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AGE-DEPENDENCY IN HOST-VECTOR MODELS: A GLOBAL ANALYSIS

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Abstract. In this paper, we introduce and analyze two structured models for the transmission of a vector-borne infectious disease. The first of these models assumes that the level of contagiousness and the rate of removal (recovery) of infected individuals depends on the infection age. In the second model the hosts population is structured with respect to the physical age of the hosts, and the susceptibility of the hosts is assumed to be age-dependent. For these models, the threshold parameter for the existence of a positive (endemic) equilibrium state is determined, and the global asymptotic stability of the equilibrium states are established by the Lyapunov's direct method.

1. Introduction

The emergence and re-emergence of vector-borne diseases, such as malaria, dengue fever, Chagas disease, yellow fever, Japanese encephalitis and many other, is driven by social and economical factors and, most impotently, the climate change, and widely recognized as a global problem [13, 14]. A vector-borne disease is an infectious disease such that the pathogenic microorganism (helminths, protozoa, bacteria or viruses) is transmitted from an infected individual to another individual by an intermediate vector. The most common and probably most important vectors are arthropods (fly, bug, tick or mosquitoes), but many domestic and wild animals can be asymptomatic carriers of parasites and pathogens as well [20].

There is a long history of mathematical modeling employed to assist in decision-making in the control of vector-borne diseases. Historically, the first differential equation model for the transmission dynamics of a vector-transmitted disease was developed by Ronald Ross in 1911 to study the strategies for the control of malaria. Usually, mathematical models of vector-born diseases are formulated in the form of a system of ordinary differential equations [2, 4, 5, 7, 8, 10, 11, 12, 26, 32, 35], or delay differential equations, where a discrete or distributed time delay models an incubation period [3, 6, 30, 31, 34, 36]. Such models were used to study the spread of malaria [3, 11, 33, 34], dengue fever [9, 10, 12],

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Japanese encephalitis [26], Chagas disease [5, 16, 35], West Nile virus [2, 4, 32] and Chikungunya disease [8, 25]. In some cases, partial differential equations or integro-differential equations are employed in the so-called age-dependent or age-structured models [1, 16, 27, 28], where there is an assumption that the host or vector population are stratified according to their epidemiological status. However, despite the existence of a massive evidence of the role of age in the disease dynamics [16, 27], the dependence of epidemiological parameters, such as the susceptibility of a host or a vector, or the probability of a disease-induced death, on host's or vector's age was mostly neglected. The most common reason for this is that the analysis of a system of the partial differential equations or integro-differential equations, compared with that of a system of ordinary differential equations, is generally a considerably more challenging task.

In this paper, we introduce two models of vector-born infectious diseases with two principally different types of the age dependencies. In the first model, we assume that the infected hosts population is stratified by their infection age and considers an infection-age dependent infectivity. In the second model the susceptible hosts population is structured according their physical age and the model is dealing with the age-dependent susceptibility of the hosts. For both these models, the age dependencies are given be general unspecified functions, and, apart of usual conditions of differentiability, we impose no constrains on these functions.

To study properties of these models, we employ the direct Lyapunov method. Recently, the direct Lyapunov method was successfully applied to age-structured SIR epidemic models with an infection–age dependent infectivity [21, 23] and with an age-dependent susceptibility of the hosts [24]. These results employed the Volterra-type Lyapunov functions of the form

$$V(x) = x - 1 - \ln x. \tag{1}$$

Below in this paper, we further develop these results and extend them to the age-structured host-vector models.

The paper is organized as follows: the basic host-vector ODE model of a vector-born infection is described in Section 2. In Section 3, we formulate a host-vector model with an unspecified infection-age dependent infectivity and establish its global asymptotic stability. In Section 4, we introduce a host-vector model with age-dependent susceptibility of the hosts, and prove its global asymptotic stability. Section 5 contains the discussion and concluding remarks.

2. A model of a vector–borne infection

We start with a description of a simple model for the spread of a vector-transmitted infectious disease which serves as a basis for the age-structured models which are introduced below. We use a SIR (Susceptible-Infectious-Recovered) model for the host population, and an SI (Susceptible-Infectious) model for the vector population. Accordingly, the host population of size N_h is partitioned

into three subpopulations, namely these of susceptible, infectious and recovered individuals, with sizes denoted by $S_h(t)$, $I_h(t)$ and $R_h(t)$, respectively. The recovered hosts are assumed to be permanently immune. The vector population of size N_v is divided into two classes, namely these of the susceptible vectors $S_v(t)$ and infectious vectors $I_v(t)$. We assume that the infected vectors never recover from the infection and carry the pathogenic microorganisms until their death, and hence there is no recovered class for the vectors. Both populations N_h and N_v are assumed to be constant.

The susceptible hosts can be infected by an infectious vector. We assume that the infective contacts (biting) occur according to the law of mass action, which, taking into consideration the constant population size assumptions for both, hosts and vectors, implies the bilinear incidence rate $\beta S_h(t)I_v(t)$. Here β is the rate of infective contacts. Susceptible vectors can acquire the pathogen after contacting (biting) an infected host. We assume that for the vectors the transmission rate is $\kappa S_v(t)I_h(t)$. Here the rate κ is not necessary equal to the rate β, as a biting a susceptible host by an infected vector, as well as biting an infected host by a susceptible vector, does not necessary warranty transmission of pathogenic microorganisms.

We assume that the rate of recruitment into the host population (or hosts' birth rate) is constant and equal to Λ. The hosts' per capita natural death rate is μ_h, and the per capita disease-inflicted death rate of the infected hosts is α_h. The infected hosts recover with the per capita recovery rate ϵ. The vector reproduction rate is $\mu_v N_v$, and the vector per capita mortality rate is μ_v.

Under these assumptions, the model is governed by the following system of non-linear ordinary differential equations:

$$\begin{aligned}
\frac{dS_h(t)}{dt} &= \Lambda - \mu_h S_h(t) - \beta S_h(t) I_v(t), \\
\frac{dI_h(t)}{dt} &= \beta S_h(t) I_v(t) - (\mu_h + \epsilon + \alpha) I_h(t), \\
\frac{dR_h(t)}{dt} &= \epsilon I_h(t) - \mu_h R_h(t), \\
\frac{dS_v(t)}{dt} &= \mu_v N_v - \kappa S_v(t) I_h(t) - \mu_v S_v, \\
\frac{dI_v(t)}{dt} &= \kappa S_v(t) I_h(t) - \mu_v I_v(t).
\end{aligned} \tag{2}$$

Here all parameters are positive constants.

3. Host-vector model with infection age of the hosts

Contageousness of infected individuals varies through the infective period. The most apparent example of such a variability is the latent state. These variations can be described by a modification of model (2), where disease age/infection

age of the infected hosts, measured as the time that passed from the moment the exposure to an infecting dose, is incorporated, and the contageousness is assumed to be dependent on this age.

Let assume that ω is the infection age; then $i_h(t,\omega)$ is a distribution of the infected hosts by their infection age at time t, $\int_{\omega_1}^{\omega_2} i_h(t,\omega)d\omega$ is the number of infected hosts with infection ages between ω_1 and ω_2 at time t, and the total host population $I_h(t)$ is

$$I_h(t) = \int_0^{+\infty} i_h(t,\omega)d\omega.$$

We assume that the infectivity of the hosts and their rates of recovery and disease-induced mortality vary with ω and denote them $\kappa(\omega)$, $\epsilon(\omega)$ and $\alpha(\omega)$, respectively. Then the force of infection of the host population (that is, the probability for a susceptible vector to become infected for a unit of time) is

$$\int_0^{+\infty} \kappa(\omega) i_h(t,\omega)d\omega.$$

A model of a vector-born infectious disease with infection age of the hosts can be formulated as

$$\begin{aligned}
\frac{dS_h(t)}{dt} &= \Lambda - \mu_h S_h(t) - \beta S_h(t) I_v(t),\\
\frac{\partial i_h(t,\omega)}{\partial t} + \frac{\partial i_h(t,\omega)}{\partial \omega} &= -(\mu_h + \epsilon(\omega) + \alpha(\omega)) i_h(t,\omega),\\
\frac{dR_h(t)}{dt} &= \int_0^{+\infty} \epsilon(\omega) i_h(t,\omega)d\omega - \mu_h R_h(t),\\
\frac{dS_v(t)}{dt} &= \mu_v N_v - \int_0^{+\infty} \kappa(\omega) S_v(t) i_h(t,\omega)d\omega - \mu_v S_v,\\
\frac{dI_v(t)}{dt} &= \int_0^{+\infty} \kappa(\omega) S_v(t) i_h(t,\omega)d\omega - \mu_v I_v(t),\\
i_h(t,0) &= \beta S_h(t) I_v(t).
\end{aligned} \tag{3}$$

Here we assume that $\epsilon(\omega)$, $\alpha(\omega)$ and $\kappa(\omega)$ are non-negative, bounded integrable functions, and that the initial conditions are

$$S_h(0) = S_{h0}, \quad i_h(0,\omega) = \sigma_{i0}(\omega), \quad R_h(0) = R_{h0}, \quad S_v(0) = S_{v0}, \quad I_v(0) = I_{v0},$$

where $\sigma_{i0}(\omega)$ is an initial distribution of the infected hosts with respect to their infection age, and S_{h0}, R_{h0}, S_{v0} and I_{h0} are initial populations of the susceptible hosts, recovered hosts, susceptibles vectors and infected vectors, respectively.

This model assumes the constant vector population size N_v and no recovery for infected vectors. Moreover, the equation for $R_h(t)$ is decoupled from the rest of the system. Therefore, omitting this equation, using the relation $N_v = S_v(t) + I_v(t)$ and denoting $\delta(\omega) = \mu_h + \epsilon(\omega) + \alpha(\omega)$, we can reduce system (3) to the following system of three equations:

$$\begin{aligned} \frac{dS_h(t)}{dt} &= \Lambda - \mu_h S_h(t) - \beta S_h(t) I_v(t), \\ \frac{\partial i_h(t,\omega)}{\partial t} + \frac{\partial i_h(t,\omega)}{\partial \omega} &= -\delta(\omega) i_h(t,\omega), \\ \frac{dI_v(t)}{dt} &= (N_v - I_v(t)) \int_0^{+\infty} \kappa(\omega) i_h(t,\omega) d\omega - \mu_v I_v(t), \\ i_h(t,0) &= \beta S_h(t) I_v(t). \end{aligned} \tag{4}$$

3.1. **Analysis of the model.** The system (4) can have two non-negative equilibrium states, namely a disease-free equilibrium state $E^0 = (S_h^0, i_h^0(\omega), I_v^0)$, where $S_h^0 = \Lambda/\mu_h$, $i_h^0(\omega) = 0$ and $I_v^0 = 0$, and an endemic equilibrium state $E^* = (S_h^*, i_h^*(\omega), I_v^*)$, where S_h^*, $i_h^*(\omega)$ and I_v^* satisfy the equalities

$$0 = \Lambda - \mu_h S_h^* - \beta S_h^* I_v^*, \tag{5}$$

$$\frac{di_h^*(\omega)}{d\omega} = -\delta(\omega) i_h^*(\omega), \tag{6}$$

$$(N_v - I_v^*) \int_0^{+\infty} \kappa(\omega) i_h^*(\omega) d\omega = \mu_v I_v^*, \tag{7}$$

$$i_h^*(0) = \beta S_h^* I_v^*. \tag{8}$$

Equality (6) yields

$$i_h^*(\omega) = i_h^*(0)\rho(\omega),$$

where

$$\rho(\omega) = \exp\left(- \int_0^{\omega} \delta(\phi) d\phi \right) \tag{9}$$

is the probability that an infected host survives to age ω. By (8),

$$i_h^*(\omega) = \beta S_h^* I_v^* \rho(\omega).$$

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Substituting this equality into (7), we obtain

$$(N_v - I_v^*) \int_0^{+\infty} \kappa(\omega) i_h^*(\omega) d\omega = (N_v - I_v^*)\, \beta S_h^* I_v^* \int_0^{+\infty} \kappa(\omega)\rho(\omega) d\omega$$

$$= \beta S_h^* I_v^* \,(N_v - I_v^*)\, \eta = \mu_v I_v^*, \tag{10}$$

and hence

$$S_h^* = \mu_v / \left(\beta \,(N_v - I_v^*)\, \eta\right). \tag{11}$$

Here

$$\eta = \int_0^{+\infty} \kappa(\omega)\rho(\omega) d\omega \tag{12}$$

is the total number of infective vectors produced by a single infective host during his infective period. Substituting (11) into (5) and then solving this with respects to I_v^*, we obtain

$$I_v^* = \frac{\Lambda N_v \eta}{\mu_v + \eta\Lambda} - \frac{\mu_h \mu_v}{\beta(\mu_v + \eta\Lambda)} = \frac{\mu_h \mu_v}{\beta(\mu_v + \eta\Lambda)} \,(R_0 - 1)\,,$$

where

$$R_0 = \frac{\beta\Lambda N_v}{\mu_h \mu_v} \cdot \frac{\Lambda N_v \beta}{\mu_h \mu_v} \eta = \frac{\beta\Lambda N_v}{\mu_h \mu_v} \int_0^{+\infty} \kappa(\omega)\rho(\omega) d\omega \tag{13}$$

is the threshold parameter for system (4). (In the literature on vector-transmitted diseases, the basic reproductive number is more often assumed to be equal to the square root of the threshold parameter, $\widehat{R_0} = \sqrt{R_0}$.) Finally, we get

$$S_h^* = \frac{\mu_v + \eta\Lambda}{\eta(\mu_h + \beta N_v)} \quad \text{and} \quad i_h^*(\omega) = \frac{\mu_h \mu_v}{\eta(\mu_h + \beta N_v)} \rho(\omega) \,(R_0 - 1)\,.$$

Thus, we just proved following Proposition:

Proposition 1. *System* (4) *always has the disease-free equilibrium state* $E^0(S_h^0, i_h^0(\omega), I_v^0)$. *If* $R_0 > 1$, *then system also has a unique endemic equilibrium state* $E^*(S_h^*, i_h^*(\omega), I_v^*)$.

Global properties of system (4) are given by following Theorem:

Theorem 2. *System* (4) *is globally asymptotically stable. That is,*

(i) *if* $R_0 \le 1$, *then the disease-free equilibrium state* E^0 *is globally asymptotically stable; and*

(ii) *if* $R_0 > 1$, *then the positive (endemic) equilibrium state* E^* *exists and is globally asymptotically stable.*

Proof. We define a positive auxiliary function

$$\xi(\omega) = \int_{\omega}^{+\infty} \kappa(\nu) \exp\left(- \int_{\omega}^{\nu} \delta(\phi) d\phi \right) d\nu. \tag{14}$$

It is easy to see that $\xi(\omega) > 0$ for all $\omega \in [0, +\infty)$, and that $\xi(0) = \eta$ holds. Furthermore,

$$\frac{d\xi(\omega)}{d\omega} = \xi(\omega) \cdot \delta(\omega) - \kappa(\omega). \tag{15}$$

(i) To prove the global asymptotic stability of disease-free equilibrium E^0 we consider a Lyapunov function

$$W(t) = AV\left(\frac{S_h(t)}{S_h^0}\right) + B \int_0^{+\infty} \xi(\omega) i_h(t,\omega) d\omega + C I_v(t),$$

where $V(x)$ is the Volterra function (1), and

$$A = S_h^0, \quad B = \frac{1}{\eta}, \quad C = \frac{1}{\eta N_v}.$$

Function $W(t)$ satisfies

$$\begin{aligned}
\frac{dW}{dt} =& \left(1 - \frac{S_h^0}{S_h(t)}\right) \frac{dS_h(t)}{dt} + \frac{1}{\eta} \int_0^{+\infty} \xi(\omega) \frac{\partial i_h(t,\omega)}{\partial t} d\omega + \frac{1}{\eta N_v} \frac{dI_v(t)}{dt} \\
=& \Lambda \left(1 - \frac{S_h^0}{S_h(t)}\right) \left(1 - \frac{S_h(t)}{S_h^0}\right) - \left(1 - \frac{S_h^0}{S_h(t)}\right) \beta S_h(t) I_v(t) \\
& - \frac{1}{\eta} \int_0^{+\infty} \xi(\omega) \left(\frac{\partial i_h(t,\omega)}{\partial \omega} + \delta(\omega) i_h(t,\omega) \right) d\omega \\
& + \frac{1}{\eta N_v} \left(N_v \int_0^{+\infty} \kappa(\omega) i_h(t,\omega) d\omega - \mu_v I_v(t) - I_v(t) \int_0^{+\infty} \kappa(\omega) i_h(t,\omega) d\omega \right) \\
=& \Lambda \left(2 - \frac{S_h(t)}{S_h^0} - \frac{S_h^0}{S_h(t)}\right) - \beta S_h(t) I_v(t) \\
& - \frac{1}{\eta} \int_0^{+\infty} \xi(\omega) \frac{\partial i_h(t,\omega)}{\partial \omega} d\omega - \frac{1}{\eta} \int_0^{+\infty} (\xi(\omega)\delta(\omega) - \kappa(\omega)) i_h(t,\omega) d\omega \\
& + \left[\beta S_h^0 - \frac{\mu_v}{\eta N_v} - \frac{1}{\eta N_v} \int_0^{+\infty} \kappa(\omega) i_h(t,\omega) d\omega \right] I_v(t).
\end{aligned}$$

(Here we used equality $\Lambda = \mu_h S_h^0$.) Integrating by parts and using $\xi(0) = \eta$, $i_h(t,0) = \beta S_h(t) I_v(t)$ and (15), we get

$$\begin{aligned}\int_0^{+\infty} \xi(\omega)\frac{\partial i_h(t,\omega)}{\partial \omega}d\omega &= \left[\xi(\omega) i_h(t,\omega)\right]_{\omega=0}^{\omega=+\infty} - \int_0^{+\infty} \xi'(\omega) i_h(t,\omega)d\omega \\ &= \left[\xi(\omega) i_h(t,\omega)\right]_{\omega=+\infty} - \eta\beta S_h(t) I_v(t) - \int_0^{+\infty} (\xi(\omega)\delta(\omega) - \kappa(\omega)) i_h(t,\omega)d\omega.\end{aligned}$$

Substituting this into dW/dt, we finally obtain

$$\begin{aligned}\frac{dW}{dt} =& \Lambda\left(2 - \frac{S_h(t)}{S_h^0} - \frac{S_h^0}{S_h(t)}\right) - \frac{1}{\eta}\left[\xi(\omega) i_h(t,\omega)\right]_{\omega=+\infty} \\ &- \frac{\mu_v}{\eta N_v}\left[1 - \frac{\beta\eta N_v S_h^0}{\mu_v}\right] I_v(t) - \frac{1}{\eta N_v}\left[\int_0^{+\infty} \kappa(\omega) i_h(t,\omega)d\omega\right] I_v(t) \\ \leq& \Lambda\left(2 - \frac{S_h(t)}{S_h^0} - \frac{S_h^0}{S_h(t)}\right) - \frac{\mu_v}{\eta N_v}\left[1 - R_0\right] I_v(t).\end{aligned}$$

That is, $\frac{dW}{dt} \leq 0$ holds for all $R_0 \leq 1$. Furthermore, for $R_0 \leq 1$ equality $\frac{dW}{dt} = 0$ holds only on the set $S_h(t) = S_h^0$. It is easy to see that this set is transversal to the phase flow everywhere but a single point E^0. Therefore, by Lyapunov-LaSalle asymptotic stability theorem, condition $R_0 \leq 1$ is necessary and sufficient to ensure the globally asymptotic stability of equilibrium state E^0.

(ii) The existence of endemic equilibrium state E^* for $R_0 > 1$ is proved above. To prove its global stability, we consider a Lyapunov function

$$L(t) = AV\left(\frac{S_h(t)}{S_h^*}\right) + B\int_0^{+\infty}\left[\xi(\omega) i_h^*(\omega) V\left(\frac{i_h(t,\omega)}{i_h^*(\omega)}\right)\right]d\omega + CV\left(\frac{I_v(t)}{I_v^*}\right),$$

where $V(x)$ is the Volterra function (1), and

$$A = S_h^*, \quad B = \frac{1}{\eta}, \quad C = \frac{\beta S_h^* I_v^*}{\mu_v}.$$

Function $L(t)$ satisfies

$$\begin{aligned}\frac{dL}{dt} =& \left(1 - \frac{S_h^*}{S_h(t)}\right)\frac{dS_h(t)}{dt} + \frac{1}{\eta}\int_0^{+\infty}\xi(\omega)\left(1 - \frac{i_h^*(\omega)}{i_h(t,\omega)}\right)\frac{\partial i_h(t,\omega)}{\partial t}d\omega \\ &+ \frac{\beta S_h^*}{\mu_v}\left(1 - \frac{I_v^*}{I_v(t)}\right)\frac{dI_v(t)}{dt} \\ =& \left(1 - \frac{S_h^*}{S_h(t)}\right)\left(\Lambda - \mu_h S_h(t) - \beta S_h(t) I_v(t)\right)\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{\eta}\int_0^{+\infty} \xi(\omega)\left(1-\frac{i_h^*(\omega)}{i_h(t,\omega)}\right)\left(\frac{\partial i_h(t,\omega)}{\partial\omega}+\delta(\omega)i_h(t,\omega)\right)d\omega\\
&+ \frac{\beta S_h^*}{\mu_v}\left(1-\frac{I_v^*}{I_v(t)}\right)\left((N_v-I_v^*)\int_0^{+\infty}\kappa(\omega)i_h(t,\omega)d\omega-\mu_v I_v(t)\right)\\
&+ \frac{\beta S_h^*}{\mu_v}\left(1-\frac{I_v^*}{I_v(t)}\right)I_v^*\left(1-\frac{I_v(t)}{I_v^*}\right)\int_0^{+\infty}\kappa(\omega)i_h(t,\omega)d\omega\\
=\ &\mu_h S_h^*\left(2-\frac{S_h^*}{S_h(t)}-\frac{S_h(t)}{S_h^*}\right)+\beta S_h^* I_v^*\left(1-\frac{S_h(t)I_v(t)}{S_h^*I_v^*}-\frac{S_h^*}{S_h(t)}+\frac{I_v(t)}{I_v^*}\right)\\
&- \frac{1}{\eta}\int_0^{+\infty} \xi(\omega)\left(1-\frac{i_h^*(\omega)}{i_h(t,\omega)}\right)\left(\frac{\partial i_h(t,\omega)}{\partial\omega}+\delta(\omega)i_h(t,\omega)\right)d\omega\\
&+ \frac{1}{\eta}\int_0^{+\infty} i_h^*(\omega)\kappa(\omega)\left(\frac{i_h(t,\omega)}{i_h^*(\omega)}-\frac{i_h(t,\omega)I_v^*}{i_h^*(\omega)I_v(t)}\right)d\omega+\beta S_h^*I_v^*\left(1-\frac{I_v(t)}{I_v^*}\right)\\
&+ \frac{\beta S_h^*I_v^*}{\mu_v}\left(2-\frac{I_v^*}{I_v(t)}-\frac{I_v(t)}{I_v^*}\right)\int_0^{+\infty}\kappa(\omega)i_h(t,\omega)d\omega,
\end{aligned}
$$

(Here we used (5), (10) and an obvious equality $N_v - I_v = N_v + I_v^* - I_v^* - I_v$.)
By (6),

$$
\begin{aligned}
i_h^*(\omega)\frac{\partial V}{\partial\omega}\left(\frac{i_h(t,\omega)}{i_h^*(\omega)}\right) &= \left(1-\frac{i_h^*(\omega)}{i_h(t,\omega)}\right)\left(i_{h\omega}(t,\omega)-\frac{i_h(t,\omega)\cdot i_{h\omega}^*(\omega)}{i_h^*(\omega)}\right)\\
&= \left(1-\frac{i_h^*(\omega)}{i_h(t,\omega)}\right)\left(i_{h\omega}(t,\omega)+\delta(\omega)i_h(t,\omega)\right),
\end{aligned}
$$

where $i_{h\omega}(t,\omega)$ and $i_{h\omega}^*(\omega)$ denote $\frac{\partial i_h(t,\omega)}{\partial\omega}$ and $\frac{di_h^*(\omega)}{d\omega}$, respectively. Hence

$$
\begin{aligned}
&\int_0^{+\infty}\xi(\omega)\left(1-\frac{i_h^*(\omega)}{i_h(t,\omega)}\right)\left(\frac{\partial i_h(t,\omega)}{\partial\omega}+\delta(\omega)i_h(t,\omega)\right)d\omega\\
&\qquad = \int_0^{+\infty}\xi(\omega)i_h^*(\omega)\frac{\partial V}{\partial\omega}\left(\frac{i_h(t,\omega)}{i_h^*(\omega)}\right)d\omega.
\end{aligned}
$$

Integrating the right-hand part of this equality by parts yields

$$\begin{aligned}
&\int_0^{+\infty} \xi(\omega) i_h^*(\omega) \frac{\partial V}{\partial \omega}\left(\frac{i_h(t,\omega)}{i_h^*(\omega)}\right) d\omega = \\
&\quad = \left[\xi(\omega) i_h^*(\omega) V\left(\frac{i_h(t,\omega)}{i_h^*(\omega)}\right)\right]_{\omega=0}^{\omega=+\infty} - \int_0^{+\infty} \frac{d}{d\omega}\left[\xi(\omega) i_h^*(\omega)\right] V\left(\frac{i_h(t,\omega)}{i_h^*(\omega)}\right) d\omega \\
&\quad = \left[\xi(\omega) i_h^*(\omega) V\left(\frac{i_h(t,\omega)}{i_h^*(\omega)}\right)\right]_{\omega=0}^{\omega=+\infty} - \int_0^{+\infty} \left[\xi'(\omega) i_h^*(\omega) + \xi(\omega) i_{h\omega}^*(\omega)\right] V\left(\frac{i_h(t,\omega)}{i_h^*(\omega)}\right) d\omega \\
&\quad = \left[\xi(\omega) i_h^*(\omega) V\left(\frac{i_h(t,\omega)}{i_h^*(\omega)}\right)\right]_{\omega=+\infty} - \eta\beta S_h^* I_v^* V\left(\frac{S_h(t) I_v(t)}{S_h^* I_v^*}\right) \\
&\qquad + \int_0^{+\infty} i_h^*(\omega)\kappa(\omega) V\left(\frac{i_h(t,\omega)}{i_h^*(\omega)}\right) d\omega.
\end{aligned}$$

(Here we used the last equation in (4), and equalities $\xi(0) = \eta$, (6), (8) and (15).) Substituting this into dL/dt, we get

$$\begin{aligned}
\frac{dL}{dt} =& \mu_h S_h^*\left(2 - \frac{S_h^*}{S_h(t)} - \frac{S_h(t)}{S_h^*}\right) - \frac{1}{\eta}\left[\xi(\omega) i_h^*(\omega) V\left(\frac{i_h(t,\omega)}{i_h^*(\omega)}\right)\right]_{\omega=+\infty} \\
&+ \beta S_h^* I_v^*\left(2 - \frac{S_h(t) I_v(t)}{S_h^* I_v^*} - \frac{S_h^*}{S_h(t)}\right) + \beta S_h^* I_v^*\left(\frac{S_h(t) I_v(t)}{S_h^* I_v^*} - 1 - \ln\frac{S_h(t) I_v(t)}{S_h^* I_v^*}\right) \\
&- \frac{1}{\eta}\int_0^{+\infty} i_h^*(\omega)\kappa(\omega)\left(\frac{i_h(t,\omega)}{i_h^*(\omega)} - 1 - \ln\frac{i_h(t,\omega)}{i_h^*(\omega)}\right) d\omega \\
&+ \frac{1}{\eta}\int_0^{+\infty} i_h^*(\omega)\kappa(\omega)\left(\frac{i_h(t,\omega)}{i_h^*(\omega)} - \frac{i_h(t,\omega) I_v^*}{i_h^*(\omega) I_v(t)}\right) d\omega \\
&+ \frac{\beta S_h^* I_v^*}{\mu_v}\left(2 - \frac{I_v^*}{I_v(t)} - \frac{I_v(t)}{I_v^*}\right)\int_0^{+\infty} \kappa(\omega) i_h(t,\omega) d\omega \\
&= \mu_h S_h^*\left(2 - \frac{S_h^*}{S_h(t)} - \frac{S_h(t)}{S_h^*}\right) - \frac{1}{\eta}\left[\xi(\omega) i_h^*(\omega) V\left(\frac{i_h(t,\omega)}{i_h^*(\omega)}\right)\right]_{\omega=+\infty} \\
&- \beta S_h^* I_v^*\left(\frac{S_h^*}{S_h(t)} - 1 - \ln\frac{S_h^*}{S_h(t)}\right)
\end{aligned}$$

$$- \frac{1}{\eta} \int_0^{+\infty} i_h^*(\omega)\kappa(\omega) \left(\frac{i_h(t,\omega)I_v^*}{i_h^*(\omega)I_v(t)} - 1 - \ln \frac{i_h(t,\omega)I_v^*}{i_h^*(\omega)I_v(t)} \right) d\omega$$

$$+ \frac{\beta S_h^* I_v^*}{\mu_v} \left(2 - \frac{I_v^*}{I_v(t)} - \frac{I_v(t)}{I_v^*} \right) \int_0^{+\infty} \kappa(\omega) i_h(t,\omega) d\omega$$

$$+ \frac{1}{\eta} \left(\beta S_h^* I_v^* \eta - \int_0^{+\infty} i_h^*(\omega)\kappa(\omega) d\omega \right) \ln \frac{I_v^*}{I_v(t)}.$$

By (10), $\int_0^{+\infty} i_h^*(\omega)\kappa(\omega)d\omega = \beta S_h^* I_v^* \eta$ holds, and hence

$$\frac{dL}{dt} = \mu_h S_h^* \left(2 - \frac{S_h^*}{S_h(t)} - \frac{S_h(t)}{S_h^*} \right) - \frac{1}{\eta} \left[\xi(\omega) i_h^*(\omega) V \left(\frac{i_h(t,\omega)}{i_h^*(\omega)} \right) \right]_{\omega=+\infty}$$

$$- \beta S_h^* I_v^* V \left(\frac{S_h^*}{S_h(t)} \right) - \frac{1}{\eta} \int_0^{+\infty} i_h^*(\omega)\kappa(\omega) V \left(\frac{i_h(t,\omega)I_v^*}{i_h^*(\omega)I_v(t)} \right) d\omega$$

$$+ \frac{\beta S_h^* I_v^*}{\mu_v} \left(2 - \frac{I_v^*}{I_v(t)} - \frac{I_v(t)}{I_v^*} \right) \int_0^{+\infty} \kappa(\omega) i_h(t,\omega) d\omega.$$

Here, $V(x) \geq 0$ for all $x > 0$, and hence the existence of positive equilibrium state E^* suffices to ensure that $\frac{dL}{dt} \leq 0$ holds everywhere in the positive region of the phase space. The equality $\frac{dL}{dt} = 0$ holds only on the set

$$M = \{(S_h, i_h(t,\omega), I_v(t)) > 0 \,|\, S_h(t) = S_h^*, i_h(t,\omega) = i_h^*(\omega), I_v(t) = I_v^*\}.$$

It is easy to see that the equilibrium state E^* is the only invariant set of the system (4) in M, and hence, by the Lyapunov-LaSalle asymptotic stability theorem, this steady state is globally asymptotically stable (in the positive region of the phase space).

This completes the proof. □

4. Host-vector model with age-dependent susceptibility

A stratification of a population by the age of individuals is a traditional concept of the population dynamics. In this section we introduce an age structure into the hosts population assuming age-dependent susceptibility of the hosts to pathogenic microorganism. Let τ be the age of a susceptible individual; then $s_h(t,\tau)$ is the density of the susceptible hosts by their age τ at time t, and $\int_{\tau_1}^{\tau_2} s_h(t,\tau)d\tau$ is the number of susceptible hosts of ages between ω_1 and age ω_2 at time t. The total

susceptible hosts population is

$$S_h(t) = \int_0^{+\infty} s_h(t,\tau)d\tau.$$

Assuming that the susceptibility of the hosts and the rate of their natural mortality vary with the age and denoting these by $\beta(\tau)$ and $\gamma_h(\tau)$, respectively, we formulate the following age-structured model of a vector-born infectious disease:

$$\begin{aligned}
\frac{\partial s_h(t,\tau)}{\partial t} + \frac{\partial s_h(t,\tau)}{\partial \tau} &= -\gamma_h(\tau)s_h(t,\tau) - \beta(\tau)s_h(t,\tau)I_v(t),\\
\frac{dI_h(t)}{dt} &= \int_0^{+\infty} \beta(\tau)s_h(t,\tau)I_v(t)d\tau - (\mu_h+\epsilon)I_h(t),\\
\frac{dR_h(t)}{dt} &= \epsilon I_h(t) - \mu_h R_h(t),\\
\frac{dS_v(t)}{dt} &= \mu_v N_v - \kappa S_v(t)I_h(t) - \mu_v S_v,\\
\frac{dI_v(t)}{dt} &= \kappa S_v(t)I_h(t) - \mu_v I_v(t),\\
s_h(t,0) &= \Lambda.
\end{aligned} \tag{16}$$

System (16) is complemented by initial conditions

$$s_h(0,\tau) = \sigma_{s0}(\tau), \quad I_h(0) = I_{h0}, \quad R_h(0) = R_{h0}, \quad S_v(0) = S_{v0}, \quad I_v(0) = I_{v0}.$$

If vector population N_v can be assumed constant, we can omit the fourth equation in (16) using relation $N_v = I_v(t) + S_v(t)$. Furthermore, the third equation in (16) can be also omitted, because variable $R_h(t)$ does not appear in the other equations. Hence we can reduce the system to the following system of three equations:

$$\begin{aligned}
\frac{\partial s_h(t,\tau)}{\partial t} + \frac{\partial s_h(t,\tau)}{\partial \tau} &= -\gamma_h(\tau)s_h(t,\tau) - \beta(\tau)s_h(t,\tau)I_v(t),\\
\frac{dI_h(t)}{dt} &= \int_0^{+\infty} \beta(\tau)s_h(t,\tau)I_v(t)d\tau - (\mu_h+\epsilon)I_h(t),\\
\frac{dI_v(t)}{dt} &= \kappa\left(N_v - I_v(t)\right)I_h(t) - \mu_v I_v(t),\\
s_h(t,0) &= \Lambda.
\end{aligned} \tag{17}$$

4.1. **Analysis of the model.** System (17) can have one or two non-negative equilibrium states. Specifically, the system always has a disease-free equilibrium state $E^0(s_h^0(\tau), I_h^0, I_v^0)$, where $s_h^0(\tau) = \Lambda \exp\left(-\int_0^\tau \gamma_h(\phi)d\phi\right)$, $i_h^0(\omega) = 0$

and $I_v^0 = 0$. The system also can have an endemic (positive) equilibrium state $E^*(s_h^*(\tau), I_h^*, I_v^*)$, where the equalities

$$\frac{ds_h^*(\tau)}{d\tau} = -\gamma_h(\tau)s_h^*(\tau) - \beta(\tau)s_h^*(\tau)I_v^*, \tag{18}$$

$$0 = I_v^* \int_0^{+\infty} \beta(\tau)s_h^*(\tau)d\tau - (\mu_h + \epsilon)I_h^*, \tag{19}$$

$$0 = \kappa\left(N_v - I_v^*\right) I_h^* - \mu_v I_v^*, \tag{20}$$

$$s_h^*(0) = \Lambda \tag{21}$$

hold. The threshold parameter for system (17) is

$$R_0 = \frac{\kappa N_v}{(\mu_h + \epsilon)\mu_v} \int_0^{+\infty} \beta(\tau)s_h^0(\tau)d\tau. \tag{22}$$

Global properties of the system are given by following Theorem:

Theorem 3. *System* (17) *is globally asymptotically stable. That is,*

(i) *if* $R_0 \leq 1$ *then disease-free equilibrium state* E^0 *is globally asymptotically stable; and*

(ii) *if* $R_0 > 1$ *then the endemic equilibrium state* E^* *exists and is globally asymptotically stable.*

Proof. (i) If $R_0 \leq 1$ we consider a Lyapunov function

$$W(t) = \int_0^{+\infty} \left[A(\tau)V\left(\frac{s_h(t,\tau)}{s_h^0(\tau)}\right)\right] d\tau + BI_h(t) + CI_v(t),$$

where $V(x)$ is, as above, the Volterra function, and

$$A(\tau) = s_h^0(\tau), \quad B = 1, \quad C = \frac{\mu_h + \epsilon}{\kappa N_v}.$$

Function $W(t)$ satisfies

$$\begin{aligned}\frac{dW}{dt} &= \int_0^{+\infty} \left(1 - \frac{s_h^0(\tau)}{s_h(t,\tau)}\right) \frac{\partial s_h(t,\tau)}{\partial t} d\tau + \frac{dI_h(t)}{dt} + C\frac{dI_v(t)}{dt} \\ &= -\int_0^{+\infty} \left(s_h(t,\tau) - s_h^0(\tau)\right) \left(\frac{s_{h\tau}(t,\tau)}{s_h(t,\tau)} + \gamma_h(\tau) + \beta(\tau)I_v(t)\right) d\tau \\ &\quad + \int_0^{+\infty} \beta(\tau)s_h(t,\tau)I_v(t)d\tau - (\mu_h + \epsilon)I_h(t)\end{aligned}$$

$$
\begin{aligned}
&+\frac{\mu_h+\epsilon}{\kappa N_v}\left(\kappa N_v I_h(t)-\mu_v I_v(t)-\kappa I_h(t)I_v(t)\right)\\
=&-\int_0^{+\infty} s_h^0(\tau)\left(\frac{s_h(t,\tau)}{s_h^0(\tau)}-1\right)\left(\frac{s_{h\tau}(t,\tau)}{s_h(t,\tau)}+\gamma_h(\tau)\right)d\tau\\
&+\frac{(\mu_h+\epsilon)\mu_v}{\kappa N_v}\left[\frac{\kappa N_v}{(\mu_h+\epsilon)\mu_v}\int_0^{+\infty}\beta(\tau)s_h^0(\tau)d\tau-1\right]I_v(t)-\frac{\mu_h+\epsilon}{N_v}I_h(t)I_v(t).
\end{aligned}
$$

Here $s_{h\tau}(t,\tau)$ denotes $\frac{\partial s_h(t,\tau)}{\partial\tau}$.

Note that, using $s_{h\tau}^0(\tau)=-\gamma_h s_h^0(\tau)$, we have

$$
\frac{\partial}{\partial\tau}V\left(\frac{s_h(t,\tau)}{s_h^0(\tau)}\right)=\left(\frac{s_h(t,\tau)}{s_h^0(\tau)}-1\right)\left(\frac{s_{h\tau}(t,\tau)}{s_h(t,\tau)}+\gamma_h(\tau)\right).
$$

Furthermore, using $s_{h\tau}^0(\tau)=-\gamma_h s_h^0(\tau)$, $s_h(t,0)=s_h^0(0)=\Lambda$ and $V(1)=0$,

$$
\begin{aligned}
&\int_0^{+\infty} s_h^0(\tau)\frac{\partial}{\partial\tau}V\left(\frac{s_h(t,\tau)}{s_h^0(\tau)}\right)d\tau\\
&\qquad=\left[s_h^0(\tau)V\left(\frac{s_h(t,\tau)}{s_h^0(\tau)}\right)\right]_{\tau=0}^{\tau=+\infty}-\int_0^{+\infty}s_{h\tau}^0(\tau)V\left(\frac{s_h(t,\tau)}{s_h^0(\tau)}\right)d\tau\\
&\qquad=\left[s_h^0(\tau)V\left(\frac{s_h(t,\tau)}{s_h^0(\tau)}\right)\right]_{\tau=+\infty}+\int_0^{+\infty}\gamma_h(\tau)s_h^0(\tau)V\left(\frac{s_h(t,\tau)}{s_h^0(\tau)}\right)d\tau.
\end{aligned}
$$

Hence, finally,

$$
\begin{aligned}
\frac{dW}{dt}=&-\left[s_h^0(\tau)V\left(\frac{s_h(t,\tau)}{s_h^0(\tau)}\right)\right]_{\tau=+\infty}-\int_0^{+\infty}\gamma_h(\tau)s_h(\tau)V\left(\frac{s_h(t,\tau)}{s_h^0(\tau)}\right)d\tau\\
&-\frac{(\mu_h+\epsilon)\mu_v}{\kappa N_v}\left[1-R_0\right]I_v(t)-\frac{\mu_h+\epsilon}{N_v}I_h(t)I_v(t).
\end{aligned}
$$

That is, $R_0\leq 1$ ensures that $\frac{dW}{dt}\leq 0$ holds in the non-negative region of the phase space. The strict equality holds only if $s_h(t,\tau)=s_h^0(\tau)$ holds simultaneously with either $I_h=0$, or $I_v=0$. It is easy to verify that equilibrium state E^0 is the only invariant set of the system in this set, and hence by Lyapunov-LaSalle asymptotic stability theorem, $R_0\leq 1$ is sufficient to ensure that this equilibrium state is globally asymptotically stable (in the non-negative region of the phase space).

Please note that the global stability of E^0 in the non-negative region implies that there is no other equilibrium states in this region, and hence no positive equilibrium state exists when $R_0\leq 1$.

(ii) We have to prove the existence of the positive equilibrium state E^* for $R_0 > 1$. Equations (18) and (21) give

(23)
$$s_h^*(\tau) = \Lambda \exp\left(-\int_0^\tau (\gamma_h(\phi) + \beta(\phi) I_v^*)\, d\phi\right) = s_h^0(\tau) \cdot \exp\left(-\int_0^\tau \beta(\phi) I_v^* d\phi\right).$$

Combining this with equations (19) and (20), we obtain the equality

$$\frac{\kappa(N_v - I_v^*)}{\mu_v(\mu_h + \epsilon)} \int_0^{+\infty} \beta(\tau) s_h^*(\tau) d\tau = \\ = \frac{\kappa(N_v - I_v^*)}{\mu_v(\mu_h + \epsilon)} \int_0^{+\infty} \beta(\tau) s_h^0(\tau) \cdot \exp\left(-\int_0^\tau \beta(\phi) I_v^* d\phi\right) d\tau = 1.$$

It is easy to see that function

$$f(I_v^*) = \frac{\kappa(N_v - I_v^*)}{\mu_v(\mu_h + \epsilon)} \int_0^{+\infty} \beta(\tau) s_h^0(\tau) \cdot \exp\left(-\int_0^\tau \beta(\phi) I_v^* d\phi\right) d\tau$$

is continuous and monotonically decreases with the growth of I_v^*, and that equalities

$$f(0) = R_0, \qquad f(N_v) = 0$$

hold. Hence, for all $R_0 > 1$ there exists $I_v^* \in (0, N_v)$ such that equality $f(I_v^*) = 1$ holds. Corresponding $s_h^*(\tau)$ and I_h^* are defined by (23) and (19), respectively.

To prove the global stability of equilibrium state E^* for $R_0 > 1$, we consider a Lyapunov function

$$L(t) = \int_0^{+\infty} \left[A(\tau) V\left(\frac{s_h(t,\tau)}{s_h^*(\tau)}\right)\right] d\tau + BV\left(\frac{I_h(t)}{I_h^*}\right) + CV\left(\frac{I_v(t)}{I_v^*}\right),$$

where $V(x)$ is the Volterra function, and

$$A(\tau) = s_h^*(\tau), \quad B = I_h^*, \quad C = \frac{I_v^*}{\mu_v} \int_0^{+\infty} \beta(\tau) s_h^*(\tau) d\tau.$$

Function $L(t)$ satisfies

$$\frac{dL}{dt} = \int_0^{+\infty} \left(1 - \frac{s_h^*(\tau)}{s_h(t,\tau)}\right) \frac{\partial s_h(t,\tau)}{\partial t} d\tau + \left(1 - \frac{I_h^*}{I_h(t)}\right) \frac{dI_h(t)}{dt} \\ + \frac{C}{I_v^*} \left(1 - \frac{I_v^*}{I_v(t)}\right) \frac{dI_v(t)}{dt} \\ = -\int_0^{+\infty} s_h^*(\tau) \left(\frac{s_h(t,\tau)}{s_h^*(\tau)} - 1\right) \left(\frac{s_{h\tau}(t,\tau)}{s_h(t,\tau)} + \gamma_h(\tau) + \beta(\tau) I_v(t)\right) d\tau$$

$$
\begin{aligned}
&+\left(1-\frac{I_h^*}{I_h(t)}\right)\left(\int_0^{+\infty}\beta(\tau)s_h(t,\tau)I_v(t)d\tau-(\mu_h+\epsilon+\alpha)I_h(t)\right)\\
&+\frac{C}{I_v^*}\left(1-\frac{I_v^*}{I_v(t)}\right)(\kappa N_v I_h(t)-\mu_v I_v(t)-\kappa I_v(t)I_h(t))\\
=&-\int_0^{+\infty}s_h^*(\tau)\left(\frac{s_h(t,\tau)}{s_h^*(\tau)}-1\right)\left(\frac{s_{h\tau}(t,\tau)}{s_h(t,\tau)}+\gamma_h(\tau)+\beta(\tau)I_v^*\right)d\tau\\
&+\int_0^{+\infty}\beta(\tau)s_h^*(\tau)I_v^*\left(\frac{s_h(t,\tau)}{s_h^*(\tau)}-1\right)\left(1-\frac{I_v(t)}{I_v^*}\right)d\tau\\
&+\int_0^{+\infty}\beta(\tau)s_h^*(\tau)I_v^*\left(1-\frac{I_h^*}{I_h(t)}\right)\left(\frac{s_h(t,\tau)I_v(t)}{s_h^*(\tau)I_v^*}-\frac{I_h(t)}{I_h^*}\right)d\tau\\
&+C\left(1-\frac{I_v^*}{I_v(t)}\right)\left(\mu_v\left(\frac{I_h(t)}{I_h^*}-\frac{I_v(t)}{I_v^*}\right)+\kappa\left(1-\frac{I_v(t)}{I_v^*}\right)I_h(t)\right)\\
=&-\int_0^{+\infty}s_h^*(\tau)\left(\frac{s_h(t,\tau)}{s_h^*(\tau)}-1\right)\left(\frac{s_{h\tau}(t,\tau)}{s_h(t,\tau)}+\gamma_h(\tau)+\beta(\tau)I_v^*\right)d\tau\\
&+\int_0^{+\infty}\beta(\tau)s_h^*(\tau)I_v^*\left(\frac{s_h(t,\tau)}{s_h^*(\tau)}-\frac{s_h(t,\tau)I_v(t)}{s_h^*(\tau)I_v^*}-1+\frac{I_v(t)}{I_v^*}\right)d\tau\\
&+\int_0^{+\infty}\beta(\tau)s_h^*(\tau)I_v^*\left(\frac{s_h(t,\tau)I_v(t)}{s_h^*(\tau)I_v^*}-\frac{I_h(t)}{I_h^*}-\frac{s_h(t,\tau)I_h^*I_v(t)}{s_h^*(\tau)I_h(t)I_v^*}+1\right)d\tau\\
&+\int_0^{+\infty}\beta(\tau)s_h^*(\tau)I_v^*\left(\frac{I_h(t)}{I_h^*}-\frac{I_v(t)}{I_v^*}-\frac{I_h(t)I_v^*}{I_h^*I_v(t)}+1\right)d\tau\\
&+\kappa C\left(2-\frac{I_v^*}{I_v(t)}-\frac{I_v(t)}{I_v^*}\right)I_h(t)\\
=&-\int_0^{+\infty}s_h^*(\tau)\left(\frac{s_h(t,\tau)}{s_h^*(\tau)}-1\right)\left(\frac{s_{h\tau}(t,\tau)}{s_h(t,\tau)}+\gamma_h(\tau)+\beta(\tau)I_v^*\right)d\tau\\
&+\int_0^{+\infty}\beta(\tau)s_h^*(\tau)I_v^*\left(\frac{s_h(t,\tau)}{s_h^*(\tau)}-\frac{s_h(t,\tau)I_h^*I_v(t)}{s_h^*(\tau)I_h(t)I_v^*}-\frac{I_h(t)I_v^*}{I_h^*I_v(t)}+1\right)d\tau
\end{aligned}
$$

$$+ \kappa C \left(2 - \frac{I_v^*}{I_v(t)} - \frac{I_v(t)}{I_v^*}\right) I_h(t).$$

Here we used (19) and (20).

Please note that, by (18),

$$\begin{aligned}\frac{\partial}{\partial\tau} V\left(\frac{s_h(t,\tau)}{s_h^*(\tau)}\right) &= \left(\frac{s_h(t,\tau)}{s_h^*(\tau)} - 1\right)\left(\frac{s_{h\tau}(t,\tau)}{s_h(t,\tau)} - \frac{s_{h\tau}^*(\tau)}{s_h^*(\tau)}\right)\\ &= \left(\frac{s_h(t,\tau)}{s_h^*(\tau)} - 1\right)\left(\frac{s_{h\tau}(t,\tau)}{s_h(t,\tau)} + \gamma_h(\tau) + \beta(\tau) I_v^*\right),\end{aligned}$$

Integrating by parts and using the equalities $s_h(t,0) = s_h^*(0) = \Lambda$, $V(1) = 0$ and (18) yields

$$\begin{aligned}&\int_0^{+\infty} s_h^*(\tau)\frac{\partial}{\partial\tau} V\left(\frac{s_h(t,\tau)}{s_h^*(\tau)}\right) d\tau\\ &\qquad = \left[s_h^*(\tau) V\left(\frac{s_h(t,\tau)}{s_h^*(\tau)}\right)\right]_{\tau=0}^{\tau=+\infty} - \int_0^{+\infty} s_{h\tau}^*(\tau) V\left(\frac{s_h(t,\tau)}{s_h^*(\tau)}\right) d\tau\\ &\qquad = \left[s_h^*(\tau) V\left(\frac{s_h(t,\tau)}{s_h^*(\tau)}\right)\right]_{\tau=+\infty} + \int_0^{+\infty} \gamma_h(\tau) s_h^*(\tau) V\left(\frac{s_h(t,\tau)}{s_h^*(\tau)}\right) d\tau\\ &\qquad + \int_0^{+\infty} \beta(\tau) s_h^*(\tau) I_v^*\left(\frac{s_h(t,\tau)}{s_h^*(\tau)} - 1 - \ln\frac{s_h(t,\tau)}{s_h^*(\tau)}\right) d\tau\end{aligned}$$

Substituting this expression into dL/dt yields

$$\begin{aligned}\frac{dL}{dt} =& -\left[s_h^*(\tau) V\left(\frac{s_h(t,\tau)}{s_h^*(\tau)}\right)\right]_{\tau=+\infty} - \int_0^{+\infty} \gamma_h(\tau) s_h^*(\tau) V\left(\frac{s_h(t,\tau)}{s_h^*(\tau)}\right) d\tau\\ &+ \int_0^{+\infty} \beta(\tau) s_h^*(\tau) I_v^*\left(-\frac{s_h(t,\tau) I_h^* I_v(t)}{s_h^*(\tau) I_h(t) I_v^*} + 1 + \ln\frac{s_h(t,\tau) I_h^* I_v(t)}{s_h^*(\tau) I_h(t) I_v^*}\right) d\tau\\ &+ \int_0^{+\infty} \beta(\tau) s_h^*(\tau) I_v^*\left(-\frac{I_h(t) I_v^*}{I_h^* I_v(t)} + 1 + \ln\frac{I_h(t) I_v^*}{I_h^* I_v(t)}\right) d\tau\\ &+ \kappa C\left(2 - \frac{I_v^*}{I_v(t)} - \frac{I_v(t)}{I_v^*}\right) I_h(t)\\ =& -\left[s_h^*(\tau) V\left(\frac{s_h(t,\tau)}{s_h^*(\tau)}\right)\right]_{\tau=+\infty} - \int_0^{+\infty} \gamma_h(\tau) s_h^*(\tau) V\left(\frac{s_h(t,\tau)}{s_h^*(\tau)}\right) d\tau\end{aligned}$$

$$- \int_0^{+\infty} \beta(\tau) s_h^*(\tau) I_v^* V \left(\frac{s_h(t,\tau) I_h^* I_v(t)}{s_h^*(\tau) I_h(t) I_v^*} \right) d\tau$$

$$- \int_0^{+\infty} \beta(\tau) s_h^*(\tau) I_v^* V \left(\frac{I_h(t) I_v^*}{I_h^* I_v(t)} \right) d\tau + \kappa C \left(2 - \frac{I_v^*}{I_v(t)} - \frac{I_v(t)}{I_v^*} \right) I_h(t).$$

That is, the derivative $\frac{dL}{dt}$ of the positive-definite function $L(t)$ is negative-definite, and equalities $L(t) = 0$ and $\frac{dL}{dt} = 0$ hold only at E^*. Hence, by Lyapunov asymptotic stability theorem, the positive endemic equilibrium state is globally asymptotically stable, when it exists.

This complete the proof. □

5. Conclusion

In this paper we introduced and analytically study the dynamics of two age-structured models of a vector-borne infection. In the first of these models, the contagiousness of an infected host an his rate of removal (recovery) are assumed to depend on the infection age. In the second model the hosts population is structured by the hosts physical age, and host's susceptibility is assumed to depend on this age. Using the direct Lyapunov method and the recently developed technique of the global analysis [21, 23, 24], the global asymptotic stability of both these models was established. In particular, it was proved that depending on a threshold parameter, which was found for both these models, each of the models either has a globally asymptotically stable positive (endemic) equilibrium state, or the infection-free equilibrium state is the only equilibrium of the model, and is, in this case, globally asymptotically stable.

It is noteworthy, that, apart from a natural requirement of differentiability, we imposed no conditions on the age-dependent functions in the models, and hence this global asymptotic stability does not depend on specific forms of these functions.

We have to stress, however, that this global stability depends on the following assumptions:

(1) the transmission at the both stages, that is from an infected host to a vector, and then from the vector to a susceptible host, is assumed to be governed by the law of mass action, and
(2) the hosts and the vectors populations are assumed to be of constant size.

The first of these assumptions does not appear to be crucial, as a natural extension of these results to models with nonlinear incidence rate appears to be fairly straightforward, using a modification of the Volterra Lyapunov function suggested in [17, 18, 19]. The assumption of constant population sizes is more restrictive. For a human hosts population, the constant population size assumption is rather common in epidemiology, and it is well justified when the epidemic

processes occur on a considerably shorted time scale than the demographic processes. This is correct for the majority of the vector-borne diseases, and hence this assumption appears to be justified. However, the second of these assumptions is more dubious, as a vector life span (and in particularly arthropod vector) may be comparable with the duration of a single epidemic outbreak. Moreover, the reproduction of the vectors is often seasonal, and, in general, can be governed by a considerably more complex law than that which was used here. Nevertheless, for a long term, the constant vector population size assumption can be a correct postulate as well. That is, while the short terms dynamics can be more complex, the averaging over a long term, under the condition that the vector population remains approximately constant for comparable subintervals of this longer time interval, gives the convergence of the level of infection to the average equilibrium levels. This assumption is also well justified if the vector life span is comparatively short, and the vector never recover from infection, that is, if their infective period ends with their death, as, for example, it is for the mosquitoes vectors of West Nile virus, dengue fever and malaria [9].

Furthermore, for different infection and for different vectors, the vector reproduction can be very different, and hence the models must be different either. Nevertheless, in the frameworks of a "general" model, this assumption appears to be the only possible option. It also should be taken into consideration that for many arthropod hosts, such as mosquito, only a fraction of a large reservoir of eggs and larvae survives to the adult stage, and this process does not depend directly on the size of the adult population. However, we believe that further studies in this direction, taking into consideration particularities of both diseases and vectors, are necessary.

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