CRM
CENTRE DE RECERCA MATEMÀTICA

Preprint núm. 1163
July 2013

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WEIGHTED NORM INEQUALITIES FOR CONVOLUTION AND RIESZ POTENTIAL

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Abstract. In this paper, we prove analogues of O'Neil's inequalities for the convolution in the weighted Lebesgue spaces. We also establish the weighted two-sided norm inequalities for the potential operator.

1. Introduction

In this paper, we study the convolution of functions on $\mathbb{R}^n$:

$$(K * f)(x) = \int_{\mathbb{R}^n} K(x-y) f(y) dy, \qquad x \in \mathbb{R}^n.$$

The following generalization of Young's inequality is due to O'Neil ([ON]): if $1 < p < q < \infty$, $\quad 0 < \tau_1 \leqslant \tau_2 \leqslant \infty$, and $\frac{1}{r} = 1 - \frac{1}{p} + \frac{1}{q}$, then

$$L^{p,\tau_1} * L^{r,\infty} \subset L^{q,\tau_2}, \tag{1.1}$$

where $L^{p,\tau} = L^{p,\tau}(\mathbb{R}^n)$ is the Lorentz space (see, e.g., [Be, Ch 4]). Throughout the paper the expression of form $X * Y \subset Z$ involving function spaces X, Y, and Z means that whenever $f \in X$, $K \in Y$, then $f * K \in Z$ and

$$\|f * K\|_Z \leqslant C \|f\|_X \|K\|_Y,$$

where the constant C being independent of f and K.

A generalization of Young's inequality for the weighted L^p spaces was obtained in [Ke]. The weighted Lebesgue space $L^p(\omega) = L^p(\omega; \mathbb{R}^n)$ consists of all measurable functions such that $\|f\|_{L^p(\omega)} = (\int_{\mathbb{R}^n} |f|^p \omega^p)^{1/p} < \infty$, where the weight ω is a nonnegative locally integrable function. Moreover, for the case of power weight $\omega(x) = |x|^s$, we write $L^p(\omega; \mathbb{R}^n) = L^p_s(\mathbb{R}^n)$.

Theorem A. [Ke] *We have*

$$L^p_\alpha(\mathbb{R}^n) * L^\theta_s(\mathbb{R}^n) \subset L^q_{-\beta}(\mathbb{R}^n), \qquad 1 < p, q, \theta < \infty, \qquad \frac{1}{q} \leqslant \frac{1}{p} + \frac{1}{\theta}, \tag{1.2}$$

2000 *Mathematics Subject Classification.* Primary 31C15; Secondary 44A35, 46E30.

Key words and phrases. Convolution, Riesz potential operator, Weights, Lorentz spaces, O'Neil-type inequalities.

This research was partially supported by the MTM2011-27637/MTM, 2009 SGR 1303, RFFI 12-01-00169, NSH 979.2012.1, and Ministry of Education and Science of the Republic of Kazakhstan (0112RK02176, 0112RK00608).

CRM Preprint Series number 1163

provided

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{\theta} + \frac{\alpha+\beta+s}{n} - 1, \tag{1.3}$$

$$\alpha < n/p', \quad \beta < n/q, \quad s < n/\theta', \tag{1.4}$$

and

$$\alpha+\beta \geqslant 0, \quad \alpha+s \geqslant 0, \quad \beta+s \geqslant 0. \tag{1.5}$$

Convolution inequalities $L^p_\alpha(\mathbb{R}^n)*L^\theta_s(\mathbb{R}^n) \subset L^q_{-\beta}(\mathbb{R}^n)$ were also studied in [Bu1]. Conditions (1.3)–(1.5) are necessary as it is shown in Section 7.

Let us now formulate the first question studied in this paper:

Problem 1. *Find sufficient conditions on weights μ and ν, so that*

$$L^p(\mu) * L^{r,\infty} \subset L^q(\nu), \qquad 1 < p \leqslant q < \infty, \quad 1 < r \leqslant \infty.$$

Further, we deal with an important example of the convolution operator $K*f$, where $K(z) = |z|^{\gamma-n}$, $z \in \mathbb{R}^n$, i.e., the operator of the fractional integration (or Riesz potential):

$$I_\gamma f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}} dy, \qquad 0 < \gamma < n. \tag{1.6}$$

For this operator, (1.1) implies $I_\gamma \colon L^{p,\tau_1}(\mathbb{R}^n) \to L^{q,\tau_2}(\mathbb{R}^n)$ for $\frac{1}{p} - \frac{1}{q} = \frac{\gamma}{n}$ and $0 < \tau_1 \leqslant \tau_2 \leqslant \infty$. In particular, this yields the Hardy-Littlewood-Sobolev theorem, i.e., $I_\gamma \colon L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$. Continuity properties of the potential operator in the Lebesgue spaces are well known, see, e.g., [BS], [So], [SW].

Analogous questions have been intensively investigated in the weighted Lebesgue spaces. For the power weights, Stein and Weiss [SW1] generalized the Hardy-Littlewood-Sobolev theorem as follows: *Suppose*

$$\gamma = \alpha+\beta+n(1/p-1/q), \ \ \alpha < n/p', \ \ \beta < n/q, \ \ 0 \leqslant \alpha+\beta, \tag{1.7}$$

then

$$I_\gamma \colon L^p_\alpha(\mathbb{R}^n) \longrightarrow L^q_{-\beta}(\mathbb{R}^n), \qquad 1 < p \leqslant q < \infty. \tag{1.8}$$

For arbitrary weights μ and ν, the solution of the problem stated as "give necessary and sufficient conditions on weights for the Riesz potential I_γ to be bounded from $L^p(\mu,\mathbb{R}^n)$ to $L^q(\nu,\mathbb{R}^n)$", can be found in [GK], [LMPT], [Ma], [MW], [Pe], [Sa1], [Sa2], [SaWh] and references therein. In particular, it is known (see [LMPT, (2.2)]) that for $1 < p \leqslant q < \infty$,

(1.9)
$$\|I_\gamma\|_{L^p(\mu,\mathbb{R}^n)\to L^q(\nu,\mathbb{R}^n)} \asymp \|I_\gamma\|_{L^{q'}(\nu^{1-q'},\mathbb{R}^n)\to L^{p',\infty}(\mu^{1-p'},\mathbb{R}^n)} + \|I_\gamma\|_{L^p(\mu,\mathbb{R}^n)\to L^{q,\infty}(\nu,\mathbb{R}^n)}.$$

Note also that 'upper' triangle case with one weight was considered in [Ma] and [MN] in terms of capacity; a non-capacity characterization was proved in [COV].

Our second goal in this paper is to study two-sided norm inequalities for the weighted potential operator

(1.10)
$$(A_\gamma f)(y) = (A_{\gamma,\mu,\nu} f)(y) = \nu(y) \int_{\mathbb{R}^n} \frac{f(x)\mu(x)}{|x-y|^{n-\gamma}} dx, \qquad y \in \mathbb{R}^n, \qquad 0 < \gamma < n,$$

in the Lorentz spaces. More precisely, we study the following question.

Problem 2. *Find upper and lower bounds of the norm of the operator* A_γ: $L^{p,\tau_1}(\mathbb{R}^n) \to L^{q,\tau_2}(\mathbb{R}^n)$, $1 < p, q < \infty$, $1 < \tau_1, \tau_2 \leqslant \infty$, *in terms of* μ *and* ν.

Note that this question is similar to the following problem: to estimate the norm of the potential operator in the weighted Lorentz space, i.e., I_γ: $L^{p,\tau_1}(\mu^{-1}, \mathbb{R}^n) \to L^{q,\tau_2}(\nu, \mathbb{R}^n)$, which is, in general, a different question. The latter case was thoroughly studied by Kerman for the power weights; see Theorems 4.1 and 4.5 in [Ke].

In this paper $F \lesssim G$ means that $F \leqslant CG$; by C we denote positive constants that may be different on different occasions. Moreover, $F \asymp G$ means that $F \lesssim G \lesssim F$.

For a Lebesgue measurable set E, $|E|$ will denote its Lebesgue measure and $\nu(E) = \int_E \nu(x)dx$ will denote its weighted measure. If E and W are any subsets of $\mathbb{R}^n$, the Minkowski sum of E and W is defined by $E + W = \Big\{x + y : x \in E \text{ and } y \in W\Big\}$. The characteristic function of a set E is denoted by χ_E. Let also $\overline{0} = (0, \dots, 0)$, $\overline{N} = (N, \dots, N)$, and $B_r(\overline{N}) = \Big\{x \in \mathbb{R}^n : |x - \overline{N}| \leqslant r\Big\}$. As usual, ψ^* is the decreasing rearrangement of ψ see, e.g., [BS].

Finally, X' is the associate space of the space X, i.e., $X' = \Big\{g : \|g\|_{X'} = \sup_{\|f\|_X \leqslant 1} |\int fg| < \infty\Big\}$.

2. Main results

First, we give a solution of Problem 1 by proving the following extension of Theorem A.

Theorem 1. *Let* $1 < p \leqslant q < \infty$ *and* $1 < r < \infty$. *Let weights* μ *and* ν *satisfy, for any* $\lambda \geqslant 1$,

$$\mu^*(\lambda t) \lesssim \frac{\mu^*(t)}{\lambda^\alpha}, \qquad t > 0, \tag{2.1}$$

$$\nu^*(\lambda t) \lesssim \frac{\nu^*(t)}{\lambda^\beta}, \qquad t > 0, \tag{2.2}$$

for some $\alpha \geqslant 0$ *and* $\beta \geqslant 0$ *such that*

$$\alpha + 1/r + 1/p > 1, \qquad \beta + 1/r + 1/q' > 1. \tag{2.3}$$

Then a sufficient condition for

$$L^p(\mu^{-1}) * L^{r,\infty} \subset L^q(\nu) \tag{2.4}$$

to hold is

$$\mathcal{G} := \sup_{|E|=|W|} \frac{\nu(E)\mu(W)}{|E|^{1+\frac{1}{r}+\frac{1}{p}-\frac{1}{q}}} < \infty, \tag{2.5}$$

where the supremum is taken over all measurable sets E *and* W *of the same positive measure. A necessary condition for (2.4) is*

$$\mathcal{S} := \sup_{|E|=|W|} \frac{\nu(E)\mu(W)}{|E|^{1+\frac{1}{p}-\frac{1}{q}}|E-W|^{\frac{1}{r}}} < \infty, \tag{2.6}$$

where the supremum is taken over all measurable sets E *and* W *of the same positive measure.*

Remark. Note that conditions (2.1) and (2.2) are fulfilled for arbitrary weights μ and ν with $\alpha = \beta = 0$. Hence, the statement of Theorem 1 holds for arbitrary μ and ν if $1/r + 1/p > 1$ and $1/r + 1/q' > 1$, cf. (2.3). Similar remarks can be applied to Theorem 2 and Corollary 2 below. Let us also note that if a function ψ is δ-regularly varying (see [BGT, Sect. 2]), that is, it satisfies $\psi(\lambda t) \lesssim \frac{\psi(t)}{\lambda^\delta}$, then it also satisfies $\psi^*(\lambda t) \lesssim \frac{\psi^*(t)}{\lambda^\delta}$.

For the power weights $\mu(x) = |x|^{-\alpha}$ and $\nu(x) = |x|^{-\beta}$, Theorem 1 implies necessary and sufficient conditions for (2.4) to hold.

Corollary 1. *Suppose* $1 < p \leqslant q < \infty$ *and* $1 < r < \infty$. *Then*

$$L^p_\alpha(\mathbb{R}^n) * L^{r,\infty}(\mathbb{R}^n) \subset L^q_{-\beta}(\mathbb{R}^n) \tag{2.7}$$

if and only if

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{r} + \frac{\alpha+\beta}{n} - 1, \quad 0 \leqslant \alpha < n/p', \quad 0 \leqslant \beta < n/q. \tag{2.8}$$

In particular, this implies the statement of Theorem A in case when $\alpha, \beta, s \geqslant 0$.

If $\alpha = \beta = 0$, (2.7) is O'Neil's inequality $L^p(\mathbb{R}^n) * L^{r,\infty}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$, $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$. Moreover, (2.7) gives the Stein-Weiss inequality (1.8) under conditions (1.7) and $\alpha, \beta \geqslant 0$.

Let us now study Problem 2. The following result gives upper and lower norm estimates of the weighted potential operator A_γ in the Lorentz spaces.

Theorem 2. *Let* $1 < p, q < \infty$, $0 < \tau_1 \leqslant \tau_2 \leqslant \infty$, *and* $0 < \gamma < n$.
(A). *Let the weights* μ *and* ν *satisfy conditions (2.1) and (2.2) for some* $\alpha \geqslant 0$ *and* $\beta \geqslant 0$ *such that*

$$\alpha + 1/p > \gamma/n, \qquad \beta + 1/q' > \gamma/n. \tag{2.9}$$

If

$$\mathcal{L} := \sup_{|E|=|W|} \frac{\nu(E)\mu(W)}{|E|^{2+\frac{1}{p}-\frac{1}{q}-\frac{\gamma}{n}}} < \infty, \tag{2.10}$$

where the supremum is taken over all measurable sets E *and* W *of the same positive measure, then* A_γ *is bounded from* $L^{p,\tau_1}(\mathbb{R}^n)$ *to* $L^{q,\tau_2}(\mathbb{R}^n)$ *and*

$$\|A_\gamma\|_{L^{p,\tau_1}(\mathbb{R}^n)\longrightarrow L^{q,\tau_2}(\mathbb{R}^n)} \leqslant C\mathcal{L},$$

where C *depends only on parameters* p, q, τ_1, τ_2, α, β, γ, *and* n.
(B). *If the operator* A_γ *is bounded from* $L^{p,\tau_1}(\mathbb{R}^n)$ *to* $L^{q,\tau_2}(\mathbb{R}^n)$, *then*

$$\mathcal{F} := \sup_{B\in M} \frac{\nu(B)\mu(B)}{|B|^{2+\frac{1}{p}-\frac{1}{q}-\frac{\gamma}{n}}} \leqslant C\,\|A_\gamma\|_{L^{p,\tau_1}(\mathbb{R}^n)\to L^{q,\tau_2}(\mathbb{R}^n)}, \tag{2.11}$$

where M *is the collection of all balls in* $\mathbb{R}^n$ *and* C *depends only on parameters* p, q, τ_1, τ_2, α, β, γ, *and* n.

Remark. Comparing $\mathcal{S}$ and $\mathcal{F}$ defined by (2.6) and (2.11), we see that

$$\mathcal{F} \leqslant \sup_{x\in\mathbb{R}^n, E \text{ convex}} \frac{\nu(E)\mu(E+x)}{|E|^{1+\frac{1}{r}+\frac{1}{p}-\frac{1}{q}}} \lesssim \mathcal{S}.$$

Taking $\tau_1 = p$ and $\tau_2 = q$ in Theorem 2, we get $\mathcal{F} \lesssim \|A_\gamma\|_{L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n)} \lesssim \mathcal{L}$. Moreover, under certain regularity conditions on weights, the left-hand side bound $\mathcal{F}$ and the right-hand side bound $\mathcal{L}$ are equivalent. In particular, this holds for radial weights $\psi(|x|)$ satisfying the following condition:

$$\psi^*(t) \lesssim \frac{1}{t}\int_{t/2}^{t} \psi(s)\,ds, \qquad t > 0. \tag{2.12}$$

Note that (2.12) holds for any monotonic function ψ on $(0,\infty)$. It is easy to give an example of a non-monotonic function satisfying condition (2.12), for example,

$$\psi(t) = \frac{|\cos nt|}{t^\alpha}, \qquad t > 0, \qquad 0 < \alpha < 1.$$

Theorem 2 implies the following relation for the norm of the weighted potential operator.

Corollary 2. *Let* $p, q, \gamma, \tau_1, \tau_2$ *satisfy all conditions of Theorem 2. Suppose functions* $\nu_0(\cdot)$ *and* $\mu_0(\cdot)$ *defined on* $(0,\infty)$ *satisfy condition (2.12). Then, for the operator*

$$(A_\gamma f)(y) = (A_{\gamma,\mu,\nu} f)(y) = \nu_0(|y|) \int\limits_{\mathbb{R}^n} \frac{f(x)\mu_0(|x|)}{|x-y|^{n-\gamma}}dx, \qquad y \in \mathbb{R}^n, \qquad 0 < \gamma < n,$$

one has

$$\|A_\gamma\|_{L^{p,\tau_1}(\mathbb{R}^n)\to L^{q,\tau_2}(\mathbb{R}^n)} \asymp \sup_{a>0} \frac{1}{a^{n(2+\frac{1}{p}-\frac{1}{q}-\frac{\gamma}{n})}} \int_0^a \nu_0(r) r^{n-1} dr \int_0^a \mu_0(r) r^{n-1} dr. \tag{2.13}$$

In particular, if $1<p\leqslant q<\infty$, *then*

$$\|A_\gamma\|_{L^{p}(\mathbb{R}^n)\to L^{q}(\mathbb{R}^n)} \asymp \sup_{a>0} \frac{1}{a^{n(2+\frac{1}{p}-\frac{1}{q}-\frac{\gamma}{n})}} \int_0^a \nu_0(r) r^{n-1} dr \int_0^a \mu_0(r) r^{n-1} dr.$$

In the case of the power weights $\mu(x)=|x|^{-\alpha}$ and $\nu(x)=|x|^{-\beta}$ we obtain necessary and sufficient conditions for boundedness of the operator A_γ.

Corollary 3. *Let* $1<p,q<\infty$, $\alpha\geqslant 0$, *and* $\beta \geqslant 0$.
(A). *Let either* $0<\tau_1\leqslant\tau_2<\infty$ *or* $1<\tau_1\leqslant\tau_2\leqslant\infty$. *Then the operator*

$$(A_{\gamma,\alpha,\beta} f)(y) = \int_{\mathbb{R}^n} \frac{f(x)}{|x|^\alpha |x-y|^{n-\gamma}|y|^\beta} dx, \qquad y\in\mathbb{R}^n, \qquad 0<\gamma<n,$$

is bounded from $L^{p,\tau_1}(\mathbb{R}^n)$ *to* $L^{q,\tau_2}(\mathbb{R}^n)$, *if and only if*

$$\gamma=\alpha+\beta+n\left(\frac{1}{p}-\frac{1}{q}\right), \qquad \alpha<\frac{n}{p'}, \qquad \beta<\frac{n}{q}. \tag{2.14}$$

(B). *Let* $0<\tau\leqslant 1$. *The operator* $A_{\gamma,\alpha,\beta}$ *is bounded from* $L^{p,\tau}(\mathbb{R}^n)$ *to* $L^{q,\infty}(\mathbb{R}^n)$ *if and only if*

$$\gamma=\alpha+\beta+n\left(\frac{1}{p}-\frac{1}{q}\right), \qquad \alpha\leqslant\frac{n}{p'}, \qquad \beta\leqslant\frac{n}{q}. \tag{2.15}$$

In particular, for the Lebesgue spaces this result is reduced to $I_\gamma\colon L^p_\alpha(\mathbb{R}^n)\to L^q_{-\beta}(\mathbb{R}^n)$, $1<p\leqslant q<\infty$, that is, (1.8). Note that necessity of conditions (1.7) was discussed in [Du].

3. Lemmas

Define

$$f^{**}(t)=\frac{1}{t}\int_0^t f^*(s)ds.$$

We will use the fact (see [BS, p. 53]) that f^{**} can be written as follows:

$$f^{**}(t)=\sup_{|E|=t}\frac{1}{|E|}\int_E |f(x)|dx.$$

Lemma 1. *Let W and E be measurable sets in $\mathbb{R}^n$, then*

(3.1)
$$\left|\int_W \mu(x)\int_E \nu(y)K(y-x)dydx\right| \leqslant \int_0^{|W|}\mu^*(s)\int_0^{|E|}\nu^*(t)K^{**}\left(\max(t,s)\right)dtds.$$

Proof. The proof is inspired by ideas from the paper [NT1]. By the Hardy-Littlewood rearrangement inequality

$$\begin{aligned}
&\int_W \mu(x)\int_E \nu(y)K(y-x)dydx \leqslant \\
&\quad \leqslant \int_0^{|W|}\mu^*(s)\left(\int_E \nu(y)K(y-\cdot)dy\right)^*(s)ds \\
&\quad \leqslant \int_0^{|W|}\mu^*(s)\left(\int_E \nu(y)K(y-\cdot)dy\right)^{**}(s)ds \\
&\quad = \int_0^{|W|}\mu^*(s)\sup_{|\eta_1|=s}\frac{1}{|\eta_1|}\int_{\eta_1}\left|\int_E \nu(y)K(y-x)dy\right|dxds \\
&\quad \leqslant \int_0^{|W|}\mu^*(s)\sup_{|\eta_1|=s}\Big(\frac{1}{|\eta_1|}\int_E|\nu(y)|\int_{\eta_1}|K(y-x)|dxdy\Big)ds.
\end{aligned}$$

We use similar estimates for the inner integral to get

$$\begin{aligned}
&\int_W \mu(x)\int_E \nu(y)K(y-x)dydx \leqslant \\
&\quad \leqslant \int_0^{|W|}\mu^*(s)\sup_{|\eta_1|=s}\int_0^{|E|}\nu^*(t)\sup_{|\eta_2|=t}\frac{1}{|\eta_2|}\frac{1}{|\eta_1|}\int_{\eta_2}\int_{\eta_1}|K(y-x)|\,dxdydtds \\
&\quad \leqslant \int_0^{|W|}\mu^*(s)\int_0^{|E|}\nu^*(t)\sup_{|\eta_1|=s}\sup_{|\eta_2|=t}\frac{1}{|\eta_2|}\frac{1}{|\eta_1|}\int_{\eta_2}\int_{\eta_1}|K(y-x)|\,dxdydtds.
\end{aligned}$$

If $t \geqslant s$, then

$$\begin{aligned}
&\sup_{|\eta_1|=s}\sup_{|\eta_2|=t}\frac{1}{|\eta_2|}\frac{1}{|\eta_1|}\int_{\eta_2}\int_{\eta_1}|K(y-x)|dxdy \leqslant \\
&\qquad \leqslant \sup_{|\eta_1|=s}\frac{1}{|\eta_1|}\int_{\eta_1}\sup_{|\eta_2|=t}\frac{1}{|\eta_2|}\int_{\eta_2}|K(y-x)|dydx = K^{**}(t).
\end{aligned}$$

Similarly, if $t < s$, then

$$\sup_{|\eta_1|=s}\sup_{|\eta_2|=t}\frac{1}{|\eta_2|}\frac{1}{|\eta_1|}\int_{\eta_2}\int_{\eta_1}|K(y-x)|dxdy \leqslant K^{**}(s)$$

and the statement follows. □

Lemma 2. *Let $\beta > 1$. Suppose*

$$\mathcal{B} = \sup_{\omega>0} \frac{1}{\omega^\beta} \int_0^\omega \mu^*(s)ds \int_0^\omega \nu^*(t)dt \int_0^\omega K^*(s)ds < \infty. \tag{3.2}$$

Then there exists C depending only on β such that

$$\sup_{\omega>0} \frac{1}{\omega^{\beta-1}} \int_0^\omega \mu^*(s) \int_0^\omega \nu^*(t) K^{**}\left(\max(t,s)\right) dtds \leqslant C\mathcal{B}. \tag{3.3}$$

Proof. For $\omega > 0$ we have

$$\begin{aligned}
&\frac{1}{\omega^{\beta-1}} \int_0^\omega \mu^*(s) \int_0^\omega \nu^*(t) K^{**}\left(\max(t,s)\right) dtds = \\
&= \frac{1}{\omega^{\beta-1}} \int_0^\omega \mu^*(s) \left(\int_0^s \nu^*(t) K^{**}(s)dt + \int_s^\omega \nu^*(t) K^{**}(t)dt \right) ds \\
&= \frac{1}{\omega^{\beta-1}} \int_0^\omega K^{**}(s) \left(\int_0^s \nu^*(t)dt \int_0^s \mu^*(t)dt \right)' ds \\
&= \frac{1}{\omega^{\beta-1}} \left[K^{**}(s) \int_0^s \nu^*(t)dt \int_0^s \mu^*(t)dt \Big|_0^\omega \right. \\
&\qquad \left. - \int_0^\omega \left(K^{**}(s)\right)' \int_0^s \nu^*(t)dt \int_0^s \mu^*(t)dtds \right] \\
&=: I_1 + I_2.
\end{aligned}$$

To estimate I_2, taking into account that

$$-\left(K^{**}(s)\right)' = \frac{1}{s}\Big(K^{**}(s) - K^*(s)\Big) \leqslant \frac{1}{s} K^{**}(s),$$

we get

$$I_2 \leqslant \frac{1}{\omega^{\beta-1}} \int_0^\omega K^{**}(s) \left(\int_0^s \nu^*(t)dt \int_0^s \mu^*(t)dt \right) \frac{ds}{s}.$$

By (3.2)

$$I_2 \leqslant \frac{\mathcal{B}}{\omega^{\beta-1}} \int_0^\omega \frac{ds}{s^{2-\beta}} \lesssim \mathcal{B}.$$

Let us estimate I_1. We have

$$\begin{aligned}
I_1 &= \frac{1}{\omega^\beta} \int_0^\omega K^*(t)dt \int_0^\omega \nu^*(t)dt \int_0^\omega \mu^*(t)dt \\
&\quad - \frac{1}{\omega^{\beta-1}} \lim_{s\to 0+} \frac{1}{s} \int_0^s K^*(t)dt \int_0^s \nu^*(t)dt \int_0^s \mu^*(t)dt \\
&\leqslant \mathcal{B} + \frac{1}{\omega^{\beta-1}} \left| \lim_{s\to 0+} \frac{1}{s} \int_0^s K^*(t)dt \int_0^s \nu^*(t)dt \int_0^s \mu^*(t)dt \right|.
\end{aligned}$$

Since
$$\frac{1}{s}\int_0^s K^*(t)dt\int_0^s \nu^*(t)dt\int_0^s \mu^*(t)dt \leqslant s^{\beta-1}\mathcal{B}, \qquad s>0,$$
then
$$\lim_{s\to 0+}\frac{1}{s}\int_0^s K^*(t)dt\int_0^s \nu^*(t)dt\int_0^s \mu^*(t)dt = 0,$$
and therefore, $I_1 \leqslant \mathcal{B}$. □

Lemma 3. *Let $1<p,q<\infty$. Let μ,ν, and K be locally integrable functions on $\mathbb{R}^n$ and satisfy, for any $\lambda \geqslant 1$,*

$$\mu^*(\lambda t) \lesssim \frac{\mu^*(t)}{\lambda^\alpha}, \qquad \nu^*(\lambda t) \lesssim \frac{\nu^*(t)}{\lambda^\beta}, \qquad K^*(\lambda t) \lesssim \frac{K^*(t)}{\lambda^\rho}, \qquad t>0, \tag{3.4}$$

for some $\alpha \geqslant 0$, $\beta \geqslant 0$, and $\rho \geqslant 0$. Let also

$$\rho+\alpha+1/p \geqslant 1, \qquad \rho+\beta+1/q' \geqslant 1. \tag{3.5}$$

If
$$\mathcal{D} := \sup_{t>0}\frac{1}{t^{2+1/p-1/q}}\int_0^t \mu^*(s)ds\int_0^t \nu^*(s)ds\int_0^t K^*(s)ds < \infty,$$
then
$$\sup_{\substack{\xi>0\\ \eta>0}}\frac{1}{\eta^{\frac{1}{q'}}}\frac{1}{\xi^{\frac{1}{p}}}\int_0^\eta \nu^*(t)\int_0^\xi \mu^*(s)K^{**}(\max(s,t))dsdt \leqslant C\mathcal{D},$$
where C depends only on p,q,α,β,ρ.

Proof. Suppose also that $\eta > \xi > 0$. We take an integer $N>1$ such that $(N-1)\xi < \eta \leqslant N\xi$. Then
$$\frac{1}{\eta^{\frac{1}{q'}}}\frac{1}{\xi^{\frac{1}{p}}}\int_0^\eta \nu^*(t)\int_0^\xi \mu^*(s)K^{**}(\max(s,t))dsdt \leqslant$$
$$\leqslant \frac{1}{((N-1)\xi)^{\frac{1}{q'}}}\frac{1}{\xi^{\frac{1}{p}}}\int_0^\eta \nu^*(t)\int_0^\xi \mu^*(s)K^{**}(\max(s,t))dsdt$$
$$\lesssim \frac{1}{((N-1)\,\xi)^{1/q'}\xi^{1/p}}\int_0^\xi \mu^*(s)\sum_{k=1}^N\int_{(k-1)\xi}^{k\xi}\nu^*(t)K^{**}(\max(s,t))dtds.$$
We divide the last expression into two terms:
$$\frac{1}{((N-1)\,\xi)^{1/q'}\xi^{1/p}}\int_0^\xi \mu^*(s)\int_0^\xi \nu^*(t)K^{**}(\max(s,t))dtds+$$
$$+\frac{1}{((N-1)\,\xi)^{1/q'}\xi^{1/p}}\int_0^\xi \mu^*(s)\sum_{k=2}^N\int_{(k-1)\xi}^{k\xi}\nu^*(t)K^{**}(t)dtds =: J_1+J_2.$$

Note that (3.4) implies

$$K^{**}(\lambda t) \lesssim \frac{K^{**}(t)}{\lambda^\rho}. \tag{3.6}$$

Indeed,

$$K^{**}(\lambda t) = \frac{1}{\lambda t}\int_0^{\lambda t} K^*(s)ds = \frac{1}{t}\int_0^t K^*(\lambda s)ds$$
$$\lesssim \frac{1}{t}\int_0^t \frac{K^*(s)}{\lambda^\rho}ds = \frac{K^{**}(t)}{\lambda^\rho}.$$

Therefore, using $\nu^*(\lambda t) \lesssim \lambda^{-\beta}\nu^*(t)$ and (3.6), we get

$$J_2 \lesssim \frac{1}{((N-1)\,\xi)^{1/q'}\xi^{1/p}} \int\limits_0^\xi \mu^*(s)ds \sum_{k=2}^N \nu^*\big((k-1)\xi\big)K^{**}\big((k-1)\xi\big)\xi$$

$$\lesssim \frac{1}{((N-1)\,\xi)^{1/q'}\xi^{1/p}} \int\limits_0^\xi \mu^*(s)\,ds \int\limits_0^\xi \nu^*(t)dtK^{**}(\xi)\sum_{k=2}^N \frac{1}{(k-1)^{\rho+\beta}}. \tag{3.7}$$

Noting that $\rho+\beta+1/q' \geqslant 1$, we get

$$J_2 \lesssim \mathcal{D}.$$

Estimating J_1, we use Lemma 2 to get $J_1 \lesssim \mathcal{D}$.

Summing the estimates for J_1 and J_2, we finally have

$$\frac{1}{\eta^{\frac{1}{q'}}}\frac{1}{\xi^{\frac{1}{p}}}\int_0^\eta \nu^*(t)\int_0^\xi \mu^*(s)K^{**}(\max(s,t))dsdt \lesssim \mathcal{D}$$

in the case when $\eta > \xi$.

If $\xi \geqslant \eta$, then we use similar estimates and the condition $\rho+\alpha+1/p \geqslant 1$. □

4. Proof of Theorem 1

Proof. We consider the operator

$$(Af)(y) = (A_{K,\mu,\nu}f)(y) = \nu(y)\int\limits_{\mathbb{R}^n} f(x)K(x-y)\mu(x)\,dx, \qquad y\in\mathbb{R}^n. \tag{4.1}$$

Let us first prove that the operator A is a (p,q) quasi-weak-type operator, i.e.,

$$\|A\|_{L^{p,1}(\mathbb{R}^n)\to L^{q,\infty}(\mathbb{R}^n)} < C^*, \tag{4.2}$$

where $C^* = C(p,q,r)\mathcal{G}\|K\|_{L^{r,\infty}(\mathbb{R}^n)}$. By Corollary 4.1 from the paper [NT2], we have

$$\|A\|_{L^{p,1}(\mathbb{R}^n)\to L^{q,\infty}(\mathbb{R}^n)} \asymp \sup_{\substack{|W|>0\\|E|>0}} \frac{1}{|E|^{\frac{1}{q'}}}\frac{1}{|W|^{\frac{1}{p}}}\Big|\int_E \nu(y)\int_W K(y-x)\mu(x)dxdy\Big| \tag{4.3}$$

and hence, it is enough to show that the latter is bounded.

Let E and W be any measurable sets from $\mathbb{R}^n$ of positive measure. Lemma 1 yields

$$\begin{aligned}
&\frac{1}{|E|^{\frac{1}{q'}}}\frac{1}{|W|^{\frac{1}{p}}}\int_E \nu(y)\int_W K(y-x)\mu(x)dxdy \leqslant \\
&\quad\leqslant \frac{1}{|E|^{\frac{1}{q'}}}\frac{1}{|W|^{\frac{1}{p}}}\int_0^{|E|}\nu^*(t)\int_0^{|W|}\mu^*(s)K^{**}(\max(s,t))dsdt \\
&\quad\leqslant \Big(\sup_{t>0}\frac{1}{t^{1-\frac{1}{r}}}\int_0^t K^*(s)ds\Big)\frac{1}{|E|^{\frac{1}{q'}}}\frac{1}{|W|^{\frac{1}{p}}}\int_0^{|E|}\nu^*(t)\int_0^{|W|}\mu^*(s)\frac{1}{\max(s,t)^{\frac{1}{r}}}dsdt \\
&\quad\lesssim \|K\|_{L^{r,\infty}}\frac{1}{|E|^{\frac{1}{q'}}}\frac{1}{|W|^{\frac{1}{p}}}\int_0^{|E|}\nu^*(t)\int_0^{|W|}\mu^*(s)\frac{1}{\max(s,t)^{\frac{1}{r}}}dsdt.
\end{aligned}$$

Since the function $\varphi(x)=|x|^{-n/r}$ satisfies $\varphi^{**}(s)\asymp s^{-1/r}$, Lemma 3 gives

$$\begin{aligned}
&\frac{1}{|E|^{\frac{1}{q'}}}\frac{1}{|W|^{\frac{1}{p}}}\int_0^{|E|}\nu^*(t)\int_0^{|W|}\mu^*(s)\frac{1}{\max(s,t)^{\frac{1}{r}}}dsdt \\
&\qquad\qquad\lesssim \quad \sup_{t>0}\frac{1}{t^{2+1/p-1/q}}\int_0^t\mu^*(s)ds\int_0^t\nu^*(s)ds\int_0^t\frac{1}{s^{\frac{1}{r}}}ds \lesssim \mathcal{G}
\end{aligned}$$

for any sets E and W with positive measure, provided

$$1<p,q,r<\infty, \qquad \alpha+1/r+1/p>1, \qquad \beta+1/r+1/q'>1. \tag{4.4}$$

We note that the expression $\mathcal{G}$ depends only on $1/p-1/q$. Further, since conditions (4.4) contain strict inequalities, one can choose two pairs (p_0,q_0) and (p_1,q_1) such that $1<p_0<p<p_1<\infty$, $1<q_0<q<q_1<\infty$, and

$$1/p-1/q=1/p_0-1/q_0=1/p_1-1/q_1,\;\; \alpha+1/r+1/p_i>1,\;\; \beta+1/r+1/q_i'>1,\;\; i=0,1.$$

Then, (4.2) holds for (p_0,q_0) and (p_1,q_1), and therefore,

$$A\colon L^{p_0,1}\to L^{q_0,\infty} \quad \text{and} \quad A\colon L^{p_1,1}\to L^{q_1,\infty},$$

where the norms are bounded by $\mathcal{G}\|K\|_{L^{r,\infty}}$ up to a constant. Then, by the interpolation theorem (see, e.g., [BS, Ch 5, §1]) we write

$$A\colon L^{p(\theta),\tau_1}\to L^{q(\theta),\tau_1}$$

and

$$\|A\|_{L^{p(\theta),\tau_1}\to L^{q(\theta),\tau_1}}\leqslant C(\theta,p_i,q_i,r,\tau_1)\,\mathcal{G}\,\|K\|_{L^{r,\infty}}$$

for

$$1/p(\theta)=(1-\theta)/p_0+\theta/p_1,\;\; 1/q(\theta)=(1-\theta)/q_0+\theta/q_1,\quad 0<\theta<1\,,\quad 0<\tau_1\leqslant\infty.$$

For some $0<\theta<1$ we have $p(\theta)=p$. Since $1/p_0-1/q_0=1/p_1-1/q_1=1/p(\theta)-1/q(\theta)$, in this case $q(\theta)$ coincides with q.

Finally,

$$\|A\|_{L^{p,\tau_1}\to L^{q,\tau_2}} \lesssim \|A\|_{L^{p,\tau_1}\to L^{q,\tau_1}} \lesssim \mathcal{G}\|K\|_{L^{r,\infty}},$$

when $\tau_1 \leqslant \tau_2$. Taking $p=\tau_1$ and $q=\tau_2$ implies (2.4).

Let us show necessity of (2.6). Suppose (2.4) holds. For measurable sets E and W such that $|E|=|W|$, we put

$$K(x) := |E-W|^{-\frac{1}{r}}\chi_{E-W}(x).$$

Then $\|K\|_{L^{r,\infty}(\mathbb{R}^n)}=1$. Using (4.3), we get

$$\begin{aligned} 1 &\gtrsim \|A\|_{L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n)} \gtrsim \|A\|_{L^{p,1}(\mathbb{R}^n)\to L^{q,\infty}(\mathbb{R}^n)} \\ &\gtrsim \frac{1}{|E|^{\frac{1}{q'}}}\frac{1}{|W|^{\frac{1}{p}}}\Big|\int_E \nu(y)\int_W K(y-x)\mu(x)dxdy\Big| \\ &= \frac{1}{|E|^{1+\frac{1}{p}-\frac{1}{q}}}\frac{1}{|E-W|^{\frac{1}{r}}}\int_E \nu(y)\int_W \mu(x)dxdy. \end{aligned}$$

Taking supremum over all E and W completes the proof. □

Proof of Corollary 1. First, we prove the sufficiency part. We take $\mu(x)=|x|^{-\alpha}$ and $\nu(x)=|x|^{-\beta}$ and note that (2.3) can be written as $\alpha/n+1/r+1/p>1$ and $\beta/n+1/r+1/q'>1$. Further, $\mathcal{G}<\infty$ only if $\frac{1}{q}=\frac{1}{p}+\frac{1}{r}+\frac{\alpha+\beta}{n}-1$ and (2.7) follows.

Second, to prove necessity of condition (2.8) for a fixed $r\in(1,\infty)$, we consider $K(x)=|x|^{\gamma-n}$ such that $\frac{1}{r}=1-\frac{\gamma}{n}$. Therefore, $0<\gamma<n$ and $K\in L^{r,\infty}(\mathbb{R}^n)$. Then the boundedness of the fractional integral from $L^p_\alpha(\mathbb{R}^n)$ to $L^q_{-\beta}(\mathbb{R}^n)$ gives (see [Du, Th. 5.1]) that

$$\frac{1}{q}=\frac{1}{p}+\frac{1}{r}+\frac{\alpha+\beta}{n}-1, \qquad \alpha<n/p', \qquad \beta<n/q. \tag{4.5}$$

These conditions can be also verified using (2.6).

We have to show that $\alpha,\beta\geqslant 0$. Let an integer N be sufficiently large. Putting $K(x):=\chi_{B_2(\overline{N})}(x)$, we have $K\in L^{r,\infty}(\mathbb{R}^n)$, $r>1$. If

$$\|f*K\|_{L^q_{-\beta}(\mathbb{R}^n)} \lesssim \|f\|_{L^p_\alpha(\mathbb{R}^n)}\|K\|_{L^{r,\infty}(\mathbb{R}^n)},$$

then the operator

$$A_{K,\alpha,\beta}f(y)=\frac{1}{|y|^\beta}\int_{\mathbb{R}^n} K(x-y)f(x)\,\frac{dx}{|x|^\alpha}$$

is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Taking into account (4.3),

$$\begin{aligned} 1 &\gtrsim \|A_{K,\alpha,\beta}\|_{L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n)} \gtrsim \|A_{K,\alpha,\beta}\|_{L^{p,1}(\mathbb{R}^n)\to L^{q,\infty}(\mathbb{R}^n)} \\ &\asymp \sup_{\substack{|E|>0\\ |W|>0,}} \frac{1}{|E|^{\frac{1}{q'}}|W|^{\frac{1}{p}}}\Big|\int_E \frac{1}{|y|^\beta}\int_W K(x-y)\,\frac{dx}{|x|^\alpha}dy\Big|. \end{aligned}$$

Then, since $\beta < n/q < n$

$$1 \gtrsim \frac{1}{|B_1(\overline{0})|^{\frac{1}{q'}}|B_2(\overline{N})|^{\frac{1}{p}}} \int_{B_1(\overline{0})} \frac{1}{|y|^\beta} \int\limits_{B_2(\overline{N})} K(x-y)\, \frac{dx}{|x|^\alpha} dy$$

$$\geqslant C_{p,q} \int_{B_1(\overline{0})} |y|^{-\beta} dy \int_{B_1(\overline{N})} |x|^{-\alpha} dx \gtrsim N^{-\alpha}.$$

This gives $\alpha \geqslant 0$. Similarly, we get $\beta \geqslant 0$.

Finally, since

$$L_s^\theta(\mathbb{R}^n) \subset L^{r,\infty}(\mathbb{R}^n) \quad \text{for} \quad \frac{1}{r} = \frac{s}{n} + \frac{1}{\theta}, \quad 0 \leqslant s < n/\theta',$$

(2.7) gives (1.2) provided (1.3), (1.4), and $\alpha, \beta, s \geqslant 0$. □

5. Boundedness of the weighted convolution operators in Lorentz spaces

In this section we deal with Problem 2 and study the weighted convolution operator $A_K = A_{K,\mu,\nu}$ given by (4.1). In particular, if $K(x) = |x|^{\gamma-n}$, then $A_K = A_\gamma$, where A_γ is defined by (1.10). The next result gives sufficient conditions for the operator A_K to be bounded from $L^{p,\tau_1}(\mathbb{R}^n)$ to $L^{q,\tau_2}(\mathbb{R}^n)$ in terms of μ, ν, and K. This implies sufficient conditions for the boundedness of the operator A_γ given in Theorem 2 (A).

Theorem 3. *Let* $1 < p, q < \infty, \quad 0 < \tau_1 \leqslant \tau_2 \leqslant \infty$. *Suppose either,*
(A) *for any* $\lambda \geqslant 1$,

$$\mu^*(\lambda t) \lesssim \frac{\mu^*(t)}{\lambda^\alpha}, \qquad \nu^*(\lambda t) \lesssim \frac{\nu^*(t)}{\lambda^\beta}, \qquad K^*(\lambda t) \lesssim \frac{K^*(t)}{\lambda^\rho}, \qquad t > 0, \tag{5.1}$$

where $\alpha \geqslant 0$, $\beta \geqslant 0, \rho \geqslant 0$, *and*

$$\rho + \alpha + 1/p > 1, \qquad \rho + \beta + 1/q' > 1,$$

or,
(B) *for any* $\lambda \geqslant 1$,

$$\mu^*(t) \lesssim \mu^*(\lambda t)\lambda^{\bar{\alpha}}, \qquad \nu^*(t) \lesssim \nu^*(\lambda t)\lambda^{\bar{\beta}}, \qquad K^*(t) \lesssim K^*(\lambda t)\lambda^{\bar{\rho}}, \qquad t > 0, \tag{5.2}$$

where $\bar{\alpha} \geqslant 0$, $\bar{\beta} \geqslant 0$, $\bar{\rho} \geqslant 0$, *and*

$$\bar{\rho} + \bar{\alpha} + 1/p < 1, \qquad \bar{\rho} + \bar{\beta} + 1/q' < 1.$$

If

$$\mathcal{D} := \sup_{t>0} \frac{1}{t^{2+1/p-1/q}} \int_0^t \mu^*(s)ds \int_0^t \nu^*(s)ds \int_0^t K^*(s)ds < \infty,$$

then the operator A_K is bounded from $L^{p,\tau_1}(\mathbb{R}^n)$ to $L^{q,\tau_2}(\mathbb{R}^n)$ and

$$\|A_K\|_{L^{p,\tau_1}(\mathbb{R}^n)\to L^{q,\tau_2}(\mathbb{R}^n)} \leqslant C\mathcal{D},$$

where C depends only on p, q, τ_1, τ_2 and the corresponding parameters from (5.1) or (5.2).

Proof. As it was shown in the proof of Theorem 1, it is sufficient to obtain the following estimate:

$$\|A_K\|_{L^{p,1}(\mathbb{R}^n)\to L^{q,\infty}(\mathbb{R}^n)} \lesssim \mathcal{D}.$$

(A) Under conditions (5.1), following the proof of Theorem 1, we have

$$\|A_K\|_{L^{p,1}(\mathbb{R}^n)\to L^{q,\infty}(\mathbb{R}^n)} \lesssim$$
$$\lesssim \sup_{|E|>0,|W|>0} \frac{1}{|E|^{\frac{1}{q'}}} \frac{1}{|W|^{\frac{1}{p}}} \int_0^{|E|} \nu^*(t) \int_0^{|W|} \mu^*(s) K^{**}(\max(s,t))dsdt \lesssim \mathcal{D},$$

where the last estimate directly follows from Lemma 3.
(B) Suppose conditions (5.2) hold. Let us verify that

$$\sup_{|E|>0,|W|>0} \frac{1}{|E|^{\frac{1}{q'}}} \frac{1}{|W|^{\frac{1}{p}}} \int_0^{|E|} \nu^*(t) \int_0^{|W|} \mu^*(s) K^{**}(\max(s,t))dsdt \lesssim \mathcal{D}. \tag{5.3}$$

Let $0 < |W| < |E|$ and $|E|/|W| =: \lambda > 1$. Note that

$$K^{**}\left(\max(\frac{s}{\lambda}, t)\right) \lesssim \lambda^{\bar{\rho}} K^{**}\left(\max(s,t)\right) \quad \text{for any} \quad s, t > 0.$$

Indeed, using monotonicity of K^{**} and condition (5.2), we get

$$\begin{aligned} K^{**}\left(\max\left(s/\lambda, t\right)\right) &= \frac{1}{\max(s/\lambda, t)} \int_0^{\max(s/\lambda,t)} K^*(u)du \\ &\leqslant \frac{\lambda}{\max(s,t)} \int_0^{\max(s,t)/\lambda} K^*(u)du \\ &= \frac{1}{\max(s,t)} \int_0^{\max(s,t)} K^*(\frac{u}{\lambda})du \leqslant C\lambda^{\bar{\rho}} K^{**}\left(\max(s,t)\right). \end{aligned}$$

In view of $\bar{\alpha} + \bar{\rho} < \frac{1}{p'}$, we have

$$\begin{aligned} &\frac{1}{|E|^{\frac{1}{q'}}} \frac{1}{|W|^{\frac{1}{p}}} \int_0^{|E|} \nu^*(t) \int_0^{|W|} \mu^*(s) K^{**}\left(\max(s,t)\right) dsdt = \\ &\qquad = \frac{\lambda^{\frac{1}{p}-1}}{|E|^{\frac{1}{q'}+\frac{1}{p}}} \int_0^{|E|} \nu^*(t) \int_0^{|E|} \mu^*(\frac{s}{\lambda}) K^{**}\left(\max(\frac{s}{\lambda}, t)\right) dsdt \\ &\qquad \lesssim \frac{\lambda^{\bar{\alpha}+\bar{\rho}+\frac{1}{p}-1}}{|E|^{\frac{1}{q'}+\frac{1}{p}}} \int_0^{|E|} \nu^*(t) \int_0^{|E|} \mu^*(s) K^{**}\left(\max(s,t)\right) dsdt \end{aligned}$$

$$\leqslant \frac{1}{|E|^{\frac{1}{q'}+\frac{1}{p}}} \int_0^{|E|} \nu^*(t) \int_0^{|E|} \mu^*(s) K^{**}\left(\max(s,t)\right) dsdt.$$

Further, Lemma 2 implies

$$\frac{1}{|E|^{\frac{1}{q'}}} \frac{1}{|W|^{\frac{1}{p}}} \int_0^{|E|} \nu^*(t) \int_0^{|W|} \mu^*(s) K^{**}\left(\max(s,t)\right) dsdt \lesssim \mathcal{D}.$$

The case $|W| > |E|$ is similar. Thus, we have shown (5.3), which concludes the proof. $\square$

6. Proof of Theorem 2 and its corollaries

Proof of Theorem 2. We start with (B). If A_γ is bounded from $L^{p,\tau_1}(\mathbb{R}^n)$ to $L^{q,\tau_2}(\mathbb{R}^n)$, then it is (p,q) weak-type operator, i.e., $A_\gamma : L^{p,1}(\mathbb{R}^n) \to L^{q,\infty}(\mathbb{R}^n)$, and therefore, by (4.3), we have

$$\begin{aligned} \|A_\gamma\|_{L^{p,\tau_1}(\mathbb{R}^n)\to L^{q,\tau_2}(\mathbb{R}^n)} &\gtrsim \|A_\gamma\|_{L^{p,\min(1,\tau_1)}(\mathbb{R}^n)\to L^{q,\infty}(\mathbb{R}^n)} \asymp \|A_\gamma\|_{L^{p,1}(\mathbb{R}^n)\to L^{q,\infty}(\mathbb{R}^n)} \\ &\asymp \sup_{|E|>0,|W|>0} \frac{1}{|E|^{1/q'}} \frac{1}{|W|^{1/p}} \int\limits_E \nu(y) \int\limits_W \mu(x)|x-y|^{\gamma-n} dxdy \\ &\geqslant \sup_{B\in M} \frac{1}{|B|^{\frac{1}{p}+\frac{1}{q'}}} \int\limits_B \nu(y) \int\limits_B \mu(x)|x-y|^{\gamma-n} dxdy \\ &\gtrsim \sup_{B\in M} \frac{1}{|B|^{2+\frac{1}{p}-\frac{1}{q}-\frac{\gamma}{n}}} \int\limits_B \nu(x)dx \int\limits_B \mu(x)dx = \mathcal{F}, \end{aligned}$$

that is, (2.11) is true.

The part (A) follows from Theorem 3 (A) for $K(x) = |x|^{\gamma-n}$. Taking $\rho = 1-\gamma/n$ implies $\|A_\gamma\|_{L^{p,\tau_1}(\mathbb{R}^n)\to L^{q,\tau_2}(\mathbb{R}^n)} \lesssim \mathcal{D} \asymp \mathcal{L}$. $\square$

Proof of Corollary 2. To show that $\mathcal{L} \lesssim \mathcal{F}$ it is enough to verify that, for a fixed positive a,

$$\sup_{|E|=a^n} \int_E \nu_0(|x|)dx \lesssim \int_{B_{a_1}(\bar{0})} \nu_0(|x|)\, dx \tag{6.1}$$

and

$$\sup_{|W|=a^n} \int_W \mu_0(|x|)dx \lesssim \int_{B_{a_1}(\bar{0})} \mu_0(|x|)\, dx, \tag{6.2}$$

where $a_1/a = C(n)$.

Let us show the first inequality. We have

$$\sup_{|E|=a^n} \int_E \nu_0(|x|)dx = \int_0^{a^n} \left(\nu_0(|\cdot|)\right)^*(t)\, dt \lesssim \int_0^{a_1^n} \nu_0^*(t^{1/n})dt \lesssim \int_0^{a_1} \nu_0^*(t) t^{n-1} dt.$$

Using condition (2.12), we get

$$\sup_{|E|=a^n}\int_E \nu_0(|x|)dx \lesssim \int_0^{a_1}\left(\frac{1}{t}\int_{t/2}^t \nu_0(s)ds\right)t^{n-1}dt$$
$$\lesssim \int_0^{a_1}\nu_0(t)t^{n-1}dt \lesssim \int_{B_{a_1}(\overline{0})}\nu_0(|x|)\,dx,$$

i.e., (6.1) follows. Using (6.1) and (6.2) we get

$$\mathcal{L} = \sup_{a>0}\sup_{|E|=|W|=a}\frac{1}{a^{2+\frac{1}{p}-\frac{1}{q}-\frac{\gamma}{n}}}\int_W \mu_0(|x|)dx\int_E \nu_0(|x|)dx$$
$$\lesssim \sup_{r>0}\frac{1}{|B_r(\overline{0})|^{2+\frac{1}{p}-\frac{1}{q}-\frac{\gamma}{n}}}\int_{B_r(\overline{0})}\mu_0(|x|)dx\int_{B_r(\overline{0})}\nu_0(|x|)dx \leqslant \mathcal{F}.$$

Finally, in view of $\mathcal{F}\leqslant\mathcal{L}\lesssim\mathcal{F}$, the result follows from Theorem 2. □

Proof of Corollary 3. (A) Suppose (2.14) holds. Then $\|A_{\gamma,\alpha,\beta}\|_{L^{p,\tau_1}(\mathbb{R}^n)\to L^{q,\tau_2}(\mathbb{R}^n)} < \infty$ follows from Corollary 2. Indeed, for $\mu(x)=|x|^{-\alpha}$ and $\nu(x)=|x|^{-\beta}$, we have $A_{\gamma,\alpha,\beta}=A_{\gamma,\mu(\cdot),\nu(\cdot)}$. Since conditions (2.14) are equivalent to

$$\sup_{a>0}\frac{1}{a^{n(2+\frac{1}{p}-\frac{1}{q}-\frac{\gamma}{n})}}\int_0^a \nu_0(r)r^{n-1}dr\int_0^a \mu_0(r)r^{n-1}dr<\infty$$

and

$$\frac{\alpha}{n}+\frac{1}{p}>\frac{\gamma}{n},\qquad \frac{\beta}{n}+\frac{1}{q'}>\frac{\gamma}{n},\qquad\qquad \text{(cf. (2.9))},$$

(2.13) implies that $A_{\gamma,\alpha,\beta}: L^{p,\tau_1}(\mathbb{R}^n)\to L^{q,\tau_2}(\mathbb{R}^n)$.

Let now the operator $A_{\gamma,\alpha,\beta}$ be bounded from $L^{p,\tau_1}(\mathbb{R}^n)$ to $L^{q,\tau_2}(\mathbb{R}^n)$. Then (2.13) gives that $\gamma=\alpha+\beta+n\left(\frac{1}{p}-\frac{1}{q}\right)$.

First, we assume that $1<\tau_1\leqslant\tau_2\leqslant\infty$ and $\tau_1\neq\infty$. Let us show that $\alpha<n/p'$ and $\beta<n/q$. Using [NT2, Th. 4.1] and taking into account that $(L^{p,\tau_1}(\mathbb{R}^n))'=L^{p',\tau_1'}(\mathbb{R}^n)$, we get

$$1\gtrsim \|A_{\gamma,\alpha,\beta}\|_{L^{p,\tau_1}(\mathbb{R}^n)\to L^{q,\tau_2}(\mathbb{R}^n)}$$
$$\gtrsim \|A_{\gamma,\alpha,\beta}\|_{L^{p,\tau_1}(\mathbb{R}^n)\to L^{q,\infty}(\mathbb{R}^n)} \asymp \sup_{|E|>0}\frac{1}{|E|^{\frac{1}{q'}}}\left\|\int_E \frac{1}{|y|^\beta|x-y|^{n-\gamma}|x|^\alpha}\,dy\right\|_{L^{p',\tau_1'}(\mathbb{R}^n)}.$$

We further choose E to be $B_1(\overline{0})$ or $B_1(\overline{3})$ and we get

$$1\gtrsim \max\left\{\frac{1}{|B_1(\overline{0})|^{\frac{1}{q'}}}\left\|\chi_{\{x:|x|>2\}}(x)\int_{B_1(\overline{0})}\frac{1}{|y|^\beta|x-y|^{n-\gamma}|x|^\alpha}\,dy\right\|_{L^{p',\tau_1'}(\mathbb{R}^n)},\right.$$

$$\frac{1}{|B_1(3)|^{\frac{1}{q'}}}\left\|\chi_{\{x:|x|<1\}}(x)\int\limits_{B_1(\overline{3})}\frac{1}{|y|^\beta|x-y|^{n-\gamma}|x|^\alpha}\,dy\right\|_{L^{p',\tau_1'}(\mathbb{R}^n)}\Bigg\}$$

$$\gtrsim \max\left\{\left\|\chi_{\{x:|x|>2\}}(x)|x|^{\gamma-n-\alpha}\right\|_{L^{p',\tau_1'}(\mathbb{R}^n)},\left\|\chi_{\{x:|x|<1\}}(x)|x|^{-\alpha}\right\|_{L^{p',\tau_1'}(\mathbb{R}^n)}\right\}$$

$$\gtrsim \max\left\{\left(\int_1^\infty t^{(1/p'-1+\gamma/n-\alpha/n)\tau_1'}\frac{dt}{t}\right)^{\frac{1}{\tau_1'}},\left(\int_0^1 t^{(1/p'-\alpha/n)\tau_1'}\frac{dt}{t}\right)^{\frac{1}{\tau_1'}}\right\}.$$

Taking into account that $\tau_1' < \infty$ and $\gamma = \alpha+\beta+n\left(\frac{1}{p}-\frac{1}{q}\right)$, we have that convergence of integrals implies $\beta < n/q$ and $\alpha < n/p'$.

The case $\tau_1 = \tau_2 = \infty$ follows from the proof above and

$$\|A_{\gamma,\alpha,\beta}\|_{L^{p,\infty}(\mathbb{R}^n)\to L^{q,\infty}(\mathbb{R}^n)} \geqslant \|A_{\gamma,\alpha,\beta}\|_{L^{p,2}(\mathbb{R}^n)\to L^{q,\infty}(\mathbb{R}^n)}.$$

Let now $0 < \tau_1 \leqslant \tau_2 < \infty$. Since the function $f(x) = (\tau_1/p)^{1/\tau_1}|W|^{-1/p}\chi_W(x)$ satisfies $\|f\|_{L^{p,\tau_1}(\mathbb{R}^n)} = 1$, we get

$$\begin{aligned}1 &\gtrsim \|A_{\gamma,\alpha,\beta}\|_{L^{p,\tau_1}(\mathbb{R}^n)\to L^{q,\tau_2}(\mathbb{R}^n)}\\ &= \sup_{\|f\|_{L^{p,\tau_1}(\mathbb{R}^n)}=1}\|A_{\gamma,\alpha,\beta}f\|_{L^{q,\tau_2}(\mathbb{R}^n)}\\ &\geqslant \sup_{|W|>0}\frac{1}{|W|^{\frac{1}{p}}}\left\|\int\limits_W\frac{1}{|y|^\beta|x-y|^{n-\gamma}|x|^\alpha}\,dx\right\|_{L^{q,\tau_2}(\mathbb{R}^n)}.\end{aligned}$$

Repeating the arguments gives $\beta < n/q$ and $\alpha < n/p'$.

(B). Let $0 < \tau \leqslant 1$ and (2.15) be true. Since

$$\begin{aligned}\|A_{\gamma,\alpha,\beta}\|_{L^{p,\tau}(\mathbb{R}^n)\to L^{q,\infty}(\mathbb{R}^n)} &\asymp \|A_{\gamma,\alpha,\beta}\|_{L^{p,1}(\mathbb{R}^n)\to L^{q,\infty}(\mathbb{R}^n)}\\ &\asymp \sup_{|E|>0,|W|>0}\frac{1}{|E|^{1/q'}}\frac{1}{|W|^{1/p}}\int\limits_E\int\limits_W\frac{1}{|y|^\beta|x-y|^{n-\gamma}|x|^\alpha}\,dx\,dy,\end{aligned}$$

using Lemmas 1 and 3, we get $\|A_{\gamma,\alpha,\beta}\|_{L^{p,1}(\mathbb{R}^n)\to L^{q,\infty}(\mathbb{R}^n)} \lesssim \mathcal{D}$, where $\mathcal{D}$ is defined in Lemma 3 and $\mu(x) = |x|^{-\alpha}$, $\nu(x) = |x|^{-\beta}$, and $K(x) = |x|^{\gamma-n}$. Note that conditions (3.4)-(3.5) hold since $\alpha/n \leqslant 1/p'$ and $\beta/n \leqslant 1/q$. Finally, $\|A_{\gamma,\alpha,\beta}\|_{L^{p,1}(\mathbb{R}^n)\to L^{q,\infty}(\mathbb{R}^n)} \lesssim \mathcal{D} < \infty$ because of $\gamma = \alpha+\beta+n\left(\frac{1}{p}-\frac{1}{q}\right)$.

To prove the inverse part, we have

$$\begin{aligned}1 &\gtrsim \|A_{\gamma,\alpha,\beta}\|_{L^{p,1}(\mathbb{R}^n)\to L^{q,\infty}(\mathbb{R}^n)}\\ &\gtrsim \max\left\{\sup_{0<\delta\leqslant 1}\frac{1}{|B_\delta(\overline{0})|^{\frac{1}{q'}}|B_1(\overline{3})|^{\frac{1}{p}}}\int\limits_{B_1(\overline{3})}\int\limits_{B_\delta(\overline{0})}\frac{dxdy}{|y|^\beta|x-y|^{n-\gamma}|x|^\alpha},\right.\end{aligned}$$

$$\sup_{0<\delta\leqslant 1}\frac{1}{|B_1(\overline{3})|^{\frac{1}{q'}}|B_\delta(\overline{0})|^{\frac{1}{p}}}\int\limits_{B_\delta(\overline{0})}\int\limits_{B_1(\overline{3})}\frac{dxdy}{|y|^\beta|x-y|^{n-\gamma}|x|^\alpha}\Bigg\}$$

$$\gtrsim \max\Bigg\{\sup_{0<\delta\leqslant 1}\frac{1}{|B_\delta(\overline{0})|^{\frac{1}{q'}}}\int\limits_{B_\delta(\overline{0})}\frac{1}{|y|^\beta}\,dy,\ \sup_{0<\delta\leqslant 1}\frac{1}{|B_\delta(\overline{0})|^{\frac{1}{p}}}\int\limits_{B_\delta(\overline{0})}\frac{1}{|x|^\alpha}\,dx\Bigg\}.$$

This gives $\alpha/n\leqslant 1/p'$ and $\beta/n\leqslant 1/q$. Necessity of the condition $\gamma=\alpha+\beta+n\left(\frac{1}{p}-\frac{1}{q}\right)$ follows from

$$\begin{aligned}1 &\gtrsim \|A_{\gamma,\alpha,\beta}\|_{L^{p,1}(\mathbb{R}^n)\to L^{q,\infty}(\mathbb{R}^n)}\\ &\gtrsim \sup_{0<\delta<\infty}\frac{1}{|B_\delta(\overline{0})|^{\frac{1}{q'}}|B_\delta(\overline{0})|^{\frac{1}{p}}}\int\limits_{B_\delta(\overline{0})}\int\limits_{B_\delta(\overline{0})}\frac{dxdy}{|y|^\beta|x-y|^{n-\gamma}|x|^\alpha}\end{aligned}$$

and the homogeneity argument. □

7. Final remarks

1. Theorem A provides sufficient conditions for weighted Young's inequality $L^p_\alpha(\mathbb{R}^n)*L^\theta_s(\mathbb{R}^n)\subset L^q_{-\beta}(\mathbb{R}^n)$ to hold. Certain necessary conditions, which differ from sufficient ones, were obtained by Bui [Bu1, Th. 2.1]. He posed the problem ([Bu1, p.32]) of finding necessary and sufficient conditions for weighted Young's inequality. We give a solution of this problem.

Theorem A′. *Suppose* $1<p,q<\infty$ *and* $1<\theta\leqslant\infty$. *Then*

(7.1) $$L^p_\alpha(\mathbb{R}^n)*L^\theta_s(\mathbb{R}^n)\subset L^q_{-\beta}(\mathbb{R}^n),\qquad \frac{1}{q}\leqslant\frac{1}{p}+\frac{1}{\theta},$$

if and only if

(7.2) $$\frac{1}{q}=\frac{1}{p}+\frac{1}{\theta}+\frac{\alpha+\beta+s}{n}-1,$$

(7.3) $$\alpha<n/p',\quad \beta<n/q,\quad s<n/\theta'\qquad \textit{if}\quad \theta<\infty$$

(7.3)′ $$\alpha<n/p',\quad \beta<n/q,\quad 0<s<n\qquad \textit{if}\quad \theta=\infty$$

and

(7.4) $$\alpha+\beta\geqslant 0,\quad \alpha+s\geqslant 0,\quad \beta+s\geqslant 0.$$

Remark. Necessity of condition (7.4) was recently proved in [MV, Rem. 4.4] and [DD, Rem. 2.1]. Our proof uses different argument. Note also that in case of $\theta=\infty$ condition (7.4) is equivalent to the following condition:

(7.4)′ $$\alpha+\beta\geqslant 0.$$

This follows from (7.2) and (7.3)$'$.

Proof. Sufficiency for the case $\theta < \infty$ was proved by Kerman (Theorem A). For the case $\theta = \infty$ see [Ke, Th. 3.4], [Bu1, Th. 2.2 (ii)], and [Be, Sect. 3].

Let us prove the necessity part. First, by scaling argument, we have necessity of condition (7.2) (see [Bu1, Th. 2.1]).

Second, let $1 < \theta \leqslant \infty$ and (7.1) hold true. Take $\widetilde{K}(x) = K(x)|x|^s \in L^\theta(\mathbb{R}^n)$, i.e., $K \in L^\theta_s(\mathbb{R}^n)$. Then the operator

$$A_{\tilde{K},\alpha,\beta,s} f(y) = \frac{1}{|y|^\beta} \int\limits_{\mathbb{R}^n} \frac{\widetilde{K}(x-y) f(x)}{|x-y|^s |x|^\alpha}\, dx$$

is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Using [NT2, Th. 4.1], we get

$$\begin{aligned}\|A_{\tilde{K},\alpha,\beta,s}\|_{L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n)} &\geqslant \|A_{\tilde{K},\alpha,\beta,s}\|_{L^p(\mathbb{R}^n)\to L^{q,\infty}(\mathbb{R}^n)} \\ &\asymp \sup_{|E|>0} \frac{1}{|E|^{\frac{1}{q'}}} \Big\| \int\limits_E \frac{\widetilde{K}(x-y)}{|y|^\beta |x-y|^s |x|^\alpha}\, dy \Big\|_{(L^p(\mathbb{R}^n))'} \\ &\gtrsim \sup_{|E|>0} \frac{1}{|E|^{\frac{1}{q'}}} \left(\int\limits_{\mathbb{R}^n} \Big| \int\limits_E \frac{\widetilde{K}(x-y)}{|y|^\beta |x-y|^s |x|^\alpha}\, dy \Big|^{p'} dx \right)^{1/p'}.\end{aligned}$$

Let an integer N be sufficiently large. Put

$$\widetilde{K}(x) := \chi_{B_2(\overline{N})}(x) \in L^\theta(\mathbb{R}^n).$$

Then

$$\begin{aligned}1 &\gtrsim \|A_{\tilde{K},\alpha,\beta,s}\|_{L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n)} \\ &\gtrsim \frac{1}{|B_1(-\overline{N})|^{\frac{1}{q'}}} \left(\int\limits_{B_1(\overline{0})} \Big| \int\limits_{B_1(-\overline{N})} \frac{\tilde{K}(x-y)}{|y|^\beta |x-y|^s |x|^\alpha}\, dy \Big|^{p'} dx \right)^{1/p'} \\ &\gtrsim \left(\int\limits_{B_1(\overline{0})} |x|^{-\alpha p'} dx \right)^{1/p'} \int\limits_{B_1(\overline{N})} \frac{dy}{|y|^{s+\beta}}.\end{aligned}$$

In the last expression, the first integral converges only if $\alpha < n/p'$ while the second integral is equivalent to $N^{-(\beta+s)}$ which is bounded by a constant only if $\beta + s \geqslant 0$.

Taking into account that

$$\|A_{\tilde{K},\alpha,\beta,s}\|_{L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n)} = \|A_{\tilde{K},\beta,\alpha,s}\|_{L^{q'}(\mathbb{R}^n)\to L^{p'}(\mathbb{R}^n)},$$

we also get that $\alpha + s \geqslant 0$ and $\beta < n/q$.

To prove necessity of conditions $\alpha+\beta \geqslant 0$ and $s < n/\theta'$, we consider the operator

$$\left(A_{\tilde{f},\alpha,\beta,s}K\right)(y) = \frac{1}{|y|^\beta} \int\limits_{\mathbb{R}^n} \frac{\tilde{f}(x-y)K(x)}{|x-y|^\alpha |x|^s}\, dx$$

from $L^\theta(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ with the kernel

$$\frac{\tilde{f}(x-y)}{|y|^\beta |x-y|^\alpha |x|^s},$$

where $\tilde{f}(x) = f(x)|x|^\alpha \in L^p(\mathbb{R}^n)$, and proceed similarly.

Finally, we have to show that if $\theta=\infty$, then $s>0$. Let $L^p_\alpha(\mathbb{R}^n) * L^\infty_s(\mathbb{R}^n) \subset L^q_{-\beta}(\mathbb{R}^n)$, $1<p\leqslant q<\infty$. We define

$$K(x) := |x|^{-s} \in L^\infty_s(\mathbb{R}^n), \qquad f(x) := \chi_{\mathbb{R}^n \setminus B_2(\overline{0})}(x)|x|^{-\alpha-n/p}(\ln|x|)^{-2/p} \in L^p_\alpha(\mathbb{R}^n).$$

Taking into account that $\alpha < n/p'$, we notice that s has to be positive in order that the integral

$$(K*f)(y) = \int\limits_{\mathbb{R}^n \setminus B_2(\overline{0})} \frac{1}{|y-x|^s |x|^{\alpha+n/p}(\ln|x|)^{2/p}} dx$$

converges. □

2. The Plancherel-Polya-Nikol'skii inequality plays an important role in the theory of function spaces (see, e.g., [MV, Tr]) and approximation theory (see, e.g., [DL, DT]). The following result provides sufficient conditions for a two-weight version of this inequality, partially answering the question posed in [MV, Rem. 4.2]. Let $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse.

Corollary 4. *Let p, q, r, μ, and ν satisfy conditions of Theorem 1. Let Q_d is a rectangular parallelepiped in $\mathbb{R}^n$ with edges of lengths $d = (d_1, ..., d_n)$. If supp $(\mathcal{F}f) \subseteq Q_d$, then*

$$\|f\|_{L^q(\nu;\mathbb{R}^n)} \lesssim \mathcal{G} \left(\prod_{j=1}^n d_j\right)^{1/r'} \|f\|_{L^p(\mu^{-1};\mathbb{R}^n)},$$

where $\mathcal{G}$ is given by (2.5).

Remark. For the power weights necessary and sufficient conditions for the Plancherel-Polya-Nikol'skii inequality were proved in [MV], see Proposition 4.1 and Remark 4.2. Plancherel-Polya-Nikol'skii's inequality with A_∞ weights were studies in [Bu2, MT]. The weighted version of Nikol'skii's inequality for trigonometric polynomials was also studied in [NT2, Cor. 5.2].

The proof of Corollary 4 immediately follows from the following result.

Corollary 5. *Let Q_d be a rectangular parallelepiped in $\mathbb{R}^n$ with edges of lengths $d = (d_1, \dots, d_n)$ and*

$$S_{Q_d}(f) := \mathcal{F}^{-1}\chi_{Q_d}\mathcal{F}f.$$

(A). *Let p, q, r and μ, ν satisfy conditions of Theorem 1. If $\mathcal{G} < \infty$, where $\mathcal{G}$ is defined by (2.5), then*

$$\|S_{Q_d}\|_{L^p(\mu^{-1};\mathbb{R}^n)\longrightarrow L^q(\nu;\mathbb{R}^n)} \leqslant C\mathcal{G}\left(\prod_{j=1}^n d_j\right)^{1/r'}, \tag{7.5}$$

where C is independent of d.

(B). *Let $1 < p, q, r < \infty$. If*

$$\|S_{Q_d}\|_{L^p(\mu^{-1};\mathbb{R}^n)\to L^q(\nu;\mathbb{R}^n)} < \infty,$$

then

$$\mathcal{K} := \sup_{\frac{\pi}{4d}\leqslant t\leqslant\frac{\pi}{2d}} \sup_{Q_t} \frac{\nu(Q_t)\mu(Q_t)}{|Q_t|^{1+\frac{1}{r}+\frac{1}{p}-\frac{1}{q}}} < \infty \tag{7.6}$$

and

$$\|S_{Q_d}\|_{L^p(\mu^{-1};\mathbb{R}^n)\to L^q(\nu;\mathbb{R}^n)} \geqslant C\mathcal{K}\left(\prod_{j=1}^n d_j\right)^{1/r'}, \tag{7.7}$$

where C is independent of d.

Proof. Taking into account that $S_{Q_d}(f) = (\mathcal{F}^{-1}\chi_{Q_d}) * f$, and

$$\mathcal{F}^{-1}\chi_{Q_d}(x) = \prod_{j=1}^n \frac{\sin(d_j x_j/2)}{x_j/2},$$

we get from Theorem 1 that

$$\begin{aligned}\|S_{Q_d}(f)\|_{L^q(\nu;\mathbb{R}^n)} &\lesssim \mathcal{G}\left\|\prod_{j=1}^n \frac{\sin(d_j x_j/2)}{x_j/2}\right\|_{L^{r,\infty}(\mathbb{R}^n)} \|f\|_{L^p(\mu^{-1};\mathbb{R}^n)} \\ &\lesssim \mathcal{G}\left(\prod_{j=1}^n d_j\right)^{\frac{1}{r'}} \|f\|_{L^p(\mu^{-1};\mathbb{R}^n)},\end{aligned}$$

i.e., (7.5) follows.

Let now $\|S_{Q_d}\|_{L^p(\mu^{-1};\mathbb{R}^n)\to L^q(\nu;\mathbb{R}^n)} < \infty$. Then, defining the operator

$$S_{Q_d,\mu,\nu}(f, y) = \nu(y)\int_{\mathbb{R}^n} \prod_{j=1}^n \frac{\sin(d_j(y_j - x_j)/2)}{(y_j - x_j)/2}\mu(x)f(x)dx,$$

we get

$$\|S_{Q_d}\|_{L^p(\mu^{-1};\mathbb{R}^n)\to L^q(\nu;\mathbb{R}^n)} = \|S_{Q_d,\mu,\nu}\|_{L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n)} \gtrsim \|S_{Q_d,\mu,\nu}\|_{L^{p,1}(\mathbb{R}^n)\to L^{q,\infty}(\mathbb{R}^n)}$$

$$\asymp \sup_{|E|>0,|W|>0} \frac{1}{|E|^{1/q'}} \frac{1}{|W|^{1/p}} \left| \int_E \nu(y) \int_W \mu(x) \prod_{j=1}^n \frac{\sin(d_j(y_j-x_j)/2)}{(y_j-x_j)/2} dxdy \right|.$$

Since for $x, y \in Q_t$ we have that $y - x \in Q_{2t}(\overline{0})$, where $Q_{2t}(\overline{0})$ is the rectangular parallelepiped with edges of lengths $2t = (2t_1, ..., 2t_n)$ centered at $\overline{0} = (0, \dots, 0)$, then

$$\begin{aligned} &\|S_{Q_d}\|_{L^p(\mu^{-1};\mathbb{R}^n)\to L^q(\nu;\mathbb{R}^n)} \gtrsim \\ &\gtrsim \sup_{\frac{\pi}{4d}\leqslant t\leqslant \frac{\pi}{2d}} \sup_{Q_t} \frac{1}{|Q_t|^{\frac{1}{p}+\frac{1}{q'}}} \int_{Q_t} \nu(y) \int_{Q_t} \mu(x) \prod_{j=1}^n \frac{\sin(d_j(y_j-x_j)/2)}{(y_j-x_j)/2} dxdy \\ &\gtrsim \prod_{j=1}^n d_j \sup_{\frac{\pi}{4d}\leqslant t\leqslant \frac{\pi}{2d}} \sup_{Q_t} \frac{1}{|Q_t|^{\frac{1}{p}+\frac{1}{q'}}} \int_{Q_t} \nu(y) \int_{Q_t} \mu(x) dxdy \\ &\gtrsim \left(\prod_{j=1}^n d_j\right)^{1/r'} \sup_{\frac{\pi}{4d}\leqslant t\leqslant \frac{\pi}{2d}} \sup_{Q_t} \frac{\nu(Q_t)\mu(Q_t)}{|Q_t|^{1+\frac{1}{r}+\frac{1}{p}-\frac{1}{q}}}. \end{aligned}$$

□

For power weights $\mu(x) = |x|^{-\alpha}$ and $\nu(x) = |x|^{-\beta}$, we obtain necessary and sufficient conditions for boundedness of the operator S_{Q_d}.

Corollary 6. *Let $1 < p, q, s < \infty$ and $\alpha \geqslant 0, \beta \geqslant 0$. Then*

$$\|S_{Q_d}(f)\|_{L^q_{-\beta}(\mathbb{R}^n)} \leqslant C \left(\prod_{j=1}^n d_j\right)^{1/s} \|f\|_{L^p_\alpha(\mathbb{R}^n)} \quad \forall \quad d_1, \dots, d_n > 0, \tag{7.8}$$

where Q_d is a rectangular parallelepiped in $\mathbb{R}^n$ with edges of lengths $d = (d_1, \dots, d_n)$ and C is independent of $d_1, \dots, d_n$, if and only if

$$\frac{1}{s} = \frac{1}{p} - \frac{1}{q} + \frac{\alpha+\beta}{n}, \quad \alpha < n/p', \quad \beta < n/q. \tag{7.9}$$

Proof. Let first (7.9) hold. It is easy to see that all conditions of Theorem 1 are fulfilled and $\mathcal{G} \lesssim 1$. Therefore, Corollary 5 (A) gives (7.8).

On the other hand, (7.8) implies

$$\sup_{d_1,\dots,d_n>0} \left(\prod_{j=1}^n d_j\right)^{-1/s} \|S_{Q_d}\|_{L^p_\alpha(\mathbb{R}^n)\to L^q_{-\beta}(\mathbb{R}^n)} \lesssim 1.$$

Then using (7.7) with $r = s'$ and $d_1 = \dots = d_n = z$ we arrive at $z^{n(\frac{1}{p}-\frac{1}{q}-\frac{1}{s})+\alpha+\beta} \lesssim \mathcal{K} \lesssim 1$ for any $z > 0$. This yields $\frac{1}{s} = \frac{1}{p} - \frac{1}{q} + \frac{\alpha+\beta}{n}$.

Using a method similar to the proof of Corollary 3, one can verify that conditions $\alpha < n/p'$ and $\beta < n/q$ hold. Let us show, for example, that $\beta < n/q$. Indeed, let $C_\xi(\overline{z})$ be a cube with the edge length ξ centered at $\overline{z} = (z_1, \dots, z_n)$.

Fix $d_1, \dots, d_n > 0$ and set $d_0 := \max_{1\leqslant i\leqslant n} d_i$. Notice that if $x \in C_\delta(\overline{0})$, where $\delta < \pi/4d_0$, and $y \in C_{\pi/(4d_0)}(\overline{\pi/d})$, then $\frac{\pi}{4} \leqslant \frac{d_i|x_i-y_i|}{2} \leqslant \frac{3\pi}{4}$, $i = 1, \dots, n$. Therefore,

$$\begin{aligned}
&\|S_{Q_d}\|_{L^p(\mu^{-1};\mathbb{R}^n)\to L^q(\nu;\mathbb{R}^n)} \gtrsim \\
&\gtrsim \left|C_{\pi/(4d_0)}(\overline{\pi/d})\right|^{-\frac{1}{p}} \Big\| |y|^{-\beta} \int\limits_{C_{\pi/(4d_0)}(\overline{\pi/d})} |x|^{-\alpha} \prod_{j=1}^{n} \frac{\sin(d_j(y_j-x_j)/2)}{(y_j-x_j)/2} dx \Big\|_{L_q(\mathbb{R}^n)} \\
&\geqslant \left|C_{\pi/(4d_0)}(\overline{\pi/d})\right|^{-\frac{1}{p}} \left(\int\limits_{C_\delta(\overline{0})} \left(|y|^{-\beta} \int\limits_{C_{\pi/(4d_0)}(\overline{\pi/d})} |x|^{-\alpha} \prod_{j=1}^{n} \frac{\sin(d_j(y_j-x_j)/2)}{(y_j-x_j)/2} dx \right)^q dy \right)^{1/q} \\
&\gtrsim \left(\int\limits_{C_\delta(\overline{0})} |y|^{-\beta q} dy \right)^{1/q}.
\end{aligned}$$

This implies $\beta < n/q$. □

References

[Be] W. Beckner, *Pitt's inequality with sharp convolution estimates* Proc. Amer. Math. Soc., **136** (2008), no.5, 1871–1885.

[BS] C. Bennett, R. Sharpley, *Interpolation of Operators.* Academic Press, 1988.

[BGT] N. H. Bingham, C. M. Goldie, J. L. Teugels, *Regular Variation.* Cambridge University Press, 1989.

[Bu1] H.-Q. Bui, *Weighted Young's inequality and convolution theorems on weighted Besov spaces.* Math. Nachr., **170** (1994), 25–37.

[Bu2] H.-Q. Bui, *Weighted Besov and Triebel spaces: Interpolation by the real method.* Hiroshima Math. J., **12** (1982), no. 3581–605.

[COV] C. Cascante, J. Ortega, I. Verbitsky, *Trace inequalities of Sobolev type in upper triangle.* Proc. London Math. Soc., **80** (2000), 391–414.

[DL] R. A. DeVore, G. G. Lorentz, *Constructive Approximation*, Springer-Verlag, 1993.

[DD] P. L. De Nápoli, I. Drelichman, *Weighted convolution inequalities for radial functions.* `arXiv:1210.1206,2012`

[DT] Z. Ditzian, S. Tikhonov, *Ul'yanov and Nikol'skii-type inequalities.* J. Approx. Theory, **133** (2005), no. 1, 133.

[Du] J. Duoandikoetxea, *Fractional integrals on radial functions with applications to weighted inequalities.* Annali Mat. Pura Appl., `DOI:10.1007/s10231-011-0237-7`.

[GK] M. Gabidzashvili, V. Kokilashvili, *Two weight weak type inequalities for fractional type integrals.* Ceskoslovenska Akademie Ved., **45** (1989), 1–11.

[Ke] R. A. Kerman, *Convolution theorems with weights.* Trans. Amer. Math. Soc., **280** (1983), 207–219.

[LMPT] M. Lacey, K. Moen, C. Pérez, R. H. Torres, *Sharp weighted bounds for fractional integral operators.* J. Funct. Anal., **259** (2010), no. 5, 1073–1097.

CRM Preprint Series number 1163

[MT] G. Mastroianni, V. Totik, *Weighted polynomial inequalities with doubling and A_∞ weights*. Constr. Approx., **16** (2000), no. 1, 37–71.

[Ma] V. G. Maz'ya, *Sobolev Spaces*. Springer-Verlag, Berlin, 1985.

[MN] V. G. Maz'ya, Y. Netrusov, *Some counterexamples for the theory of Sobolev spaces on bad domains*. Potential Anal., **4** (1995), 47–65.

[MV] M. Meyries, M. Veraar, *Sharp embedding results for spaces of smooth functions with power weights*. Studia Mathematica, **208** (2012), 257–293.

[MW] B. Muckenhoupt, R. Wheeden, *Weighted norm inequalities for fractional integrals*. Trans. Amer. Math. Soc., **192** (1974), 261–274.

[NT1] E. Nursultanov, S. Tikhonov, *Convolution inequalities in Lorentz spaces*. J. Fourier Anal. Appl., **17** (2011), 486–505.

[NT2] E. Nursultanov, S. Tikhonov, *Net spaces and boundedness of integral operators*. J. Geom. Anal., **21** (2011), 950–981.

[ON] R. O'Neil, *Convolution operators and $L(p, q)$ spaces*. Duke Math. J., **30** (1963), 129–142.

[Pe] C. Pérez, *Two weighted inequalities for potential and fractional type maximal operators*. Indiana Univ. Math. J., **43** (1994), 2663–683.

[Sa1] E. Sawyer, *A two weight weak type inequality for fractional integrals*. Trans. Amer. Math. Soc., **281** (1984), 339–345.

[Sa2] E. Sawyer, *A characterization of a two-weight norm inequality for fractional and Poisson integrals*. Trans. Amer. Math. Soc. **308** (1988), 2533–545.

[SaWh] E. Sawyer, R. Wheeden, *Weighted inequlities for fractional integrals on Euclidean and homogeneous spaces*. Amer. J. Math, **114** (1992), 813–874.

[So] S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*. Providence, RI, 1991.

[SW] E. Stein, G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*. Princeton, N.J., 1971.

[SW1] E. Stein, G. Weiss, *Fractional integrals on n-dimensional Euclidean space*. J. Math. Mech., **7** (1958), 503–514.

[Tr] H. Triebel, *Theory of Function Spaces*, Birkhäuser Verlag, Basel, 1983.

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