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A MODULUS OF SMOOTHNESS FOR SOME BANACH FUNCTION SPACES

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Abstract. Based on the Steklov operator, we considered a modulus of smoothness of functions in some Banach function spaces, which can be not translation invariant, and obtained the main properties of it. A constructive characterization of the Lipschitz class are obtained using the Jackson type direct theorem and inverse theorem of trigonometric approximation. As application, several examples of related (weighted) function spaces are given.

1. Introduction and the Main results

Celebrated theorem of Jackson and Bernstein-Stechkin, on constructive characterization of the Lipschitz classes, states that[1] (see e.g., [12, Theorem 3.3, Ch.7]) *a necessary and sufficient condition for* $f \in L^p$, $1 \leq p \leq \infty$, *to belong to the Lipschitz class of order* $\alpha > 0$,

$$Lip(\alpha, p) := \{f \in L^p : \omega_{\lfloor \alpha \rfloor + 1}(f, \delta)_p \lesssim \delta^\alpha, \quad \delta > 0\},$$

is that

$$\inf_{T_n \in \mathbb{T}_n} \|f - T_n\|_{L^p} =: E_n(f)_{L^p} \lesssim n^{-\alpha},$$

for all $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$,
where $\mathbb{T}_n$ is the class of trigonometrical polynomials

$$T_n(x) = \sum_{k=0}^{n} (a_k \cos kx + b_k \sin kx), \quad a_k, b_k \in \mathbb{R},$$

of degree at most $n \in \mathbb{N}$, $\lfloor x \rfloor := \max\{n \in \mathbb{N} : n \leq x\}$,

$$\omega_r(f, \delta)_{L^p} := \sup_{0 \leq h \leq \delta} \|(I - T_h)^r f\|_{L^p}$$

is the modulus of smoothness of order $r \in \mathbb{N}$ and $T_h f(\cdot) := f(\cdot + h)$, $h \in \mathbb{R}$, is the translation operator.

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[1]Here and in what follows, $A \lesssim B$ will mean that, there exists a positive constant C, independent of essential parameters, such that the inequality $A \leq CB$ is hold.

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By this equivalence, functions in the Lipschitz classes are characterized with respect to only its best approximation orders. To obtain this equivalence, it is necessary to relate the best approximation order $E_n(f)_p$ with the modulus of smoothness $\omega_r\left(f,\frac{1}{n}\right)_{L^p}$.

Direct and inverse inequalities of trigonometric approximation constitute this relation:

$$E_n(f)_{L^p} \lesssim \omega_r\left(f,\frac{1}{n}\right)_{L^p}, \quad r\in\mathbb{N}, \tag{1.1}$$

$$\omega_r\left(f,\frac{1}{n}\right)_{L^p} \lesssim \frac{1}{n^r}\sum_{j=0}^{n}(j+1)^{r-1}E_j(f)_{L^p} \tag{1.2}$$

for any $n\in\mathbb{N}$, with constants depending only on r. We note that, the inequalities (1.1) and (1.2) are hold true (see [14]) for more general Homogeneous Banach spaces X (in short HBS), that is, the class of measurable functions, defined on $T:=[0,2\pi]$, such that the translation operator T_h is a continuous isometry and $\|f(-\cdot)\|_X=\|f(\cdot)\|_X$ holds.

Here the definition of the modulus of smoothness $\omega_r(f,\cdot)_X$ is depend heavily on the translation invariance of the space X considered. If the space X is not translation invariant (for example when the Lebesgue spaces with a weight) the modulus of smoothness $\omega_r(f,\cdot)_X$ may not be well defined.

The main purpose of this work is to define a modulus of smoothness, $\Omega_r(\cdot,\delta)_X$, that can be used also for such spaces X that can be not invariant under the translation operator T_h. Also, X may be some weighted spaces.

We suppose that,

(I) X is a Banach function space (see [8], shortly BFS) on T,

(II) $\mathbb{T}_n$ is a dense subset of X,

(III) the Steklov operator

$$f(x)\mapsto \sigma_h f(x):=\frac{1}{2h}\int_{x-h}^{x+h} f(t)dt, \quad x\in T, f\in X,$$

is uniformly bounded (in h) on X.

We define the classical de la Valleé Poussin mean

$$V_n f(x):=\frac{1}{n+1}\sum_{i=n}^{2n} S_n(x,f), \quad x\in T,$$

where $S_n(\cdot,f)$ is the nth partial sum of the Fourier series of $f\in X\subset L^1$.

The modulus of smoothness in X, satisfying the property (III), is defined as

$$\Omega_r(f,\delta)_X:=\sup_{0\le h\le\delta}\|(I-\sigma_h)^r f\|_X, \quad r\in\mathbb{N},$$

where I is the identity operator on T.

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The following theorem is the main result of this work and estimates the best approximation error $E_n(f)_X := \inf_{T_n\in\mathbb{T}_n} \|f - T_n\|_X$ from above by the modulus of smoothness $\Omega_r(f, \frac{1}{n})_X$.

Theorem 1.1. *Let X be satisfy the conditions (I)-(III), and $f \in X$. If (IV) the operator $f \mapsto V_n$ be uniformly bounded (in n) on X and (V) $E_n(g)_X \lesssim n^{-2}\|g''\|_X$ for any $g \in X'' := \{\rho \in X : \rho'' \in X\}$, then we have the following Jackson-Stechkin type estimate*

$$E_n(f)_X \lesssim \Omega_r\left(f, \frac{1}{n}\right)_X, \quad r \in \mathbb{N}, \tag{1.3}$$

for $n \in \mathbb{N}$, with some constant depending only on r and X.

In Approximation Theory, inequalities of type (1.3) is known as the direct theorem of trigonometric approximation. When $X = L^2$, the inequality (1.3) was proved in [1]. When X is a HBS, (1.3) can be obtained Theorem 10.7 of [15]. When X is the Lebesgue spaces with a weight ω, satisfying the Muckenhoupt's condition A_p, $1 < p < \infty$, the inequality (1.3) in the form

$$E_n(f)_{p,\omega} \lesssim \widetilde{\Omega}_r\left(f, \frac{1}{n}\right)_{p,\omega} := \sup_{0\le h_i\le 1/n} \left\| \prod_{i=1}^{r}(I - \sigma_{h_i})f \right\|_{p,\omega}, \quad r \in \mathbb{N}, \tag{1.4}$$

was proved in [16] (see also e.g., Theorem 2 of [18]). Considering Example 5 of §2, clearly (1.3) improves the inequality (1.4) for $r \ge 2$. Similarly (1.3) also improves direct theorems obtained in [3, 4, 5, 17, 18] for $r \ge 2$.

The weak inverse of the Jackson type estimate (1.3) is given in the following theorem.

Theorem 1.2. *Let X be satisfy the properties (I)-(III), and $f \in X$. If the inequality (VI) $\|T_n'\|_X \lesssim n\|T_n\|_X$ holds for any $T_n \in \mathbb{T}_n$, then we have*

$$\Omega_r\left(f, \frac{1}{n}\right)_X \lesssim \frac{1}{n^{2r}} \sum_{j=0}^{n} (j+1)^{2r-1} E_j(f)_X, \quad r \in \mathbb{N},$$

for $n \in \mathbb{N}$, with some constant depending only on r and X.

Theorems 1.1 and 1.2 gives the following Marchaud type inequality.

Corollary 1.3. *Under the conditions of Theorems 1.1 and 1.2, we have*

$$\Omega_r(f,\delta)_X \lesssim \delta^{2r} \int_\delta^1 u^{-2r-1}\Omega_k(f,u)_X\, du, \quad 0 < \delta < 1,$$

for $r, k \in \mathbb{N}$ with $r < k$.

Theorem 1.4. *Under the conditions of Theorems 1.1 and 1.2, if*

$$E_n(f)_X \lesssim n^{-\beta}, \quad n \in \mathbb{N}$$

for some $\beta > 0$, then, for a given $r \in \mathbb{N}$, we have

$$\Omega_r(f,\delta)_X \lesssim \begin{cases} \delta^{\beta} & , \quad r > \beta/2; \\ \delta^{\beta} \log \frac{1}{\delta} & , \quad r = \beta/2; \\ \delta^{2r} & , \quad r < \beta/2. \end{cases}$$

Definition 1.5. Let $\beta > 0$, $r := \lfloor \beta/2 \rfloor + 1$, and X be a BFS satisfying the condition (III). We define $Lip(\beta, X) := \{f \in X : \Omega_r(f,\delta)_X \lesssim \delta^{\beta}, \quad \delta > 0\}$.

The following result gives a constructive characterization of the Lipschitz classes $Lip(\beta, X)$. As a corollary of Theorems 1.1, 1.2, and 1.4 and Definition 1.5 we have

Corollary 1.6. *Let $\beta > 0$. Under the conditions of Theorems 1.1 and 1.2, the following conditions are equivalent.*

$$\text{(i)} \quad f \in Lip(\beta, X); \quad \text{(ii)} \quad E_n(f)_X \lesssim n^{-\beta}, \quad n \in \mathbb{N}.$$

Some examples for the space X is given in the following section §2. In section §3, we give the proofs of the results.

2. Applications

In this section we will collect some definitions of the function classes that are suitable for the method given in the previous section.

Non-weighted setting: Let $\mathcal{M}$ be the set of all measurable, scalar valued, functions on T and let $\mathcal{M}^+$ be the subset of functions from $\mathcal{M}$ whose values lie in $[0,\infty]$. By χ_E, we denote the characteristic function of a measurable set $E \subset T$.

A mapping $\rho\colon \mathcal{M}^+ \to [0,\infty]$ is called a function norm if for all constants $a \geq 0$, for all functions f, g, f_n, $(n \in \mathbb{N})$, and for all measurable subsets E of T, the following properties are hold.

(i) $\rho(f) = 0$ iff $f = 0$ a.e.; $\rho(af) = a\rho(f)$, and $\rho(f+g) \leq \rho(f) + \rho(g)$;

(ii) If $0 \leq g \leq f$ a.e., then $\rho(g) \leq \rho(f)$;

(iii) if $0 \leq f_n \uparrow f$ a.e., then $\rho(f_n) \uparrow \rho(f)$;

(iv) If a set E of T have a finite Lebesgue measure $|E|$, then $\rho(\chi_E) < \infty$ holds;

(v) If a set E of T satisfies $|E| < \infty$, then there exists a positive constant C, depending only on E and ρ, such that $\int_E f(x)\,dx \leq C\rho(f)$ holds.

For a function norm ρ, the class of functions $X := X(\rho) = \{f \in \mathcal{M} : \rho(|f|) < \infty\}$ is called BFS. For each $f \in X$, we define the norm

$$\|f\|_X := \rho(|f|), \quad f \in X.$$

A BFS X, equipped with norm $\|\cdot\|_X$, is a Banach space [8, pp. 3-5, Theorems 1.4 and 1.6]. If ρ is a function norm, its associate norm ρ^a is defined on $\mathcal{M}^+$ by $\rho^a(g) := \sup\left\{\int_T f(x)\,g(x)\,dx : f \in \mathcal{M}^+, \rho(f) \leq 1\right\}$, $g \in \mathcal{M}^+$. If ρ is a function norm, then ρ^a is itself a function norm [8, p.8, Th. 2.2]. The BFS

$X(\rho^a)$, determined by the function norm ρ^a, is called the associate space of $X = X(\rho)$ and is denoted by X^a. It is well-known (see e.g, [8, p.9]) that

$$\|f\|_X = \sup\left\{\int_T |f(x)g(x)|\,dx : g \in X^a, \quad \|g\|_{X^a} \leq 1\right\}. \tag{2.1}$$

The distirbution function μ_f of a measurable function f is defined as the lebesgue measure of the set $\{x \in T : |f(x)| > \lambda\}$ for $\lambda \geq 0$. A Banach function norm is said to be rearrangement invariant (shortly r.i.) if $\rho(f) = \rho(g)$ for every pair of functions f,g, which are equimeasurable, that is, $\mu_f(\lambda) = \mu_g(\lambda)$. If ρ is r.i. function norm, the BFS $X(\rho)$ is called r.i. BFS. Let X be BFS. A function $f \in X$ is said to have absolutely continuous norm if $\lim_{n\to\infty} \|f\chi_{A_n}\|_X = 0$ for every decreasing sequence of measurable sets $\{A_n\}_{n\geq 1}$ with $\chi_{A_n} \to 0$ a.e. If every $f \in X$ has this property, we will say that X has absolutely continuous norm.

Remark 2.1. ([8, Lemma 6.3, Theorem 6.10, Ch.3]) Let X be a r.i. BFS. The following conditions are equivalent:

(i) The set of trigonometric polynomials $\mathbb{T}_n$ is a dense subset of X.

(ii) Translation operator, T_h, is uniformly bounded (in h) on X.

(iii) X has absolutely continuous norm.

(iv) Fourier series of $f \in X$ converges in norm in X.

(v) Partial sum operator $S_n(\cdot, f)$ is uniformly bounded (in n) on X.

We note also that, when X is separable r.i. BFS, the condition (i) of Remark 2.1 is equivalent to
" (vi) X has non-trivial (i.e., $0 < \alpha_X,\ \beta_X < 1$) Boyd indices α_X, β_X."
(see Chapter 3, Corollary 6.11 of [8]).

Now, we give some examples of BFS.

(1) Lebesgue spaces: We define the Lebesgue functionals

$$\rho_p(f) := \left(\int_T f(x)^p\,dx\right)^{1/p},\ 0 < p < \infty \text{ and } \rho_\infty(f) := essup_{x\in T} f(x).$$

Then $\rho_p(|f|)$ is Banach function norm for $1 \leq p \leq \infty$. We set $L^p := X(\rho_p)$ and $\|f\|_p := \rho_p(|f|)$. In this case the property (I) was proved in Theorem 1.2 of [8]. Property (II) is well known and it can be found in any monograph on Approximation Theory (see e.g., Chapter 1, part 1.4.1 of [26]). Property (III) is a consequence of integral Minkowski's inequality ([26, (12) on p. 592]) and translation invariance of L^p, $1 \leq p \leq \infty$. Property (V) is known from e.g., [12, (2.17) p.206]. (VI) was proved in [7] for $p = \infty$; in [27] for $1 \leq p \leq \infty$ and in [6] $0 \leq p \leq \infty$. For (IV), one can see [25, §3, Theorem 1] and [24, (12)].

(2) (a) Lorentz spaces L^{pq}: Let $0 < p, q \leq \infty$ and $\mathcal{M}_0$ be the subset of functions from $\mathcal{M}$, such that finite a.e. on T. We set for $f \in \mathcal{M}_0$ that

$$\|f\|_{p,q} := \left(\int_0^\infty \left[t^{1/p} f^* (t) \right]^q \frac{dt}{t} \right)^{1/q}, \quad 0 < p < \infty,$$

$$\|f\|_{p,\infty} := \sup_{x \in (0,\infty)} t^{1/p} f^* (t), \quad \|f\|_{(p,\infty)} := \sup_{x \in (0,\infty)} t^{1/p} f^{**} (t),$$

$$\|f\|_{(p,q)} := \left(\int_0^\infty \left[t^{1/p} f^{**} (t) \right]^q \frac{dt}{t} \right)^{1/q}, \quad 0 < p < \infty,$$

where f^* is the decreasing rearrangement of the function f ([8, Ch. 2, Section 1]) and $f^{**} (t) := \frac{1}{t} \int_0^t f^* (s) \, ds$, $t > 0$. The class of functions $\left\{ f \in \mathcal{M}_0 : \|f\|_{p,q} < \infty \right\}$ is denoted by L^{pq}. It is known that L^{pq} coincides with L^p for $0 < p \leq \infty$ and $\|f\|_{p,p} = \|f\|_p$ when $f \in L^p$. On the other hand, if $1 \leq q \leq p < \infty$ or $q = p = \infty$, then $\|\cdot\|_{p,q}$ is a r.i. Banach function norm. If $1 < p < \infty, 1 \leq q \leq \infty$ or $q = p = \infty$, then $\|\cdot\|_{(p,q)}$ is a r.i. Banach function norm (see Ch.4, Theorem 4.3 and Lemma 4.5 of [8]).

(b) Lorentz Λ and M spaces: Let X be a r.i. BFS, on $(\mathbb{R}^+, dx)$ and suppose that X has been renormed so that its fundamental function φ_X is concave. The Lorentz space $\Lambda (X)$ consists of all functions f in $\mathcal{M}_0^+(\mathbb{R}^+, dx)$ for which

$$\|f\|_{\Lambda(X)} := \int_0^\infty f^* (s) \, d\varphi_X (s) < \infty.$$

The Lorentz space $M (X)$ consists of all functions f in $\mathcal{M}_0^+(\mathbb{R}^+, dx)$ for which

$$\|f\|_{M(X)} := \sup_{t \in (0,\infty)} f^{**} (t) \, \varphi_X (t) < \infty.$$

The Lorentz spaces $\Lambda (X)$ and $M (X)$ are r.i. BFS (see Ch.2, Theorem 5.13 of of [8]).

(c) Zygmund spaces $L (\log L)$ and $L_{\exp}$ are r.i. BFS (see Ch.4, part 6 of [8]).

If X is a r.i. BFS and has absolutely continuous norm, the properties (I)-(II) and (IV) can be obtained from Remark 2.1 and the properties (III) and (V)-(VI) were obtained in [17, Lemma 2.2, Theorem 1.2, Lemma 2.5].

(3) Orlicz spaces: A function φ is called Young function if φ is even, continuous, nonnegative in $\mathbb{R}$, increasing on $(0, \infty)$ such that $\varphi (0) = 0$, $\lim_{x \to \infty} \varphi (x) = \infty$. A Young function φ is said to satisfy Δ_2 condition (shortly $\varphi \in \Delta_2$) if there is a constant $C > 0$ such that $\varphi (2x) \leq C\varphi (x)$ for all $x \in \mathbb{R}$. Two Young functions φ and φ_1 are said to be equivalent (shortly $\varphi \sim \varphi_1$) if there are $C, C' > 0$ such that $\varphi_1 (Cx) \leq \varphi (x) \leq \varphi_1 (C'x)$ holds for any $x > 0$. A nonnegative function $M\colon [0, \infty) \to [0, \infty)$ is said to be quasiconvex if there exist a convex Young function Φ and a constant $C \geq 1$ such that $\Phi (x) \leq M (x) \leq \Phi (Cx)$ holds for any $x \geq 0$. Let φ be a quasiconvex Young function. We denote by $\tilde{L}_\varphi (T)$, the class of Lebesgue measurable functions $f\colon T\rightarrow\mathbb{R}$, satisfying the condition $\int_T \varphi\left(\left|f\left(x\right)\right|\right)dx<\infty$. The linear span of the Orlicz class $\tilde{L}_{\varphi}\left(T\right)$, denoted by $\varphi(L)$, becomes a normed space with the Orlicz norm

$$\left\Vert f\right\Vert_{\varphi}:=\sup\left\{ \int\limits_T \left|f\left(x\right)g\left(x\right)\right|dx:\int\limits_T \varphi^{a}\left(\left|g\right|\right)dx\leq 1\right\}, \tag{2.2}$$

where $\varphi^{a}\left(y\right):=\sup_{x\geq 0}\{xy-\varphi\left(x\right)\}$, $(y\geq 0)$, is the complementary function of φ. It can be easily seen that, $\varphi(L)\subset L^{1}\left(T\right)$ and $\varphi(L)$ becomes a Banach space with the Orlicz norm. The Banach space $\varphi(L)$ is called Orlicz space. In this case the condition (I) can be changed with "(I') X is a Banach space with a norm satisfying the integral Minkowski inequality". So the Orlicz norm (2.2) satisfies this property. Under the conditions φ^{α} is quasiconvex function for some $\alpha\in(0,1)$ and $\varphi\in\Delta_2$, the property (II) is a consequence of [20, Lemma 3] and the properties (III)-(VI) were proved in [5, Lemma 2, Lemma 3, Theorem 1, Lemma 5].

The examples given in 1)-3) above are HBS and, in these cases, the inequalities (1.1) and (1.2) can be obtained also by the method developed in [15, §10]. On the other hand, the examples given below are not translation invariant, in general, and in these cases, the method given in [15] is not applicable. The aim of this work is arise from this fact.

The following example is demonstrate a function class that is not rearrangement invariant.

(4) Variable exponent Lebesgue spaces: Let $\mathcal{P}$ be, the class of 2π-periodic, Lebesgue measurable functions $p=p\left(x\right):T\rightarrow(1,\infty)$ such that $essup_{x\in T}p\left(x\right)<\infty$. We define class $L_{2\pi}^{p(\cdot)}$ of 2π-periodic, measurable functions f, defined on T, satisfying

$$\int\limits_T \left|f\left(x\right)\right|^{p(x)}dx<\infty.$$

The class $L_{2\pi}^{p(\cdot)}$ is a Banach space ([21, Theorem 2.5]) with the norm

$$\left\Vert f\right\Vert_{p(\cdot)}:=\inf\left\{\alpha>0:\int\limits_T \left|\frac{f\left(x\right)}{\alpha}\right|^{p(x)}dx\leq 1\right\}.$$

The variable exponent $p\left(\cdot\right)$, defined on T, is said to be satisfy Dini-Lipschitz property DL_{γ} of order γ if

$$\sup_{x_1,x_2\in T}\{\left|p\left(x_1\right)-p\left(x_2\right)\right|:\left|x_1-x_2\right|\leq\delta\}\left(\ln\frac{1}{\delta}\right)^{\gamma}\leq C,\quad 0<\delta<1. \tag{2.3}$$

If $p(\cdot)$ satisfies the properties $1 < essinf_{x\in T} p(x)$, $essup_{x\in T} p(x) < \infty$ and the Dini-Lipschitz condition (2.3) of order ≥ 1, the property (I) follows from Theorem 3.2.13 of [13]; (II)-(IV) follows from [23, Theorem 6.2, Lemma 3.1, Theorem 6.1] and (V)-(VI) follows from [3, Theorem 1, Lemma 1].

Weighted case: A function $\omega\colon T \to [0,\infty]$ will be called weight if ω is measurable and positive a.e. A 2π-periodic weight function ω belongs to the Muckenhoupt class A_p, $p > 1$, if

$$\sup_J \left(\frac{1}{|J|}\int\limits_J \omega(x)\,dx\right)\left(\frac{1}{|J|}\int\limits_J \omega^{1/(1-p)}(x)\,dx\right)^{p-1} \leq C,$$

with a finite positive constant C, independent of J, where J is any subinterval of T.

(5) Weighted Lebesgue spaces: For a weight ω, we denote by $L^p(T,\omega)$, the class of measurable functions, defined on T, such that $\omega f \in L^p(T)$. We set $\|f\|_{p,\omega} := \|\omega f\|_p$ for $f \in L^p(T,\omega)$. If $\omega \in A_p$ and $1 < p < \infty$, the properties (II)-(VI) are known from [18] and (I) is a consequence of [19, Lemma 2.5(b)].

(6) (i) Weighted Orlicz spaces $\varphi_\omega(L)$: Let φ be a quasiconvex Young function. We denote by $\tilde{L}_{\varphi,\omega}(T)$, the class of Lebesgue measurable functions $f\colon T \to \mathbb{R}$, satisfying the condition

$$\int_T \varphi(|f(x)|)\,\omega(x)\,dx < \infty.$$

The linear span of the weighted Orlicz class $\tilde{L}_{\varphi,\omega}$, denoted by $\varphi_\omega(L)$, becomes a normed space with the Orlicz norm

$$\|f\|_{\varphi,\omega} := \sup\left\{\int\limits_T |f(x)\,g(x)|\,\omega(x)\,dx : \int\limits_T \varphi^a(|g|)\,\omega(x)\,dx \leq 1\right\}. \qquad (2.4)$$

For a quasiconvex function φ, we define the indice $p(\varphi)$ of φ as

$$\frac{1}{p(\varphi)} := \inf\{p : p > 0,\ \varphi^p \text{ is quasiconvex}\}.$$

If $\omega \in A_{p(\varphi)}$, then it can be easily seen that, $\varphi_\omega(L) \subset L^1(T)$ and $\varphi_\omega(L)$ becomes a Banach space with the Orlicz norm. The Banach space $\varphi_\omega(L)$ is called weighted Orlicz space. In this case also the condition (I) can be changed with the condition (I'). So the Orlicz norm (2.4) satisfies the property (I'). If the conditions φ^α quasiconvex for some $\alpha \in (0,1)$, $\varphi \in \Delta_2$ and $\omega \in A_{p(\varphi)}$ are fulfilled, then the properties (II)-(VI) were proved in [5].

(ii) Weighted Orlicz spaces $L_{M,\omega}$: A convex and continuous function $M\colon [0,\infty) \to [0,\infty)$, for which $M(0) = 0$, $M(x) > 0$ for $x > 0$ and

$$\lim_{x\to 0} \frac{M(x)}{x} = 0, \quad \lim_{x\to\infty} \frac{M(x)}{x} = \infty$$

is called a N-function. The complementary Young function N of M is defined by

$$N(y) := \max\{xy - M(x) : x \geq 0\}$$

for $y \geq 0$.

Let M be a N-function. We denote by L_M the linear space of 2π-periodic measurable functions $f\colon T \to \mathbb{R}$, such that

$$\int_T M(\lambda |f(x)|)\, dx < \infty$$

holds for some $\lambda > 0$. Equipped with the norm

$$\|f\|_M := \sup\left\{ \int_T |f(x)\, g(x)|\, dx : \int_T N(|g(x)|)\, dx \leq 1 \right\},$$

where N is the complementary function, L_M becomes a Banach space, called the Orlicz space generated by M.

Let $M^{-1}\colon [0,\infty) \to [0,\infty)$ be the inverse of the Young function M and let

$$h(t) := \limsup_{x\to\infty} \frac{M^{-1}(x)}{M^{-1}(x/t)}, \quad t > 0.$$

The numbers β_M and α_M defined by

$$\beta_M := \lim_{t\to\infty} \frac{\log h(t)}{\log t}, \quad \alpha_M := \lim_{t\to 0^+} \frac{\log h(t)}{\log t}$$

are called the upper and lower Boyd indices of the Orlicz space L_M, respectively.

Let ω be a weight function. We denote by $L_{M,\omega}$, the space of the measurable functions $f\colon T \to \mathbb{R}$ such that $f\omega \in L_M$. The norm on $L_{M,\omega}$ is defined by $\|f\|_{M,\omega} := \|f\omega\|_M$. The normed space $L_{M,\omega}$ is called a weighted Orlicz space. If M is an N-function, $L_{M,\omega}$ has non-trivial Boyd indices α_M, β_M and $\omega \in A_{1/\alpha_M} \cap A_{1/\beta_M}$, then the properties (I)-(VI) were proved in [18]. We note also that the spaces $L_{M,\omega}$ and $\varphi_\omega(L)$ are different, in general (see [9]).

7. Weighted variable exponent Lebesgue spaces: By $L_\omega^{p(\cdot)}$, we will denote the class of Lebesgue measurable functions $f\colon T \to \mathbb{R}$, satisfying the condition $\omega f \in L_{2\pi}^{p(\cdot)}$. The weighted variable exponent Lebesgue space $L_\omega^{p(\cdot)}$ is a Banach space with the norm $\|f\|_{p(\cdot),\omega} := \|\omega f\|_{p(\cdot)}$.

For given $p \in \mathcal{P}$, the class of weights ω, satisfying the condition (see [10])

$$\|\omega \chi_Q\|_{p(\cdot)} \left\|\omega^{-1} \chi_Q\right\|_{p'(\cdot)} \lesssim |Q|,$$

for all balls Q in T, will be denoted by $A_{p(\cdot)}$. Here $p'(x) := p(x)/(p(x)-1)$ is the conjugate exponent of $p(x)$. The variable exponent $p(x)$ is said to be satisfy log-Hölder continuous on T if there exists a constant $C \geq 0$ such that

$$|p(x_1) - p(x_2)| \lesssim \frac{1}{\log(e + 1/|x_1 - x_2|)} \text{ for all } x_1, x_2 \in T.$$

If $1 < essinf_{x\in T}p(x)$, $essup_{x\in T}p(x) < \infty$, $1/p$ is Log-Hölder continuous on T, and $\omega^{p_0} \in A_{(p(\cdot)/p_0)'}$ for some $p_0 \in (1, essinf_{x\in T}p(x))$, then the properties (I)-(VI) were obtained in [4].

8. Weighted r.i. BFS: For a weight ω, we denote by $X(T,\omega)$, the class of measurable functions, defined on T, such that $\omega f \in X(T)$. We set $\|f\|_{X,\omega} := \|\omega f\|_X$ for $f \in X(T,\omega)$. If $X(T)$ is a reflexive r.i. BFS that having non-trivial Boyd indices α_X, β_X such that $\omega \in A_{1/\alpha_X} \cap A_{1/\beta_X}$, then the properties (I)-(VI) were obtained in [17].

3. Proofs of the results

The following two lemmas are required for the proof of Theorem 1.1. If $A \lesssim B$ and $B \lesssim A$, simultaneously, we will write $A \approx B$.

Lemma 3.1. *Let X be satisfy the conditions (I)-(III), $f \in X$, $r \in \mathbb{N}$ and $t, l > 0$. Then*

$$\Omega_r(f, lt)_X \lesssim (1 + [l])^2 \, \Omega_r(f, t)_X$$

holds, with some constant depending only on r and X.

Proof. Let $r = 1$ and $t > 0$. Then there exists a $n \in \mathbb{N}$ such that $(1/n) < t \leq (2/n)$. We define the operator

$$\left(U_{1/n} f\right)(x) := 3n^3 \int_0^{1/n} \int_0^t \int_{-u}^u f(x+s)\, ds du dt, \quad x \in T, f \in X.$$

In this case (see [3], p.14)

$$\frac{d^2}{dx^2} U_{1/n} f(x) = C n^2 \left(I - \sigma_{1/n}\right) f(x)$$

holds for almost every $x \in T$, with some constant $C \in \mathbb{R}$.

Hence using the uniform boundedness of the operator $f \mapsto \sigma_{1/n} f$ (for fixed $n \in \mathbb{N}$) in X we get that $\frac{d^2}{dx^2} U_{1/n} f \in X$ and hence $U_{1/n} f \in X''$. On the other hand, from (2.1), we obtain

$$\left\|U_{1/n} f\right\|_X = \left\| 3n^3 \int_0^{1/n} \int_0^t \int_{-u}^u f(x+s)\, ds du dt \right\|_X$$

$$\lesssim n^3 \int_0^{1/n}\int_0^t 2u \left\|\sigma_u f\right\|_X dudt \lesssim 3n^3 \left\|f\right\|_X \int_0^{1/n}\int_0^t 2ududt = \left\|f\right\|_X$$

and hence $f - U_{1/n}f \in X$. Then

$$\begin{aligned}\inf_{g\in X''}\left\{\left\|f-g\right\|_X + t^2\left\|g''\right\|_X\right\} =: K_2\left(f,t,X,X''\right) &\leq 2K_2\left(f,1/n,X,X''\right)\\ &\lesssim \left\|f-U_{1/n}f\right\|_X + n^{-2}\left\|\frac{d^2}{dx^2}U_{1/n}f\right\|_X\\ &=: I_1 + I_2. \qquad (3.1)\end{aligned}$$

We estimate I_1. Using (2.1), we obtain

$$\begin{aligned}\left\|f-U_{1/n}f\right\|_X &\lesssim n^3\int_0^{1/n}\int_0^t 2u\left\|\left(I-\sigma_u\right)f\right\|_X dudt\\ &\lesssim \sup_{0\leq u\leq 1/n}\left\|\left(I-\sigma_u\right)f\right\|_X 3n^3\int_0^{1/n}\int_0^t 2ududt\\ &\lesssim \sup_{0\leq u\leq 1/n}\left\|\left(I-\sigma_u\right)f\right\|_X = \Omega_1(f,1/n)_X. \qquad (3.2)\end{aligned}$$

For the estimate I_2, we find

$$\begin{aligned}\frac{1}{n^2}\left\|\frac{d^2}{dx^2}U_{1/n}f\right\|_X = \left\|n^{-2}\frac{d^2}{dx^2}U_{1/n}f\right\|_X &= \left\|C\left(I-\sigma_{1/n}\right)f\right\|_X\\ &\lesssim \sup_{0\leq u\leq 1/n}\left\|\left(I-\sigma_u\right)f\right\|_X = \Omega_1\left(f,1/n\right)_X. \qquad (3.3)\end{aligned}$$

Now (3.1)-(3.3) give

$$K_2\left(f,t,X,X''\right) \lesssim \Omega_1\left(f,1/n\right)_X \leq \Omega_1\left(f,t\right)_X.$$

On the other hand, for $g \in X''$,

$$\left(I-\sigma_h\right)g\left(x\right) = \frac{1}{2h}\int_{-h}^{h}\left(g\left(x\right)-g\left(x+t\right)\right)dt = -\frac{1}{8h}\int_0^h\int_0^t\int_{-u}^{u} g''\left(x+s\right)dsdudt.$$

Therefore

$$\begin{aligned}\left\|\left(I-\sigma_h\right)g\right\|_X &= \frac{1}{8h}\sup_{v\in X^a}\left\{\int_T\left|\int_0^h\int_0^t\int_{-u}^{u} g''\left(x+s\right)dsdudt\right|\left|v\left(x\right)\right|dx : \left\|v\right\|_{X^a}\leq 1\right\}\\ &\leq \frac{1}{8h}\int_0^h\int_0^t 2u\left\|\frac{1}{2u}\int_{-u}^{u} g''\left(x+s\right)ds\right\|_X dudt\end{aligned}$$

$$\lesssim \frac{1}{8h}\int_0^h\int_0^t 2u\left\|g''\right\|_X dudt = h^2\left\|g''\right\|_X$$

and we obtain

$$\Omega_1(g,t)_X \lesssim t^2\left\|g''\right\|_X \tag{3.4}$$

for $g\in X''$.

Then for $g\in X''$,

$$\Omega_1(f,t)_X \lesssim \|f-g\|_X + t^2\left\|g''\right\|_X$$

and taking infimum on $g\in X''$

$$\Omega_1(f,t)_X \lesssim K_2(f,t;X,X'').$$

We obtained that $\Omega_1(f,t)_X \approx K_2(f,t;X,X'')$. Using the last equivalence we have

$$\begin{aligned}\Omega_1(f,lt)_X &\lesssim \inf_{g\in X''}\left\{\|f-g\|_{p,\omega}+(lt)^2\left\|g''\right\|_{p,\omega}\right\}\\ &\lesssim (1+[l])^2\inf_{g:\ g\in X''}\left\{\|f-g\|_{p,\omega}+t^2\left\|g''\right\|_X\right\}\\ &\lesssim (1+[l])^2\,\Omega_1(f,t)_X.\end{aligned}$$

For $r>1$

$$\begin{aligned}\Omega_r(f,lt)_X &= \sup_{0\le h\le lt}\|(I-\sigma_h)^r f\|_X = \sup_{0\le h\le lt}\|(I-\sigma_h)(I-\sigma_h)^{r-1}f\|_X\\ &= \sup_{0\le h\le lt}\|(I-\sigma_h)U\|_X \lesssim (1+[l])^2\sup_{0\le h\le t}\|(I-\sigma_h)U\|_X\\ &= (1+[l])^2\sup_{0\le h\le t}\|(I-\sigma_h)(I-\sigma_h)^{r-1}f\|_X\\ &= (1+[l])^2\sup_{0\le h\le t}\|(I-\sigma_h)^r f\|_X\end{aligned}$$

and the result follows. □

Lemma 3.2. *Let X be satisfy the conditions (I)-(III), $f\in X$ and $n,m,r\in\mathbb{N}$. Then there exists a number $\delta\in(0,1)$, depending only on X, such that*

$$\Omega_r(f,t)_X \lesssim C\delta^{mr}\|f\|_X + C'\Omega_{r+1}(f,t)_X$$

holds for any $t\in(0,1/n)$, where the constant $C>0$ depending only on r and X; the constant $C'>0$ depending only on r,m and X.

Proof. For any $h>0$, there exists a constant $\mathtt{C}>1$, such that

$$\|\sigma_h f\|_X \le \mathtt{C}\|f\|_X.$$

We set $\delta := \mathtt{C}/(1+\mathtt{C})$. Now, for any $h\in(0,1/n)$ we prove firstly that

$$\|(I-\sigma_h)^r f\|_X \le \delta^r\|\left(I-\sigma_h^2\right)^r f\|_X + C\Omega_{r+1}(f,h)_X. \tag{3.5}$$

To prove (3.5), we observe

$$I - \sigma_h = 2^{-1}(I - \sigma_h)(I + \sigma_h) + 2^{-1}(I - \sigma_h)^2$$

and

$$\sigma_h (I - \sigma_h) = 2^{-1}(I - \sigma_h)(I + \sigma_h) - 2^{-1}(I - \sigma_h)^2 .$$

Hence, for $g \in X$,

$$\|(I - \sigma_h) g\|_X + \|\sigma_h (I - \sigma_h) g\|_X \le \|(I - \sigma_h)(I + \sigma_h) g\|_X + \\ + \|(I - \sigma_h)^2 g\|_X . \tag{3.6}$$

On the other hand

$$\|(I - \sigma_h)^r f\|_X = \delta\left((1/\mathtt{C})\|(I - \sigma_h)^r f\|_X + \|(I - \sigma_h)^r f\|_X\right)$$

$$\begin{aligned}
&\le \delta\left(\|(I - \sigma_h)^r f\|_{p,\omega} + \|(I - \sigma_h)^r f\|_X\right) \\
&= \delta\left(\|(I - \sigma_h)(I - \sigma_h)^{r-1} f\|_X + \|(I - \sigma_h)^r f\|_X\right) \\
&= \delta\left(\|\left(\sigma_h (I - \sigma_h) + (I - \sigma_h)^2\right)(I - \sigma_h)^{r-1} f\|_X + \|(I - \sigma_h)^r f\|_X\right) \\
&\le \delta\left(\|\sigma_h (I - \sigma_h)(I - \sigma_h)^{r-1} f\|_X + \|(I - \sigma_h)^2 (I - \sigma_h)^{r-1} f\|_X\right) + \\
&\quad + \delta\|(I - \sigma_h)^r f\|_X \\
&\le \delta\left(\|\sigma_h (I - \sigma_h)^r f\|_X + \|(I - \sigma_h)^{r+1} f\|_{p,\omega} + \|(I - \sigma_h)^r f\|_X\right) .
\end{aligned} \tag{3.7}$$

Taking $g := (I - \sigma_h)^{r-1} f$ in (3.6), we have

$$\|\sigma_h (I - \sigma_h)^r f\|_X + \|(I - \sigma_h)^r f\|_X \le \|(I - \sigma_h)^r (\sigma_h + I) f\|_{p,\omega} + \|(I - \sigma_h)^{r+1} f\|_X$$

and using this in (3.7)

$$\begin{aligned}
\|(I - \sigma_h)^r f\|_X &\le \delta\left(\|\sigma_h (I - \sigma_h)^r f\|_{p,\omega} + \|(I - \sigma_h)^{r+1} f\|_X + \|(I - \sigma_h)^r f\|_X\right) \\
&\le \delta\left(\|(I - \sigma_h)^r (\sigma_h + I) f\|_X + \|(I - \sigma_h)^{r+1} f\|_X\right) + \\
&\quad + \delta\|(I - \sigma_h)^{r+1} f\|_X \\
&\le \delta\|(I - \sigma_h)^r (\sigma_h + I) f\|_X + 2\delta\|(I - \sigma_h)^{r+1} f\|_X .
\end{aligned} \tag{3.8}$$

Repeating r times the last inequality we have

$$\begin{aligned}
\|(I - \sigma_h)^r f\|_X &\le \delta\|(I - \sigma_h)^r (\sigma_h + I) f\|_X + 2\delta\|(I - \sigma_h)^{r+1} f\|_X \\
&\le \delta^2\|(I - \sigma_h)^r (\sigma_h + I)^2 f\|_X + 2\delta^2\|(I - \sigma_h)^{r+1} (\sigma_h + I) f\|_X \\
&\quad + 2\delta\|(I - \sigma_h)^{r+1} f\|_X \\
&\le \ldots \le \delta^r\|(I - \sigma_h)^r (\sigma_h + I)^r f\|_X \\
&\quad + 2\sum_{k=1}^{r} \delta^k\|(I - \sigma_h)^{r+1} (\sigma_h + I)^{k-1} f\|_X \\
&= \delta^r\|\left(I - \sigma_h^2\right)^r f\|_{p,\omega} + 2\sum_{k=1}^{r} \delta^k\|(I - \sigma_h)^{r+1} (\sigma_h + I)^{k-1} f\|_X .
\end{aligned}$$

Hence

$$\| (I-\sigma_h)^r f\|_X \le \delta^r \| \left(I-\sigma_h^2\right)^r f\|_X + C\,(r,X)\,\Omega_{r+1}\,(f,h)_X$$

and the proof of (3.5) is finished. Using (3.5) we obtain

$$\begin{aligned}
\| (I-\sigma_h)^r f\|_X &\le \delta^r \| \left(I-\sigma_h^2\right)^r f\|_X + C\,(r,X)\,\Omega_{r+1}\,(f,h)_X \\
&\le \delta^{2r} \| \left(I-\sigma_h^4\right)^r f\|_X + (\delta^r+1)\,C\,(r,X)\,\Omega_{r+1}\,(f,h)_X \\
&\le \dots \\
&\le \delta^{mr} \| \left(I-\sigma_h^{2m}\right)^r f\|_X + C\,(r,X,m)\,\Omega_{r+1}\,(f,h)_X\,. \qquad (3.9)
\end{aligned}$$

Taking supremum, the last inequality imply the result

$$\Omega_r\,(f,t)_X \lesssim \delta^{mr}\|f\|_X + \Omega_{r+1}\,(f,t)_X\,,$$

because $\| \left(I-\sigma_h^{2m}\right)^r f\|_X \lesssim \|f\|_X$. □

Proof of Theorem 1.1. The case $r=1$. Let $n\in\mathbb{N}$ and $f\in X$ be fixed. We will use the operator $U_{1/n}f$. Using (IV), (3.2) and (3.3) we have

$$\begin{aligned}
E_n(f)_X &= E_n(f-U_{1/n}f+U_{1/n}f)_X \le E_n(f-U_{1/n}f)_X + E_n(U_{1/n}f)_X \\
&\lesssim \|f-U_{1/n}f\|_X + n^{-2}\left\|\frac{d^2}{dx^2}U_{1/n}f\right\|_X \lesssim \Omega_1\left(f,\frac{1}{n}\right)_X \qquad (3.10)
\end{aligned}$$

for any $n\in\mathbb{N}$.

The case $r\ge 2$. Following the idea given in [11], we will use induction on r. We know that the Jackson type estimate (1.3) holds for $r=1$, (see (3.10)). We suppose that the inequality (1.3) holds for some $r=2,3,4,\dots$. We have to verify the fulfilment of inequality (1.3) for $r+1$. We will use the classical de la Vallée Poussin mean V_nf and show that

$$\|f-V_{\lfloor n/2\rfloor}f\|_X \lesssim \Omega_{r+1}\left(f,\frac{1}{n}\right)_X.$$

We set $u(\cdot):=f(\cdot)-V_{\lfloor n/2\rfloor}f(\cdot)$. We have $a_i(V_nf)=a_if$ and $b_i(V_nf)=b_if$, $i\in\{0\}\cup\mathbb{N}$, where a_if,b_if and $a_i(V_nf)$, $b_i(V_nf)$ are the Fourier coefficients of the functions f and V_nf, respectively. These equalities yield $V_m(V_nf)=V_mf$ for $m,n\in\{0\}\cup\mathbb{N}$. And hence we get $V_{\lfloor n/4\rfloor}(u)=0$. Since V_nf is near best approximant for f, i.e., $\|f-V_nf\|_X\lesssim E_n(f)_X$, using induction hypothesis and Lemma 3.1

$$\|u\|_X = \|u-V_{\lfloor n/4\rfloor}(u)\|_X \le CE_{\lfloor n/4\rfloor}(u)_X \le C\Omega_r\left(u,\frac{1}{\lfloor n/4\rfloor}\right)_X \le \mathbf{C}\Omega_r\left(u,\frac{1}{n}\right)_{p,\omega}.$$

We know, from Lemma 3.2, that

$$\Omega_r\left(u,\frac{1}{n}\right)_X \le C\delta^{mr}\|u\|_X + C'\Omega_{r+1}\left(u,\frac{1}{n}\right)_X.$$

Choosing m such that $C\mathbf{C}\delta^{mr} < 1/2$, we get

$$\|u\|_X \leq \mathbf{C}\Omega_r\left(u, \frac{1}{n}\right)_X \leq C\mathbf{C}\delta^{mr}\|u\|_X + C\Omega_{r+1}\left(u, \frac{1}{n}\right)_X.$$

Therefore

$$\|u\|_X \lesssim \Omega_{r+1}\left(u, \frac{1}{n}\right)_X.$$

From uniform boundedness of operator $f \longmapsto V_n f$ in X, we have

$$\Omega_{r+1}\left(u, \frac{1}{n}\right)_X \lesssim \Omega_{r+1}\left(f, \frac{1}{n}\right)_X$$

and the result

$$E_n(f)_X \lesssim \|f - V_{\lfloor n/2\rfloor}f\|_{p,\omega} = \|u\|_X \lesssim \Omega_{r+1}\left(u, \frac{1}{n}\right)_X \lesssim \Omega_{r+1}\left(f, \frac{1}{n}\right)_X$$

holds for $r \in \mathbb{N}$. The proof is completed. □

Proof of Theorem 1.2. Let $T_n \in \mathbb{T}_n$, $n \in \{0\} \cup \mathbb{N}$, be the best approximating trigonometric polynomial for $f \in X$. From (3.4) we get

$$\Omega_r(g, \delta)_X \lesssim \delta^{2r}\left\|g^{(2r)}\right\|_X, \quad r \in \mathbb{N},$$

for $g^{(2r)} \in X$ and $\delta > 0$. On the other hand, for any $m \in \mathbb{N}$

$$\Omega_X^r(f, \delta) \leq \Omega_X^r(f - T_{2^{m+1}}, \delta) + \Omega_X^r(T_{2^{m+1}}, \delta) \tag{3.11}$$

and

$$\Omega_X^r(f - T_{2^{m+1}}, \delta) \lesssim \|f - T_{2^{m+1}}\|_X \lesssim E_{2^{m+1}}(f)_X. \tag{3.12}$$

Then

$$\begin{aligned}
\Omega_X^r(T_{2^{m+1}}, \delta) &\lesssim \delta^{2r}\left\|T_{2^{m+1}}^{(2r)}\right\|_X \\
&\lesssim \delta^{2r}\left\{\left\|T_1^{(2r)} - T_0^{(2r)}\right\|_X + \sum_{i=1}^{m}\left\|T_{2^{i+1}}^{(2r)} - T_{2^i}^{(2r)}\right\|_X\right\} \\
&\lesssim \delta^{2r}\left\{E_0(f)_X + \sum_{i=1}^{m} 2^{(i+1)2r}E_{2^i}(f)_X\right\} \\
&\lesssim \delta^{2r}\left\{E_0(f)_X + 2^{2r}E_1(f)_X + \sum_{i=1}^{m} 2^{(i+1)2r}E_{2^i}(f)_X\right\}.
\end{aligned}$$

Applying here, the inequality

$$2^{(i+1)2r}E_{2^i}(f)_X \lesssim \sum_{k=2^{i-1}+1}^{2^m} k^{2r-1}E_k(f)_X, \quad i \geq 1, \tag{3.13}$$

we get

$$\Omega_X^r\left(T_{2^{m+1}},\delta\right)\lesssim\delta^{2r}\left\{E_0\left(f\right)_X+2^{2r}E_1\left(f\right)_X+\sum_{k=2}^{2^m}k^{2r-1}E_k\left(f\right)_X\right\}$$
$$\lesssim\delta^{2r}\left\{E_0\left(f\right)_X+\sum_{k=1}^{2^m}k^{2r-1}E_k\left(f\right)_X\right\}. \qquad (3.14)$$

Since

$$E_{2^{m+1}}\left(f\right)_X\lesssim\frac{1}{n^{2r}}\sum_{k=2^{m-1}+1}^{2^m}k^{2r-1}E_k\left(f\right)_X,$$

choosing m as $2^m\leq n<2^{m+1}$, from (3.11)-(3.14), we obtain the result. □

Proof of Theorem 1.4. Let $f\in X$ and

$$E_n\left(f\right)_X\lesssim n^{-\beta},\quad n=1,2,3,\dots$$

for some $\beta>0$. We suppose that $\delta>0$ and $n:=\lfloor 1/\delta\rfloor$. From Theorem 1.2 we get

$$\Omega_r\left(f,\delta\right)_X\leq\Omega_r\left(f,\frac{1}{n}\right)_X\lesssim\frac{1}{n^{2r}}\sum_{j=0}^{n}(j+1)^{2r-1}E_j(f)_X$$
$$\lesssim\delta^{2r}\left(E_0(f)_X+\sum_{j=1}^{n}j^{2r-1}E_j(f)_X\right)$$
$$\lesssim\delta^{2r}\left(E_0(f)_X+\sum_{j=1}^{n}j^{2r-1-\beta}\right).$$

If $2r>\beta$, then we get $\Omega_r\left(f,\delta\right)_X\lesssim\delta^\beta$. If $2r=\beta$, then $\sum_{j=1}^{n}j^{2r-1-\beta}=\sum_{j=1}^{n}j^{-1}\leq 1+\log(1/\delta)$ and hence $\Omega_r\left(f,\delta\right)_X\lesssim\delta^\beta\log(1/\delta)$. If $2r<\beta$, then the series $\sum_{j=0}^{n}j^{2r-1-\beta}$ is convergent and

$$\Omega_r\left(f,\delta\right)_X\lesssim\delta^{2r}\left(E_0(f)_X+\sum_{j=1}^{n}j^{2r-1-\beta}\right)\lesssim\delta^{2r}$$

holds. □

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