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Realization and characterization of modulus of smoothness in weighted Lebesgue spaces

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REALIZATION AND CHARACTERIZATION OF MODULUS OF SMOOTHNESS IN WEIGHTED LEBESGUE SPACES

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Abstract. We obtain a characterization of modulus of smoothnes of fractional order in the Lebesgue spaces L_{ω}^{p}, $1<p<\infty$, with weights ω satisfying the Muckenhoupt's A_p condition. Also, a realization result and equivalence between modulus of smoothness and the Peetre K-functional are proved in L_{ω}^{p} for $1<p<\infty$ and $\omega\in A_p$.

1. Introduction

One of the main problems of the constructive function theory is finding a relationship between structural characteristics and differential properties of functions. For studying the structural properties of a function one of the measures is given by the modulus of smoothness. This concept has various applications in analysis such as approximation theory, function spaces and interpolation theory. The modulus of smoothness of fractional order $r\geq 0$

$$\omega_r(f,\delta)_p:=\sup_{0\leq h\leq\delta}\left\|(T_h-I)^r f\right\|_p,\ f\in L^p,\ 1\leq p\leq\infty,$$

was defined by P. L. Butzer, H. Dyckhoff, E. Görlich, R. L. Stens [7] and R. Taberski [30], where I is the identity operator and $T_h f(\cdot):=f(\cdot+h)$ is the translation operator. It is well known that there is a close relation between decreasing order of modulus of smoothness of a function f and certain approximation properties of this function. Some of useful properties of the modulus of smoothness $\omega_r(\cdot,\delta)_p$ are hold when $\delta\searrow 0$ (near the origin). For $f\in L^p$, the decreasing order $\omega_r(f,\delta)_p\searrow 0$ can be described in terms of the function class Φ_r.

Definition 1. *We say that a function φ belongs to the class Φ_r $(r\in\mathbb{R})$ if it satisfies the conditions:* (a) *$\varphi(t)$ is nonnegative and bounded on $(0,\infty)$,* (b) *$\varphi(t)\to 0$ as $t\to 0$,* (c) *$\varphi(t)$ is nondecreasing, and* (d) *$\varphi(t)t^{-r}$ is non-increasing.*

The class Φ_r describes completely the class of all majorants for the modulus of smoothness $\omega_r(\cdot,\delta)_p$ in the space L^p, $1\leq p\leq\infty$, namely,

Theorem 1 ([31])**.** *Suppose that $r>0$ and $1\leq p\leq\infty$. In this case*

(a) *If $f\in L^p$, then there exists a function $\varphi\in\Phi_r$ such that*

$$\varphi(t)\approx\omega_r(f,t)_p$$

holds for any $t\in(0,\infty)$, where equivalence constants are depend only on r and p.

(b) *If $\varphi\in\Phi_r$, then there exist $f\in L^p$ and a positive real number t_0 such that*

$$\omega_r(f,\delta)_p\approx\varphi(\delta)$$

holds for any $\delta\in(0,t_0]$, where the equivalence constants are depend only on r and p.

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Here and in what follows, $A \lesssim B$ means that there exists a constant c, independent of essential parameters, such that the inequality $A \leq cB$ is hold. If $A \lesssim B$ and $B \lesssim A$ simultaneously, we will write $A \approx B$.

For $\omega_r(\cdot,\delta)_p$, $r \in \mathbb{N}$, some results of type (a) and (b) in Theorem 1 was investigated in papers [5, 24, 26]. For more information about fractional order modulus of smoothness we refer to the book [27] and the papers [6, 30, 31].

An important property of modulus of smoothness is that $\omega_r(\cdot,\delta)_p$ is equivalent to the Peetre K-functional

$$K_r(f,t,L^p) := \inf_{g,g^{(r)}\in L^p}\left\{\|f-g\|_p + t^r\left\|g^{(r)}\right\|_p\right\}, \quad t,r>0,\ f\in L^p,\ 1\leq p\leq\infty. \tag{1.1}$$

The following equivalence was proved in [7]

$$\omega_r(f,t)_p \approx K_r(f,t,p), t>0,$$

for $r>0$, $f\in L^p$, $1\leq p\leq\infty$. In the case of $0<p<1$, the K-functional is $\equiv 0$ (see e.g. [10]). In this case the realization functional

$$R_r(f,1/n,L^p) := \|f-t_n^*\|_p + \frac{1}{n^r}\left\|(t_n^*)^{(r)}\right\|_p, \ r>0,\ f\in L^p,\ 0<p\leq\infty,\ n\in\mathbb{N},$$

can be used. Here and in what follows, by t_n^* we will denote the best (or near best) approximating trigonometric polynomial for f, i.e., $\|f-T\|_{p,\omega} = E_n(f)_{p,\omega} := \inf_{T\in\mathcal{T}_n}\|f-T\|_{p,\omega}$ (respectively, $\|f-T\|_{p,\omega} \lesssim E_n(f)_{p,\omega}$).

Realization result has a lot of applications ([9]). In particular, it is used to get Ul'yanov type inequalities (see e.g., [11]).

Theorem 2. *If $r>0$, $0<p\leq\infty$, $f\in L^p$, then the equivalence*

$$\omega_r(f,1/n)_p \approx R_r(f,1/n,L^p)$$

holds for $n=1,2,3,\ldots$.

Theorem 2 was proved, in the equivalent form, in [10], for integer order modulus of smoothness. In case of the fractional order modulus of smoothness, Theorem 2 was proved in [28], [7] ($1\leq p\leq\infty$) and [19] ($0<p\leq\infty$).

The main goal of this paper is to obtain analog of Theorems 1 and 2 for the weighted Lebesgue spaces L^p_ω, where the weight ω belongs to the Muckenhoupt class A_p, ($1<p<\infty$). A 2π-periodic weight (i.e., a measurable and almost everywhere positive function) ω belongs to the Muckenhoupt class A_p, $1<p<\infty$, (see e.g., [23]) if

$$\left(\frac{1}{|J|}\int_J \omega(x)\,dx\right)\left(\frac{1}{|J|}\int_J \omega^{\frac{1}{1-p}}(x)\,dx\right)^{p-1} \leq c \tag{1.2}$$

with a finite constant c independent of J, where J is any subinterval of $\mathbb{T}:=[0,2\pi]$ and $|J|$ denotes the length of J. The least constant c satisfying (1.2) is called the A_p constant of ω and denoted by $A[p]:=A[p](\omega)$. The Muckenhoupt weights play special role in Harmonic Analysis because these are precisely the weights for which some singular integrals and maximal operators are bounded in the weighted Lebesgue spaces. We refer to the monograph [13] for a complete account on the theory of Muckenhoupt weights. Throughout this work by c we denote positive constants which can be different at different places.

The weighted Lebesgue space L^p_ω, $1<p<\infty$, is the collection of measurable functions $f\colon\mathbb{T}\to\mathbb{R}$ having the norm $\|f\|_{p,\omega} := \left\{\int_{\mathbb{T}}|f(x)|^p\,\omega(x)\,dx\right\}^{1/p} < \infty$.

If $\omega \in A_p$, $1 < p < \infty$, then there exists a real number $a > 1$ such that

$$L^\infty \hookrightarrow L^p_\omega \hookrightarrow L^a \tag{1.3}$$

with constants dependent only on p and $A[p]$. The left hand side of (1.3) is seen from (1.2) and the right hand side of (1.3) was proved in [15, 21].

Since the weighted Lebesgue spaces L^p_ω are, in general, not translation invariant, we need different definition of modulus of smoothness in L^p_ω. For the case $\omega \equiv 1$, there are many different types of modulus of smoothness (see for example [8, 32, 33]). For the Muckenhoupt weights, in 1986 E. A. Gadjieva ([12]) continued investigations in [33], and defined a modulus of smoothness which is constructed by means of the Steklov operator

$$\sigma_h f(x) := \frac{1}{2h}\int_{x-h}^{x+h} f(t)\,dt, \quad x \in \mathbb{T}, \quad h > 0. \tag{1.4}$$

For another approach, see [21, 22]. The Steklov mean satisfies the inequality $\sigma_h f(x) \leq Mf(x)$ a.e. on $\mathbb{T}$, where M is the Hardy-Littlewood maximal function. It is well known that necessary and sufficient condition for boundedness of M in L^p_ω, $1 < p < \infty$ is $\omega \in A_p$. Hence, under the conditions $f \in L^p_\omega$ $(1 < p < \infty,\ \omega \in A_p)$, we have

$$\|\sigma_h f\|_{p,\omega} \lesssim \|f\|_{p,\omega}, \tag{1.5}$$

with some constant independent of f and h. We note that for the case $\omega \equiv 1$ we have (1.5) for the range $1 \leq p \leq \infty$.

We set $B_h := (I - \sigma_h)$, $[x] := \max\{a \in \mathbb{N} : a \leq x\}$, $\{x\} := x - [x]$ and $\prod_{i=k}^{l} B_{h_i} f := f$ when $l < k$. For $1 < p < \infty$, $\omega \in A_p$ and $f \in L^p_\omega$, we define

$$\Omega_r(f,\delta)_{p,\omega} := \sup_{0 \leq h_i, t \leq \delta} \left\| \prod_{i=1}^{[r]} B_{h_i} B_t^{\{r\}} f \right\|_{p,\omega} \quad r > 0, \tag{1.6}$$

where $\Omega_0(f,\delta)_{p,\omega} := \|f\|_{p,\omega}$. In case $\omega \equiv 1$ and $1 \leq p \leq \infty$, we write $\Omega_r(\cdot,\delta)_p := \Omega_r(\cdot,\delta)_{p,\omega}$.

Using binomial expansion, for $t > 0$ and $0 \leq r < 1$, we have

$$B_t^r f(\cdot) = \sum_{k=0}^{\infty} (-1)^k [C]_k^r \left(\sigma_t^k f\right)(\cdot), \quad f \in L^p_\omega. \tag{1.7}$$

Here as usual $[C]_0^r := 1$ and $[C]_k^r := \prod_{j=1}^{k} \frac{r-j+1}{j}$ are the binomial coefficients, $\sigma_t^0 := I$ and $\sigma_t^i := \sigma_t\left(\sigma_t^{i-1}\right)$ for $i = 1, 2, 3, \ldots$.

From [27], we get $|[C]_k^r| \lesssim k^{-r-1}$, $(k \in \mathbb{N})$ and hence,

$$C_r := \sum_{k=0}^{\infty} |[C]_k^r| < \infty. \tag{1.8}$$

Then using (1.7) and (1.8) we have

$$\|B_t^r f\|_{p,\omega} \lesssim \|f\|_{p,\omega} \quad 0 \leq r < 1 \quad t > 0 \tag{1.9}$$

for some constant dependent only on r and $A[p]$. Hence, from (1.9), (1.5) and (1.6) we get for $r > 0$, $\delta \geq 0$, $1 < p < \infty$, $\omega \in A_p$ and $f \in L^p_\omega$

$$\Omega_r(f,\delta)_{p,\omega} \lesssim \|f\|_{p,\omega}, \tag{1.10}$$

with some constant dependent only on r and $A[p]$. We note that for the case $\omega \equiv 1$ we have (1.10) for the range $1 \leq p \leq \infty$. Our main result is the following.

Theorem 3. *Let $r > 0$, $1 < p < \infty$ and $\omega \in A_p$.*

(a) *If* $f \in L^p_\omega$, *then there exists a function* $\varphi \in \Phi_{2r}$ *such that*

$$\varphi(t) \approx \Omega_r(f,t)_{p,\omega}$$

holds for any $t \in (0,\infty)$, *where equivalence constants are depend only on* r *and* $A[p]$.

(b) *If* $\varphi \in \Phi_{2r}$, *then there exist* $f \in L^p_\omega$ *and a real number* $t_0 > 0$ *such that*

$$\Omega_r(f,\delta)_{p,\omega} \approx \varphi(\delta), \delta \in (0,t_0],$$

and

$$\Omega_r(f,\delta)_{p,\omega} \approx \varphi(t_0), \delta \in (t_0,2\pi),$$

hold, where the equivalence constants are depend only on r *and* $A[p]$.

The following theorem includes the realization result and an equivalence of the modulus of smoothness (1.6) and the Peetre K-functional (1.1).

Theorem 4. *If* $r > 0$, $f \in L^p_\omega$, $1 < p < \infty$ *and* $\omega \in A_p$, *then the equivalence*

$$\Omega_r(f,1/n)_{p,\omega} \approx R_{2r}(f,1/n,L^p_\omega) \tag{1.11}$$

holds for $n = 1,2,3,\ldots$, *where the equivalence constants are depend only on* r *and* $A[p]$. *Furthermore, we have*

$$\Omega_r(f,\delta)_{p,\omega} \approx K_{2r}(f,\delta,L^p_\omega), \quad \delta \geq 0, \tag{1.12}$$

where the equivalence constants are depend only on r *and* $A[p]$.

(1.12) implies the following result.

Corollary 1. (a) *If* $r > 0$, $f \in L^p_\omega$, $1 < p < \infty$ *and* $\omega \in A_p$, *then*

$$\Omega_r(f,\lambda\delta)_{p,\omega} \lesssim (1+[\lambda])^{2r}\,\Omega_r(f,\delta)_{p,\omega}, \quad \delta,\lambda > 0,$$

and

$$\Omega_r(f,\delta)_{p,\omega}\,\delta^{-2r} \lesssim \Omega_r(f,\delta_1)_{p,\omega}\,\delta_1^{-2r}, \quad 0 < \delta_1 \leq \delta.$$

(b) *We have*

$$\Omega_r(f,\cdot)_p \approx \omega_{2r}(f,\cdot)_p \text{ for } 1 < p < \infty \text{ and } f \in L^p$$

and therefore, Theorems 3 and 4 reduce the Theorems 1 and 2 (with $2r$*).*

The rest of the work is organized as follows. In §2, some polynomial inequalities, required for realization theorem, are obtained. Also, some estimates are obtained for characterization theorem. In §3 we give the proof of the main results.

2. Polynomial Inequalities

Before the statement of the polynomial inequalities we give some preliminary explanations. Let $\mathcal{T}_n$ denote the class of trigonometrical polynomials of degree not greater than n. We take a trigonometric polynomial $T \in \mathcal{T}_n$

$$T(x) = \sum_{k=0}^{n}(a_k \cos kx + b_k \sin kx) = \sum_{k=0}^{n} A_k(x,T), \quad a_k \in \mathbb{R} \ (k = 0,1,\ldots)$$

and we define its conjugate $\widetilde{T}$ by

$$\widetilde{T}(x) = \sum_{k=1}^{n}(a_k \sin kx - b_k \cos kx) =: \sum_{k=1}^{n} A_k\left(x,\widetilde{T}\right).$$

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We define the *derivative* $T^{(\beta)}$ *of fractional order* $\beta>0$ *for* the polynomial $T\in\mathcal{T}_n$ as

$$T^{(\beta)}(\cdot):=\sum_{k=0}^{n}k^{\beta}A_k\left(\cdot+\frac{\beta\pi}{2k},T\right)=\sum_{k=1}^{n}k^{\beta}A_k\left(\cdot+\frac{\beta\pi}{2k},T\right)=:\sum_{k=1}^{n}A_k\left(\cdot,T^{(\beta)}\right).$$

We note from (1.3) that we have $L^p_\omega\subset L^1$ for $1<p<\infty$ and $\omega\in A_p$. Hence for a given $f\in L^p_\omega$, $1<p<\infty$ and $\omega\in A_p$, we define the corresponding Fourier series

$$f(x)\sim\sum_{k=0}^{n}\left(a_k(f)\cos kx+b_k(f)\sin kx\right):=\sum_{k=0}^{\infty}A_k(x,f) \tag{2.1}$$

where

$$a_k(f):=\pi^{-1}\int_{\mathbb{T}}f(x)\cos kxdx\ \ (k=0,1,2,...)\ ,\ b_k(f)=\pi^{-1}\int_{\mathbb{T}}f(x)\sin kxdx\ \ (k=1,2,...)\,.$$

Since the corresponding Fourier series (2.1) of $f\in L^p_\omega$ converges in L^p_ω norm for $1<p<\infty$ and $\omega\in A_p$, then we can write

$$f(x)\overset{p,\omega}{=}\sum_{k=0}^{\infty}A_k(x,f)\ \text{ a.e. on }\mathbb{T}. \tag{2.2}$$

Using (2.2) and (1.4) we find

$$\sigma_h f(x)\overset{p,\omega}{=}\sum_{k=0}^{\infty}\frac{\sin kh}{kh}A_k(x,f) \tag{2.3}$$

a.e. on $\mathbb{T}$ with $(\sin 0)/0:=1$ and hence consecutively

$$\sigma_h^2 f(x)\overset{p,\omega}{=}\sum_{k=1}^{\infty}\left(\frac{\sin kh}{kh}\right)^2A_k(x,f)\,,\ \dots\,,\ \ \sigma_h^k f(x)\overset{p,\omega}{=}\sum_{k=1}^{\infty}\left(\frac{\sin kh}{kh}\right)^kA_k(x,f)\ \text{ all a.e. on }\mathbb{T}.$$

From the relations (1.7), (2.2), (2.3), (1.8) and the last equalities we obtain

$$B_h^r f(x)\overset{p,\omega}{=}\sum_{k=0}^{\infty}\left(1-\frac{\sin kh}{kh}\right)^rA_k(x,f)$$

a.e. on $\mathbb{T}$. In particular, if $f(x)=\cos nx$, for $x\in\mathbb{T}$ and $n\in\mathbb{N}$ then from the last equality

$$B^r_{\frac{\pi}{n}}\cos nx\overset{p,\omega}{=}\left(1-\frac{\sin\pi}{\pi}\right)^r\cos nx=\cos nx\text{ a.e. on }\mathbb{T}.$$

The following two lemma are required for realization result. We note that all lemmas of this section are new also in the case $\omega\equiv 1$.

Lemma 1. *Let* $r>0$, $1<p<\infty$, $\omega\in A_p$ *and* $T_n\in\mathcal{T}_n$, $n=1,2,\ldots$. *Then*

$$\Omega_r\left(T_n,\frac{1}{n}\right)_{p,\omega}\lesssim\frac{1}{n^{2r}}\left\|T_n^{(2r)}\right\|_{p,\omega}$$

holds with some constant depending only on r,p *and* $A[p]$.

Proof of Lemma 1. Setting

$$\left(1-\frac{\sin x}{x}\right)_*:=\begin{cases}1-\frac{\sin x}{x} & ,\ x>0\\ 0 & ,\ x=0\end{cases}$$

we have

$$\left(1-\frac{\sin x}{x}\right)_*\leq x^2\text{ for }x\in\mathbb{R}^+\cup\{0\}\,.$$

For $0<t,h_i\leq\frac{1}{n}$

$$\left\|\prod_{i=1}^{[r]}B_{h_i}B_t^{\{r\}}T_n\right\|_{p,\omega}=$$

$$=\left\|\sum_{k=0}^{n}\left(1-\frac{\sin kh_1}{kh_1}\right)_*..\left(1-\frac{\sin kh_{[r]}}{kh_{[r]}}\right)_*\left(1-\frac{\sin kt}{kt}\right)_*^{\{r\}}A_k\left(x,T_n\right)\right\|_{p,\omega}$$

$$=\left\|\sum_{k=1}^{n}\frac{\left(1-\frac{\sin kh_1}{kh_1}\right)\left(kh_1\right)^2}{\left(kh_1\right)^2}..\frac{\left(1-\frac{\sin kh_{[r]}}{kh_{[r]}}\right)\left(kh_{[r]}\right)^2}{\left(kh_{[r]}\right)^2}\left(\frac{\left(1-\frac{\sin kt}{kt}\right)}{\left(kt\right)^2}\right)^{\{r\}}\left(kt\right)^{2\{r\}}A_k\left(x,T_n\right)\right\|_{p,\omega}$$

$$\leq n^{-2r}\left\|\sum_{k=1}^{n}\frac{\left(1-\frac{\sin kh_1}{kh_1}\right)}{\left(kh_1\right)^2}k^2\ldots\frac{\left(1-\frac{\sin kh_{[r]}}{kh_{[r]}}\right)}{\left(kh_{[r]}\right)^2}k^2\left(\frac{\left(1-\frac{\sin kt}{kt}\right)}{\left(kt\right)^2}\right)^{\{r\}}k^{2\{r\}}A_k\left(x,T_n\right)\right\|_{p,\omega}$$

$$\leq n^{-2r}\left\|\sum_{k=1}^{n}k^{2r}\frac{\left(1-\frac{\sin kh_1}{kh_1}\right)}{\left(kh_1\right)^2}\ldots\frac{\left(1-\frac{\sin kh_{[r]}}{kh_{[r]}}\right)}{\left(kh_{[r]}\right)^2}\left(\frac{\left(1-\frac{\sin kt}{kt}\right)}{\left(kt\right)^2}\right)^{\{r\}}A_k\left(x,T_n\right)\right\|_{p,\omega}.$$

Using Marcinkiewicz multiplier theorem for Lebesgue spaces with Muckenhoupt weight (see e.g, [20]) we have

$$\left\|\prod_{i=1}^{[r]}B_{h_i}B_t^{\{r\}}T_n\right\|_{p,\omega}\lesssim n^{-2r}\left\|\sum_{k=1}^{n}k^{2r}A_k\left(x,T_n\right)\right\|_{p,\omega}.$$

For $k=1,2,3,\ldots$ we observe

$$\begin{aligned}A_k\left(x,T_n\right)&=a_k\cos k\left(x+\frac{r\pi}{k}-\frac{r\pi}{k}\right)+b_k\sin k\left(x+\frac{r\pi}{k}-\frac{r\pi}{k}\right)\\&=\cos r\pi\left[a_k\cos k\left(x+\frac{r\pi}{k}\right)+b_k\sin k\left(x+\frac{r\pi}{k}\right)\right]\\&\quad+\sin r\pi\left[a_k\sin k\left(x+\frac{r\pi}{k}\right)-b_k\cos k\left(x+\frac{r\pi}{k}\right)\right]\\&=A_k\left(x+\frac{r\pi}{k},T_n\right)\cos r\pi+A_k\left(x+\frac{r\pi}{k},\widetilde{T_n}\right)\sin r\pi\end{aligned}$$

and hence

$$\left\|\prod_{i=1}^{[r]}B_{h_i}B_t^{\{r\}}T_n\right\|_{p,\omega}\lesssim$$

$$\lesssim\frac{1}{n^{2r}}\left\|\sum_{k=1}^{n}k^{2r}\left[A_k\left(x+\frac{r\pi}{k},T_n\right)\cos r\pi+A_k\left(x+\frac{r\pi}{k},\widetilde{T_n}\right)\sin r\pi\right]\right\|_{p,\omega}$$

$$\lesssim n^{-2r}\left(\left\|\sum_{k=0}^{n}k^{2r}A_k\left(x+\frac{r\pi}{k},T_n\right)\right\|_{p,\omega}+\left\|\sum_{k=1}^{n}k^{2r}A_k\left(x+\frac{r\pi}{k},\widetilde{T_n}\right)\right\|_{p,\omega}\right).$$

Since

$$A_k\left(x,T_n^{(2r)}\right)=k^{2r}A_k\left(x+\frac{r\pi}{k},T_n\right),\quad k=1,2,3,\ldots$$

we obtain

$$\begin{aligned}\Omega_r\left(T_n,\frac{1}{n}\right)_{p,\omega} &= \sup_{0\le t,h_i\le 1/n}\left\|\prod_{i=1}^{[r]}B_{h_i}B_t^{\{r\}}T_n\right\|_{p,\omega}\\ &\lesssim n^{-2r}\left(\left\|\sum_{k=1}^{n}k^{2r}A_k\left(x+\frac{r\pi}{k},T_n\right)\right\|_{p,\omega}+\left\|\sum_{k=1}^{n}k^{2r}A_k\left(x+\frac{r\pi}{k},\widetilde{T_n}\right)\right\|_{p,\omega}\right)\\ &\lesssim n^{-2r}\left(\left\|T_n^{(2r)}\right\|_{p,\omega}+\left\|\widetilde{T_n}^{(2r)}\right\|_{p,\omega}\right)=cn^{-2r}\left(\left\|T_n^{(2r)}\right\|_{p,\omega}+\left\|\widetilde{T_n^{(2r)}}\right\|_{p,\omega}\right)\\ &\lesssim n^{-2r}\left\|T_n^{(2r)}\right\|_{p,\omega}.\end{aligned}$$

and the proof is completed. □

The following lemma is an improvement of Bernstein inequality.

Lemma 2. *Let $r>0$, $1<p<\infty$, $\omega\in A_p$ and $T_n\in\mathcal{T}_n$, $n=1,2,\ldots$. Then*

$$\frac{1}{n^{2r}}\left\|T_n^{(2r)}\right\|_{p,\omega}\lesssim\Omega_r\left(T_n,\frac{1}{n}\right)_{p,\omega}$$

holds with some constant depending only on r,p and $A[p]$.

Proof of Lemma 2. Let

$$T(x)=\frac{a_0}{2}+\sum_{k=1}^{n}\left(a_k\cos kx+b_k\sin kx\right).$$

Then

$$\begin{aligned}n^{-2r}\left\|T_n^{(2r)}\right\|_{p,\omega} &= n^{-2r}\left\|\sum_{k=1}^{n}k^{2r}A_k\left(x+\frac{r\pi}{k},T_n\right)\right\|_{p,\omega}\\ &= n^{-2r}\left\|\sum_{k=1}^{n}k^{2r}\left(\cos r\pi A_k\left(x,T_n\right)-\sin r\pi A_k\left(x,\widetilde{T_n}\right)\right)\right\|_{p,\omega}\\ &\le n^{-2r}\left\|\sum_{k=1}^{n}k^{2r}\cos r\pi A_k\left(x,T_n\right)\right\|_{p,\omega}\\ &\quad+n^{-2r}\left\|\sum_{k=1}^{n}k^{2r}\sin r\pi A_k\left(x,\widetilde{T_n}\right)\right\|_{p,\omega}\\ &= \left\|\sum_{k=1}^{n}\cos r\pi\left(\frac{\left(\frac{k}{n}\right)^2}{\left(1-\frac{\sin\frac{k}{n}}{\frac{k}{n}}\right)}\right)^r\left(1-\frac{\sin\frac{k}{n}}{\frac{k}{n}}\right)^r A_k\left(x,T_n\right)\right\|_{p,\omega}\\ &\quad+\left\|\sum_{k=1}^{n}\sin r\pi\left(\frac{\left(\frac{k}{n}\right)^2}{\left(1-\frac{\sin\frac{k}{n}}{\frac{k}{n}}\right)}\right)^r\left(1-\frac{\sin\frac{k}{n}}{\frac{k}{n}}\right)^r A_k\left(x,\widetilde{T_n}\right)\right\|_{p,\omega}.\end{aligned}$$

Using Marcinkiewicz multiplier theorem ([20]) for Lebesgue spaces with Muckenhoupt weight we have

$$\begin{aligned} n^{-2r}\left\Vert T_n^{(2r)}\right\Vert_{p,\omega} &\lesssim \left\Vert \sum_{k=1}^{n}\left(1-\frac{\sin\frac{k}{n}}{\frac{k}{n}}\right)^{r} A_k\left(x,T_n\right)\right\Vert_{p,\omega} \\ &\quad+\left\Vert \sum_{k=1}^{n}\left(1-\frac{\sin\frac{k}{n}}{\frac{k}{n}}\right)^{r} A_k\left(x,\widetilde{T_n}\right)\right\Vert_{p,\omega} \\ &= \left\Vert \sum_{k=1}^{n}\left(1-\frac{\sin\frac{k}{n}}{\frac{k}{n}}\right)^{r} A_k\left(x,T_n\right)\right\Vert_{p,\omega} \\ &\quad+\left\Vert \left(\sum_{k=1}^{n}\left(1-\frac{\sin\frac{k}{n}}{\frac{k}{n}}\right)^{r} A_k\left(x,T_n\right)\right)^{\sim}\right\Vert_{p,\omega}. \end{aligned}$$

In the last step we used the linearity property of conjugate operator. Thus from boundedness of conjugate (see e.g, [14]) operator we get

$$\begin{aligned} n^{-2r}\left\Vert T_n^{(2r)}\right\Vert_{p,\omega} &\lesssim \left\Vert \sum_{k=1}^{n}\left(1-\frac{\sin\frac{k}{n}}{\frac{k}{n}}\right)^{r} A_k\left(x,T_n\right)\right\Vert_{p,\omega} \\ &\quad+\left\Vert \sum_{k=1}^{n}\left(1-\frac{\sin\frac{k}{n}}{\frac{k}{n}}\right)^{r} A_k\left(x,T_n\right)\right\Vert_{p,\omega} \\ &\lesssim \left\Vert B_{1/n}^{r}T_n\right\Vert_{p,\omega} = c\left\Vert B_{1/n}^{[r]+\{r\}}T_n\right\Vert_{p,\omega} \\ &\lesssim \sup_{\substack{0<h_i,u\le 1/n\\ i=1,2,\ldots,r}}\left\Vert \prod_{i=1}^{[r]} B_{h_i}B_u^{\{r\}}T_n\right\Vert_{p,\omega} \lesssim \Omega_r\left(T_n,1/n\right)_{p,\omega}. \end{aligned}$$

Then

$$\frac{1}{n^{2r}}\left\Vert T_n^{(2r)}\right\Vert_{p,\omega}\lesssim \Omega_r\left(T_n,1/n\right)_{p,\omega}$$

as required. □

For our characterization theorem we need the following lemma.

Lemma 3. *Let $r>0$ and $n\in\mathbb{N}$.*

(a) *If $f(x)=\cos x$, $1<p<\infty$ and $\omega\in A_p$, then there exist a $t_1>0$ such that*

$$\Omega_r\left(f,\delta\right)_{p,\omega}\approx\delta^{2r}$$

holds for all $\delta\in(0,t_1]$, where constants are depend only r and $A[p]$.

(b) *If $f(x)=\cos nx$ and $1\le p\le\infty$, then for any $\delta\in(0,\pi/n)$ we have*

$$\Omega_r\left(f,\delta\right)_p\le(2\pi)^{1/p}6^{-2r}\left(n\delta\right)^{2r}$$

with $(1/\infty):=0$.

(c) *If $f(x)=\cos nx$, then*

$$\left\Vert B_{\pi/n}^{r}f\right\Vert_1=4.$$

(d) *If $f(x)=\cos nx$, then for any $\delta\in(0,\pi/n)$*

$$\left\Vert B_\delta^r f\right\Vert_1\ge 2^{r+2}3^{-r}\pi^{-2r}\left(n\delta\right)^{2r}.$$

Proof of Lemma 3. (a) Let $f(x)=\cos x$ and $\delta\geq 0$. Then

$$\Omega_r(f,\delta)_{p,\omega}=\sup_{0\leq u,h_i\leq\delta}\left\|\prod_{i=1}^{[r]}B_{h_i}B_u^{\{r\}}(\cos x)\right\|_{p,\omega}$$
$$=\sup_{0\leq u,h_i\leq\delta}\left\|\left(1-\frac{\sin h_1}{h_1}\right)\dots\left(1-\frac{\sin h_{[r]}}{h_{[r]}}\right)\left(1-\frac{\sin u}{u}\right)^{\{r\}}\cos x\right\|_{p,\omega}.$$

Since

$$\left(1-\frac{\sin x}{x}\right)\leq 6^{-1}x^2 \text{ for } x\in\mathbb{R}^+$$

we have

$$\Omega_r(f,\delta)_{p,\omega}\leq\sup_{0\leq u,h_i\leq\delta}6^{-2r}h_1^2\dots h_{[r]}^2u^{2\{r\}}\|\cos x\|_{p,\omega}$$
$$\lesssim 6^{-2r}\delta^{2r}\|\cos x\|_\infty\lesssim 6^{-2r}\delta^{2r}.$$

On the other hand, using

$$\left(1-\frac{\sin x}{x}\right)\geq(2/3)\sin^2(x/2),\qquad x>0,$$

and

$$\sin x\geq(2/\pi)x,\qquad 0\leq x\leq\pi/2,$$

we get, for $0\leq\delta\leq\pi/2$,

$$\sup_{0\leq u,h_i\leq\delta}\left\|\left(1-\frac{\sin h_1}{h_1}\right)\dots\left(1-\frac{\sin h_{[r]}}{h_{[r]}}\right)\left(1-\frac{\sin u}{u}\right)^{\{r\}}\cos x\right\|_{p,\omega}$$
$$\geq\sup_{0\leq h\leq\delta}\left\|\left(1-\frac{\sin h}{h}\right)^r\cos x\right\|_{p,\omega}\geq\left\|\left(1-\frac{\sin\delta}{\delta}\right)^r\cos x\right\|_{p,\omega}$$
$$\geq(2/3)^r(2/\pi)^{2r}\delta^{2r}2^{-2r}\|\cos x\|_{p,\omega}\gtrsim(1/6)^r(2/\pi)^{2r}\delta^{2r}\|\cos x\|_1$$
$$\gtrsim 4\cdot(1/6)^r(2/\pi)^{2r}\delta^{2r}.$$

Now, taking $t_1=\pi/2$, the proof of (a) is completed.

(b) Let $f(x)=\cos nx$, $1\leq p\leq\infty$, $\delta\in(0,\pi/n)$ and $h_1,\dots h_{[r]},u\in[0,\delta]$. Then

$$\Omega_r(f,\delta)_p=\sup_{0\leq u,h_i\leq\delta}\left\|\prod_{i=1}^{[r]}B_{h_i}B_u^{\{r\}}\cos nx\right\|_p$$
$$=\sup_{0\leq u,h_i\leq\delta}\left\|\left(1-\frac{\sin nh_1}{nh_1}\right)\dots\left(1-\frac{\sin nh_{[r]}}{nh_{[r]}}\right)\left(1-\frac{\sin nu}{nu}\right)^{\{r\}}\cos nx\right\|_p$$
$$\leq\sup_{0\leq u,h_i\leq\delta}6^{-2r}n^{2r}h_1^2\dots h_{[r]}^2u^{2\{r\}}\|\cos nx\|_p$$
$$\leq 6^{-2r}(n\delta)^{2r}\|1\|_p\leq(2\pi)^{1/p}6^{-2r}\delta^{2r}.$$

(c) Let $f(x)=\cos nx$. Then

$$\left\|B_{\pi/n}^r\cos nx\right\|_1=\left\|\left(1-\frac{\sin\pi}{\pi}\right)^r\cos nx\right\|_1=\|\cos nx\|_1=4.$$

(d) Let $f(x) = \cos nx$ and $\delta \in (0, \pi/n)$. Then

$$\begin{aligned}\|B_\delta^r f\|_1 &= \left\| \left(1 - \frac{\sin n\delta}{n\delta}\right)^r \cos nx \right\|_1 \\ &\geq (2/3)^r (2/\pi)^{2r} (n\delta)^{2r} 2^{-2r} \|\cos nx\|_1 = 2^{r+2} 3^{-r} \pi^{-2r} (n\delta)^{2r}\end{aligned}$$

and the proof is completed. □

3. Proof of Theorems 3 and 4

Proof of Theorem 3. Part (a): We define

$$\varphi(t) := t^{2r} \inf_{0<\zeta\leq t} \frac{\Omega_r(f,\zeta)_{p,\omega}}{\zeta^{2r}}, \ t > 0.$$

It is easy to see that ([29]) the function φ satisfies the properties (i) $\varphi(t)$ is nonnegative, bounded on $(0, \infty)$, (ii) $\varphi(t) \to 0$ as $t \to 0$, (iii) $\varphi(t)$ is nondecreasing, (iv) $\varphi(t) t^{-2r}$ is non-increasing. Hence $\varphi \in \Phi_{2r}$. From definition we have

$$\varphi(t) = t^{2r} \inf_{0<\zeta\leq t} \frac{\Omega_r(f,\zeta)_{p,\omega}}{\zeta^{2r}} \leq t^{2r} \frac{\Omega_r(f,t)_{p,\omega}}{t^{2r}} = \Omega_r(f,t)_{p,\omega}.$$

Also

$$\Omega_r(f,t)_{p,\omega} = t^{2r} \frac{\Omega_r(f,t)_{p,\omega}}{t^{2r}}$$

and taking infimum we get

$$\Omega_r(f,t)_{p,\omega} \lesssim t^{2r} \inf_{0<\zeta\leq t} \frac{\Omega_r(f,\zeta)_{p,\omega}}{\zeta^{2r}} = c\varphi(t).$$

Part (b): Following some ideas from [31] we will consider the following two possibility. The first is (A) $\lim_{t\to 0} (\varphi(t)/t^{2r}) = C \in [0, \infty)$ and the second is (B) $\lim_{t\to 0} (\varphi(t)/t^{2r}) = \infty$.

We consider the case (A). In this case from definition of Φ_{2r} and $\lim_{t\to 0} (\varphi(t)/t^{2r}) = C$ we have

$$\varphi(t) \lesssim t^{2r}, \text{ for } t \in (0, \pi) \tag{3.1}$$

and there exists a $t_0 > 0$ such that

$$\varphi(t) \geq Ct^{2r}, \text{ for } t \in (0, t_0]. \tag{3.2}$$

Now we define $f(x) = C \cos x$. Then $f \in L_\omega^p$. Using (a) of Lemma 3, we have that there is a $t_1 > 0$ and constants $c, C > 0$, depending only r and $A[p]$, such that

$$c\delta^{2r} \leq \Omega_r(f,\delta)_{p,\omega} \lesssim \delta^{2r}$$

for all $\delta \in (0, t_1]$. Hence using (3.1) and (3.2)

$$c\varphi(t) \leq \Omega_r(f,\delta)_{p,\omega} \lesssim \varphi(t)$$

for $\delta \in (0, t_2]$, where $t_2 := \min\{\pi, t_0, t_1\}$. We have been proved the case (A).

We consider the case (B). From $\lim_{t\to 0} (\varphi(t)/t^{2r}) = \infty$ and $\lim_{t\to 0} \varphi(t) = 0$ we obtain that $\lim_{t\to 0} (t^{2r}/\varphi(t)) = 0$. We take $a \geq 2$ and fix it. Later we will state the exact condition on a when necessary. Following the construction given in [25] and [31], it is possible to find a sequence $\{n_v\}_{v=1}^{\infty}$ such that $n_v = 2^{m_v}$, $m_1 = 2$ and

$$m_{v+1} := \min\left\{ m \in \mathbb{N} : \max\left\{ \frac{\varphi(2^{-m})}{\varphi(2^{-m_v})}, \frac{2^{2m_v r}\varphi(2^{-m_v})}{2^{2mr}\varphi(2^{-m})} \right\} \leq \frac{1}{a} \right\}.$$

From this construction we have $m_{v+1} > m_v$, $n_{v+1} \geq 2n_v$ $(v = 1, 2, \ldots)$ and

$$\varphi\left(\frac{1}{n_{v+1}}\right) \leq \frac{1}{a}\varphi\left(\frac{1}{n_v}\right), \tag{3.3}$$

$$n_v^{2r}\varphi\left(\frac{1}{n_v}\right) \leq \frac{1}{a}n_{v+1}^{2r}\varphi\left(\frac{1}{n_{v+1}}\right). \tag{3.4}$$

We take $\chi = 2^l$, $l \in \mathbb{N}$, $\chi > 2\pi$ and fix χ. Hence

$$\sum_{v=1}^{\infty}\varphi\left(\frac{1}{n_v}\right) \leq \varphi\left(\frac{1}{n_1}\right)\sum_{v=1}^{\infty}\left(\frac{1}{a}\right)^{v-1} < +\infty.$$

We define

$$f(x) := \sum_{v=1}^{\infty}\varphi\left(\frac{1}{n_v}\right)\cos(\chi n_v x).$$

Then $f \in L_\omega^p$ (in fact $f \in L^\infty$). From (1.3) we get $\|f\|_{p,\omega} \lesssim \|f\|_\infty$ and hence $\Omega_r(f,\delta)_{p,\omega} \lesssim \Omega_r(f,\delta)_\infty$. If we prove $\Omega_r(f,\delta)_\infty \lesssim \varphi(\delta)$ we will have the lower estimate of Theorem 3 (b). Let $\delta \in (0, 1/n_1]$. For any $h_i \in (0, 1/n_1]$ with $i = 1, 2, 3, \ldots, [r]+1$ there are natural numbers N_i such that $n_{N_i+1}^{-1} < h_i \leq n_{N_i}^{-1}$. We set $h_{\max} := \max\{h_i : i = 1, 2, 3, \ldots, [r]+1\}$. Then there is a $i_0 \in \{1, 2, 3, \ldots, [r]+1\}$ such that $h_{\max} = h_{i_0}$. Also we set $N_{\max} := N_{i_0}$. In this case

$$\begin{aligned}
&\left\|(I-\sigma_{h_1})\ldots\left(I-\sigma_{h_{[r]}}\right)\left(I-\sigma_{h_{[r]+1}}\right)^{\{r\}}f\right\|_\infty \\
&\leq \left\|\sum_{v=1}^{N_{\max}}\varphi\left(\frac{1}{n_v}\right)\left((I-\sigma_{h_1})\ldots\left(I-\sigma_{h_{[r]}}\right)\left(I-\sigma_{h_{[r]+1}}\right)^{\{r\}}\right)\cos(\chi n_v x)\right\|_\infty \\
&\quad+\left\|\sum_{v=N_{\max}+1}^{\infty}\varphi\left(\frac{1}{n_v}\right)\left((I-\sigma_{h_1})\ldots\left(I-\sigma_{h_{[r]}}\right)\left(I-\sigma_{h_{[r]+1}}\right)^{\{r\}}\right)\cos(\chi n_v x)\right\|_\infty \\
&=: I_1 + I_2.
\end{aligned}$$

We estimate I_1. Using (b) of Lemma 3, (3.4) and (d) of the definition of Φ_r

$$\begin{aligned}
I_1 &\leq \sum_{v=1}^{N_{\max}}\varphi\left(\frac{1}{n_v}\right)\left\|\left((I-\sigma_{h_1})\ldots\left(I-\sigma_{h_{[r]}}\right)\left(I-\sigma_{h_{[r]+1}}\right)^{\{r\}}\right)\cos(\chi n_v x)\right\|_\infty \\
&\lesssim \sum_{v=1}^{N_{\max}}\varphi\left(\frac{1}{n_v}\right)(\chi n_v)^{2r}h_1^2\ldots h_{[r]}^2 h_{[r]+1}^{2\{r\}} \lesssim (\chi h_{\max})^{2r}\sum_{v=1}^{N_{\max}}n_v^{2r}\varphi\left(\frac{1}{n_v}\right) \\
&\lesssim (\chi h_{\max})^{2r}n_{N_{\max}}^{2r}\varphi\left(\frac{1}{n_{N_{\max}}}\right)\sum_{v=1}^{N_{\max}}\left(\frac{1}{a}\right)^{N_{\max}-v} \\
&\lesssim (h_{\max}n_{N_{\max}})^{2r}\varphi\left(\frac{1}{n_{N_{\max}}}\right) \lesssim \varphi(h_{\max}).
\end{aligned}$$

We estimate I_2. Using (1.10) and (3.3)

$$\begin{aligned}
I_2 &\leq \sum_{v=N_{\max}+1}^{\infty}\varphi\left(\frac{1}{n_v}\right)\left\|\left((I-\sigma_{h_1})\ldots\left(I-\sigma_{h_{[r]}}\right)\left(I-\sigma_{h_{[r]+1}}\right)^{\{r\}}\right)\cos(\chi n_v x)\right\|_\infty \\
&\lesssim \sum_{v=N_{\max}+1}^{\infty}\varphi\left(\frac{1}{n_v}\right) \lesssim \varphi\left(\frac{1}{n_{N_{\max}+1}}\right)\sum_{v=N_{\max}+1}^{\infty}\left(\frac{1}{a}\right)^{v-N_{\max}-1}
\end{aligned}$$

$$\lesssim \varphi\left(\frac{1}{n_{N_{\max}+1}}\right) \lesssim \varphi\left(h_{\max}\right).$$

Now for $i = 1, 2, 3, \ldots, [r] + 1$ we have $n_{N_i+1}^{-1} < h_i \le n_{N_i}^{-1}$ and

$$\left\| (I - \sigma_{h_1}) \ldots \left(I - \sigma_{h_{[r]}}\right) \left(I - \sigma_{h_{[r]+1}}\right)^{\{r\}} f \right\|_\infty \lesssim \varphi\left(h_{\max}\right).$$

Thus

$$\Omega_r (f, \delta)_\infty = \sup_{\substack{0 \le h_i \le \delta \\ i=1,\ldots,[r]+1}} \left\| (I - \sigma_{h_1}) \ldots \left(I - \sigma_{h_{[r]}}\right) \left(I - \sigma_{h_{[r]+1}}\right)^{\{r\}} f \right\|_\infty \lesssim \varphi\left(\delta\right).$$

From this we obtain $\Omega_r (f, \delta)_{p,\omega} \lesssim \varphi (\delta)$. For reverse of the last inequality we will use $\|f\|_1 \lesssim \|f\|_{p,\omega}$ (see for example (1.3)). We will prove

$$\Omega_r (f, \delta)_1 \gtrsim \varphi (\delta)$$

and this will give the inequality $\Omega_r (f, \delta)_{p,\omega} \gtrsim \varphi (\delta)$. We choose an integer i such that $n_{i+1}^{-1} = 2^{-m_{i+1}} < \delta \le 2^{-m_i} = n_i^{-1}$. From the definition of m_i at least one of the following conditions hold.

$$2^{2r(m_{i+1}-1)} \varphi\left(\frac{1}{2^{m_{i+1}-1}}\right) < a 2^{2rm_i} \varphi\left(\frac{1}{2^{m_i}}\right), \tag{3.5}$$

$$\varphi\left(\frac{1}{2^{m_{i+1}-1}}\right) > \frac{1}{a} \varphi\left(\frac{1}{2^{m_i}}\right). \tag{3.6}$$

For the first case (3.5), we decompose

$$f(x) = \sum_{v=1}^{i-1} \varphi\left(\frac{1}{n_v}\right) \cos(\chi n_v x) + \varphi\left(\frac{1}{n_i}\right) \cos(\chi n_i x) + \sum_{v=i+1}^{\infty} \varphi\left(\frac{1}{n_v}\right) \cos(\chi n_v x)$$
$$=: f_1 + f_2 + f_3.$$

We estimate f_2. For $\frac{1}{\chi n_i} < \delta < \frac{\pi}{\chi n_i}$, using (d) of Lemma 3

$$\|(I - \sigma_\delta)^r f_2\|_1 = \varphi\left(\frac{1}{n_i}\right) \|(I - \sigma_\delta)^r \cos(\chi n_i x)\|_1 \ge \frac{2^{r+2}}{3^r \pi^{2r}} (\chi\delta)^{2r} n_i^{2r} \varphi\left(\frac{1}{n_i}\right).$$

We estimate f_1. For $\frac{1}{\chi n_i} < \delta < \frac{\pi}{\chi n_i}$, using (b) of Lemma 3 and (3.4)

$$\|(I - \sigma_\delta)^r f_1\|_1 \le \sum_{v=1}^{i-1} \varphi\left(\frac{1}{n_v}\right) \|(I - \sigma_\delta)^r \cos(\chi n_v x)\|_1$$
$$\le \frac{4}{6^{2r}} (\chi\delta)^{2r} \sum_{v=1}^{i-1} n_v^{2r} \varphi\left(\frac{1}{n_v}\right) \le \frac{4}{6^{2r}} (\chi\delta)^{2r} n_{i-1}^{2r} \varphi\left(\frac{1}{n_{i-1}}\right) \sum_{v=1}^{i-1} \left(\frac{1}{a}\right)^{i-1-v}$$
$$\le \frac{4 \cdot 2}{6^{2r}} (\chi\delta)^{2r} n_{i-1}^{2r} \varphi\left(\frac{1}{n_{i-1}}\right) \le \frac{4 \cdot 2}{6^{2r} a} (\chi\delta)^{2r} n_i^{2r} \varphi\left(\frac{1}{n_i}\right).$$

We estimate f_3. For $\frac{1}{\chi n_i} < \delta < \frac{\pi}{\chi n_i}$, using (1.10), $C_r \le 2^{[r]+1}$, (3.3), $1 < \delta\chi n_i$ and $\|\cos(\chi n_v x)\|_1 = 4$ we have

$$\begin{aligned}\|(I-\sigma_\delta)^r f_3\|_1 &\le \sum_{v=i+1}^{\infty} \varphi\left(\frac{1}{n_v}\right) \|(I-\sigma_\delta)^r \cos(\chi n_v x)\|_1 \\ &\le 4\cdot 2^{[r]+1} \sum_{v=i+1}^{\infty} \varphi\left(\frac{1}{n_v}\right) \le 4\cdot 2^{[r]+1} \varphi\left(\frac{1}{n_{i+1}}\right) \sum_{v=i+1}^{\infty} \left(\frac{1}{a}\right)^{v-i-1} \\ &\le 2\cdot 4\cdot 2^{[r]+1} \varphi\left(\frac{1}{n_{i+1}}\right) \le \frac{2\cdot 4\cdot 2^{[r]+1}}{a} \varphi\left(\frac{1}{n_i}\right) \\ &\le \frac{2\cdot 4\cdot 2^{[r]+1}}{a} (\chi n_i \delta)^{2r} \varphi\left(\frac{1}{n_i}\right).\end{aligned}$$

Hence for $\frac{1}{\chi n_i} < \delta < \frac{\pi}{\chi n_i}$ we have

$$\begin{aligned}\|(I-\sigma_\delta)^r f\|_1 &\ge \|(I-\sigma_\delta)^r f_2\|_1 - \|(I-\sigma_\delta)^r f_1\|_1 - \|(I-\sigma_\delta)^r f_3\|_1 \\ &\ge \varphi\left(\frac{1}{n_i}\right) (\chi n_i \delta)^{2r} \left(\frac{2^{r+2}}{3^r \pi^{2r}} - \frac{4\cdot 2}{6^{2r} a} - \frac{2\cdot 4\cdot 2^{[r]+1}}{a}\right) \\ &= \varphi\left(\frac{1}{n_i}\right) (\chi n_i \delta)^{2r}\, 4 \left(\frac{2^r}{3^r \pi^{2r}} - \frac{1}{6^{2r} a} - \frac{2^{[r]+1}}{a}\right).\end{aligned}$$

We choose a such a way that $a \ge 2$ and

$$\frac{2^r}{3^r \pi^{2r}} - \frac{1}{a}\left(\frac{1}{6^{2r}} + 2^{[r]+1}\right) > 0,$$
$$1 - \frac{1}{a}\left(\frac{\pi^{2r+1}}{6^{2r}} + 2\cdot 2^{[r]+1}\right) > 0.$$

Since $\frac{\pi}{\chi n_i} < \frac{1}{n_{i+1}} < \frac{1}{n_i}$, we get for $\frac{1}{\chi n_i} < \delta < \frac{\pi}{\chi n_i}$

$$\|(I-\sigma_\delta)^r f\|_1 \gtrsim (\chi n_i \delta)^{2r} \varphi\left(\frac{1}{n_i}\right) \gtrsim \varphi(\delta).$$

Thus for $0 < \delta < \frac{\pi}{\chi n_i}$

$$\begin{aligned}\Omega_r(f,\delta)_1 &\ge \sup_{0<h\le\delta} \|(I-\sigma_h)^r f\|_1 \ge \sup_{\chi^{-1} n_i^{-1} < h \le \delta} \|(I-\sigma_h)^r f\|_1 \\ &\ge \|(I-\sigma_\delta)^r f\|_1 \gtrsim \varphi(\delta).\end{aligned}$$

In case $\frac{\pi}{\chi n_i} \le \delta \le \frac{1}{n_i}$ we get

$$\Omega_r(f,\delta)_1 \ge \Omega_r\left(f, \frac{\pi}{\chi n_i}\right)_1.$$

If we prove

$$\Omega_r\left(f, \frac{\pi}{\chi n_i}\right)_1 \gtrsim \varphi\left(\frac{1}{n_i}\right) \tag{3.7}$$

then this will give the proof of the case (3.5) completely.

$$\Omega_r(f,\delta)_1 \ge \Omega_r\left(f, \frac{\pi}{\chi n_i}\right)_1 \gtrsim \varphi\left(\frac{1}{n_i}\right) \gtrsim \varphi(\delta).$$

Now we prove (3.7). Since

$$f=\sum_{v=1}^{i-1}\varphi\left(\frac{1}{n_v}\right)\cos\left(\chi n_v x\right)+\varphi\left(\frac{1}{n_i}\right)\cos\left(\chi n_i x\right)+\sum_{v=i+1}^{\infty}\varphi\left(\frac{1}{n_v}\right)\cos\left(\chi n_v x\right)$$
$$=f_1+f_2+f_3$$

we have

$$\left\|\left(I-\sigma_{\frac{\pi}{\chi n_i}}\right)^r f\right\|_1\geq\left\|\left(I-\sigma_{\frac{\pi}{\chi n_i}}\right)^r f_2\right\|_1-\left\|\left(I-\sigma_{\frac{\pi}{\chi n_i}}\right)^r f_1\right\|_1-\left\|\left(I-\sigma_{\frac{\pi}{\chi n_i}}\right)^r f_3\right\|_1.$$

Using (c) of Lemma 3

$$\left\|\left(I-\sigma_{\frac{\pi}{\chi n_i}}\right)^r f_2\right\|_1=\varphi\left(\frac{1}{n_i}\right)\left\|\left(I-\sigma_{\frac{\pi}{\chi n_i}}\right)^r\cos\left(\chi n_i x\right)\right\|_1=4\varphi\left(\frac{1}{n_i}\right).$$

By (b) of Lemma 3, $\sum_{v=1}^{i-1}\left(\frac{1}{a}\right)^{i-1-v}\leq 2$ and (3.4) we get

$$\begin{aligned}\left\|\left(I-\sigma_{\frac{\pi}{\chi n_i}}\right)^r f_1\right\|_1&\leq\sum_{v=1}^{i-1}\varphi\left(\frac{1}{n_v}\right)\left\|\left(I-\sigma_{\frac{\pi}{\chi n_i}}\right)^r\cos\left(\chi n_v x\right)\right\|_1\\&\leq\frac{2\pi}{6^{2r}}\left(\chi\frac{\pi}{\chi n_i}\right)^{2r}\sum_{v=1}^{i-1}n_v^{2r}\varphi\left(\frac{1}{n_v}\right)=\frac{2\pi}{6^{2r}}\left(\frac{\pi}{n_i}\right)^{2r}\sum_{v=1}^{i-1}n_v^{2r}\varphi\left(\frac{1}{n_v}\right)\\&\leq\frac{2\pi}{6^{2r}}\left(\frac{\pi}{n_i}\right)^{2r}n_{i-1}^{2r}\varphi\left(\frac{1}{n_{i-1}}\right)\sum_{v=1}^{i-1}\left(\frac{1}{a}\right)^{i-1-v}\\&\leq\frac{4\pi}{6^{2r}}\left(\frac{\pi}{n_i}\right)^{2r}n_{i-1}^{2r}\varphi\left(\frac{1}{n_{i-1}}\right)\leq\frac{4\pi^{2r+1}}{6^{2r}a}\varphi\left(\frac{1}{n_i}\right).\end{aligned}$$

Taking into account (1.10), $C_r\leq 2^{[r]+1}$, (3.3) and $\left\|\cos\left(\chi n_v x\right)\right\|_1=4$

$$\begin{aligned}\left\|\left(I-\sigma_{\frac{\pi}{\chi n_i}}\right)^r f_3\right\|_1&\leq\sum_{v=i+1}^{\infty}\varphi\left(\frac{1}{n_v}\right)\left\|\left(I-\sigma_{\frac{\pi}{\chi n_i}}\right)^r\cos\left(\chi n_v x\right)\right\|_1\\&\leq 4\cdot 2^{[r]+1}\sum_{v=i+1}^{\infty}\varphi\left(\frac{1}{n_v}\right)\leq 4\cdot 2^{[r]+1}\varphi\left(\frac{1}{n_{i+1}}\right)\sum_{v=i+1}^{\infty}\left(\frac{1}{a}\right)^{v-i-1}\\&\leq 2\cdot 4\cdot 2^{[r]+1}\varphi\left(\frac{1}{n_{i+1}}\right)\leq\frac{2\cdot 4\cdot 2^{[r]+1}}{a}\varphi\left(\frac{1}{n_i}\right).\end{aligned}$$

Hence

$$\begin{aligned}\left\|\left(I-\sigma_{\frac{\pi}{\chi n_i}}\right)^r f\right\|_1&\geq\left\|\left(I-\sigma_{\frac{\pi}{\chi n_i}}\right)^r f_2\right\|_1-\left\|\left(I-\sigma_{\frac{\pi}{\chi n_i}}\right)^r f_1\right\|_1-\left\|\left(I-\sigma_{\frac{\pi}{\chi n_i}}\right)^r f_3\right\|_1\\&\geq\varphi\left(\frac{1}{n_i}\right)\left(4-\frac{4\pi^{2r+1}}{6^{2r}a}-\frac{2\cdot 4\cdot 2^{[r]+1}}{a}\right)\\&=4\varphi\left(\frac{1}{n_i}\right)\left(1-\frac{1}{a}\left(\frac{\pi^{2r+1}}{6^{2r}}+2^{[r]+2}\right)\right)=c\left(r\right)\varphi\left(\frac{1}{n_i}\right)\end{aligned}$$

and we proved (3.7). We prove the remaining case (3.6). Since $\varphi\left(t\right)t^{-2r}\downarrow$ we have

$$\varphi\left(\frac{1}{2^{m_{i+1}-1}}\right)\leq 2^{-2r}\varphi\left(\frac{1}{2^{m_{i+1}}}\right)$$

and hence

$$\varphi\left(\frac{1}{n_{i+1}}\right)>\frac{2^{2r}}{a}\varphi\left(\frac{1}{n_i}\right).$$

Using the last inequality and (3.7)

$$\Omega_r(f,\delta)_1 \geq \Omega_r\left(f,\frac{1}{n_{i+1}}\right)_1 \geq \Omega_r\left(f,\frac{\pi}{\chi n_{i+1}}\right)_1 \gtrsim \varphi\left(\frac{1}{n_{i+1}}\right)$$
$$\gtrsim \varphi\left(\frac{1}{n_i}\right) \gtrsim \varphi(\delta).$$

The proof of Theorem 3 is completed. □

It is well known that the class of trigonometric polynomials is a dense (see e.g., [18, Lemma 3]) subset of L^p_ω, $1<p<\infty$ for any weight $\omega\in A_p$. Hence the approximation problems are make sense in L^p_ω for $1<p<\infty$ and $\omega\in A_p$.

The following Jackson type theorem relates the best approximation error $E_n(f)_{p,\omega}$ with the modulus of smoothness $\Omega_r(f,n^{-1})_{p,\omega}$.

Theorem 5 ([12, 2])**.** *If $r>0$, $f\in L^p_\omega$, $1<p<\infty$ and $\omega\in A_p$, then*

$$E_n(f)_{p,\omega} \lesssim \Omega_r\left(f,\frac{1}{n}\right)_{p,\omega}$$

holds for $n=1,2,3,\ldots$ with some constant depending only on r and p.

This theorem was proved in [12] for the integer order case, in [1, 2] for the fractional case. We note that integer order case was considered in a more general setting in the case of weighted Orlicz spaces [4, 16, 17].

Proof of Theorem 4. We prove (1.11). Let T_n be the near best approximating trigonometric polynomial to f. From Theorem 5

$$\|f-T_n\|_{p,\omega} \lesssim E_n(f)_{p,\omega} \lesssim \Omega_r\left(f,\frac{1}{n}\right)_{p,\omega}.$$

Thus using Lemma 2

$$\frac{1}{n^{2r}}\left\|T_n^{(2r)}\right\|_{p,\omega} \lesssim \Omega_r(T_n,1/n)_{p,\omega}$$
$$\lesssim \Omega_r(T_n-f,1/n)_{p,\omega}+\Omega_r(f,1/n)_{p,\omega}$$
$$\lesssim \|f-T_n\|_{p,\omega}+\Omega_r(f,1/n)_{p,\omega} \lesssim \Omega_r(f,1/n)_{p,\omega}$$

and

$$\|f-T_n\|_{p,\omega}+\frac{1}{n^{2r}}\left\|T_n^{(2r)}\right\|_{p,\omega} \lesssim \Omega_r(f,1/n)_{p,\omega}.$$

On the other hand from Lemma 1

$$\Omega_r(f,1/n)_{p,\omega} \leq \Omega_r(f-T_n,1/n)_{p,\omega}+\Omega_r(T_n,1/n)_{p,\omega}$$
$$\lesssim \|f-T_n\|_{p,\omega}+\frac{1}{n^{2r}}\left\|T_n^{(2r)}\right\|_{p,\omega} = R_{2r}(f,1/n).$$

Thus, (1.11) is proved. The proof of the equivalence (1.12) follows from properties of modulus of smoothness, K-functional and the following lemma. □

Lemma 4. *Let $1<p<\infty$, $\omega\in A_p$, $f\in L^p_\omega$ and $\beta>0$ be a real number. Then for any $0<t<2$, the following inequality is hold*

$$\Omega_\beta(f,t)_{p,\omega} \lesssim t^{2\beta}\|f^{(2\beta)}\|_{p,\omega}$$

with some constant depending only on β and $A[p]$.

Proof of Lemma 4. Since $0 < t < 2$ there exists some $n = 1, 2, 3, \dots$ so that $(1/n) < t \leq (2/n)$ holds. Using Lemma 1 we have

$$\Omega_\beta (f, t)_{p,\omega} \leq \Omega_\beta (f - T_n, t)_{p,\omega} + \Omega_\beta (T_n, t)_{p,\omega} \lesssim E_n (f)_{p,\omega} + t^{2\beta} \|T_n^{(2\beta)}\|_{p,\omega}. \tag{3.8}$$

On the other hand using Theorem 1 of [4] and Theorem 5 we get

$$E_n (f)_{p,\omega} \lesssim \frac{1}{n^{2\beta}} E_n \left(f^{(2\beta)}\right)_{p,\omega} \lesssim \frac{1}{n^{2\beta}} \Omega_\beta \left(f^{(2\beta)}, 1/n\right)_{p,\omega} \lesssim t^{2\beta} \|f^{(2\beta)}\|_{p,\omega} \tag{3.9}$$

and the result follows from (3.8), (3.9), Theorem 1 of [3] and Theorem 5. □

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