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EXPLICIT FOCAL BASIS FOR SOME PLANAR RIGID POLYNOMIAL DIFFERENTIAL SYSTEMS

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Abstract. In general the center–focus problem cannot be solved, but in the case that the singularity has purely imaginary eigenvalues there are algorithms to solving it. The present paper implements one of these algorithms for the polynomial differential systems of the form

$$\dot{x} = -y + xf(x)g(y), \quad \dot{y} = x + yf(x)g(y),$$

where $f(x)$ and $g(y)$ are arbitrary polynomials. These differential systems have constant angular speed and are also called rigid systems. More precisely, this paper gives focal bases of these systems, and then necessary and sufficient conditions in order to have an uniform isochronous center. In particular, the existence of a focus with the highest order is also studied.

1. Introduction and statement of the main result

Consider $x_0 \in \mathbb{R}^2$ as the singularity of an analytic differential system in the plane. The singularity x_0 is a *center* if there exists an open neighborhood U of x_0 such that all the solutions in $U \setminus \{x_0\}$ are periodic. Without loss of generality we can assume that the singular point is located at the origin de coordinates. Thus, after a linear change of variables and a rescaling of the time variable (if necessary) the analytic differential system with a center at the origin can be written in one of the following three forms

$$\text{(1a)} \qquad \dot{x} = -y + F_1(x, y), \qquad \dot{y} = x + F_2(x, y);$$

$$\text{(1b)} \qquad \dot{x} = y + F_1(x, y), \qquad \dot{y} = F_2(x, y);$$

$$\text{(1c)} \qquad \dot{x} = F_1(x, y), \qquad \dot{y} = F_2(x, y);$$

where $F_1(x, y)$ and $F_2(x, y)$ are real analytic functions without constant and linear terms, and defined in a neighborhood of the center. A center of an analytic system in the plane is called *linear*, *nilpotent*, or *degenerate* if, after an affine change of variables and a constant rescaling of the time it can be written as systems (1a), (1b) or (1c), respectively.

A center is *rigid* if its angular speed is constant. The nilpotent and degenerate centers cannot be rigid. Thus, the systems with a rigid center at the origin are

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linear. Therefore, a planar analytic differential system with a center at the origin – after a linear change of variables, and a rescaling of the time variable – is rigid if and only if it can be written in the form

$$\begin{aligned} \dot{x} &= -y + xF(x,y), \\ \dot{y} &= \quad x + yF(x,y), \end{aligned} \tag{2}$$

where $F(x, y)$ is an analytic function, which vanishes at the origin. Observe that, in the polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, system (2) becomes

$$\begin{aligned} \dot{r} &= rF(r\cos\theta, r\sin\theta), \\ \dot{\theta} &= 1, \end{aligned}$$

and then its angular speed is constant when it has a center at the origin.

A classical problem in the qualitative theory of planar differential equations is to distinguishing between a focus and a center, and it is called the *center–focus problem*. This problem is unsolved, but in the case that the singularity located at the origin is linear (see (1a)) it is of center or focus type, and there are standard algorithms to distinguish between a focus and a center. These algorithms go back to Poincaré [25], Liapunov [22, 21] and other authors: [6, 13, 15, 19, 20]. The results given by Liapunov [21] define functions in the coefficients of the analytic differential system which determine the stability of the focus, or if we have a center when all these functions are zero, such a functions are called now the *Liapunov constants* or *focal values*. This paper describes these constants for some rigid systems, and solves the center–focus problem on a class of systems of the form (2). Moreover, by using the order of a weak focus we provide an upper bound for the number of local limit cycles bifurcating in a small neighborhood of the center–focus.

The so called *isochronicity problem* consists in determining the conditions under which a singular point of a planar differential system with purely imaginary eigenvalues is a center and all its periodic orbits in a neighborhood of it have the same period. For the rigid systems the center–focus problem is equivalent to the isochronicity problem. This is one of the reasons for which differential systems of the form (2) has already been studied by several authors. In [10] is characterized the case $F(x, y) = H_p(x, y)$, where $H_p(x, y)$ is a homogeneous polynomial of degree $p \in \mathbb{N}$ (see also [9]). This result was recently improved in [17] by studying their limit cycles when $F(x, y) = a + H_p(x, y)$ and $a \in \mathbb{R}$. In [8] the case $F(x, y) = H_1(x, y) + H_2(x, y)$ is analyzed, and it is proved that all the center of this class are reversible. It was improved in [1] where the authors research the cases $F(x, y) = H_1(x, y) + H_p(x, y)$ and $F(x, y) = H_2(x, y) + H_{2p}(x, y)$. In all these cases it is determined the maximum number of limit cycles which can bifurcate from the weak focus localized at the origin of system (2) (see also [28]). In [18] the case $F(x, y) = H_0(x, y) + H_p(x, y) + H_q(x, y)$ is studied and complements [5, 7, 4] (see also [2, 3, 27]). In [23] and [4] the authors also study the existence of polynomial commutators for (2). In short in all these studies F is at most a sum of three homogeneous polynomials.

The present paper solve the center–focus problem for (2) when $F(x,y)$ is the product $f(x)g(y)$ of two polynomials with arbitrary degree. The strategy consist in describe some focal basis of those systems, and the existence of a focus with the highest order is also studied (see Corollary 5 and Remark 3).

Consider the following real polynomial differential system

$$\begin{aligned} \dot{x} &= -y + xf(x)g(y), \\ \dot{y} &= x + yf(x)g(y), \end{aligned} \tag{3}$$

where

$$\begin{aligned} f(x) &= a_0 + a_1 x + \ldots + a_N x^N, \\ g(y) &= b_0 + b_1 y + \ldots + b_M y^M, \end{aligned}$$

and

$$N \in \{2n, 2n+1\} \subset \mathbb{N} \cup \{0\} \quad \text{and} \quad M \in \{2m, 2m+1\} \subset \mathbb{N} \cup \{0\}.$$

Remark 1. *It is not difficult to see that one of following conditions*

$$g(-y) = -g(y) \text{ for all } y, \quad \text{or} \quad f(-x) = -f(x) \text{ for all } x, \tag{4}$$

implies that system (3) *is reversible. More precisely, if the function g (respectively f) is odd, the system is invariant under the changes $(x, y, t) \mapsto (x, -y, -t)$ (respectively $(x, y, t) \mapsto (-x, y, -t)$) and then it is symmetric with respect to the $x-$axis (respectively $y-$axis). This symmetry forces that any center–focus singularity at the origin of coordinates must be a center. Moreover, system* (3) *with $a_0 b_0 = 0$ has either a center or a weak focus at the origin. Therefore, one of the conditions, mentioned in* (4) *implies that system* (3) *has a reversible center at the origin.*

We recall that a singular point of a planar polynomial differential system is a *strong focus* if its eigenvalues are of the form $\alpha + i\beta$ with $\alpha \neq 0$, and it is a *weak focus* if $\alpha = 0$ and it is not a center.

In general the rigid system (2) with $F(0,0) = 0$ has a center–focus singularity at the origin, and this is a reversible center as long as $F(x, -y) = -F(x, y)$ for all x, y or $F(-x, y) = -F(x, y)$ for all x, y.

Our main result is the following one, where for a definition of a focal basis see subsection 2.1. Here $\mathbb{N}$ dentoes the set of all positive integers.

Theorem 1. *Consider the polynomial differential system* (3) *with $N \in \{2n, 2n+1\} \subset \mathbb{N} \cup \{0\}$ and $M \in \{2m, 2m+1\} \subset \mathbb{N} \cup \{0\}$. Then the origin of system* (3) *admits*

$$\mathcal{A} = \{a_0, a_2, \ldots, a_{2n-2}, a_{2n}\} \quad \text{and} \quad \mathcal{B} = \{b_0, b_2, \ldots, b_{2m-2}, b_{2m}\}$$

as focal bases.

Theorem 1 improves the main result of Dias and Mello [12], where the authors assume that either $f(x) = 1$ or $g(y) = 1$ (see also [11]).

In section 2 we describe the standard methods for computing the Liapunov constants in the case of rigid systems, and we prove a particular case of Theorem 1 (see Proposition 3). Finally, section 3 is devoted to prove Theorem 1 and some consequences.

2. Preliminary definitions and results

In the next subsection we recall some definitions and some classical results in order to prove a weaker version of the main result (see Lemma 2 and Proposition 3).

2.1. **The Poincaré map and the Liapunov constants.** Consider system (2) in polar coordinates with F a polynomial of degree $\delta \in \mathbb{N} \cup \{0\}$, and not necessarily with a zero of F at the origin. The homogeneous decomposition of F allows to write system (2) as

$$\frac{dr}{d\theta} = F(0,0)r + \sum_{j=2}^{1+\delta} R_j(\theta) r^j, \tag{5}$$

where $R_j(\theta)$ is a homogeneous polynomial of degree $j-1$ in the variables $\cos\theta$ and $\sin\theta$. In particular, $R_j(\theta)$ is a $2\pi-$periodic function.

Let $r(\theta, \rho)$ be the solution of system (5) satisfying that $r(0, \rho) = \rho$. Since $r(\theta, 0) \equiv 0$ we obtain that $r(\theta, \rho)$ can be expanded in a convergent power series of $\rho \geq 0$ sufficiently small. By (5), it take the form

$$r(\theta; \rho) = \exp(F(0,0)\theta)\rho + \sum_{i \geq 2} u_i(\theta)\rho^j, \text{ with } u_j(0) = 0.$$

Moreover the continuous dependence of the solutions on the parameters imply that every trajectory of system (2) in a sufficiently small neighborhood of the origin crosses every ray $\theta = c$, $0 \leq c < 2\pi$. Consequently, it is sufficient to consider all the trajectories passing through a small segment. In this context it is suitable to recall the following definitions and properties.

(i) The function $P(\rho) = r(2\pi, \rho)$ defined in a convenient interval $[0, \rho_0]$ with $\rho_0 > 0$ is called the *Poincaré map* or *first return map*.

(ii) $D(\rho) = P(\rho) - \rho$ is the *displacement map*, and it satisfies

$$D(\rho) = [\exp(F(0,0)2\pi) - 1]\rho + \sum_{k \geq 2} u_k(2\pi)\rho^k.$$

Its zeros provides the periodic solutions near the origin of coordinates.

(iii) The coefficients of the displacement map are named the *Liapunov constants* of system (2).

(iv) A *focal basis* associated to system (2) is a basis of the ideal generate by the Liapunov constants in the ring of real polynomials on the coefficients of $F(x, y)$.

By the Hilbert Basis Theorem the ideals of this ring are finitely generated. Furthermore it is clear that system(2) has a center at the origin if and only if $D(\rho) \equiv 0$, and also that the Liapunov constants control the behavior of the Poincaré map in a neighborhood of the origin. The geometry of this Poincaré map implies that the first nonzero Liapunov constant corresponds to an odd power. This nonzero constant does not depend of the ray where we take the segment in the definition of the respective displacement map, and it defines the *order of the weak focus*. Thus the order is ℓ when the power of the first non–zero coefficient of $D(\rho)$ is $k = 2\ell + 1$). The coefficients $u_{2\ell+1}(2\pi)$ are called the *Liapunov quantities* of system (2), and they span the same ideal associated with the definition of focal basis. Thus, if all the Liapunov quantities vanish the singular point is a center.

Remark 2. *A classical method to compute the Liapunov quantities of system* (1a) *uses a formal power series* $V(x, y) = \dfrac{x^2 + y^2}{2} + \sum_{\tilde{q}\geq 3} H_{\tilde{q}}(x, y)$*, where* $H_{\tilde{q}}$ *are homogeneous polynomials of degree* $\tilde{q}$ *(see for instance* [21]*). It is well know that there exist some* $V(x, y)$ *such that* $\dot{V}$ *(its rate of change along the orbits of system* (2)*) takes the form*

$$\dot{V} = \eta_2 r^2 + \eta_4 r^4 + \ldots + \eta_{2k} r^{2k} + \eta_{2k+2} r^{2k+2} + \ldots$$

where $r^2 = x^2 + y^2$*. Thus system* (2) *has a center at the origin if and only if the focal values* $\eta_{2k} = 0, \forall k$*. Moreover, the stability of the origin is determined by the sign of the first nonzero coefficient of* $\dot{V}$*, and it is proportional to the respective Liapunov quantity of* (2)*. More precisely,* $F(0, 0) = 0$ *and the Frommer's Theorem* [15] *shows that if* $\eta_2 = \cdots = \eta_{2\ell} = 0$ *and* $\eta_{2\ell+2} \neq 0$ *then*

$$u_{2\ell+1}(2\pi) = 2\pi\eta_{2\ell+2}.$$

Therefore the characterization of a center joint to the stability and the order of the weak focus is independent of the method used to describe the Liapunov constants.

The problem of computing the Liapunov constants for determining a center goes back to Liapunov [21, 22] and Poincaré [25]. The major difficulty with these Liapunov constants is their high complexity, and to find them explicitly becomes a computational problem. However there are several ways to compute them [14, 26], and different application to solve the center–focus problem [16, 24]. For some differential equations of the form (5) it is possible to describe the behavior of the expansion $\sum_{j\geq 1} u_j(\theta)\rho^j$ of $r(\theta; \rho)$, because $u'_k(\theta)$ is the coefficient of ρ^k in

$\frac{d}{d\theta}\sum_{j\geq 1} u_j(\theta)\rho^j$. Thus if we replace $r(\theta;\rho)$ in (5) we get

$$u_k'(\theta) = \text{coefficient of } \rho^k \text{ in } \sum_{j=1}^{\delta+1} R_j(\theta)\Big[\sum_{i\geq 1} u_i(\theta)\rho^i\Big]^j, \quad \text{for all } k \geq 1, \tag{6}$$

with $R_1(\theta) = F(0,0)$. For instance, (6) induces the Initial Value Problem

$$u_1'(\theta) = F(0,0)u_1(\theta), \quad u_1(0) = 1. \tag{7}$$

Its solution is $u_1(\theta) = \exp(F(0,0)\theta)$. In general, observe that given $j > k$, the right side of (6) does not add any element in the coefficient of ρ^k, and that given $1 \leq j \leq k$ the coefficient of ρ^k in the expression $\Big[\sum_{i\geq 1} u_i(\theta)\rho^i\Big]^j$ corresponds to all the possible ways of obtaining k by adding j indices $i_1, i_2, \ldots, i_j$ of $u_i(\theta)$ (repetitions are allowed). Consequently, the coefficient of ρ^k in $\Big[\sum_{i\geq 1} u_i(\theta)\rho^i\Big]^j$ is

$$\sum_{\substack{i_1+i_2+\cdots+i_j=k \\ \text{every } i_\ell \geq 1}} u_{i_1}(\theta)u_{i_2}(\theta)\cdots u_{i_j}(\theta).$$

Therefore, if $k \geq 2$ the function $u_k'(\theta)$ must satisfy the linear equation

$$u_k'(\theta) = F(0,0)u_k(\theta) + \sum_{j=2}^{k} R_j(\theta) \sum_{\substack{i_1+i_2+\cdots+i_j=k \\ \text{every } i_\ell \geq 1}} u_{i_1}(\theta)u_{i_2}(\theta)\cdots u_{i_j}(\theta),$$

with the initial condition $u_k(0) = 0$. In particular, if $F(0,0) = 0$

$$\begin{aligned} &u_1(\theta) = 1, \text{ and} \\ &u_k(\theta) = \sum_{j=2}^{k} \int_0^\theta R_j(t) \sum_{\substack{i_1+i_2+\cdots+i_j=k \\ \text{every } i_\ell \geq 1}} u_{i_1}(t)u_{i_2}(t)\cdots u_{i_j}(t)dt. \end{aligned} \tag{8}$$

2.2. **First result.** In this subsection we prove a particular case of the main result. To this end the next formulaes

$$\int_0^{2\pi} \cos^{2\tilde{p}+1}\theta \sin^{\tilde{q}} d\theta = \int_0^{2\pi} \cos^{\tilde{p}}\theta \sin^{2\tilde{q}+1} d\theta = 0, \quad \forall \tilde{p}, \tilde{q} \in \mathbb{N}\cup\{0\}, \tag{9}$$

will be needed.

Lemma 2. *Consider the system*

$$\begin{aligned} \dot{x} &= -y + x(1 + b_1 y + \ldots + b_M x^M)(a_1 x + \ldots + a_N x^N), \\ \dot{y} &= x + y(1 + b_1 y + \ldots + b_M x^M)(a_1 x + \ldots + a_N x^N), \end{aligned} \tag{10}$$

where $N \in \{2n, 2n+1\} \subset \mathbb{N}\cup\{0\}$ *and* $M \in \{2m, 2m+1\} \subset \mathbb{N}$. *The following statements hold.*

(a) *The solution* $r(\theta,\rho)$ *of system* (10) *with* $r(0,\rho)=\rho$ *satisfies* (8). *The displacement function has the form*

$$D(\rho)=a_2\pi\rho^3+\sum_{i\geq 4}u_i(2\pi)\rho^i,$$

and $u_2(\theta)=a_1\sin\theta$.

(b) *Suppose that* $a_2=\cdots=a_{2\ell}=0$ *for some* $\ell\leq n-1$, *then there is a constant* $A_\ell\neq 0$ *such that the map in (a) can be written as*

$$D(\rho)=a_{2\ell+2}A_\ell\rho^{2\ell+3}+\sum_{i\geq 2\ell+4}u_i(2\pi)\rho^i.$$

Moreover, there are $C_{k,j}\in\mathbb{R}$, $P(j,k)\in 2\mathbb{N}\cup\{0\}$ *and* $I(j,k)\in 2\mathbb{N}+1$ *such that*

$$u_{2s}(\theta)=\sum_{j\in F_{2s}}C_{2s,j}\cos^{P(j,2s)}\theta\sin^{I(j,2s)}\theta,\qquad 1\leq s\leq \ell+1;$$
$$u_{2s+1}(\theta)=\sum_{j\in F_{2s+1}}C_{2s+1,j}\cos^{P(j,2s+1)}\theta\sin^{P(j,2s+1)}\theta,\qquad 1\leq s\leq \ell;$$

where F_k *is a finite set for every* $2\leq k\leq 2\ell+2$.

(c) *System* (10) *has a center at the origin if and only if* $a_2=\cdots=a_{2n}=0$.

Proof. System (10) satisfies (2) with $F(0,0)=0$, and equation becomes

$$\frac{dr}{d\theta}=\sum_{j=2}^{M+N+1}\left(\sum_{s+t=j-1}a_sb_t\cos^s\theta\sin^t\theta\right)r^j=\sum_{j=2}^{M+N+1}\left(\sum_{s+t=j-1}R_j(\theta)\right)r^j, \tag{11}$$

with $a_0=0$ and $b_0=1$. As $R_2(\theta)=a_1\cos\theta$ and $R_3(\theta)=a_2\cos^2\theta+a_1b_1\cos\theta\sin\theta$, (8) implies that $u_2(\theta)=a_1\sin\theta$ and

$$u_3(\theta)=\int_0^\theta(a_2\cos^2 t+a_1(2a_1+b_1)\cos t\sin t)dt.$$

Therefore (a) holds.

We shall obtain (b) by induction. The first step starts with (a) and $a_2=0$. A direct computation shown that $D(\rho)=\frac{3\pi}{4}a_4^2\rho^5+\sum_{i\geq 6}u_i(2\pi)\rho^i$, $u_3(\theta)=\dfrac{2a_1^2+b_1a_1}{2}\sin^2\theta$ and $u_4(\theta)=-\frac{1}{6}\Big(\big(3a_1^3-a_3+4a_1^2b_1+a_1b_2\big)\cos(2\theta)-3a_1^3-5a_3-4a_1^2b_1-a_1b_2\Big)\sin\theta$. Since $\cos(2\theta)=\cos^2\theta-\sin^2\theta$, the first induction step is proved. In general, if $a_2=\cdots=a_{2\ell}=0$, with $\ell\geq 2$ the induction hypothesis implies that

$$D(\rho)=a_{2\ell}A_{\ell-1}\rho^{2\ell+1}+\sum_{i\geq 2\ell+2}u_i(2\pi)\rho^i.$$

Furthermore, there is a finite collection of numbers $C_{k,j} \in \mathbb{R}$, $P(j,k) \in 2\mathbb{N} \cup \{0\}$ and $I(j,k) \in 2\mathbb{N}+1$ such that

$$(12)\quad \begin{aligned} u_{2s}(\theta) &= \sum_{j \in F_{2s}} C_{2s,j} \cos^{P(j,2s)}\theta \sin^{I(j,2s)}\theta, & 1 \le s \le \ell; \\ u_{2s+1}(\theta) &= \sum_{j \in F_{2s+1}} C_{2s+1,j} \cos^{P(j,2s+1)}\theta \sin^{P(j,2s+1)}\theta, & 1 \le s \le \ell-1; \end{aligned}$$

where F_k is a finite set, for every $2 \le k \le 2\ell$. Since the first nonzero Liapunov constant corresponds to an odd number we have

$$(13)\quad D(\rho) = u_{2\ell+3}(2\pi)\rho^{2\ell+3} + \sum_{i \ge 2\ell+4} u_i(2\pi)\rho^i,$$

where $2\ell+3 \le 2n+1$ and

$$u_{2\ell+3}(\theta) = \int_0^\theta R_{2\ell+3}(t)dt + \sum_{j=2}^{2\ell+2} \int_0^\theta R_j(t) \sum_{\substack{i_1+i_2+\cdots+i_j=2\ell+3 \\ \text{every } i_\ell \ge 1}} u_{i_1}(t)\cdots u_{i_j}(t)dt.$$

In this formula the functions R_j's are given in (11) with the induction assumptions, that is

$$(14)\quad R_{2\ell+3}(\theta) = a_{2\ell+2}\cos^{2\ell+2}\theta + \sum_{\sigma=0}^{\ell+1} a_{2\sigma+1} b_{2\ell+1-2\sigma} \cos^{2\sigma+1}\theta \sin^{2\ell+1-2\sigma}\theta$$

and

$$(15)\quad R_j(\theta) = \sum_{\sigma=0}^{\left[\frac{j-2}{2}\right]} a_{2\sigma+1} b_{j-2\sigma-2} \cos^{2\sigma+1}\theta \sin^{j-2\sigma-2}\theta, \quad \text{for all } 2 \le j \le 2\ell+2.$$

Thus, (9) implies that

$$u_{2\ell+3}(2\pi) = a_{2\ell+2}\int_0^{2\pi} \cos^{2\ell+2} t dt + I_{2\ell+3}(2\pi) = a_{2\ell+2}A_\ell + I_{2\ell+3}(2\pi),$$

where

$$I_{2\ell+3}(\theta) = \sum_{j=2}^{2\ell+2} \int_0^\theta R_j(t) \sum_{\substack{i_1+i_2+\cdots+i_j=2\ell+3 \\ \text{every } i_\ell \ge 1}} u_{i_1}(t)\cdots u_{i_j}(t)dt.$$

Therefore statement (b) is directly obtained from (13) and the next claim.

Claim 1. *Suppose that* $a_2 = \cdots = a_{2\ell} = 0$ *with* $\ell \ge 2$. *Then*

(i) $u_{2\ell+1}(\theta)$ *finitely expands on even powers of* $\cos\theta$ *and* $\sin\theta$ *as in (b);*
(ii) $u_{2\ell+2}(\theta)$ *satisfies (b); and*
(iii) $\int_0^{2\pi} \cos^{2\ell+2} t dt \ne 0$ *and* $I_{2\ell+3}(2\pi) = 0$.

Proof of Claim 1. By (8) $u_{2\ell+1}(\theta)=\int_0^\theta R_{2\ell+1}(t)dt+\sum_{j=2}^{2\ell}\int_0^\theta S_j^{2\ell+1}(t)dt$, and $u_{2\ell+2}(\theta)$ $=\int_0^\theta R_{2\ell+2}(t)dt+\sum_{j=2}^{2\ell+1}\int_0^\theta S_j^{2\ell+2}(t)dt$, where

$$S_j^k(t)=R_j(t)\sum_{\substack{i_1+i_2+\cdots+i_j=k\\ \text{every } i_\ell\geq 1}} u_{i_1}(t)\cdots u_{i_j}(t),$$

for every $k\in\{2\ell+1,2\ell+2\}$ and $j\geq 2$. In both cases (15) implies

$$\int_0^\theta R_{2\ell+1}(t)dt=\sum_{\sigma=0}^{\left[\frac{2\ell-1}{2}\right]} a_{2\sigma+1}b_{2\ell-2\sigma-1}\int_0^\theta \cos^{2\sigma+1}t\sin^{2\ell-2\sigma-1}tdt,$$

and

$$\int_0^\theta R_{2\ell+2}(t)dt=\sum_{\sigma=0}^{\ell} a_{2\sigma+1}b_{2\ell-2\sigma}\int_0^\theta \cos^{2\sigma+1}t\sin^{2\ell-2\sigma}tdt.$$

Thus the first terms of $u_{2\ell+1}(\theta)$ and $u_{2\ell+2}(\theta)$ have the stated form because for every $p,q\in\mathbb{N}\cup\{0\}$ we have

$$\int_0^\theta \cos^{2\hat{p}+1}(t)\sin^{2\hat{q}+1}(t)dt=\sum_{a=0}^{\hat{q}}\binom{\hat{q}}{a}(-1)^a\frac{1-\cos^{2a+2\hat{p}+2}\theta}{2a+2\hat{p}+2}, \tag{16}$$

and

$$\int_0^\theta \cos^{2\check{p}+1}(t)\sin^{2\check{q}}(t)dt=\sum_{a=0}^{\check{p}+\check{q}}\binom{\check{p}+\check{q}}{a}(-1)^a\frac{\sin^{2a+1}\theta}{2a+1}, \tag{17}$$

In general the arguments to prove (i) and (ii) reduce to show the following claim.

Claim 2. *Suppose that $j\geq 2$. Then $S_j^{2\ell+1}(t)$ (respectively $S_j^{2\ell+2}(t)$) admits a finite expansion such that every term has the form of the integrant function in* (16) *(respectively* (17)*).*

Proof of Claim 2. Here we only do the proof for the function $S_j^{2\ell+1}(t)$, because the proof for the function $S_j^{2\ell+2}(t)$ is similar. From (15) we have

$$S_2^{2\ell+1}(t)=a_1\cos t\sum_{\substack{i_1+i_2=2\ell+1\\ \text{every } i_\ell\geq 1}} u_{i_1}(t)u_{i_2}(t).$$

Since the terms have the form $u_{even}(t)u_{odd}(t)$, equation (12) directly implies that $\cos t u_{even}(t)u_{odd}(t)$ satisfies the claim. Then, the second term of $u_{2\ell+1}(\theta)$ has the form of (b). In general, the terms of $S_j^{2\ell+1}(t)$, with $j-1\geq 2$ admit two cases:

(a.1) $R_j(t)u_{i_1}(t)\cdots u_{i_{j-1}}(t)u_{even}(t)$ with $i_1+\cdots+i_{j-1}$ odd; and

(a.2) $R_j(t)u_{i_1}(t)\cdots u_{i_{j-1}}(t)u_{odd}(t)$ with $i_1+\cdots+i_{j-1}$ even.

Consider (a.1) with $j = even$. By (12) and (16),

$$R_{j-1}(t)u_{i_1}(t)\cdots u_{i_{j-1}}(t) \approx_m \cos^{odd} t \sin^{odd} t, \tag{18}$$

where $A(t) \approx_m \cos^{odd} t \sin^{odd} t$, means the existence of a finite decomposition $A(t) = \sum_k A_k(t)$ and a finite set of constants $p_k, q_k \in \mathbb{N} \cup \{0\}$, $c_k \in \mathbb{R}$ (independent of t) such that $A_k(t) = c_k \cos^{2p_k+1} t \sin^{2q_k+1} t$, for all t and k. Notice that this notation naturally extends to all the possible cases $\approx_m \cos^{odd} t \sin^{even} t$, $\approx_m \cos^{even} t \sin^{odd} t$, etc. In this way, equation (15) implies that $R_{j-1}(t) = R_{odd}(t) \approx_m \cos^{odd} t \sin^{odd} t$, $R_j(t) = R_{even}(t) \approx_m \cos^{odd} t \sin^{even} t$ and also that $\sin t R_{odd}(t) \approx_m R_{odd+1} t$, for all $3 \leq odd \leq 2\ell - 1$, then

$$R_j(t)u_{i_1}(t)\cdots u_{i_{j-1}}(t) \approx_m \cos^{odd} t \sin^{even} t.$$

But (12) gives $u_{even}(t) \approx_m \cos^{even} t \sin^{odd} t$, thus

$$R_j(t)u_{i_1}(t)\cdots u_{i_{j-1}}(t)u_{even}(t) \approx_m \cos^{odd} t \sin^{odd} t. \tag{19}$$

Therefore, (a.1) with $j = even$ satisfies the claim. The case (a.1) with $j = odd$, also verifies (18), but now $R_{j-1}(t) = R_{even}(t) \approx_m \cos^{odd} t \sin^{even} t$, $R_j(t) = R_{odd}(t) \approx_m \cos^{odd} t \sin^{odd} t$ and $\sin t R_{even}(t) \approx_m R_{even+1}(t)$ for all $2 \leq even \leq 2\ell - 2$. However, as $u_{even}(t) \approx_m \cos^{even} t \sin^{odd} t$, (a.1) with $j = odd$ also satisfies (19). This concludes the proof of the claim in the case (a.1).

To obtain the claim in the case (a.2), observe that (12) and (17) give

$$R_{j-1}(t)u_{i_1}(t)\cdots u_{i_{j-1}}(t) \approx_m \cos^{odd} t \sin^{even} t. \tag{20}$$

Thus, the conditions $R_{even}(t) \approx_m \cos^{odd} t \sin^{even} t \approx_m \cos^{odd} t$ and $R_{odd}(t) \approx_m \cos^{odd} t \sin^{odd} t \approx_m \cos^{odd} t \sin t$ imply that

$$R_j(t)u_{i_1}(t)\cdots u_{i_{j-1}}(t) \approx_m \cos^{odd} t \sin^{odd} t.$$

But $u_{odd}(t) \approx_m \cos^{even} t \sin^{even} t$, consequently if $2\ell \geq j \geq 2$ then

$$R_j(t)u_{i_1}(t)\cdots u_{i_{j-1}}(t)u_{odd}(t) \approx_m \cos^{odd} t \sin^{odd} t.$$

This proves the claim in the case (a.2). Therefore Claim 2 holds. □

From Claim 2 statement (i) of Claim 1 follows.

The proof of (ii) is similar, and uses (i). The details are left to the reader. The first part of (iii) is directly obtained from the equality

$$\int_0^{2\pi} \cos^{2\bar{p}} \theta \sin^{2\bar{q}} \theta d\theta = 2\frac{\Gamma(\bar{p}+\frac{1}{2})\Gamma(\bar{q}+\frac{1}{2})}{\Gamma(\bar{p}+\bar{q}+1)} \quad \forall\, \bar{p}, \bar{q} \in \mathbb{N}, \tag{21}$$

where Γ is the Gamma Function which satisfies $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, and also that

$$\Gamma(m) = m!, \quad \Gamma\left(m+\frac{1}{2}\right) = \frac{1\cdot 3\cdot 5\cdot 7\cdots(2m-1)}{2^m}\Gamma\left(\frac{1}{2}\right), \quad \text{for all } m \in \mathbb{N}.$$

In particular $\int_0^{2\pi} \cos^{2\ell+2} t dt \neq 0$. The last part of (iii) follows from (9). The integrant function of $I_{2\ell+3}(\theta)$ is $\sum_{j=2}^{2\ell+2} S_j^{2\ell+3}(t)$, with $u_{2\ell+1}(\theta)$ and $u_{2\ell+2}(\theta)$ as in (i) and (ii), respectively. Thus a similar idea as in the last proofs shows that $\sum_{j=2}^{2\ell+2} S_j^{2\ell+3}(t) \approx_m \cos^{odd} t \sin^{odd} t$ and then $I_{2\ell+3}(2\pi) = 0$. Therefore (iii) holds. This proves Claim 1. □

From Claim 1 it follows the proof of statement (b).

To obtain (c) consider the center assumption $D(\rho) \equiv 0$. By using (a) and (b) it is not difficult to prove that $a_2 = \cdots = a_{2n} = 0$. The converse follows from Remark 1, and so (c) holds. Therefore the lemma is proved. □

Remark 3. *Notice that a direct application of statement (b) of Lemma 2 shows that the assumptions $a_2 = a_4 = \cdots = a_{2n-2} = 0$ and $a_{2n} \neq 0$ imply that* (10) *has a weak focus or order n, located at the origin.*

Proposition 3. *Consider the following system*

$$\begin{aligned} \dot{x} &= -y + x(b_0 + b_1 y)(a_0 + a_1 x + \cdots + a_N x^N), \\ \dot{y} &= x + y(b_0 + b_1 y)(a_0 + a_1 x + \cdots + a_N x^N), \end{aligned} \tag{22}$$

where $N \in \{2n, 2n+1\} \subset \mathbb{N} \cup \{0\}$. The origin is a center for system (22) *if and only if one of the following conditions holds:*

(i) $b_0 = 0$.
(ii) $a_0 = a_2 = \cdots = a_{2n} = 0$.

Moreover, the center is reversible.

Proof. Suppose that one of the hypotheses, (i) or (ii) holds. A direct application of Remark 1 shows that (22) has a reversible center at the origin.

Conversely, assume that (22) has a center. Since the eigenvalues of the linearization at the singularity are $a_0 b_0 \pm i$, the product $a_0 b_0$ must be zero. If $b_0 \neq 0$ we can rewrite (22) as

$$\begin{aligned} \dot{x} &= -y + x(1 + \bar{b}_1 y)(\bar{a}_1 x + \cdots + \bar{a}_N x^N), \\ \dot{y} &= x + y(1 + \bar{b}_1 y)(\bar{a}_1 x + \cdots + \bar{a}_N x^N), \end{aligned}$$

where $\bar{b}_1 = \frac{b_1}{b_0}$ and $\bar{a}_j = a_j b_0$ (because $a_0 = 0$). Statement (c) of Lemma 2 with $M = 1$ implies $\bar{a}_2 = \cdots = \bar{a}_{2n} = 0$. The last part follows from Remark 1 and concludes the proof. □

3. The focal basis and rigid systems

Using the notation of system (3) in the next lemma we consider the case $m \geq 1$ because in Lemma 2 we have studied the case $m = 0$.

Lemma 4. *Let $f(x) = a_0 + a_1 x + \cdots + a_N x^N$ be a real polynomial map such that $N \in \{2n, 2n+1\} \subset \mathbb{N} \cup \{0\}$, and consider the following system*

$$\begin{aligned}\dot{x} &= -y + x\Big(\overbrace{b_1 y + \cdots + b_{2p-1} y^{2p-1}}^{\textit{odd powers}} + \overbrace{b_{2p} y^{2p} + \cdots + b_M y^M}^{\textit{no power restrictions}}\Big) f(x),\\ \dot{y} &= x + y\Big(b_1 y + \cdots + b_{2p-1} y^{2p-1} + b_{2p} y^{2p} + \cdots + b_M y^M\Big) f(x),\end{aligned} \tag{23}$$

where $M \in \{2m, 2m+1\} \subset \mathbb{N}$ and $1 \leq p \leq m$. Suppose that $b_{2p} \neq 0$. Then the following holds.

(a) *The first Liapunov constants of system (23) are*

$$0, 0, \ldots, 0, a_0 b_{2p} B_p,$$

where the constant $B_p \neq 0$.

(b) *Suppose that $a_0 = a_2 = \cdots = a_{2\ell} = 0$, for some $\ell \leq n-1$. Then there is a constant $\tilde{A}_\ell \neq 0$ such that the Liapunov constants of system (23), as in (a) satisfy*

$$0, 0, \ldots, 0, a_0 b_{2p} B_p = 0, 0, \ldots, b_{2p} a_{2\ell+2} \tilde{A}_\ell.$$

(c) *System (23) has a center at the origin if and only if $a_2 = \cdots = a_{2n} = 0$.*

Proof. In the new coordinates $X = -y$ and $Y = x$, system (23) becomes

$$\begin{aligned}\dot{X} &= -Y + X\left(\sum_{i=0}^{M} b_i (-X)^i\right) f(Y),\\ \dot{Y} &= X + Y\left(\sum_{i=0}^{M} b_i (-X)^i\right) f(Y),\end{aligned} \tag{24}$$

where $b_0 = b_2 = \cdots = b_{2p-2} = 0$. In polar coordinates $X = r\cos\theta$ and $Y = r\sin\theta$ this system becomes

$$\frac{dr}{d\theta} = \sum_{j=2}^{M+N+1}\left(\sum_{t=0}^{j-1} (-1)^t a_{j-1-t} b_t \cos^t\theta \sin^{j-1-t}\theta\right) r^j = \sum_{j=2}^{M+N+1} R_j(\theta) r^j. \tag{25}$$

Therefore

$$R_j(\theta) = \sum_{\tau=0}^{\left[\frac{j-1}{2}\right]} a_{j-1-2\tau} b_{2\tau} \cos^{2\tau}\theta \sin^{j-1-2\tau}\theta \tag{26}$$
$$- \sum_{\tau=0}^{\left[\frac{j-2}{2}\right]} a_{j-2\tau-2} b_{2\tau+1} \cos^{2\tau+1}\theta \sin^{j-2\tau-2}\theta, \quad \text{for all } j \geq 2.$$

Thus $R_{2p+1}(t) = a_0 b_{2p} \cos^{2p}(t) - \sum_{\tau=0}^{\left[\frac{2p-1}{2}\right]} a_{2p-2\tau-1} b_{2\tau+1} \cos^{2\tau+1} t \sin^{2p-2\tau-1} t$, and from (8) and (9) we obtain that

$$u_{2p+1}(2\pi) = a_0 b_{2p} \int_0^{2\pi} \cos^{2p}(t)dt + I_{2p+1}(2\pi) = a_0 b_{2p} B_p + I_{2p+1}(2\pi),$$

where $I_{2p+1}(\theta) = \sum_{j=2}^{2p} \int_0^\theta S_j^{2p+1}(t)dt$. More precisely, as (8) induces the general equality $u_k(\theta) = \int_0^\theta R_k(t)dt + \sum_{j=2}^{k-1} \int_0^\theta S_j^k(t)dt$, statement (a) is directly obtained from Remark 2 and the next claim.

Claim 3. *The following statements hold.*

(i) $\int_0^{2\pi} R_j(t)dt = 0$ *for all* $2 \leq j \leq 2p$, $u_1(\theta) = 1$ *and* $u_2(\theta) = -a_0 b_1 \sin\theta$.

(ii) $u_j(2\pi) = 0$ *for all* $2 \leq j \leq 2p$ *and*

$$\begin{aligned} u_{2s}(\theta) &\approx_m \cos^{even}\theta \sin^{odd}\theta \quad && \text{if} \quad 1 \leq s \leq p, \\ u_{2s+1}(\theta) &\approx_m \cos^{even}\theta \sin^{even}\theta \quad && \text{if} \quad 0 \leq s \leq p-1. \end{aligned}$$

(iii) $\int_0^{2\pi} \cos^{2p} t dt \neq 0$ *and* $I_{2p+1}(2\pi) = 0$.

Proof of Claim 3. The first part of (i) is obtained form (9) because

$$R_j(\theta) = - \sum_{\tau=0}^{\left[\frac{j-2}{2}\right]} a_{j-2\tau-2} b_{2\tau+1} \cos^{2\tau+1}\theta \sin^{j-2\tau-2}\theta, \quad \text{for all } 2 \leq j \leq 2p. \tag{27}$$

As (26) gives $R_2(t) = -a_0 b_1 \cos t$, the last equality follows from (8). Therefore (i) holds.

To obtain (ii) observe that the case $p = 1$ is obtained directly from (i). In the general case $p \geq 2$ the argument proceeds by induction. The firs step is given by

$u_1(\theta)=1$, $u_2(\theta)=-a_0b_1\sin\theta$ and $u_2(2\pi)=0$. The induction assumption is

$$
\begin{array}{lll}
u_j(2\pi)=0 & \text{if} & 2\le j\le 2s-2\le 2p-2.\\
u_{2\sigma}(\theta)\approx_m \cos^{even}\theta\sin^{odd}\theta & \text{if} & 1\le\sigma\le s-1\le p-1,\\
u_{2\sigma+1}(\theta)\approx_m \cos^{even}\theta\sin^{even}\theta & \text{if} & 0\le\sigma\le s-1\le p-2.
\end{array}
\tag{28}
$$

Now the argument reduces to show the following claim.

Claim 4. *The following statements hold.*

(I) $2\le j\le 2s\Rightarrow S_j^{2s}(t)\approx_m \cos^{odd}(t)\sin^{even}(t)$.

(II) $2\le j\le 2s+1\Rightarrow S_j^{2s+1}(t)\approx_m \cos^{odd}(t)\sin^{odd}(t)$.

(III) $u_{2s-1}(2\pi)=0$.

Proof of Claim 4. In fact $R_2(t)=-a_0b_1\cos(t)$ gives

$$S_2^{2s}(t)=-a_0b_1\cos t\sum_{\substack{i_1+i_2=2s\\ \text{every } i_\ell\ge 1}} u_{i_1}(t)u_{i_2}(t).$$

The terms are either $u_{even}(t)u_{even}(t)$ or $u_{odd}(t)u_{odd}(t)$. Thus (28) implies that $\cos t u_{even}(t)u_{even}(t)$ and $\cos t u_{odd}(t)u_{odd}(t)$ have the stated form, and $S_2^{2s}(t)$ satisfies the claim. In general, the terms of $S_j^{2s}(t)$, with $j-1\ge 2$ admit two cases:

(a.1) $R_j(t)u_{i_1}(t)\cdots u_{i_{j-1}}(t)u_{even}(t)$ with $i_1+\cdots+i_{j-1}$ even.

(a.2) $R_j(t)u_{i_1}(t)\cdots u_{i_{j-1}}(t)u_{odd}(t)$ with $i_1+\cdots+i_{j-1}$ odd.

Consider (a.1). By (28) and (17),

$$R_{j-1}(t)u_{i_1}(t)\cdots u_{i_{j-1}}(t)\approx_m \cos^{odd}t\sin^{even}t,$$

Since the R_j's satisfy (27), $R_{j-1}(t)\sin t\approx_m R_j(t)$ and then

$$R_j(t)u_{i_1}(t)\cdots u_{i_{j-1}}(t)\approx_m \cos^{odd}t\sin^{odd}t.$$

Moreover, (28) gives $u_{even}(t)\approx_m \cos^{even}t\sin^{odd}t$. Therefore

$$R_j(t)u_{i_1}(t)\cdots u_{i_{j-1}}(t)u_{even}(t)\approx_m \cos^{odd}t\sin^{even}t,$$

and $S_j^{2s}(t)$ satisfies (I) if $(a.1)$ holds. If (a.2) holds then equations (28) and (16) imply

$$R_{j-1}(t)u_{i_1}(t)\cdots u_{i_{j-1}}(t)\approx_m \cos^{odd}t\sin^{odd}t.$$

Consequently

$$R_j(t)u_{i_1}(t)\cdots u_{i_{j-1}}(t)\approx_m \cos^{odd}t\sin^{even}t.$$

and $u_{odd}(t)\approx_m \cos^{even}t\sin^{even}t$ imply that

$$R_j(t)u_{i_1}(t)\cdots u_{i_{j-1}}(t)u_{odd}(t)\approx_m \cos^{odd}t\sin^{even}t.$$

Therefore, $S_j^{2s}(t)$ always satisfies (I). The proof of (II) for $S_j^{2s+1}(t)$ is similar to the proof of (I). Moreover the properties of $S_j^{2s-1}(t)$ and (i) imply (III). The details are left to the reader. Therefore the claim is proved. □

From Claim 4 statement (ii) follows.

To prove (iii) observe that equation (21) directly gives $\int_0^{2\pi} \cos^{2p}(t)dt \neq 0$. A similar argument used in the proof of the (ii) shows that the integrant function of $I_{2p+1}(\theta) = \sum_{j=2}^{2p} \int_0^{\theta} S_j^{2p+1}(t)dt$ complies (9), and consequently $I_{2p+1}(2\pi) = 0$. This concludes the proof of Claim 3. □

By using Remark 2 and Claim 3, it is not difficult to obtain statement (a).

To prove (b) consider the next claim.

Claim 5. *The following statements hold.*

(i) *If* $2 \leq j \leq 2p + 2\ell + 2$, *then*

$$R_j(\theta) \in \{\approx_m \cos^{odd}\theta \sin^{even}\theta, \approx_m \cos^{even}\theta \sin^{odd}\theta\},$$

and so $\int_0^{2\pi} R_j(t)dt = 0$.

(ii) $u_j(2\pi) = 0$ *for all* $2 \leq j \leq 2p + 2\ell + 2$

$$\begin{aligned} u_{2s}(\theta) &\approx_m \cos^{even}\theta \sin^{odd}\theta, && \text{if} \quad 1 \leq s \leq p + \ell + 1, \\ u_{2s+1}(\theta) &\approx_m \cos^{even}\theta \sin^{even}\theta, && \text{if} \quad 0 \leq s \leq p + \ell. \end{aligned}$$

(iii) *By* (26),

$$\begin{aligned} R_{2p+2\ell+3}(\theta) = {} & b_{2p} a_{2\ell+2} \cos^{2p}\theta \sin^{2\ell+2}\theta \\ & - \sum_{\tau=0}^{p+\ell} a_{2p+2\ell-2\tau+1} b_{2\tau+1} \cos^{2\tau+1}\theta \sin^{2p+2\ell-2\tau+1}\theta. \end{aligned}$$

Consequently

$$u_{2p+2\ell+3}(2\pi) = b_{2p} a_{2\ell+2} \tilde{A}_\ell + I_{2p+2\ell+3}(2\pi),$$

where $\tilde{A}_\ell = \int_0^{2\pi} \cos^{2p}(t) \sin^{2\ell+2}(t)dt$ *is different from zero and the value* $I_{2p+2\ell+3}(2\pi) = 0$.

The proof of Claim 5 is similar to the proof of Claim 3 and we do not prove it. Claim 5 directly implies statement (b). In the first part, the hypothesis $a_0 = \cdots = a_{2\ell} = 0$ is used to describe R_{2p+odd} with $2 \leq 2p + odd \leq 2p + 2\ell + 2$. The details are left to the reader. Therefore statement (b) holds.

Statement (c) follows from an application of (a), (b) and Remark 1. Therefore the lemma is proved. □

Corollary 5. *Set* $f(x) = a_0 + a_1 x + \cdots + a_N x^N$. *Suppose that* $a_0 = a_2 = \cdots = a_{2n-2} = 0$ *and* $a_{2n} \neq 0$. *Then the system*

$$
\begin{aligned}
\dot{x} &= -y + x\big(y^{2m-1} + y^{2m}\big) f(x), \\
\dot{y} &= x + y\big(y^{2m-1} + y^{2m}\big) f(x),
\end{aligned}
\tag{29}
$$

with $m \geq 1$ *has at the origin a weak focus of order* $m+n$. *Consequently, there are at most* $m+n$ *small limit cycles in a suitable neighborhood of the origin.*

Proof. This is a direct consequence of the proofs of the last claims, under the assumptions $p = m$, $\ell = n-1$ and $b_{2m} = 1$. The origin is a center–focus singularity of (29). Moreover equation (29) and a particular solution $r(\theta;\rho)$ of it imply that its displacement function is

$$
D(\rho) = \left(a_{2n} \int_0^{2\pi} \cos^{2m}(t) \sin^{2n}(t) dt \right) \rho^{2m+2n+1} + \sum_{i \geq 2m+2n+2} u_i(2\pi)\rho^i .
$$

Therefore, the order of the weak focus associated to (29) is $m+n$, and it is located at the origin. The last part is consequence of the well–know property that the order of a weak focus is an upper bound for the number of local limit cycles bifurcating in a small neighborhood of the focus. □

Remark 4. *By using Lemmas 2 and 4 it is not difficult to prove that the set* $\{a_0, a_2, \ldots, a_{2n}\}$ *is a focal basis for both systems* (10) *and* (23).

Proof of Theorem 1. If some $b_{even} \neq 0$, by using the smallest even j with $b_j \neq 0$. The result is directly obtained from Remark 4.

If some $a_{even} \neq 0$, the new coordinates $X = -y$ and $Y = x$, imply that system (3) becomes

$$
\dot{X} = -Y + X \left(\sum_{i=0}^{N} a_i (-Y)^i \right) g(X),
$$

$$
\dot{Y} = X + Y \left(\sum_{i=0}^{N} a_i (-Y)^i \right) g(X).
$$

Thus the proof of Lemmas 4 and 2 imply that for the smallest *even* with $a_{even} \neq 0$ the following two properties.

(i) The first nontrivial Liapunov constant has the form $a_{even} b_0 B_{even}$, where B_{even} is a nonzero constant.

(ii) Assuming that $b_0 = b_2 = \cdots = b_{2\ell} = 0$, for some $\ell \leq m-1$, then the first nontrivial Liapunov constant has the form $a_{even} b_{2\ell+2} \tilde{B}_{even}$, where $\tilde{B}_{even}$ is a nonzero constant.

Consequently, Remark 2 helps to obtain that the set $\{b_0, b_2, \ldots, b_{2m}\}$ is a focal basis for (3), and the theorem holds.

If $a_0 = a_2 = \cdots = a_{2n} = 0$ and $b_0 = b_2 = \cdots = b_{2m} = 0$, the result follows from Remark 1. This concludes the proof of the theorem. □

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