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SYMPLECTIC TOPOLOGY OF b-SYMPLECTIC MANIFOLDS

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Abstract. A Poisson manifold (M^{2n}, π) is b-symplectic if $\bigwedge^n \pi$ is transverse to the zero section. In this paper we apply techniques of Symplectic Topology to address global questions pertaining to b-symplectic manifolds. The main results provide constructions of: b-symplectic submanifolds à la Donaldson, b-symplectic structures on open manifolds by Gromov's h-principle, and of b-symplectic manifolds with a prescribed singular locus, by means of surgeries.

1. Introduction and statement of main results

A Poisson structure on a manifold M can be described as a bivector $\pi \in \mathfrak{X}^2(M)$ which obeys the partial differential equation $[\pi, \pi] = 0$, where $[\cdot, \cdot]$ is the Schouten bracket of multivector fields. The image of the induced bundle map $\pi^\sharp : T^*M \to TM$ is an involutive distribution, of possibly varying rank, each of whose integral submanifolds carries an induced symplectic form.

Symplectic structures are those Poisson structures whose underlying foliation has M^{2n} as its only leaf; equivalently, they are those Poisson structures for which $\pi^\sharp$ is invertible – i.e., for which $\bigwedge^n \pi$ does not meet the zero section.

In [12], this non-degeneracy condition has been relaxed in a very natural way, by demanding that $\bigwedge^n \pi$ be *transverse* to the zero section instead of avoiding it:

Definition 1. *A Poisson manifold* (M^{2n}, π) *is of* b-**symplectic type** *if* $\bigwedge^n \pi$ *is transverse to the zero section* $M \subset \bigwedge^{2n} TM$.

Such structures were first defined, in the case of dimension two, by Radko [24], who called them *topologically stable* Poisson structures. Poisson structures of b-symplectic type have also appeared under the names *log symplectic* [11], [4] and *b-log symplectic structures* [17, 18].

Symplectic structures are those Poisson structures of b-symplectic type whose **singular locus** $Z(\pi) := (\bigwedge^n \pi)^{-1} M \subset M$ is empty. Quite crucially for what

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follows is that general Poisson structures of b-symplectic type do not stray too far from being symplectic.

To explain what we mean by this, observe that the transversality condition $\bigwedge^n \pi \pitchfork M$ ensures that the singular locus $Z = Z(\pi)$ is a codimension-one submanifold of M, which by the Poisson condition is itself foliated in codimension one by symplectic leaves of π. Those vector fields $v \in \mathfrak{X}(M)$ which are tangent to Z form the space of all sections of a vector bundle ${}^bT(M,Z) \to M$, called the **b-tangent bundle** [20], which is canonically identified with TM outside Z. ${}^bT(M,Z)$ has a canonical structure of Lie algebroid, and a Poisson structure of b-symplectic type π on M with singular locus Z can be described alternatively by ω a closed, non-degenerate section of $\bigwedge^2 \left({}^bT^*(M,Z)\right)$, in complete analogy with the symplectic case. This viewpoint motivates the nomenclature adopted in [12, 23] [1], and to which we adhere. Henceforth, we will refer to Poisson structures of b-symplectic type as **b-symplectic** structures.

With this perspective, it is rather unsurprising that many tools from Symplectic Topology can be adapted to this b-setting. In fact, the purpose of this paper is to use such tools to address global questions in b-symplectic geometry.

1.1. **Statement of the main results.** As will be recalled below, b-manifolds fit into a category where morphisms $f\colon (M,Z) \to (M'Z')$, called b-maps, are maps $f\colon M \to M'$ transverse to Z', and pulling back Z' to Z; b-submanifolds are those b-maps which correspond to inclusions of submanifolds. Sections of $\bigwedge^p \left({}^bT^*(M,Z)\right)$ can be pulled back along b-maps, and we define a b-submanifold $W \hookrightarrow (M,Z)$ of a b-symplectic (M,ω) to be a **b-symplectic submanifold** if $\omega|_W$ is b-symplectic.

One natural instance where b-symplectic manifolds show up is when one glues cosymplectic cobordisms ([4]; cf. Proposition 2). As it turns out, every compact, orientable, b-symplectic manifold can be written as a concatenation of such cobordisms (Proposition 3). The link between symplectic manifolds with cosymplectic boundary, on the one hand, and b-symplectic manifolds, on the other, underlies our approach to the two main problems which will occupy us in this note, namely, proving the existence of b-symplectic submanifolds, and what we call *realization* problems.

Problem 1. *Do b-symplectic manifolds have closed b-symplectic submanifolds of any possible dimension?*

Donaldson showed in [5] that such submanifolds always exist when $Z(\omega) = \varnothing$. Our first main result answers in the affirmative the general existence problem for submanifolds:

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[1]Closed, non-degenerate sections of $\bigwedge^2 \left({}^bT^*(M,Z)\right)$ were introduced in [23] in the case $Z = \partial M$.

Theorem 1. *Every (M, ω) compact b-symplectic manifold without boundary has closed b-symplectic submanifolds $W \hookrightarrow (M, \omega)$ of any even dimension.*

In particular, any compact, 4-dimensional b-symplectic manifold contains topologically stable Poisson surfaces.

We next turn to our next big concern, which we call

Problem 2 (Realization problems)**.**

1. *Which manifolds M carry a structure of b-symplectic manifold?*
2. *Which b-cosymplectic manifolds Z appear as singular loci of compact b-symplectic manifolds?*

For compact manifolds without boundary, the answer to (1) is unknown, even in the symplectic case. For *open* manifolds (i.e., whose connected components either have non-empty boundary or are non-compact), we show:

Theorem 2. *Let M be an orientable, open manifold. Then M is b-symplectic if and only if $M \times \mathbb{C}$ is almost-complex.*

In fact, the story here is completely analogous to the symplectic case: supporting a b-symplectic structure imposes restrictions on the de Rham cohomology of a compact manifold without boundary [4, 17], but these do not apply to open manifolds. There, the existence of b-symplectic structures becomes a purely homotopical question, and we show that they abide by a version of the h-principle of Gromov [10]. In some very special cases, the finer control granted by having an h-principle description allows one to even prescribe the singular locus Z of the ensuing b-symplectic manifold. However, the case where Z bounds a compact region in M, these techniques break down completely, and we have our stage set for the Realization problem (2).

We first briefly recall some concepts.

Definition 2. *A* **cosymplectic structure** *on a manifold Z^{2n-1} consists of a pair of closed forms $(\theta, \eta) \in \Omega^1(Z) \times \Omega^2(Z)$, such that $\theta \wedge \eta^{n-1}$ is a volume form.*

As it is well-known [12], a corank-one Poisson structure (Z, π_Z) is the singular locus of a b-symplectic manifold if and only if it comes from a cosymplectic structure, in the way described in Section 6.

Two cosymplectic structures (θ, η), (θ', η') on Z are b**-equivalent** if $\theta' = \theta$ and $\eta' = \eta + d(f\theta)$ for some $f \in C^\infty(Z)$. An equivalence class of cosymplectic structures will be called a b**-cosymplectic structure** on Z; the name is justified by the fact that the singular locus of a b-symplectic manifold carries a canonical b-cosymplectic structure ([12]; cf. Proposition 1). As we shall see, a given cosymplectic manifold lies in the b-cosymplectic class of the singular locus of a *compact*, orientable b-symplectic manifold without boundary if and only if it is **symplectically fillable** (Lemma 6). Problem 2.2 can be thus rephrased as that of determining those cosymplectic manifolds which admit symplectic fillings.

Symplectic fillings of contact manifolds –and more generally symplectic cobordisms with concave/convex boundaries– are central to Symplectic Topology, whereas the case of cosymplectic (or flat) boundaries has received comparatively little attention. In this respect Eliashberg has shown that when Z is a **symplectic mapping torus** –that is, when Z is the suspension of a symplectomorphism $\varphi\colon (F,\sigma) \to (F,\sigma)$– and has dimension 3, then it is symplectically fillable [6].

We extend Eliashberg's result as follows: Firstly, we prove that for symplectic mapping tori the symplectic fillability question is decided merely in terms of the symplectic isotopy class of φ, and we exhibit one class of symplectomorphisms φ which yield symplectically fillable symplectic mapping tori; namely, Dehn twists τ_l around parametrized Lagrangian spheres $l \subset (F,\sigma)$ (see Definition 9) and their inverses τ_l^{-1}:

Theorem 3. *Let Z be a compact symplectic mapping torus. Assume that φ is symplectically isotopic to*

$$\tau_{l_1} \cdots \tau_{l_m} \tau_{l_{m+1}}^{-1} \cdots \tau_{l_{m'}}^{-1},$$

where $l_i : \mathbb{S}^{n-1} \hookrightarrow (F,\sigma)$, $i = 1,\dots,m'$ are parametrized Lagrangian spheres.

Then there exists a compact b-symplectic manifold without boundary, the b-cosymplectic class of whose singular locus is represented by Z.

Secondly, we observe that symplectic fillability of all cosymplectic manifolds would be a consequence of symplectic fillability of all symplectic mapping tori, hence solving the cosymplectic realization problem in dimension 3.

Theorem 4. *Any compact cosymplectic manifold of dimension* 3 *is b-equivalent to the singular locus of a compact b-symplectic manifold without boundary.*

While this project was being completed the authors learned of research by G. Cavalcanti which has some overlap with theirs. More precisely, the idea of constructing b-symplectic manifolds without boundary by gluing cosymplectic cobordisms appeared independently in [4].

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Conventions. *By a* manifold *M, we mean a smooth manifold with (possibly empty) boundary ∂M; maps are always assumed to be smooth. A submanifold $W \subset M$ will be said to be* closed *if it is closed as a subspace of M. If $\partial M \neq \varnothing$, we shall always assume that $W \pitchfork \partial M$, and $\partial W = W \cap \partial M$. Under this convention, a submanifold $W \subset M$ which does not meet the boundary ∂M has itself empty boundary, $\partial W = \varnothing$. Cosymplectic structures are considered only on manifolds with empty boundary.*

2. Cosymplectic cobordisms and b-symplectic structures

We summarize below the basic facts and conventions about b-symplectic manifolds and cosymplectic cobordisms which will be of use in this note. The aim is to establish a correspondence (see Section 3) between both structures, so as to reduce some problems in b-symplectic geometry to problems on symplectic geometry. For a more detailed account on b-manifolds and b-symplectic structures we refer the reader to [11, 12, 18, 20, 23].

2.1. **b-manifolds and b-symplectic structures.** The category of **b-manifolds** has as objects pairs (M, Z), where $Z \subset M$ is a closed submanifold of codimension one with empty boundary, and as morphisms $f\colon (M, Z) \to (M', Z')$ those maps $f\colon M \to M'$ transverse to Z', and pulling back Z' to Z.

The Lie subalgebra $\mathfrak{X}(M, Z) \subset \mathfrak{X}(M)$ consisting of those vector fields v which are tangent to Z can be identified with the space of smooth sections of the **b-tangent bundle** ${}^bT(M, Z) \to M$. By its very construction, ${}^bT(M, Z)$ comes equipped with a bundle map ${}^bT(M, Z) \to TM$ covering id_M, which is identical outside Z. Its restriction to Z defines an epimorphism ${}^bT(M, Z)|_Z \to TZ$, whose kernel ${}^bN(M, Z)$ has a canonical trivialization: if one expresses Z locally as $x_1 = 0$ in a coordinate chart $(x_1, ..., x_n)$, then $x_1 \frac{\partial}{\partial x_1}$ is independent of choices along Z, and determines the canonical nowhere-vanishing section $\nu \in \Gamma(Z, {}^bN(M, Z))$.

The bundle dual to ${}^bT(M, Z)$ will be denoted by ${}^bT^*(M, Z)$; sections of $\bigwedge^p \left({}^bT^*(M, Z)\right)$ will be called b-forms (of degree p) on (M, Z), and we write ${}^b\Omega^p(M, Z)$ for the space of all such forms.

Since $\mathfrak{X}(M, Z) \subset \mathfrak{X}(M)$ is a Lie subalgebra, ${}^bT(M, Z)$ has a natural structure of Lie algebroid, and as such, it carries a differential ${}^bd\colon {}^b\Omega^p(M, Z) \to {}^b\Omega^{p+1}(M, Z)$, ${}^bd^2 = 0$, determined by the usual Koszul-type formula:

$$({}^bd\omega)(v_0, ..., v_p) = \sum (-1)^i v_i \omega(v_0, ..., \widehat{v_i}, ..., v_p) +$$
$$+ \sum (-1)^{i+j} \omega([v_i, v_j], v_0, ..., \widehat{v_i}, ..., \widehat{v_j}, ..., v_p), \quad v_i \in \mathfrak{X}(M, Z).$$

Note that ${}^bd$ agrees with d outside Z, and that we have a short exact sequence of chain complexes:

$$0 \longrightarrow (\Omega^\bullet(M), d) \longrightarrow ({}^b\Omega^\bullet(M, Z), {}^bd) \xrightarrow{\flat} (\Omega^{\bullet-1}(Z), d) \longrightarrow 0, \tag{1}$$

where $\flat$ maps a b-form ω to its contraction with the canonical ν.

A b-map $f\colon (M, Z) \to (M', Z')$ gives rise to homomorphisms

$$f_*\colon {}^bT(M, Z) \to {}^bT(M', Z'), \quad (f|_Z)_*\colon TZ \to TZ',$$

covering f and $f|_Z$, respectively, and mapping the canonical sections to one another: $f_*\nu = \nu'$. One checks that the assignment $f \mapsto (f_*, (f|Z)_*)$ is functorial, that b-forms $\omega' \in {}^b\Omega^\bullet(M', Z')$ pull back under a b-map f, and ${}^bdf^*\omega' = f^*({}^bd\omega')$.

Mimicking the usual terminology, a b-form $\omega \in {}^b\Omega^2(M,Z)$ will be called non-degenerate if ω^n is nowhere vanishing, and **symplectic** if it is non-degenerate and closed, ${}^b d\omega = 0$.

We quote from [12] (cf. Lemma 2):

Lemma 1. *There is a bijective correspondence between symplectic forms on (M,Z), and Poisson structures of b-symplectic type with singular locus Z.*

We can thus speak unambiguously of *b-symplectic manifolds*.

When M is oriented, with volume form $\mu \in \Omega^m(M)$, one assigns to a b-symplectic structure $\pi = \omega^{-1}$ the function $f_\pi \in C^\infty(M)$, determined by $\bigwedge^n \pi = f_\pi \mu^{-1}$; then $f_\pi \pitchfork 0$ and $Z(\pi) = f_\pi^{-1}(0)$. Hence $Z(\pi)$ is coorientable, and $M \smallsetminus Z(\pi)$ decomposes as a disjoint union of points where the orientation induced by π and by μ agree or disagree: $M \smallsetminus Z(\pi) = \{f_\pi < 0\} \coprod \{f_\pi > 0\}$.

Example 1. (Radko's sphere) *The b-form $\omega = \frac{1}{h} dh \wedge d\theta$ on $\mathbb{S}^2$, where h, θ stand for cylindrical coordinates, is b-symplectic. Its symplectic leaves are either points in the equator $\mathbb{S}^1 \subset \mathbb{S}^2$, or components of $\mathbb{S}^2 \smallsetminus \mathbb{S}^1$.*

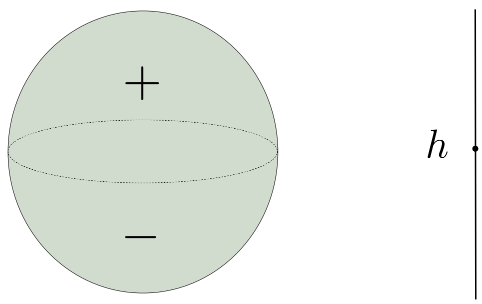

Figure 1. Radko sphere $\mathbb{S}^2$ with the equator as critical hypersurface and the upper and lower hemisphere as positive and negative symplectic leaf, respectively.

2.2. **Cobordisms.** Cosymplectic structures appear naturally in Symplectic Geometry as hypersurfaces $Z \subset (M,\omega)$ which are transverse to symplectic vector fields. To wit, if $v \in \mathfrak{X}(M)$ is such a vector field, $(\eta,\theta) := (\omega|_Z, \omega(v)|_Z)$ defines a cosymplectic structure on Z, and clearly every cosymplectic structure (η,θ) on Z can be realized in this manner: take for instance $\mathrm{pr}^*\eta + dt \wedge \mathrm{pr}^*\theta$ on $Z \times \mathbb{R}$, where $\mathrm{pr}\colon Z \times \mathbb{R} \to Z$ denotes the first projection, and the symplectic vector field $\partial/\partial t$.

If Z is a codimension-one, coorientable submanifold in a symplectic manifold (M,ω), we say that a cosymplectic structure (η,θ) on Z is compatible with ω if $\eta = \omega|_Z$.

As we shall shortly see (Lemma 1), if (M,ω) is a b-symplectic manifold, and $Z(\omega)$ coorientable, on the boundary of the complement of an open tubular neighborhood $C \subset M$ of $Z(\omega)$, there exists a one-form θ for which $(\omega|_{\partial(M\setminus C)}, \theta)$ is a cosymplectic structure.

Definition 3. *A* b**-cosymplectic cobordism** *is a compact b-symplectic manifold (M,ω), together with a compatible cosymplectic structure θ on its boundary ∂M. If its singular locus is empty we refer to it as a* **cosymplectic cobordism**.

A b-cosymplectic subcobordism of (M,ω,θ) is a b-symplectic submanifold W which intersects the kernel of θ transversally, so $(W, \omega_{|W}, \theta_{|\partial W})$ itself is a b-cosymplectic cobordism.

The normal bundle of the boundary of a b-cosymplectic cobordism is endowed with an orientation as follows: $v \in \Gamma(N(\partial M))$ is declared positive if $\iota_v(\omega^n)|_{\partial M}$ is a positive multiple of the volume form $\theta \wedge (\omega^{n-1}|_Z)$. A connected component X of the boundary ∂M is called *incoming* if positive normal vectors along X point into M, and *outgoing* otherwise; this gives a natural splitting $\partial M = \partial_{\text{in}} M \coprod \partial_{\text{out}} M$ into incoming and outgoing components.

Definition 4. *Let (Z_0,η_0,θ_0), (Z_1,η_1,θ_1) be cosymplectic manifolds. We say that Z_0 is* **cosymplectic cobordant** *to Z_1 if there exists a cosymplectic cobordism (M,ω,θ) and diffeomorphisms of cosymplectic manifolds*

$$\varphi_0 : (\partial_{\text{in}} M, \omega|_{\partial_{\text{in}} M}, \theta) \xrightarrow{\sim} (Z_0,\eta_0,\theta_0), \quad \varphi_1 : (\partial_{\text{out}} M, \omega|_{\partial_{\text{out}} M}, \theta) \xrightarrow{\sim} (Z_1,\eta_1,\theta_1).$$

A cosymplectic manifold (Z,η,θ) will be called **symplectically fillable** if it is cosymplectic cobordant to the empty set.

3. Composition and factorization of b-cosymplectic cobordisms

Here we describe a gadget we shall exploit in our construction of b-symplectic submanifolds, and which clarifies the relation between b-symplectic manifolds and cosymplectic cobordisms.

The construction goes essentially as follows: if we are given cosymplectic cobordisms sharing a common cosymplectic boundary, we can *glue* the cobordisms provided the orientations be the right ones, and there results a cosymplectic cobordism.

Not too surprisingly, if the orientations are 'wrong', one can still glue the cobordisms, but now the boundaries along which one glues are converted into a singular locus, and the upshot is a b-cosymplectic cobordism.

In the same train of thought, we describe an inverse procedure of *cutting* b-symplectic manifolds into cosymplectic cobordisms. These operations are well-behaved enough to allow us to turn problems in b-symplectic geometry into problems about cosymplectic cobordisms.

3.1. **Collars.** We must now take a technical detour to develop normal forms for b-cosymplectic cobordisms around singular loci and boundaries. Normal forms will be crucial in our strategy to construct b-symplectic submanifolds; they will also be useful to prove that the composition of b-cosymplectic cobordisms is canonically a b-symplectic cobordism. Normal forms are obtained by variations of standard arguments in symplectic Geometry, so we will be omitting unnecessary details; perhaps the relevant observation is that cosymplectic boundaries and coorientable singular loci can be treated almost on equal footing.

Let (M, ω, θ) be a b-cosymplectic cobordism. There are distinguished closed, coorientable, codimension-one submanifolds of M: the components of the boundary ∂M, and the coorientable components of the singular locus $Z = Z(\omega)$; we shall refer to those as distinguished submanifolds. If $W \hookrightarrow (M, \omega, \theta)$ is a b-cosymplectic subcobordism, by our conventions W is automatically transverse to all distinguished submanifolds $X \subset M$, and $X \cap W \subset W$ is distinguished.

For each distinguished X and $\epsilon > 0$ we let $I(X, \varepsilon) \subset X \times \mathbb{R}$ denote:

$$I(X,\varepsilon) := \begin{cases} (x,t), X \times (\varepsilon, 0] & \text{if } X \subset \partial_{\text{out}} M; \\ (x,t), X \times [0, \varepsilon) & \text{if } X \subset \partial_{\text{in}} M; \\ (x,t), X \times (-\varepsilon, \varepsilon) & \text{if } X \subset Z(\omega). \end{cases}$$

A **collar** for a distinguished $X \subset M$ is an embedding $c : I(X, \varepsilon) \hookrightarrow M$, for some $\varepsilon > 0$, which extends the identity map id_X.

If c is a collar for $X \subset Z$, then for any $\eta \in \Omega^*(Z)$, its pull back $\mathrm{pr}^*\eta$ –in principle a form on $I(X, \varepsilon)$– also defines a b-form on $I(X, \varepsilon)$. The reason is that the restriction of $\mathrm{pr}^*\eta$ to $I(X, \varepsilon)\backslash X$ (uniquely) extends to a b-form on $I(X, \varepsilon)$. We shall abuse notation an interpret $\mathrm{pr}^*\eta$ both as a form and a b-form as needed.

Definition 5. *A collar $c : I(X, \varepsilon) \hookrightarrow M$ for a distinguished $X \subset M$ of a b-cosymplectic cobordism is called* **adapted** *if:*

- *c pulls back ω to a model form $\mathrm{pr}^*\eta + dt \wedge \mathrm{pr}^*\theta$ when $X \subset \partial M$;*
- *c, regarded as a b-map $c\colon\ (I(X, \varepsilon), X) \to (M, X)$, pulls back ω to a model b-form $\mathrm{pr}^*\eta + d\log|t| \wedge \mathrm{pr}^*\theta$, where (η, θ) denotes a cosymplectic structure on $X \subset Z$.*

Given a b-symplectic subcobordism W, the adapted collar c is called a W**-collar** *if the restriction of c to $I(X \cap W, \epsilon_{|X\cap W})$ is an adapted collar for $X \cap W$.*

Here is the technical result we need:

Lemma 2. *Let (M, ω, θ) be a cosymplectic cobordism and W a b-symplectic subcobordism:*

1. *W-collars exist for every distinguished X.*
2. *If $c, c' : I(X, \epsilon) \hookrightarrow M$ are W-collars, then there exist $0 < \delta \leq \epsilon$ and a W-collar $\overline{c} : I(X, \delta) \hookrightarrow M$, agreeing with c on $I(X, \delta) \diagdown I(X, \frac{2\delta}{3})$, and with c' on $I(X, \frac{\delta}{3})$.*

Proof.

(1) It is routine to check that a collar $c : I(X,\varepsilon) \hookrightarrow M$ is adapted if and only if $L(v)\omega = 0$, where $v = c_*(\frac{\partial}{\partial t})$ if $X \subset \partial M$, and $v = c_*(t\frac{\partial}{\partial t})$ if $X \subset Z(\omega)$. It is a W-collar if and only if v is tangent to W.

Let X be a distinguished manifold, and choose a tubular neighborhood $p\colon E \to X$ with the property that $p^{-1}(W \cap X) = W \cap E$. Regard p as a b-vector bundle $p\colon (E, W \cap E) \to (X, W \cap X)$, and consider the b-vector field $v := \omega^{-1}(p^*\theta) \in \mathfrak{X}(E)$. Observe that it is vertical and b-symplectic, $L(v)\omega = 0$.

When $X \subset \partial M$ we can define the collar $c\colon X \times (-\varepsilon, \varepsilon) \to M$ by $c(x,t) = \phi^t(x)$, where ϕ^t denotes the time-t flow of v. By the characterization in the first paragraph c is a W-collar.

In the case $X \subset Z(\omega)$, $v = e^f\mathcal{E}$, where $\mathcal{E}$ denotes the Euler vector field of E and f a function vanishing along X. Hence v has the same linear part along X as $\mathcal{E}$, and its local flow $\phi\colon E \times \mathbb{R} \supset \mathrm{dom}(\phi) \to E$ is defined for all negative times, $\{(x,s) \in E \times \mathbb{R} : s < \varepsilon\} \subset \mathrm{dom}(\phi)$, for some $\varepsilon > 0$; since $\lim_{s \to -\infty} \phi^s(x) = p(x)$ for all $x \in E$, by [22] there is a uniquely determined, fibered diffeomorphism $c : I(X,\varepsilon) \hookrightarrow E$ fixing X pointwise and pushing forward $t\partial/\partial t$ to v; since v is tangent to W, c is a W-collar.

(2) Let v, v' be the distinguished b-vector fields determined by the W-collars c, c'. The affine combination $v_s := (1-s)v + sv'$, $s \in [0,1]$, defines a smooth family of b-symplectic vector fields, all tangent to W, which correspond to W-collars $c_s\colon I(X,\delta) \to M$, for some $\varepsilon > 0$, connecting $c|_{I(X,\delta)}$ to $c'|_{I(X,\delta)}$.

The equality $\frac{d}{ds}(c_s^*\omega) = 0$ implies that the b-vector fields $w_s := \frac{dc_s}{ds}$, $s \in [0,1]$, are all b-symplectic, so $\omega(w_s) =: \alpha_s$ are closed one-forms defined on the image of c_s. But since $c_s|_X = \mathrm{id}_X$ for all $s \in [0,1]$, we see that w_s vanishes along X, so there is a smooth family $s \mapsto f_s \in C^\infty(\mathrm{im}\, c_s)$ with $\alpha_s = df_s$. Choose a family of functions $\varrho_s\colon M \to [0,1]$, ϱ_s identically one on $c_s(I(X,\delta/3))$, and with support in $c_s(I(X,2\delta/3))$. The isotopy ψ^s generated by the time-dependent b-vector field $\overline{w}_s := -\omega^{-1}d(\varrho_s f_s)$ satisfies $\psi_s^*\omega = \omega$, $\psi_{s*} = \mathrm{id}$ at points $x \in X$, and $\psi_1 c = c'$ around X.

The adapted collar $\overline{c} := \psi_1 c$ then does the required job. □

Recall from the Introduction that two cosymplectic structures $(\eta,\theta), (\eta',\theta')$ are called b-**equivalent** if $\theta' = \theta$ and $\eta' = \eta + d(f\theta)$, for some $f \in C^\infty(Z)$. Item (1) in Lemma 2 for $W = \varnothing$ recovers[2] the following result from [12]:

[2]Firstly, non-coorientable components of the singular locus can be dealt with going to the coorientable covering space. Secondly, distinguished submanifolds of cosymplectic cobordism are by definition compact. Collars for non-compact distinguished submanifolds are defined by replacing ϵ by an strictly positive function of the submanifold. The proof of Lemma 2 also produces adapted collars this more general setting.

Proposition 1. *There is a canonical b-cosymplectic structure on the singular locus of a b-symplectic manifold.*

3.2. **Composition.**

Proposition 2. *Let $(M_0, \omega_0, \theta_0)$ and $(M_1, \omega_1, \theta_1)$ be b-cosymplectic cobordisms, and suppose*

$$\varphi_i \colon X \to \partial M_i, \quad i = 0, 1,$$

embed X as a sum of connected components, and induce the same cosymplectic structure on X:

$$\varphi_0^* \theta_0 = \varphi_1^* \theta_1, \quad \varphi_0^* \eta_0 = \varphi_1^* \eta_1.$$

Then the space

$$M := M_0 \cup_X M_1 = \frac{M_0 \coprod M_1}{\varphi_0(x) \sim \varphi_1(x)}$$

carries a canonical isomorphism type of b-cosymplectic cobordism.

Moreover, the data of b-cosymplectic subcobordisms $W_i \subset (M_i, \omega_i, \theta_i)$, satisfying $\varphi_0^{-1} W_0 = \varphi_1^{-1} W_1$, gives rise to a well-defined isotopy class of b-cosymplectic subcobordisms $W \subset (M, \omega, \theta)$.

We will refer to M as the composition of the b-cosymplectic cobordisms $(M_0, \omega_0, \theta_0)$ and $(M_1, \omega_1, \theta_1)$ along φ.

Proof. Decompose X as $X = X_- \coprod X_+$, where X_+ stands for the sum of those connected components Y of X with the property that $\varphi_0(Y)$ is incoming if and only if $\varphi_1(Y)$ is incoming, and X_- for the sum of those Y with $\varphi_0(Y)$ incoming if and only if $\varphi_1(Y)$ is outgoing.

As will shortly become apparent, by the inductive nature of the recipe we can assume without loss of generality that X connected. Two are then the cases to consider: $X = X_-$ and $X = X_+$.

Let $X{=}X_+$ and assume without loss of generality that $\varphi_0(X)$ and $\varphi_1(X)$ are both incoming. Fix $c_i \colon X \times [0, 1) \to M_i$ any adapted W_i-collars, $c_i^* \omega_i = \mathrm{pr}^* \eta + dt \wedge \mathrm{pr}^* \theta$, and choose an even function $f \colon [-1, 0) \cup (0, 1] \to \mathbb{R}$, with

$$df \neq 0, \quad df|_{U_0} = \frac{dt}{t}, \quad df|_{U_{\pm 1}} = dt,$$

where U_0 denotes a neighborhood of the end 0 and $U_{\pm 1}$ a neighborhood of $\{-1, +1\}$. Assign to f the b-symplectic form $\omega_f := \mathrm{pr}^* \eta + df(t) \wedge \mathrm{pr}^* \theta$ on $(X \times [-1, +1], X \times 0)$. Because f is even the reflection along X,

$$\iota : X \times [-1, +1] \xrightarrow{\sim} X \times [-1, +1], \quad (x, t) \mapsto (x, -t),$$

defines a b-symplectic involution: $\iota^* \omega_f = \omega_f$. Define

$$M := (M_0 \diagdown c_0(X \times [0, 1] \diagdown U_{\pm 1})) \cup_{c_0} X \times [-1, +1] \cup_{c_1 \iota} (M_1 \diagdown c_1(X \times [0, 1] \diagdown U_{\pm 1}))$$

and observe that the b-symplectic forms

$$\omega_0|_{(M_0 \diagdown c_0(X \times [0,1] \diagdown U_{\pm 1}))}, \quad \omega_f, \quad \omega_1|_{(M_1 \diagdown c_1(X \times [0,1] \diagdown U_{\pm 1}))}$$

glue into a b-symplectic form ω, having X as its singular locus; moreover, the compatible cosymplectic structures θ_0, θ_1 induce a compatible cosymplectic structure $(\partial M, \omega, \theta)$, and

$$Z(\omega) = Z(\omega_0) \coprod X \coprod Z(\omega_1).$$

By our choice of collars, the recipe glues the submanifolds W_i into an embedded submanifold $W \subset (M, \omega, \theta)$.

The indeterminacy in the composition lies in the choices of W_i-collars c_0, c_1, and of function f, and we momentarily write $M = M(c_0, c_1, f)$ to highlight the particular choices.

Suppose f' is another function with the same properties as f; observe then that $\omega_{f'} - \omega_f$ is an honest two-form on $X \times [-1, +1]$, with $\omega_{f'} - \omega_f = d\alpha$, where $\alpha \in \Omega^1(X \times [-1, +1])$ is a one-form with zero germ along $X \times t$ for $t = 0, \pm 1$. The usual Moser argument now applies: the family $s \mapsto \omega_s := \omega_f + s d\alpha$ is b-symplectic, and the time-dependent b-vector field $v_s := -\omega_s^{-1}\alpha$ generates a b-isotopy ϕ^s which is stationary around X and $X \times \{\pm 1\}$, and stabilizes ω_s: $\phi^{s*}\omega_s = \omega_0$. In particular, $\phi^{1*}\omega_{f'} = \omega_f$, and so ϕ^1 induces a b-symplectomorphism $M(c_0, c_1, f) \xrightarrow{\sim} M(c_0, c_1, f')$ which is identical outside the collars. So the isomorphism type of $M(c_0, c_1, f)$, as well as the isotopy class of W, are independent of the particular f we pick.

Now suppose we choose a different pair c_0', c_1' of collars. Then $c_0' c_0^{-1}$ and $c_1' c_1^{-1}$ induce a homeomorphism $h\colon M(c_0, c_1, f) \to M(c_0', c_1', f)$, mapping W onto W', and restricting to diffeomorphisms

$$h_0 : X \times (-1, 0] \xrightarrow{\sim} X \times (-1, 0], \quad h_1 : X \times [0, +1) \xrightarrow{\sim} X \times [0, +1)$$

which fix $X = X \times 0$ pointwise.

According to Lemma 2, c_0 can be modified to another adapted W_0-collar $\overline{c_0}\colon I(X, \delta) \to M$, which agrees with c_0 on $X \times (-\delta, 2\delta/3]$, and with c' on $X \times [-\delta/3, 0]$. Similarly, find a W_1-collar $\overline{c_1}$ agreeing with c_1' on $X \times [0, \delta/3)$, and with c_1 on $X \times [2\delta/3, \delta)$. Then $c_0'\overline{c_0}^{-1}$, $c_1'\overline{c_1}^{-1}$ and $h|_{M(c_0,c_1,f) \setminus c_0[-2\delta/3,0] \cup c_1[0,2\delta/3]}$ glue into an isomorphism $\overline{h} : M(c_0, c_1, f) \xrightarrow{\sim} M(c_0', c_1', f)$, $\overline{h}W = W'$.

The case $X{=}X_-$ –which can be proved following the same pattern– is a well-known result in Symplectic Geometry. □

Remark 1. *Note that a general composition of b-cosymplectic cobordisms need* not *be orientable, even if each of the b-cobordisms is orientable. However, we can ensure the orientability of the composition, provided both b-cosymplectic cobordisms be orientable and either $X_- = \varnothing$ or $X_+ = \varnothing$; this will be the case in our applications to the realization of cosymplectic manifolds as singular loci.*

Thus b-symplectic manifolds appear quite naturally when dealing with cosymplectic cobordisms. We should perhaps stress that, to our knowledge, this is the sole general construction of compact b-symplectic manifolds without boundary:

Corollary 1. *The double $M \cup_{\partial M} M$ of a cosymplectic cobordism (M, ω, θ) carries a canonical isomorphism type of oriented b-symplectic manifold without boundary.*

3.3. **Factorization.** Note also that, by the same token, we can *factorize* b-cosymplectic cobordism (M, ω, θ) into *cosymplectic* cobordisms.

Proposition 3. *Every b-cosymplectic cobordism (M, ω, θ) with coorientable singular locus is a composition of cosymplectic cobordisms.*

Proof. The construction associates to a connected component $X \subset Z(\omega)$ a pair of b-cosymplectic cobordisms $(M_0, \omega_0, \theta_0)$ and $(M_1, \omega_1, \theta_1)$, and embeddings $\varphi_i : X \hookrightarrow \partial M_i$, where

$$(M_0 \smallsetminus \partial M_0) \coprod (M_1 \smallsetminus \partial M_1) = M \smallsetminus X, \quad \partial M_0 \coprod \partial M_1 = \partial M \coprod X \coprod X$$
$$Z(\omega_0) \coprod Z(\omega_1) = Z(\omega) \smallsetminus X, \quad \theta_i = \varphi_i^* \flat(\omega), \quad \varphi_0^* \omega_0 = \varphi_1^* \omega_1,$$

in such a way that M is recovered as the composition of M_0 and M_1 along $\varphi = (\varphi_0, \varphi_1)$. Repeating the procedure for connected components $X_i \subset Z(\omega_i)$ (and so forth) one ultimately achieves the situation where none of the b-cosymplectic cobordisms has singular points, which is what is claimed.

So let $X \subset Z$ connected be given. Choose an adapted collar $c \colon X \times (-\varepsilon, \varepsilon) \to M$, $c^* \omega = d \log |t| \wedge \mathrm{pr}^* \theta + \mathrm{pr}^* \eta$, and $f \colon \mathbb{R} \to \mathbb{R}$ a function satisfying $f'(t) \neq 0$, $f'(t) = 1$ for $t \leqslant \varepsilon/3$, and $f'(t) = 1/t$ for $t \geqslant 2\varepsilon/3$. Then define $f_+ := f|_{[0,\varepsilon)}$ and $f_-(t) := f_+(-t)$. Consider the *symplectic* forms

$$\omega_- := \mathrm{pr}^* \eta + df_- \wedge \mathrm{pr}^* \theta \in \Omega^2(X \times (-\varepsilon, 0]), \quad \omega_+ := \mathrm{pr}^* \eta + df_+ \wedge \mathrm{pr}^* \theta \in \Omega^2(X \times [0, \varepsilon)).$$

Now, $M \smallsetminus X$ consists of two connected components, the closures of which we denote M_0 and M_1; we convene that M_0 meets $c(X \times -\varepsilon/2)$ while M_1 meets $c(X \times \varepsilon/2)$.

The b-symplectic forms $\omega|_{M_0 \smallsetminus \mathrm{im}\, c}$ and $c_* \omega_-$ glue into a well-defined b-symplectic form ω_0 on M_0, just as $\omega|_{M_1 \smallsetminus \mathrm{im}\, c}$ and $c_* \omega_+$ glue into a b-symplectic ω_1 on M_1; singular points of ω_i are precisely those singular points of ω lying in M_i but not in X. We take for $\varphi_i : X \hookrightarrow \partial M_i$ the identity, and set $\theta_i \in \Omega^1(\partial M_i)$ equal to $\flat(\omega)$ along X and to θ along $\partial M_i \smallsetminus X$.

It is now straightforward to check that the composition of the b-cosymplectic cobordisms $(M_0, \omega_0, \theta_0)$ and $(M_1, \omega_1, \theta_1)$ recovers (M, ω, θ). □

4. Submanifolds of b-symplectic manifolds

In this section we prove:

Theorem 1. *Every (M, ω) compact b-symplectic manifold without boundary has closed b-symplectic submanifolds $W \hookrightarrow (M, \omega)$ of any even dimension.*

In the symplectic case (i.e., when $Z(\omega) = \varnothing$), the existence of symplectic submanifolds of compact symplectic manifolds is due to Donaldson, and follows from the *approximately holomorphic* techniques developed in [5] and [1]. In a nutshell, it is there shown how, for an integral, compact, symplectic manifold (M, ω), equipped with a compatible almost-complex structure, one may construct sections $s_k \in \Gamma(M, L^{\otimes k})$ of the tensor powers of the prequantum line bundle $L \to M$ of ω, in such a way that, as k grows large, the zero set of these define submanifolds $W_k \subset M$ which become as close to being J-complex as we like, meaning that

$$\mathrm{dist}(TW_k, J(TW_k)) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty.$$

(The distance here is measured according the induced metric on the Grassmannian of real, codimension-two distributions on M.) As a consequence, W_k is a symplectic submanifold for $k \gg 0$.

The techniques have been tweaked to take into account a given real hypersurface and free, finite group actions. We record below for future reference the version which will be of use to us:

Theorem 5. *Let (M, ω, θ) be a cosymplectic cobordism. If ω is* rational, *$[\omega] \in H^2(M; \mathbb{Q})$, and J is a compatible almost-complex structure on M whose Levi distribution on ∂M coincides with* $\ker\theta$, *$T\partial M \cap JT\partial M = \ker\theta$, then:*

1. *through every point of M there pass a sequence of submanifolds $W_k \subset (M, \omega, \theta)$ of codimension two which, as $k \to \infty$, become as close to being J-complex as we like;*
2. *if θ and J are invariant under a given a free symplectic involution ι of an open U around ∂M, one can require that the submanifolds W_k be invariant under ι as well.*

Proof. See e.g. [21, §2, 2] for the existence of cosymplectic subcobordisms of rational cosymplectic cobordisms, and [14, §4.4], where it is explained how the 'globalization process' of Donaldson can be made compatible with free, semi-local actions of finite groups. □

The Theorem above suggests an obvious way to prove Theorem 1: factorizing a given b-symplectic manifold into cosymplectic cobordisms, applying Theorem 4 to produce symplectic subcobordisms, and then use Proposition 2 to glue these subcobordisms into a b-symplectic submanifold of our original manifold.

We need a technical lemma first, whose proof is deferred until the end of the section.

Lemma 3. *Let (M, ω) be a compact b-symplectic manifold, with $Z(\omega)$ coorientable.*

Then for any choice of adapted collar c, there is a sequence of b-symplectic forms $\omega_n \in {}^b\Omega^2(M, Z(\omega))$ such that:

1. *$Z(\omega_n) = Z(\omega)$;*
2. *$\omega_n \to \omega$;*
3. *$[\flat\omega_n] \in H^1(Z(\omega); \mathbb{Q})$;*

(4) $[\omega_n|_{M\smallsetminus Z(\omega)}] \in H^2(M\smallsetminus Z(\omega);\mathbb{Q})$;
(5) *c restricts on* $I(Z(\omega),\varepsilon/27)$ *to an adapted collar for each* ω_n.

Proof of Theorem 1. **CASE ONE:** $Z(\omega)$ **coorientable.**

Choose an adapted collar c around $Z(\omega)$, $c^*\omega = d\log|t| \wedge \mathrm{pr}^*\theta + \mathrm{pr}^*\eta$. Approximate ω by the sequence ω_n provided by Lemma 3; shrinking the collar if need be, we may assume without loss of generality that c is adapted to each ω_n, $c^*\omega_n = d\log|t| \wedge \mathrm{pr}^*\theta_n + \mathrm{pr}^*\eta_n$.

Using an auxiliary Riemmanian metric g_Z on $Z(\omega)$, construct almost-complex structures $J'\colon \ker\theta \to \ker\theta$ and $J'_n\colon \ker\theta_n \to \ker\theta_n$, compatible respectively with η, η_n, and with J'_n converging to J' as $n\to\infty$. Define then almost-complex structures J, J_n on $Z(\omega)\times(-\varepsilon,\varepsilon)$ by

$$J := J' + \left(dt\otimes\widetilde{v} + \mathrm{pr}^*\theta\otimes\frac{\partial}{\partial t}\right),\quad J := J' + \left(dt\otimes\widetilde{v}_n + \mathrm{pr}^*\theta_n\otimes\frac{\partial}{\partial t}\right),$$

where $\widetilde{v},\widetilde{v}_n$ are the horizontal lifts (for the horizontal connection) of the vector fields $v, v_n \in \mathfrak{X}(Z(\omega))$ determined by

$$\theta(v) = 1,\quad \eta(v) = 0,\quad \theta_n(v_n) = 1,\quad \eta_n(v_n) = 0.$$

Observe that J is compatible with ω, and J_n is compatible with ω_n, and $J_n \to J$.

Using the common collar c, factorize M as a composition of cosymplectic cobordisms (M_i,ω_i,θ_i):

$$M = M_0 \cup_{X_1} M_1 \cup_{X_2} \cdots \cup_{X_k} M_k,$$

where k is the number of connected components of $Z(\omega)$.

Let $(\widehat{M},\widehat{\omega},\widehat{\theta})$ stand for the cosymplectic cobordism $\coprod_{i=0}^k (M_i,\omega_i,\theta_i)$. Observe that, being adapted to c, each ω_n also endows $\widehat{M}$ with the structure of a cosymplectic cobordism $\coprod_{i=0}^k (M_i,\omega_{ni},\theta_{ni})$. The adapted collar c induces adapted collars $\widehat{c}\colon \partial\widehat{M}\times I(\partial\widehat{M},\varepsilon)\to\widehat{M}$ of $\partial\widehat{M}$, and there are induced almost-complex structures $\widehat{J},\widehat{J}_n$ on $\operatorname{im}\widehat{c}$, induced by J, J_n, which are compatible with $\widehat{w},\widehat{w}_n$, respectively. Out of these compatible pairs $(\widehat{J},\widehat{\omega}),(\widehat{J}_n,\widehat{\omega}_n)$, construct the corresponding Riemannian metrics $(\widehat{g},\widehat{g}_n)$ on $\operatorname{im}\widehat{c}$. Since $\widehat{g}_n\to\widehat{g}$, we can extend these metrics to metrics on the whole $\widehat{M}$, and retain the property that $\widehat{g}_n\to\widehat{g}$. Note also that the reflection $(x,t)\mapsto(x,-t)$ on $Z(\omega)\times(-\varepsilon,\varepsilon)$ induces a *free* involution $\iota\colon\widehat{c}\to\widehat{c}$, which preserves all structures defined on $\operatorname{im}\widehat{c}$: $\widehat{\omega},\widehat{\omega}_n,\widehat{J},\widehat{J}_n,\widehat{g},\widehat{g}_n,\widehat{\theta}$ and $\widehat{\theta}_n$.

Using $\widehat{g},\widehat{g}_n$, we construct, in the usual fashion, almost-complex structures $\widetilde{J},\widetilde{J}_n$ compatible with $\widetilde{\omega},\widetilde{\omega}_n$. Note that:

- $\widetilde{J}\to\widetilde{J}_n$;
- the hypotheses of Theorem 5 apply to each $(\widehat{M},\widehat{\omega}_n,\widehat{\theta}_n,\widetilde{J}_n,\iota)$.

Applying Theorem 5, we find ι-invariant submanifolds $\widehat{W}_{n,k} \subset (\widehat{M}, \widehat{\omega}_n, \widehat{\theta}_n)$, with

$$\lim_{k\to\infty} \operatorname{dist}(T\widehat{W}_{n,k}, \widetilde{J}_n T\widehat{W}_{n,k}) = 0.$$

Since $\widetilde{J}_n \to \widetilde{J}$, we conclude that, for large enough n and k, $\widehat{W}_{n,k}$ is a cosymplectic subcobordism of $(\widehat{M}, \widehat{\omega}, \widehat{\theta})$.

Let n, k be large enough so that the conclusion above hold true, and set $\widehat{W} := \widehat{W}_{n,k}$. Choose an adapted $\widehat{W}$-collar $\widetilde{c}$ for $\partial\widehat{M}$, and compose the cosymplectic cobordism $(\widehat{M}, \widehat{\omega}, \widehat{\theta})$ to get (M, ω, θ) back. According to Proposition 2, there ensues a closed b-symplectic submanifold $W \subset (M, \omega, \theta)$.

CASE TWO: $Z(\omega)$ **not coorientable.**

When indicate the necessary modifications of the above proof for the case where $Z(\omega)$ is not coorientable.

Let $p\colon \widetilde{M} \to M$ be the orientation covering of M, thought of as a $\mathbb{Z}_2$-principal bundle, with involution $\tau\colon \widetilde{M} \to \widetilde{M}$. Then $Z(p^*\omega) \subset \widetilde{M}$ is coorientable. Choose a $\mathbb{Z}_2$-equivariant tubular neighborhood $\widetilde{M} \supset E \to X$ of X and proceed as in Lemma 2 to find an adapted collar $c\colon I(Z(p^*\omega), \varepsilon) \hookrightarrow \widetilde{M}$ which is $\mathbb{Z}_2$-equivariant; that is, $\tau\circ c = c\circ(\tau|_X \times -\operatorname{id})$. Note that this action commutes with the involution $\iota = \operatorname{id} \times -\operatorname{id}$ of the collar.

Using a such equivariant collar, all of the objects $\widehat{\omega}, \widehat{\omega}_n, \widehat{J}, \widehat{J}_n, \widehat{g}, \widehat{g}_n, \widehat{\theta}$ and $\widehat{\theta}_n$ constructed in Case One can be additionally assumed to be invariant with respect to the symplectic involution $\widehat{\tau}$ of $(\widehat{M}, \widehat{\omega}, \widehat{\theta})$, $(\widehat{M}, \widehat{\omega}_n, \widehat{\theta}_n)$ arising from τ. Theorem 5 can be slightly modified to produce τ-invariant subcobordisms $\widehat{W}_{n,k} \subset (\widehat{M}, \widehat{\omega}_n, \widehat{\theta}_n)$, which are as close to being $\widetilde{J}_n$-complex as we desire, as $k \to \infty$. Again we deduce that, for $n, k \gg 0$, $\widehat{W} := \widehat{W}_{n,k}$ will also be a b-cosymplectic subcobordism in $(\widehat{M}, \widehat{\omega}, \widehat{\theta})$; choose a $\widehat{\tau}$-equivariant $\widehat{W}$-collar, and compose $\widehat{M}$ into a τ-invariant b-symplectic manifold, equivariantly isomorphic to the double cover of M with the pullback b-symplectic structure, carrying the (composed) closed, τ-invariant, b-symplectic submanifold $\widetilde{W}$. □

Remark 2. *The arguments above prove this slightly more general statement: every compact, b-cosymplectic cobordism has b-cosymplectic subcobordisms of all even codimensions.*

Proof of Lemma 3. Choose an adapted collar $c : (-\varepsilon, \varepsilon) \times Z(\omega) \hookrightarrow M$, $c^*\omega = \operatorname{pr}^*\eta + d\log|t| \wedge \operatorname{pr}^*\theta$, and let $\varrho\colon (-\varepsilon, \varepsilon) \to [0, 1]$ denote a function satisfying

$$\varrho(t) = 0, \text{ for } |t| \leqslant \varepsilon/3, \quad \varrho(t) = 1, \text{ for } |t| \geqslant 2\varepsilon/3.$$

Choose an approximation $\theta_n \to \flat\omega$ by rational, closed one-forms, and let $\widetilde{\theta}_n$ denote the one-form $\widetilde{\theta}_n := \operatorname{pr}^*\theta_n + \varrho(t)\operatorname{pr}^*(\flat\omega - \theta_n)$. Then $\widetilde{\theta}_n \to \operatorname{pr}^*\flat\omega$ as

$n \to \infty$, and so the closed b-forms $\omega_n' := d\log|t| \wedge \widetilde{\theta}_n + \mathrm{pr}^*\eta$ converge to ω, and $c^*\omega_n|_{I(Z(\omega),\varepsilon/3)} = d\log|t| \wedge \mathrm{pr}^*\theta_n + \mathrm{pr}^*\eta$.

Choose another function $\mu\colon (-\varepsilon/3, \varepsilon/3) \to \mathbb{R}$, with

$$\mu(t) = 1 \text{ for } |t| \leqslant \varepsilon/9, \quad \mu(t) = 0 \text{ for } |t| \geqslant 2\varepsilon/9,$$

so that the b-form $d(\mu(t)\log|t|) \wedge \mathrm{pr}^*\theta_n$ has support in the collar, and equals $d\log|t| \wedge \mathrm{pr}^*\theta_n$ on $(-\varepsilon/9, \varepsilon/9) \times Z(\omega)$.

Then $\varpi_n := \omega - c_*\left(d(\mu(t)\log|t|) \wedge \mathrm{pr}^*\theta_n\right)$ is an honest, closed two-form on M, which we can approximate by rational forms $\varpi_{n,m} \in H^2(M;\mathbb{Q})$.

Choose an odd function $\nu\colon (-\varepsilon/9, \varepsilon/9) \to (-\varepsilon/9, \varepsilon/9)$ satisfying

$$\nu(t) = 0 \text{ for } |t| \leqslant \varepsilon/27, \quad \nu(t) = t \text{ for } |t| \geqslant 2\varepsilon/27.$$

and let $\psi\colon (-\varepsilon/9, \varepsilon/9) \times Z(\omega) \to (-\varepsilon/9, \varepsilon/9) \times Z(\omega)$ be the map $\psi(t,x) = (\nu(t), x)$. Define closed, rational forms, $\varpi_{n,m}' \in \Omega^2(M)$ by

$$\varpi_{n,m}'(t,x) = \begin{cases} \varpi_n(t,x) + (c\psi c^{-1})^*(\varpi_{n,m} - \varpi_n)(t,x) & \text{for } |t| \leqslant 2\varepsilon/27 \\ \quad \varpi_{n,m}(t,x) & \text{for } |t| \geqslant 2\varepsilon/27. \end{cases}$$

The $\varpi_{n,m}'$ converge to ϖ_n, for each n, and $c^*\varpi_{n,m}' = \mathrm{pr}^*\left(\varpi_n|_{Z(\omega)}\right)$ on $(-\varepsilon/27, \varepsilon/27) \times Z(\omega)$. Hence the sequence

$$\omega_n := c_*\left(d(\mu(t)\log|t|) \wedge \mathrm{pr}^*\theta_n\right) + \varpi_{n,n}'$$

of closed b-forms satisfies the conditions required by the Lemma. □

5. h-principle

In this section, we use standard h-principle arguments to provide a complete answer to the Realization Problem (1) under the additional assumption that M be open, and defer a partial answer to (2) to Section 6.

A necessary condition for a manifold M^{2n} to be symplectic is that it carry a non-degenerate two-form, or, equivalently, an almost-complex structure. If M is compact, we have a further necessary condition, namely, that there be a degree-two cohomology class $\tau \in H^2(M)$ with $\tau^n \neq 0$.

For *open* manifolds M – that is, those manifolds, none of whose connected components is compact without boundary – a classical theorem of Gromov [10] states that the sole obstruction to the existence of a symplectic structure is that M be almost-complex. More precisely, given any non-degenerate two-form $\omega_0 \in \Omega^2(M)$ and any degree-two cohomology class $\tau \in H^2(M)$, there is a path $\omega\colon [0,1] \to \Omega^2(M)$ of non-degenerate two-forms connecting ω_0 to ω_1, $d\omega_1 = 0$, $[\omega_1] = \tau$.

We consider now the case of b-symplectic structures. Recall that b-symplectic manifolds need *not* be oriented as usual manifolds, so in particular they may fail to be almost-complex. However:

Lemma 4. *If an orientable M admits a b-symplectic structure ω, then $M \times \mathbb{C}$ is almost-complex.*

We follow the argument in [2, §4].

Proof. Let $c : I(Z(\omega), \varepsilon) \hookrightarrow M$ be an adapted collar, $c^*\omega = d\log|t| \wedge \mathrm{pr}^*\theta + \mathrm{pr}^*\eta$. Split $TZ(\omega)$ into a direct sum $TZ(\omega) = \langle v \rangle \oplus \ker\theta$, where v is as usual the vector field $\eta(v) = 0$, $\theta(v) = 1$, and fix an almost-complex structure $J'\colon \ker\theta \to \ker\theta$ compatible with η. Let now $E \subset TI(X,\varepsilon)$ denote the rank-two subbundle spanned by $\frac{\partial}{\partial t}$ and $\widetilde{v}$, where $\widetilde{v} \in \mathfrak{X}(I(X,\varepsilon))$ is the unique vector field tangent to the fibers of pr_1 and satisfying $\mathrm{pr}_{2*}\widetilde{v} = v$, and observe that $TI(X,\varepsilon) = \mathrm{pr}^*\ker\theta \oplus E$. Define an almost-complex structure J_{in} on $T\left(I(X,\varepsilon) \setminus X\right)$ by

$$J_{\mathrm{in}} := \mathrm{pr}^* J' \oplus J''\colon (\mathrm{pr}^*\ker\theta \oplus E)|_{I(X,\varepsilon)\setminus X} \to (\mathrm{pr}^*\ker\theta \oplus E)|_{I(X,\varepsilon)\setminus X},$$

where $J''\colon E|_{I(X,\varepsilon)\setminus X} \to E|_{I(X,\varepsilon)\setminus X}$ denotes any almost-complex structure with

$$J'' = \begin{cases} \frac{dt}{t} \otimes \widetilde{v} - \mathrm{pr}^*\theta \otimes t\frac{\partial}{\partial t} & \text{outside } I(Z(\omega, 2\varepsilon/3)); \\ -dt \otimes \widetilde{v} - \mathrm{pr}^*\theta \otimes \frac{\partial}{\partial t} & \text{on } I(Z(\omega, \varepsilon/3)) \cap \{t < 0\} \\ dt \otimes \widetilde{v} - \mathrm{pr}^*\theta \otimes \frac{\partial}{\partial t} & \text{on } I(Z(\omega, \varepsilon/3)) \cap \{t > 0\}. \end{cases}$$

Observe that J_{in} is compatible with $c^*\omega|_{I(Z(\omega),\varepsilon)\setminus I(Z(\omega),2\varepsilon/3)}$.

Now choose an almost-complex structure J_{out} on $M \setminus cI(Z(\omega), \varepsilon/3)$, compatible with ω. Since the space of almost-complex structures compatible with ω on $M \setminus cI(Z(\omega), 2\varepsilon/3)$ is contractible, J_{out} can be chosen so that $c_* J_{\mathrm{in}}$ and J_{out} glue into a well-defined almost-complex structure J on $M \setminus Z(\omega)$.

Now define an almost-complex structure $\widehat{J}$ on $I(X, \varepsilon/3) \times \mathbb{R}^2$ as follows: on the pullback of $\ker\theta$, $\widehat{J}$ acts as J'. On $\mathrm{pr}^*_{I(X,\varepsilon/3)} E \oplus \mathrm{pr}^*_{\mathbb{R}^2} T\mathbb{R}^2$, $\widehat{J}$ can be described in terms of the basis $\{\frac{\partial}{\partial t}, \widetilde{v}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\}$ as

$$\begin{pmatrix} 0 & -\cos(\vartheta(x,t)) & 0 & -\sin(\vartheta(x,t)) \\ \cos(\vartheta(x,t)) & 0 & -\sin(\vartheta(x,t)) & 0 \\ 0 & \sin(\vartheta(x,t)) & 0 & -\cos(\vartheta(x,t)) \\ \sin(\vartheta(x,t)) & 0 & \cos(\vartheta(x,t)) & 0 \end{pmatrix},$$

where $\vartheta \in C^\infty(I(Z(\omega), \varepsilon))$ is a function satisfying

$$\vartheta_{|Z(\omega)\times[-\varepsilon,-\varepsilon/3]} = -\pi, \quad \vartheta_{|Z(\omega)\times[-\varepsilon/6,\varepsilon/6]} = \tfrac{6\pi t}{\varepsilon}, \quad \vartheta_{|Z(\omega)\times[\varepsilon/3,\varepsilon]} = \pi.$$

Then $\widetilde{J}\colon T(M \times \mathbb{R}^2) \to T(M \times \mathbb{R}^2)$ defined by

$$\widetilde{J} := \begin{cases} J \oplus -i & \text{on } (M \setminus cI(Z(\omega), \varepsilon/3)) \cap \{t > 0\}; \\ (c \times \mathrm{id})_* \widehat{J} & \text{on } cI(Z(\omega), \varepsilon/3); \\ J \oplus i & \text{on } (M \setminus cI(Z(\omega), \varepsilon/3)) \cap \{t < 0\}. \end{cases}$$

is a well-defined almost-complex structure on $M \times \mathbb{R}^2$. □

Just as in the symplectic case, if we demand that M be compact, obstructions in cohomology appear:

(1) as explained in [17], if M^{2n} is compact b-symplectic, then there exists a cohomology class $\tau \in H^2(M)$ with $\tau^{n-1} \neq 0$;
(2) furthermore, if M is compact, orientable and b-symplectic, a non-trivial $\vartheta \in H^2(M)$ must exist squaring to zero: $\vartheta^2 = 0$ [4].

None of these obstructions appear when M is open, so one wonders if, in that case, $M \times \mathbb{C}$ being almost-complex is sufficient to ensure that M carries a b-symplectic structure. We answer the question in the affirmative:

Theorem 2. *Let M be an orientable, open manifold. Then M is b-symplectic if and only if $M \times \mathbb{C}$ is almost-complex.*

We need to introduce the analogs of non-degenerate two-forms, which we do in two stages:

Definition 6. *A* **transversally non-degenerate** *bivector $\pi \in \mathfrak{X}^2(M^{2n})$ is a bivector whose top exterior power $\bigwedge^n \pi$ is transverse to the zero section.*

In $\mathfrak{X}^2_{\pitchfork}(M) \subset \mathfrak{X}^2(M)$ we collect all such transversally non-degenerate bivectors. To every $\pi \in \mathfrak{X}^2_{\pitchfork}(M)$ there corresponds a b-manifold $(M, Z(\pi))$, $Z(\pi) := (\bigwedge^n \pi)^{-1} M \subset M$.

Since b-bivectors $\mathfrak{X}^2(M, Z(\pi))$ sit inside the space of all bivectors $\mathfrak{X}^2(M)$, it makes sense to ask if a given transversally non-degenerate π is a b-bivector in the b-manifold it defines. (As a simple coordinate check shows, the space of b-bivectors on a b-manifold (M, Z) neither contains nor is contained in the space of transversally non-degenerate bivectors having Z as singular locus).

Definition 7. *A bivector $\pi \in \mathfrak{X}^2(M)$ is* b-**serious** *if it is transversally non-degenerate and a b-bivector in $(M, Z(\pi))$.*

Lemma 5. *A transversally non-degenerate π is b-serious if and only if $\pi^\sharp_x(T^*_x M|_{Z(\pi)}) \subset T_x Z(\pi)$ for all $x \in Z(\pi)$. The subspace $Y \subset Z$ of points where the condition is satisfied is a sum of connected components.*

Proof. The first claim is immediate from the definitions. For the second, let π be transversally non-degenerate, and define Y as in the statement, $Y := \{x \in Z(\pi) : \pi(T^*_x M) \subset T_x Z\}$. It is clearly a closed subset of $Z(\pi)$. Choose a collar c of $Z(\pi)$, and write $c^*\pi$ as $\frac{\partial}{\partial t} \wedge v + \nu$, where $v \in \mathfrak{X}(Z(\pi) \times (-\varepsilon, \varepsilon))$, $\nu \in \mathfrak{X}^2(Z(\pi) \times (-\varepsilon, \varepsilon))$ vanish identically on dt, $v(dt) = 0 = \nu(dt)$.

Suppose $c(x) \in Y$, that is, $v_x = 0$. Then $\bigwedge^n \pi = n \frac{\partial}{\partial t} \wedge v \wedge \bigwedge^{n-1} \nu$ vanishes at x. On the other hand, $\bigwedge^n \pi$ is transverse to M at x, so $\bigwedge^{n-1} \nu_x \neq 0$; hence $\bigwedge^{n-1} \nu \neq 0$ on some neighborhood $U \subset M$ of x. But $\bigwedge^n \pi|_{U \cap Z(\pi)} = 0$ then implies $v_{x'}$ for all $x' \in U \cap Z(\pi)$, so $U \cap Z(\pi) \subset Y$. Hence Y is open and closed in $Z(\pi)$. □

So, to check whether a given transversally non-degenerate π is b-serious, it suffices to check whether π is tangent to $Z(\pi)$ at one single point in each connected component of $Z(\pi)$.

Corollary 2. *Let $t \mapsto \pi_t$ be a smooth family of transversally non-degenerate bivectors. Then either all π_t are b-serious or none is. In particular, a transversally non-degenerate π can only be homotopic through such bivectors to one which is Poisson if it is b-serious.*

Proof. Let $\widetilde{\pi} \in \mathfrak{X}^2(M \times \mathbb{R}^2)$ be the bivector $\widetilde{\pi} := \pi_t + \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial s}$. Then $\bigwedge^n \pi_t \pitchfork M$ for all t if and only if $\bigwedge^{n+1} \widetilde{\pi} \ \pitchfork\ M \times \mathbb{R}^2$, and $\widetilde{\pi}(T^*(M \times \mathbb{R}^2)) \subset TZ(\widetilde{\pi})$ if $\pi_t(T^*M) \subset TZ(\pi_t)$ for all t. This proves the first claim. The second then follows from Lemma 1. ☐

So it is clear from the start that being b-serious is a necessary condition for a transversally non-degenerate bivector to be deformed to a b-symplectic structure through such bivectors. In the sequel we show that the condition is also sufficient if the manifold is open:

Theorem 6. *On an* open *manifold M, a transversally non-degenerate bivector π_0 is homotopic in $\mathfrak{X}^2_{\pitchfork}(M)$ to a Poisson bivector π_1 if and only if π_0 is b-serious. Moreover, one can arrange that $Z(\pi_1)$ be non-empty if $Z(\pi_0)$ is non-empty.*

This statement is a result of checking that 1-jets of Poisson bivectors of b-symplectic type forms a microflexible differential relation, invariant under the pseudogroup of local diffeomorphisms of M, cf. [10]. We opted instead to follow the somewhat more visual scheme of proof of [7].

Proof. Take $\pi_0 \in \mathfrak{X}^2_{\pitchfork}(M)$ b-serious, $\pi_0 \in \mathfrak{X}(M, Z_0)$.

Observe that the b-differential ${}^b d : {}^b\Omega^p(M, Z_0) \longrightarrow {}^b\Omega^{p+1}(M, Z_0)$ can be factored as a composition ${}^b d = \widetilde{\mathrm{symb}}({}^b d) \circ j_1$, where j_1 denotes the 1-jet map

$$j_1 : \Gamma(M, \bigwedge^p {}^bT^*(M, Z_0)) \longrightarrow \Gamma(M, J_1 \bigwedge^p {}^bT^*(M, Z_0))$$

and

$$\widetilde{\mathrm{symb}}({}^b d) : \Gamma(M, J_1 \bigwedge^p {}^bT^*(M, Z_0)) \longrightarrow \Gamma(M, \bigwedge^{p+1} {}^bT^*(M, Z_0))$$

is induced by a bundle map

$$\mathrm{symb}({}^b d) : J_1 \bigwedge^p {}^bT^*(M, Z_0) \longrightarrow \bigwedge^{p+1} {}^bT^*(M, Z_0).$$

As one easily checks, $\mathrm{symb}({}^b d)$ is an epimorphism with contractible fibres; in particular, we can lift ω_0 to $\widetilde{\omega}_0 \in \Gamma(M, J_1 \bigwedge^p {}^bT^*(M, Z_0))$.

Now, since M is an open manifold, there exists a a subcomplex K of a smooth triangulation of M, of positive codimension, with the property that, for an arbitrarily small open $U \subset M$ around K, there exists an isotopy of open embeddings $g_t : M \hookrightarrow M$, $h_0 = \mathrm{id}_M$, with $g_1(M) \subset U$ and $g_t|_K = \mathrm{id}_K$. We will refer to K as a core of M, and say that g_t compresses M into U. Note in passing that one can always find a core K of M meeting Z_0.

Fix then a core K of M, and a compression of M into an open U around K. The Holonomic Approximation theorem of [7] then says that we can find

- an isotopy h_t of M mapping K into U;
- an open $V \subset U$ around $h_1(K)$;
- a section $\alpha \in \Gamma(V, {}^bT^*(M, Z_0))$

such that $j_1\alpha$ is so C^0-close to $\widetilde{\omega}_0$ that we can find a homotopy

$$\widetilde{\omega}(t) \in \Gamma(V, J_1{}^bT^*(M, Z_0)),$$

connecting $\widetilde{\omega}_0|_V$ to $j_1\alpha$, and with $\widetilde{\mathrm{symb}}({}^bd)\widetilde{\omega}_t$ non-degenerate b-forms on V.

Now regard the compression g_t as a smooth family of b-maps

$$g_t : (M, Z_t) \longrightarrow (M, Z_0), \quad Z_t := g_t^{-1}Z_0,$$

and set $\omega_1 := {}^bd(g_1^*\alpha) \in {}^b\Omega^2(M, Z_1)$. Observe now that $\widehat{\omega}_t^1 := g_t^*\widetilde{\omega}_0$ connects $\widetilde{\omega}_0$ to $g_1^*(\widetilde{\omega}_0|_V)$, and $\widehat{\omega}_t^2 := g_1^*\widetilde{\omega}(t)$ connects $g_1^*(\widetilde{\omega}_0|_V)$ to a lift of ω_1. Let $\widehat{\omega}_t$ denote the concatenation of $\widehat{\omega}_t^1$ and $\widehat{\omega}_t^2$:

$$\widehat{\omega}_t := \begin{cases} \widehat{\omega}_{2t}^1 & 0 \leqslant t \leqslant 1/2, \\ \widehat{\omega}_{2t-1}^2 & 1/2 \leqslant t \leqslant 1. \end{cases}$$

Then $t \mapsto \pi_t := \widehat{\omega}_t^{-1} \in \mathfrak{X}(M, Z_t)$ defines a homotopy of b-serious bivectors between π_0 and a Poisson π_1. □

A few remarks are in order:

- If ω_0 could be C^0-approximated by a closed $\omega_1 \in {}^b\Omega^2(M, Z_0)$, we would be done. However, such an approximation is severely obstructed, in that it would imply that Z_0 admits a structure of cosymplectic manifold; it is easy (say, be means of a folded two-form [2]) to construct b-serious bivectors whose singular loci cannot be cosymplectic.
- We get around this problem by changing the topology of Z_0 rather drastically; observe in particular that Z_1 may be disconnected even if Z_0 is connected. One should perhaps think of Z_1 as Z_0 with those places 'blown to infinity' where ω_0 cannot be approximated by closed b-forms.
- Of course, when M is itself almost-complex, Gromov's theorem allows us to produce an honest symplectic structure. The construction above guarantees that the singular locus can be made non-empty, regardless of whether M is almost-complex or merely stably so.

- When M is an open 4-manifold of finite type, being orientable is enough to ensure that M is almost-complex ([13, 4.1]). There is some work done ([3] and references therein) in the direction of telling apart stably almost-complex from almost-complex manifolds (at least in the case of closed manifolds of low dimension), but so far these seem uncharted waters.
- If in the statement of Theorem 5 we further assume that:
  - $Z = Z(\pi_0)$ is a regular fibre $f^{-1}(0)$ of a proper Morse function $f\colon M \to \mathbb{R}$, unbounded from above and from below, and
  - ω_0 is already ${}^b d$-closed around Z,

  then one can impose that the homotopy π_t above be stationary around Z [7, 7.2.4].

This last comment can be regarded as a sufficient condition to realize a given cosymplectic structure on Z on a *given* manifold M:

Corollary 3. *Any given cosymplectic structure on the regular fibre Z of a proper Morse function $f\colon M \to \mathbb{R}$ can be realized as the singular locus of a b-symplectic structure, provided f be unbounded from above and from below.*

Proof of Theorem 2. It remains to show that the existence of an almost-complex structure on $M \times \mathbb{C}$ ensures the existence of a b-serious bivector. But according to [3], this former guarantees the existence of a folded symplectic form $\phi \in \Omega^2(M)$, namely, a closed two-form the top power of which is transverse to the zero section, thus defining a smooth folding locus $Z = (\phi^n)^{-1}M \subset M$, along which ϕ^n does not vanish anywhere. Let now g denote any Riemannian metric on M, and denote by π the bivector $\bigwedge^2 g^\flat(\phi) \in \mathfrak{X}^2(M)$. Then π is b-serious, and has singular locus Z. □

6. Prescribing the singular locus of a b-symplectic manifolds

We now address the second realization problem introduced earlier, namely, that of determining which cosymplectic manifolds (Z, η, θ) appear as singular loci of b-symplectic structures on compact manifolds without boundary.

We say that a b-symplectic manifold (M, ω) realizes a corank-one Poisson structure $(Z(\omega), \pi_{Z(\omega)})$ if $\pi = \omega^{-1}$ restricts to $\pi_{Z(\omega)}$ along $Z(\omega)$. Of course, we know that if ω realizes a given Poisson structure π_Z, then:

(1) $\operatorname{im} \pi_Z = \ker \flat\omega$, so $\operatorname{im} \pi_Z$ is given by the kernel of a *closed* one-form;
(2) π_Z is calibrated, i.e., the symplectic structure on the leaves of π_Z is the restriction of a *closed* two-form η, whose cohomology class is determined by ω.

That is: if a Poisson structure π_Z appears as the singular locus of a b-symplectic manifold, then it must come from a cosymplectic structure (η, θ) on Z, in the sense that

$$\pi_Z^\sharp \theta = 0, \quad \pi_Z(\xi, \xi') = \eta(\pi_Z^\sharp \xi, \pi_Z^\sharp \xi').$$

Observe that the expression above assigns a unique corank-one Poisson bivector π_Z cosymplectic structure (η, θ).

Thus, the realization problem pertains to the realm of cosymplectic manifolds, so we restate.

Definition 8. *A cosymplectic manifold is said to be* **realized** *by a b-symplectic manifold* (M, ω) *if it represents the canonical b-cosymplectic structure on* $Z(\omega) = Z$.

Every connected cosymplectic manifold Z is b-equivalent to a connected component of the singular locus of some connected, b-symplectic manifold without boundary, compact if Z is compact – just double $Z \times [0,1]$ with the obvious symplectic structure $dt \wedge \mathrm{pr}^* \theta + \mathrm{pr}^* \eta$. However:

Lemma 6. *A compact cosymplectic manifold* (Z, η, θ) *can be realized by a compact, orientable b-symplectic manifold* (M, ω) *without boundary if and only if it is symplectically fillable.* (M, ω) *can be chosen connected if* (Z, η, θ) *is connected.*

Proof. If (M, ω, θ) is a symplectic filling of (Z, η, θ), then (Z, η, θ) is realized in the double of the symplectic filling.

Conversely, suppose $(Z(\omega), \eta, \theta)$ represents the canonical b-cosymplectic structure on the singular locus of a compact, orientable b-symplectic (M, ω). Factorize M as a composition of cosymplectic cobordisms $M_0 \cup_{Z(\omega)} M_1$ as in Proposition 3, to obtain M_0 a symplectic filling of its cosymplectic boundary ∂M_0. Note that, again by the recipe of factorization, ∂M_0 is b-equivalent to $(Z(\omega), \eta, \theta)$. Hence it suffices to show that there is a cosymplectic cobordism between any two b-equivalent cosymplectic structures, which follows from Lemma 7 below. □

Lemma 7. *Two cosymplectic structures* (η_0, θ_0), (η_1, θ_1) *on* Z *are cobordant if there is a homotopy* (η_t, θ_t) *of cosymplectic structures joining them, and* $[\eta_0] = [\eta_1]$. *In that case, the cobordism can be chosen to be* $M = Z \times [0,1]$.

Proof. Subdivide $[0,1]$ into $0 = t_0 < t_1 < \cdots < t_N = 1$ so that $\theta_{t_{i+1}}|_{\ker \eta_t} > 0$,for all $t \in [t_i, t_{i+1}]$. It suffices to show that $(Z, \eta_{t_i}, \theta_{t_i})$ is cobordant to $(Z, \eta_{t_{i+1}}, \theta_{t_{i+1}})$ for each i, so we may as well assume that $N = 1$. Now, $\mathrm{pr}_1^* \eta_0 + \theta_t \wedge dt$ is then a symplectic form on $M = Z \times [0,1]$ defining a cosymplectic cobordism between (Z, η_0, θ_0) and (Z, η_0, θ_1). Hence we may assume without loss of generality $\theta_0 = \theta_1$.

Let $\mathcal{F}$ denote the codimension-two foliation $\mathrm{pr}_1^* \ker \theta \cap \ker dt$ on M, which is transverse to ∂M and induces there the foliation determined by $\mathrm{pr}_1^* \theta$. We employ a suitable adaptation of Thurston's trick for $\mathcal{F}$.

Let $\eta_1 - \eta_0 = d\alpha$, and choose a monotone function $\varrho\colon [0,1] \to [0,1]$, taking the value 0 around zero, and 1 around 1. Define $\omega' = \mathrm{pr}_1^* \eta_0 - d(\varrho \, \mathrm{pr}_1^*(\alpha))$ which is symplectic on the leaves of $\mathcal{F}$. Since M is compact, for $K > 0$ large enough, the form

$$\omega = \omega' + K\,\mathrm{pr}_1^*\,\theta \wedge dt$$

is symplectic and restricts to η_i on Z_i. Hence (M, ω, θ) is the desired cobordism. □

For a particular kind of cosymplectic manifolds, symplectic mapping tori, much more can be said.

Symplectic mapping tori and symplectic fillings.

We shall regard a mapping torus as a foliated bundle with base $\mathbb{S}^1$. The corresponding holonomy representation is generated by a diffeomorphism $\varphi \in \mathrm{Diff}(F)$. Conversely, the suspension any such φ defines a mapping torus $Z(\varphi)$ with fiber diffeomorphic to F.

Henceforth we will assume mapping tori to be compact, so we can equivalently define them as fibrations $Z \to \mathbb{S}^1$ with an Ehresmann connection. We shall also identify $\mathbb{S}^1$ with $\mathbb{R}/\mathbb{Z}$.

A **symplectic mapping torus** is a symplectic bundle over $\mathbb{S}^1$, that is, a bundle over the circle endowed with a closed two form η which is symplectic on each fiber. Its kernel defines an Ehresmann connection, and its holonomy φ preserves the symplectic structure of the fiber, i.e., $\varphi \in \mathrm{Symp}(F, \sigma)$. Conversely, the suspension of any $\varphi \in \mathrm{Symp}(F, \sigma)$ canonically defines a symplectic mapping torus $(Z(\varphi), \eta_\varphi)$.

A symplectic mapping torus becomes a cosymplectic manifold upon the choice of a defining closed 1-form for the fibration; this is equivalent to the choice of a period $\lambda > 0$, as it is convened that the pullback of the oriented generator of $H^1(\mathbb{S}^1; \mathbb{Z})$ has period 1. The result is a cosymplectic manifold with compact codimension one foliation, which can be characterized as a cosymplectic manifold (Z, η, θ) whose period lattice $[\theta](H_1(Z; \mathbb{Z})) \subset \mathbb{Z}$ has rank 1. Conversely, one can speak of the symplectic mapping torus associated to a cosymplectic manifold with compact foliation.

We shall abuse notation and regard a symplectic mapping torus as a cosymplectic manifold $(Z(\varphi), \eta_\varphi, \theta_\varphi)$ by declaring θ_φ to have period 1. Having this convention in mind, Lemma 7 implies that a cosymplectic manifold with compact foliation is symplectically fillable if and only if its associated symplectic mapping torus is symplectically fillable.

To address the symplectic fillability of symplectic mapping tori, we need to recall how to compare symplectic mapping tori defining the same Poisson structure: let $\varphi, \varphi' \in \mathrm{Symp}(F, \sigma)$ belong to the same connected component. A choice of symplectic isotopy ϕ^s, $s \in [0, 1]$, connecting the identity to $\varphi'\varphi^{-1}$ produces an obvious Poisson isomorphims between $(Z(\varphi), \pi_\varphi)$ and $(Z(\varphi'), \pi_{\varphi'})$. The two-forms $\eta := \eta_\varphi, \eta' := \eta_{\varphi'}$ can then be regarded as forms in $Z(\varphi)$, and they **calibrate** the

Poisson structure $\pi := \pi_\varphi$, in the sense that they are closed and

$$\pi(\xi,\xi') = \eta(\pi^\sharp\xi, \pi^\sharp\xi') = \eta'(\pi^\sharp\xi, \pi^\sharp\xi').$$

Their difference can be written

$$\eta' - \eta = \alpha \wedge \theta_\varphi, \tag{2}$$

The one-form α is closed on fibers, so it gives rise to symplectic vector fields $v_s \in \mathfrak{X}(F)$. Of course, ϕ^s is the isotopy integrating v_s.

It is well-known that the pullback to the fiber $[\alpha_{|F}] \in H^1(F;\mathbb{R})$ coincides with the image of ϕ^s by the Flux homomorphism, and therefore $\eta' - \eta$ is exact if and only if ϕ^1 is a Hamiltonian diffeomorphism [16], meaning that there is a Hamiltonian isotopy joining the identity to ϕ^1.

The full interpretation of (2) at the cohomological level is provided by the Wang long exact sequence, which asserts that [8]

$$[\eta'] = [\eta] + \gamma \cup [\theta_\varphi],$$

where $\gamma \in H^1(Z(\varphi);\mathbb{R})$ is any extension of $[\alpha_{|F}] \in H^1(F;\mathbb{R})^{\varphi^*}$.

Now we are ready to show that, for a symplectic mapping torus, being symplectically fillable is a property of the symplectic isotopy class of the return map, i.e., it only depends on the underlying Poisson structure (it is independent of the calibration). This is a key result to provide a partial generalization to arbitrary dimensions of a theorem of Eliashberg, which proves all 3-dimensional symplectic mapping tori are symplectically fillable [6].

Proposition 4. *Suppose φ_0 and φ_1 are symplectically isotopic. There is then a cosymplectic cobordism (M,ω) from $(Z(\varphi_0),\eta_{\varphi_0},\theta_{\varphi_0})$ to $(Z(\varphi_1),\eta_{\varphi_1},\theta_{\varphi_1})$, which can be chosen to be a symplectic bundle, with fiber (F,σ), and base the torus with two disks removed.*

Thus, whether $(Z(\varphi),\eta_\varphi,\theta_\varphi)$ is symplectically fillable depends only on the *symplectic* isotopy class of φ.

Proof. Firstly, we construct the cobordism M.

Let Σ be the oriented 2-torus with two open disks removed and let $\varsigma\colon [0,1] \to \Sigma$ be an embedded arc connecting the component ∂M_0 to the component ∂M_1. Select a collar $c\colon [0,1]\times(-\varepsilon,\varepsilon) \to \Sigma$ for ς with coordinates s,t, respectively, and such that $\frac{\partial}{\partial s}, \frac{\partial}{\partial t}$ is sent to a positive basis; orient $\partial\Sigma_0$ and $\partial\Sigma_1$ so that $\frac{\partial}{\partial t}$ restricts to a positive vector.

Define $\widetilde{\Sigma}$ to be the surface with corners

$$\widetilde{\Sigma} = \Sigma \diagdown c(\varsigma\times(-\varepsilon,\varepsilon)) \coprod c(\varsigma\times(-\epsilon,0]) \coprod c(\varsigma\times[0,\epsilon)),$$

and $\widetilde{M}$ to be the manifold with corners $F\times\widetilde{\Sigma}$. The projection onto the second and third factors followed by c produces collars

$$c_-\colon F\times\varsigma\times(-\varepsilon,0] \to \widetilde{M}, \quad c_+\colon F\times\varsigma\times[0,\varepsilon,) \to \widetilde{M}$$

of $F\times\varsigma_- := F\times\varsigma \subset c(\varsigma\times(-\epsilon,0])$ and $F\times\varsigma_+ := F\times\varsigma\subset c(\varsigma\times[\epsilon,0))$, respectively.

The cobordism M we look for M is obtained upon gluing part of the boundary of $\widetilde{M}$ according to the recipe

$$M=\widetilde{M}/F\times\varsigma_-\ni(x,s)\sim(\varphi_0(x),s)\in F\times\varsigma_+;$$

in order to induce forms on M out of forms on $\widetilde{M}$ we fix the atlas associated to the collars c_-,c_+ (strictly speaking we allow the coordinate t to go past 0 (and s past 0,1) so we enlarge $\widetilde{M}$ beyond $F\times\varsigma_-$ and $F\times\varsigma_+$).

Note that M carries an obvious bundle structure $p: M\to\Sigma$.

The second step is endowing the bundle $p: M\to\Sigma$ with a symplectic form using Thurston's trick.

Since $\eta\in\Omega^2(F)$ is invariant under φ_0, its pullback to $\widetilde{M}$ induces a closed two-from η_M on M. It is clear that η_M makes each fiber a symplectic manifold.

We need to correct η_M by a closed two-form trivial on fibers. This correction will be the wedge of two closed one-forms.

The first closed one-form is required to be an extension of the Flux class $[\alpha_{|F}]\in H^1(F;\mathbb{R})^{\varphi_0^*}$ associated to φ_0,φ_1 and any choice of isotopy for $\varphi_1\varphi_0^{-1}$. It can be built as follows: let $\tau\in\Omega^1(F)$ be a representative of $[\alpha_{|F}]$, and let $\widetilde{\tau}$ be its pullback to $\widetilde{M}$. The form $\widetilde{\tau}$ is not φ_0 invariant, but it becomes so after a correction by an exact 1-form. The reason is that $\varphi_0^*\tau-\tau=df$, so it is straightforward to find $f_+\in C^\infty(F\times c(\varsigma\times[0,\varepsilon)))$ with has compact support, is independent of the t coordinates near $F\times\varsigma_+$, and such that $\tau+df_+$ is ϕ-invariant. Therefore it induces a closed one-form τ_M on M.

The second closed one-form comes from the base. We choose $\beta\in\Omega^1(\Sigma)$ a closed one-form with the following property: its pullback to $\partial\Sigma_0$ is zero, and its pullback to $\partial\Sigma_1$ is nowhere vanishing and is cohomologous to the (positive) generator of $H^1(\partial\Sigma_1;\mathbb{Z})$.

It is clear that $\eta_M+\tau_M\wedge p^*\beta$ is a closed one-form making each fiber symplectic, so by Thurston's argument

$$w:=\eta_M+\tau_M\wedge p^*\beta+Kp^*\mu$$

is symplectic on M, where μ is a positive area form on Σ and $K>0$ is large enough.

Upon restricting ω to the boundary, we obtain symplectic bundles $\partial M_0\to\partial\Sigma_0$ and $\partial M_1\to\partial\Sigma_1$, so that $\partial M_0=\partial_{\rm in}M$ and $\partial_{\rm out}M$. By construction $\partial_{\rm in}M$ is isomorphic to $(Z(\varphi_0),\eta_{\varphi_0},\theta_{\varphi_0})$ and $\partial_{\rm out}M$ is isomorphic to $(Z(\varphi_1'),\eta_{\varphi_1'},\theta_{\varphi_1'})$, where φ_1' is Hamiltonian isotopic to φ_1 (this is because the calibrations η_{φ_1} and $\eta_{\varphi_1'}=\omega_{|\partial_{\rm out}M}$ are cohomologous). We can now apply Lemma 7 to attach a cosymplectic cylinder $\partial_{\rm out}M\times[1-\varepsilon,1]\to$, whose outgoing boundary has the correct calibration η_{φ_1}. The resulting composition $M\cup_{\partial_{\rm out}M}\partial_{\rm out}M\times[1-\varepsilon,1]$ is the cosymplectic cobordism we sought. □

Note that, in particular, if φ is symplectically isotopic to the identity, $(Z(\varphi), \eta_\varphi, \theta_\varphi)$ is symplectically fillable.

Dehn twists.

There is another class of symplectomorphisms φ which we can 'cap off': Dehn twists. We briefly recall the construction of those maps, and refer the reader to [25] for further details.

The norm function $\mu\colon (T^*\mathbb{S}^{n-1}\setminus\mathbb{S}^{n-1}) \to \mathbb{R}$, $\mu(\xi) = \|\xi\|$, associated to the round metric $\langle\cdot,\cdot\rangle$ on the $(n-1)$-sphere $\mathbb{S}^{n-1}$, is the moment map of a Hamiltonian $\mathbb{S}^1$-action on $(T^*\mathbb{S}^{n-1}\setminus\mathbb{S}^{n-1})$. Upon identifying $T^*\mathbb{S}^{n-1} \subset \mathbb{R}^n \times \mathbb{R}^n$ as $T^*\mathbb{S}^{n-1} = \{(u,v) : \langle u,v\rangle = 0, \|u\| = 1\}$, we can write

$$e^{2\pi it}\cdot(u,v) = \left(\cos(2\pi t)u + \sin(2\pi t)v\|v\|^{-1}, \cos(2\pi t)v - \sin(2\pi t)\|v\|u\right),$$

Then $e^{\pi}\cdot(u,v) = (-u,-v)$ extends by the antipodal map to a symplectomorphism $T^*\mathbb{S}^{n-1} \to T^*\mathbb{S}^{n-1}$.

Choose now a function $r\colon \mathbb{R} \to \mathbb{R}$, satisfying:

(1) $r(t) = 0$ for $|t| \geqslant C > 0$;
(2) $r(t) - r(-t) = kt$ for $k \in \mathbb{Z}$ and $|t| \ll 1$,

and let ϕ^t denote the flow of the Hamiltonian vector field of $r(\mu)$.

Observe that $\phi^{2\pi}$ extends to a symplectomorphism $\psi\colon T^*\mathbb{S}^{n-1} \to T^*\mathbb{S}^{n-1}$, supported on the compact subspace $T(\varepsilon) \subset T^*\mathbb{S}^{n-1}$ the subspace of cotangent vectors of length $\leqslant C$. We call this a **model Dehn twist**.

We can graft this construction onto manifolds using Weinstein's Lagrangian neighborhood theorem. If $l : \mathbb{S}^{n-1} \hookrightarrow (F,\sigma)$ embeds $\mathbb{S}^{n-1}$ as a Lagrangian sphere, there are neighborhoods $\mathbb{S}^{n-1} \subset U \subset T^*\mathbb{S}^{n-1}$ and $l(\mathbb{S}^{n-1}) \subset V \subset F$ and a symplectomorphism $\varphi\colon (U,\omega_{\text{can}}) \to (V,\omega)$ extending l. If ψ is a model Dehn twist, supported inside U, we produce a symplectomorphism $\tau\colon (F,\sigma) \to (F,\sigma)$, supported in V, by

$$\tau(x) := \begin{cases} \varphi\circ\psi\circ\varphi^{-1}(x) & \text{if } x \in V; \\ x & \text{if } x \in F\setminus V. \end{cases}$$

Definition 9. *A symplectomorphism of the form above will be called a* **Dehn twist** *around $l := l(\mathbb{S}^{n-1})$, and it will be denoted by τ_l.*

We also recall that any two Dehn twists around a parametrized Lagrangian sphere l are Hamiltonian isotopic if $n > 2$, and symplectically isotopic if $n = 2$ [25].

Proof of theorem 3. By Lemma 6, is it enough to show that $Z(\varphi)$ is symplectically fillable.

Embed $l\colon \mathbb{S}^{n-1} \hookrightarrow Z(\varphi)$, $n > 2$, as a Lagrangian sphere landing inside a single leaf of $Z(\varphi)$, and observe that the normal bundle to l in $Z(\varphi)$ is trivial and carries a canonical framing. The cobordism M obtained from $Z(\varphi)\times[0,1]$ by attaching a n-handle along $l\times 1$ carries the structure of a cosymplectic cobordism, as follows from [19, Proposition 4], with $\partial_{\mathrm{in}}M \simeq Z(\varphi)$ and $\partial_{\mathrm{out}}M \simeq Z(\tau_l^{-1}\varphi)$ according to [19, Theorem 3]. We call this the trace of a positive Lagrangian surgery along l. Negative Lagrangian surgeries can be similarly defined, by attaching a n-handle to $l\times 1$ according to the opposite of the canonical framing, and one obtains a cosymplectic cobordism M with $\partial_{\mathrm{in}}M \simeq Z(\varphi)$ and $\partial_{\mathrm{out}}M \simeq Z(\tau_l\varphi)$.

By hypothesis, φ is symplectically isotopic to $\tau_{l_1}\cdots\tau_{l_m}\tau_{l_{m+1}}^{-1}\cdots\tau_{l_{m'}}^{-1}$, where $l_i\colon \mathbb{S}^{n-1}\hookrightarrow (F,\sigma)$, $i=1,\dots,m'$ are parametrized Lagrangian spheres. By Proposition 4, we can assume that φ equals this composition.

By the discussion above,

$$Z(\varphi),\quad Z(\tau_{l_1}^{-1}\varphi),\quad \cdots,\quad Z(\tau_{l_{m'}}\cdots\tau_{l_{m+1}}\tau_{l_m}^{-1}\cdots\tau_{l_1}^{-1}\varphi) = Z(\mathrm{id}_F)$$

are all cosymplectic cobordant, and $Z(\mathrm{id}_F)$ is symplectically fillable, since it bounds $(F\times\mathbb{D}^2, \mathrm{pr}_1^*\sigma + \mathrm{pr}_2^*\, dy_1\wedge dy_2)$.

This concludes the proof when $n>2$. The proof when $n=2$ is is a well-known fact in Symplectic Topology (the same approach as for the case $n>2$ also works; the latter case is technically much harder and finer in some sense, since Dehn twists in these dimensions are defined up to Hamiltonian isotopy) □

Proof of Theorem 4. Let (Z,η,θ) be a 3-dimensional cosymplectic manifold. By Lemma 6 all we must show is that (Z,η,θ) is symplectically fillable. By Lemma 7, we may assume without loss of generality that (Z,η,θ) is a symplectic mapping torus. But, according to Elishberg [6], all 3-dimensional symplectic mapping tori are symplectically fillable.

Alternatively, recall that every symplectic transformation on a closed surface is symplectically isotopic to a word on Dehn twists [15], so Theorem 3 yields the desired symplectic filling. □

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