutions for the given opportunities. The author is also grateful to prof. Manuel Ritoré, whose lecture unexpectedly raised questions solved in the paper, and to prof. Alexander Borisenko for many fruitful discussions and remarks on the paper.

References

[1] A. Borisenko, V. Miquel, *Total curvatures of convex hypersurfaces in hyperbolic space.* Illinois J. Math., **43** (1999), no. 1, 61–78.
[2] A. Borisenko, V. Miquel, *Comparison theorems for convex hypersurfaces in Hadamard manifolds.* Ann. Glob. Anal. Geom., **21** (2002), No.2, 191-202.
[3] A. Borisenko, K. Drach, *Closeness to spheres of hypersurfaces with normal curvature bounded below.* Sb. Math., **204** (2013), no. 11 1565-–1583.
[4] W. Blaschke, Kreis und Kugel, – de Gruyter, Berlin, 1956.
[5] H. Karcher, *Umkreise und Inkreise konvexer Kurven in der spharischen und der hyperbolischen Geometrie.* Math. Ann., **177** (1968), 122–132.
[6] R. Howard, *Blaschke's rolling theorem for manifolds with boundary.* Manuscripta Math., **99** (1999),no. 4, 471–483.
[7] A.D. Milka, *On a theorem of Schur and Schmidt*, Ukrain. Geom. Sb., **8** (1970), 95–102. (Russian)
[8] A. Borisenko, K. Drach, *Comparison theorems for support functions of hypersurfaces*, in preparation.
[9] A.V. Pogorelov, *Extrinsic geometry of convex surfaces.* Amer. Math. Soc., Providence, 1973.

Geometry Department, V.N. Karazin Kharkiv National University
Svobody Sq. 4, 61022, Kharkiv, Ukraine
and
Department of Mathematical Analysis and Optimization
Sumy State University
Rimskogo - Korsakova str. 2
40007, Sumy, Ukraine
E-mail address: drach@karazin.ua, kostya.drach@gmail.com

CENTRE DE RECERCA MATEMÀTICA

Facultat de Ciències, UAB
08193 Bellaterra
Barcelona, Spain
www.crm.cat