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1. Introduction

Let $1 \leqslant p < \infty$ and $0 < r \leqslant \infty$. The Lorentz space $L_{p,r}[0,1]$ is the set of all measurable functions defined on $[0,1]$ satisfying: for $0 < r < \infty$,

$$\|f\|_{L_{p,r}} = \left(\int_0^1 \left(t^{\frac{1}{p}} f^*(t) \right)^r \frac{dt}{t} \right)^{\frac{1}{r}},$$

and for $r = \infty$,

$$\|f\|_{L_{p,\infty}} = \sup_t t^{\frac{1}{p}} f^*(t).$$

Here

$$f^*(t) = \inf\{\sigma : \mu\{x : |f(x)| > \sigma\} \leqslant t\}$$

is a non-increasing rearrangement of the function $f(x)$.

The Hardy-Littlewood inequality for the Lorentz spaces $L_{p,r}$ was proved by Stein (see [1],[2]).

Let $2 < p < \infty$, $0 \leqslant r \leqslant \infty$, and $f \overset{\text{a.e.}}{=} \sum_{k=1}^{\infty} a_k \varphi_k(x)$, where $\Phi = \{\varphi_k\}_{k=1}^{\infty}$ is an orthonormal system of uniformly bounded functions. Then we have

$$\|f\|_{L_{p,r}} \leqslant C \left(\sum_{k=1}^{\infty} k^{\frac{r}{p'}} (a_k^*)^r \frac{1}{k} \right)^{1/r}.$$

If $1 < p < 2$, $0 \leqslant r \leqslant \infty$, then we have

$$\left(\sum_{k=1}^{\infty} k^{\frac{r}{p'}} (a_k^*)^r \frac{1}{k} \right)^{1/r} \leqslant C \|f\|_{L_{p,r}}.$$

Bochkarev [3] showed that if we formally substitute $p = 2$ in these inequalities then these relations are not true. In this paper, we prove the following statement.

Theorem. *Let $\{\varphi_k\}_{k=1}^{\infty}$ be an orthonormal system of complex-valued functions on $[0,1]$, such that*

$$\sup_{x \in [0,1]} |\varphi_k(x)| \leqslant M, \ k = 1, 2, \dots$$

Let $f \in L_{2,r}, \quad 2 < r \leqslant \infty$. *Then the inequality*

$$\sup_{n\in\mathbb{N}} \frac{1}{|n|^{\frac{1}{2}}(\log(n+1))^{\frac{1}{2}-\frac{1}{r}}} \sum_{k=1}^{n} a_k^* \leqslant C\|f\|_{L_{2,r}}. \tag{1}$$

holds, where a_n *are the Fourier coefficients of the system* $\{\varphi_k\}_{k=1}^{\infty}$.

Definition. *An orthonormal system* $\Phi = \{\varphi_k\}_{k=1}^{\infty}$ *is called to be a regular system if there is a constant B, such that*

1) *for any set* $e \subset [0,1]$ *and* $k \in \mathbb{N}$, *we have*

$$\left|\int_e \varphi_k(x)dx\right| \leqslant B\min(|e|, 1/k); \tag{2}$$

2) *for any set* $\omega \subset \mathbb{N}$ *and* $t \in (0,1]$, *we have*

$$\left(\sum_{k\in\omega} \varphi_k(\cdot)\right)^*(t) \leqslant B\min(|\omega|, 1/t), \tag{3}$$

where $\left(\sum_{k\in\omega} \varphi_k(\cdot)\right)^*(t)$ *is a non-increasing rearrangement of* $\sum_{k\in\omega}\varphi_k(x)$, $|e|$ *is the length of the segment* e, $|\omega|$ *is the number of elements in* ω.

It can be seen that a regular uniformly bounded system, as in (2) satisfies

$$\sup_{x\in[0,1]} |\varphi_n(x)| \leqslant B, n = 1, 2, \dots$$

At the same time, regular systems cover all the trigonometric systems, the multiplicative systems with limited generator. The definition of a regular system was introduced by Nursultanov [4].

Let M be a fixed net in $\mathbb{Z}$ containing all singletons. For a finite set A of $\mathbb{Z}$ we define $[A]_M$ by

$$[A]_M = \min\{l : A = \bigcup_{k=1}^{l} I_k, I_k \in M\}.$$

Theorem ([5]). *Let M be the set of all segments of* $\mathbb{Z}$. *Then for any function* $f \in L_{2,\infty}$, *we have the inequality*

$$\sup_{A} \frac{1}{|A|^{\frac{1}{2}}(\ln(1+[A]_M))^{\frac{1}{2}}} \left|\sum_{m\in A} \hat{f}(m)\right| \leqslant C\|f\|_{L_{2,\infty}}, \tag{4}$$

where C *does not depend on the choice of* f *in* $L_{2,\infty}$.

In this paper, we study the analogues of the Hardy Littlewood series for regular systems in Lorentz spaces $L_{2,r}, \quad l_{2,r}, L_{2,r,q}$.

Let M be a set in $\mathbb{Z}$. For a finite set $A \subset \mathbb{Z}$ we define $\langle A\rangle_M$ by

$$\langle A\rangle_M = \min\{l : A = \bigsqcup_{k=1}^{l} I_k, I_k \in M\}.$$

Here the disjoint union is the union of disjoint sets, i.e,

$$\bigsqcup_{k=1}^{l} I_k = \bigcup_{k=1}^{l} I_k, I_k \cap I_j = \varnothing,$$

for $k \neq j$.

The number $\langle A\rangle_M$ characterizes the structural properties of A, namely, it is the union of sets in M.

The main result of this paper is as follows.

Theorem 1. *Let M be the set of all arithmetic progressions in $\mathbb{Z}$. Then for any function $f \in L_{2,r}$, $2 < r < \infty$, we have the inequality*

$$\sup_{A\subset\mathbb{Z}} \frac{1}{|A|^{1/2}\log_2(1+\langle A\rangle_M)^{\frac{1}{2}-\frac{1}{r}}} \left|\sum_{m\in A} \hat{f}(m)\right| \leqslant 24B\|f\|_{L_{2,r}}. \tag{5}$$

Note that the inequality (1) is equivalent to the inequality

$$\sup_{A\subset\mathbb{Z}} \frac{1}{|A|^{\frac{1}{2}}\log_2(1+|A|)^{\frac{1}{2}-\frac{1}{r}}} \left|\sum_{m\in A} \hat{f}(m)\right| \leqslant C\|f\|_{L_{2,r}},$$

hence (5) implies (1).

The converse is not true. If $\{a_k\}_{k=1}^{\infty}$ is a monotone sequence, then we have

$$\sup_{A\subset\mathbb{Z}} \frac{1}{|A|^{\frac{1}{2}}\log_2(1+\langle A\rangle_M)^{\frac{1}{2}-\frac{1}{r}}} \left|\sum_{k\in A} a_k\right| = \sup_{A\subset\mathbb{Z}} \frac{1}{|A|^{\frac{1}{2}}} \left|\sum_{k\in A} a_k\right| \leqslant 24B\|f\|_{L_{2,r}}.$$

Now we take $\{a_k\}_{k=1}^{\infty}$ to be the following sequence:

$$a_k = \frac{(\ln k)^{\frac{1}{2}-\frac{1}{r}}}{k^{\frac{1}{2}}}.$$

Then

$$\sup_{n\in\mathbb{N}} \frac{1}{|n|^{\frac{1}{2}}(\log(n+1))^{\frac{1}{2}-\frac{1}{r}}} \sum_{k=1}^{n} \frac{(\ln k)^{\frac{1}{2}-\frac{1}{r}}}{k^{\frac{1}{2}}} < \infty$$

but we have

$$\sup_{A\subset\mathbb{Z}} \frac{1}{|A|^{\frac{1}{2}}\ln(1+\langle A\rangle_M)^{\frac{1}{2}-\frac{1}{r}}} \sum_{k\in A} \frac{(\ln k)^{\frac{1}{2}-\frac{1}{r}}}{k^{\frac{1}{2}}} = +\infty$$

Theorem 2. *Let $G_n = \{A = \bigsqcup_{k=1}^{m} [a_k, b_k] : 1 \leqslant m \leqslant n\}$, $2 < r < \infty$ and let $\Phi = \{\varphi_k\}_{k=1}^{\infty}$ be a regular system, and $a = \{a_k\}_{k=1}^{\infty} \in l_{2,r}$. Then the following inequality holds:*

$$\sup_{A\subset G_n} \frac{1}{|A|^{1/2}(\ln(n+1))^{1/2-1/r}} \left|\int_A f(x)dx\right| \leqslant 24B\|a\|_{l_{2,r}}.$$

To prove Theorem 1, we establish the following lemma.

Lemma 1. *Let* $\frac{1}{4} < q < 2 < r < 4$, *let* $\Phi = \{\varphi_k\}_{k=1}^{\infty}$ *be a regular system, and let* $f \overset{a.e.}{=} \sum\limits_{m=1}^{\infty} \hat{f}(m)\varphi_m(x)$. *Then for any finite subset* A *of* $\mathbb{Z}$, *the inequality*

$$\frac{1}{|A|^{1/q}} \left| \sum_{m\in A} \hat{f}(m) \right| \leqslant C\|f\|_{L_{q,r}} \tag{6}$$

holds, where

$$C = 2\sqrt{2}B\left(\frac{q}{2-q}\right)^{\frac{1}{q'}\left(\frac{r-2}{r}\right)}. \tag{7}$$

and $\hat{f}(m), m \in \mathbb{Z}$, *are the Fourier coefficients of* f.

Proof. We use the fact that a regular system is uniformly bounded,

$$\left| \sum_{m\in A} \hat{f}(m) \right| \leqslant B|A|\|f\|_{L_1},$$

and from the Parseval identity we have

$$\left| \sum_{m\in A} \hat{f}(m) \right| \leqslant B|A|^{1/2}\|f\|_{L_2}.$$

Let $\tau \in (0,1)$. Define

$$f_1(x) = \begin{cases} f(x), & |f(x)| \leqslant f^*(\tau), \\ 0, & \text{otherwise}, \end{cases}$$
$$f_0(x) = f(x) - f_1(x).$$

Then, for any $0 < \tau < 1$, we have

$$\hat{f}(m) = \hat{f}_0(m) + \hat{f}_1(m),$$

and

$$\left| \sum_{m\in A} \hat{f}(m) \right| \leqslant \left| \sum_{m\in A} \hat{f}_0(m) \right| + \left| \sum_{m\in A} \hat{f}_1(m) \right|$$
$$\leqslant B\left(|A| \int_0^{\tau} f^*(s)ds + |A|^{1/2}\left(\int_{\tau}^{1} (f^*(s))^2 ds \right)^{1/2} \right).$$

Using Holder's inequality, we obtain

$$\left| \sum_{m\in A} \hat{f}(m) \right| \leqslant B\left(|A| \left(\int_0^{\tau} (t^{\frac{1}{q}} f^*(t))^r \frac{dt}{t} \right)^{1/r} \left(\int_0^{\tau} (t^{1/q'})^{r'} \frac{dt}{t} \right)^{1/r'} + \right.$$

$$+|A|^{1/2}\left(\int_\tau^\infty (t^{\frac{1}{q}}f^*(t))^{2p}\frac{dt}{t}\right)^{\frac{1}{2p}}\left(\int_\tau^\infty (t^{1-\frac{2}{q}})^{p'}\frac{dt}{t}\right)^{\frac{1}{2p'}}\Bigg)$$

$$\leqslant B\|f\|_{L_{q,r}}\left(|A|\tau^{1-\frac{1}{q}}\left(1-\frac{1}{q}\right)^{\frac{1}{r}-1}+|A|^{1/2}\tau^{\frac{1}{2}-\frac{1}{q}}\left(\frac{r(2-q)}{q(r-2)}\right)^{1/r-1/2}\right).$$

Now let $\tau=\left(\frac{q(r-2)}{r(2-q)}\right)^{\frac{r-2}{r}}\left(\frac{q}{q-1}\right)^{\frac{2(1-r)}{r}}/|A|$ and using the $\frac{1}{4}<q<2<r<4$ then we can estimate

$$\left|\sum_{m\in A}\hat{f}(m)\right|\leqslant 2\sqrt{2}B|A|^{1/q}\left(\frac{q}{2-q}\right)^{\frac{1}{q'}\left(\frac{r-2}{r}\right)}\|f\|_{L_{q,r}}.$$

The method of the proof in this paper is based on a study of the constants in inequalities of the form (6), namely their dependence on the relevant parameters. Therefore, estimate (7) becomes important.

This approach was earlier used in [5].

Lemma 2. *Let $2<p<4$, $p'=p/(p-1)$, $2<r<\infty$, let $\Phi=\{\varphi_k\}_{k=1}^\infty$ be a regular system, and let $f\overset{a.e.}{=}\sum_{m=1}^{\infty}\hat{f}(m)\varphi_m(x)$, $A=\bigcup_{k=1}^{N}I_k$, where I_k are segments in $\mathbb{Z}$. Then we have*

$$\frac{1}{|A|^{1/p}N^{\frac{1}{p'}-\frac{1}{p}}}\left|\sum_{m\in A}\hat{f}(m)\right|\leqslant 3\sqrt{2}B\left(\frac{p-2}{p}\right)^{\frac{1}{p}\left(\frac{2-r}{r}\right)}\|f\|_{L_{p,r}}.$$

Proof. From the regularity properties we obtain

$$\left|\sum_{m\in A}\hat{f}(m)\right|=\left|\int_0^1 f(x)D_A(x)dx\right|$$

$$\leqslant\sum_{k=1}^{N}\int_0^1|f(x)|\,|D_{I_k}(x)|\,dx$$

$$\leqslant B\sum_{k=1}^{N}\int_0^1|f(x)|\min\left(|I_k|,\frac{1}{|x|}\right)dx. \tag{8}$$

From the Parseval identity we get

$$\left|\sum_{m\in A}\hat{f}(m)\right|\leqslant B|A|^{1/2}\|f\|_{L_2}. \tag{9}$$

Let $B_\tau=\{x\in[0,1]:|D_n(x)|>D_n^*(\frac{\tau}{2})\}$

Consider the function

$$f_1(x)=\begin{cases} f(x), & |f(x)|\leqslant f^*(\tau),\ x\in[0,1]\backslash B_\tau\\ 0, & \text{otherwise,}\end{cases}$$

$$f_0(x) = f(x) - f_1(x).$$

Then, using the (8) and (9) we get

$$\begin{aligned}\left|\sum_{m\in A}\hat{f}(m)\right| &\leqslant \left|\sum_{m\in A}\hat{f}_0(m)\right| + \left|\sum_{m\in A}\hat{f}_1(m)\right| \\ &\leqslant B\left(|A|^{1/2}\|f_0\|_{L_2} + \sum_{k=1}^{N}\int_0^1 |f_1(x)|\min\left(|I_k|, \frac{1}{|x|}\right)dx\right) \\ &= B(I_1 + I_2).\end{aligned}$$

We have

$$\begin{aligned}I_1 &= \left(\int_0^1 |f_0(x)|^2 dx\right)^{1/2} \\ &= \left(\int_0^{\frac{\tau}{2}} |f(x)|^2 dx + \int_{1-\frac{\tau}{2}}^{1} |f(x)|^2 dx + \int_{\frac{\tau}{2}}^{1-\frac{\tau}{2}} |f_0(x)|^2 dx\right)^{1/2} \\ &\leqslant 2\left(\int_0^{\tau} |f^*(t)|^2 dt\right)^{1/2}.\end{aligned}$$

Using that $(D_{I_k}(x)\chi_{[\frac{\tau}{2},1-\frac{\tau}{2}]}(x))^*(t) \leqslant \frac{1}{t+\tau}$, we get

$$\begin{aligned}I_2 &= \sum_{k=1}^{N}\int_0^1 |f_1(x)||D_{I_k}(x)|dx \\ &= \sum_{k=1}^{N}\int_{[0,1]\setminus B_\tau} |f_1(x)||D_{I_k}(x)|dx \\ &\leqslant N\int_0^{1-\tau} f_1^*(t)\frac{1}{t+\tau}dt \\ &\leqslant N\int_0^{1-\tau} f^*(t+\tau)\frac{dt}{t+\tau} \\ &\leqslant N\int_{\tau}^{1} f^*(t)\frac{dt}{t}.\end{aligned}$$

Therefore,

$$\left|\sum_{m\in A}\hat f(m)\right| \leqslant 2B|A|^{1/2}\left(\int_0^\tau (f^*(t))^2dt\right)^{1/2} + BN\int_\tau^1 f^*(t)\frac{dt}{t}.$$

Applying Holder's inequality, we obtain

$$\left|\sum_{m\in A}\hat f(m)\right| \leqslant 2B|A|^{1/2}\left(\int_0^\tau (f^*(t)t^{\frac{1}{p}})^r\frac{dt}{t}\right)^{1/r}\left(\int_0^\tau (t^{1-2/p})^{\frac{r}{r-2}}\frac{dt}{t}\right)^{\frac{r-2}{2r}}$$
$$+ NB\left(\int_\tau^1 \left(f^*(t)t^{1/p}\right)^r\frac{dt}{t}\right)^{1/r}\left(\int_\tau^1 t^{-r'/p}\frac{dt}{t}\right)^{1/r'}$$
$$\leqslant B\|f\|_{L_{p,r}}\left(2|A|^{1/2}\left(\frac{r(p-2)}{p(r-2)}\right)^{\frac{2-r}{2r}}(\tau)^{1/2-1/p} + Np^{\frac{r-1}{r}}(\tau)^{-1/p}\right).$$

Now let $\tau = \frac{N^2}{|A|\left(\frac{r(p-2)}{p(r-2)}\right)^{\frac{2-r}{r}}p^{2\left(\frac{1-r}{r}\right)}}$. Then we have the following estimate:

$$\left|\sum_{m\in A}\hat f(m)\right| \leqslant 3\sqrt{2}B|A|^{1/p}\left(\frac{p-2}{p}\right)^{\frac{1}{p}(\frac{2}{r}-1)}N^{\frac{1}{p'}-\frac{1}{p}}\|f\|_{L_{p,r}}.$$

Lemma 3. *Let $A = \bigcup_{k=1}^N I_k$, where I_k are segments, $\Phi = \{\varphi_k\}_{k=1}^\infty$ is a regular system, $f = \sum_{m=0}^\infty \hat f(m)\varphi_m(x)$ a.e. Then for any $2 < p < 4$ and $2 < r < \infty$ we have the inequality*

$$\frac{1}{|A|^{1/2}}\left|\sum_{m\in A}\hat f(m)\right| \leqslant C\|f\|_{L_{2,r}}, \tag{10}$$

where

$$C = 5\sqrt{2}B\left(\frac{2p}{p-2}\right)^{\frac{1}{p}(1-\frac{2}{r})}\left(\frac{p+2}{p}\right)^{1/4}N^{\frac{1}{2}-\frac{1}{p}}.$$

Proof. Let $\tau > 0$,

$$f_1(x) = \begin{cases} f(x), & f(x) \leqslant f^*(\tau)\\ 0, & \text{otherwise,}\end{cases}$$
$$f_0(x) = f(x) - f_1(x).$$

Then, using Lemma 1, 2 and $2 < r < \infty$, we get

$$\left|\sum_{m\in A}\hat f(m)\right| \leqslant \left|\sum_{m\in A}\hat f_0(m)\right| + \left|\sum_{m\in A}\hat f_1(m)\right|$$

$$\leqslant 2\sqrt{2}B\left(\int_0^\infty (t^{1/p'}f_0^*(t))^r\frac{dt}{t}\right)^{1/r}|A|^{1/p'}\left(\frac{p'}{2-p'}\right)^{\frac{1}{p}(\frac{r-2}{r})}$$

$$+3\sqrt{2}B\left(\int_0^\infty (t^{1/p}f_1^*(t))^r\frac{dt}{t}\right)^{1/r}|A|^{1/p}\left(\frac{p-2}{p}\right)^{\frac{1}{p}(\frac{2-r}{r})}N^{1/p'-1/p}$$

$$\leqslant 2\sqrt{2}B\left(\int_0^\tau (t^{1/p'}f^*(t))^r\frac{dt}{t}\right)^{1/r}|A|^{1-\frac{1}{p}}\left(\frac{2p}{p-2}\right)^{\frac{1}{p}(\frac{r-2}{r})}$$

$$+3\sqrt{2}B\left(\int_0^\infty (t^{1/p}f^*(t+\tau))^r\frac{dt}{t}\right)^{1/r}|A|^{1/p}\left(\frac{2p}{p-2}\right)^{\frac{1}{p}(\frac{r-2}{r})}N^{\frac{1}{p'}-\frac{1}{p}}.$$

$$\left|\sum_{m\in A}\hat{f}(m)\right|\leqslant\sqrt{2}B\left(\frac{2p}{p-2}\right)^{\frac{1}{p}\left(1-\frac{2}{r}\right)}\left(2|A|^{1-1/p}\left(\int_0^\tau (t^{1/2}f^*(t))^r\frac{dt}{t}\right)^{1/r}\sup_{0\leqslant t\leqslant\tau}t^{\frac{1}{2}-\frac{1}{p}}+\right.$$

$$+3|A|^{1/p}N^{\frac{1}{p'}-\frac{1}{p}}\left(\int_0^\tau (t^{1/p}(t+\tau)^{-1/2})^r\frac{dt}{t}\left(\sup_{0<s<\infty}s^{1/2}f^*(s)\right)^r\right.$$

$$\left.\left.+\int_\tau^\infty (t^{1/p}f^*(t+\tau))^r\frac{dt}{t}\right)^{1/r}\right)$$

$$\leqslant 2\sqrt{2}B|A|^{1-1/p}\left(\frac{2p}{p-2}\right)^{\frac{1}{p}\left(1-\frac{2}{r}\right)}\tau^{\left(\frac{1}{p'}-\frac{1}{2}\right)}\|f\|_{L_{2,r}}$$

$$+3\sqrt{2}B|A|^{1/p}\left(\frac{p-2}{2p}\right)^{\frac{1}{p}(\frac{2-r}{r})}\left(\frac{p+2}{2}\right)^{1/r}N^{\frac{1}{p'}-\frac{1}{p}}\tau^{(\frac{1}{p}-\frac{1}{2})}\|f\|_{L_{2,r}}.$$

Taking $\tau^{\frac{1}{2}-\frac{1}{p}}=\left(\left(\frac{N}{|A|}\right)^{1/p'-1/p}\left(\frac{p+2}{p}\right)^{1/2}\right)^{1/2}$, we obtain

$$\left|\sum_{m\in A}\hat{f}(m)\right|\leqslant 5\sqrt{2}B|A|^{1/2}N^{\frac{1}{2}-\frac{1}{p}}\left(\frac{2p}{p-2}\right)^{\frac{1}{p}(1-\frac{2}{r})}\left(\frac{p+2}{2}\right)^{1/4}\|f\|_{L_{2,r}}.$$

Proof of Theorem 1. Let $A\subset\mathbb{Z}$ such that $1\leqslant N=\langle A\rangle_M\leqslant 8$.

In the right part of the constant (10) we put $p=3$. Then

$$5\sqrt{2}B\left(\frac{2p}{p-2}\right)^{1/p(1-2/r)}\left(\frac{p+2}{2}\right)^{1/4}N^{\frac{1}{2}-\frac{1}{p}}\leqslant 20B.$$

Therefore, from Lemma 3, we get

$$\frac{1}{|A|^{1/2}\log_2(1+\langle A\rangle_M)^{\frac{1}{2}-\frac{1}{r}}}\left|\sum_{m\in A}\hat{f}(m)\right|\leqslant\frac{1}{|A|^{1/2}}\left|\sum_{m\in A}\hat{f}(m)\right|\leqslant 20B\|f\|_{L_{2,r}}.$$

Now let $A \subset \mathbb{Z}$ so that $N = \langle A\rangle_M > 8$. Put $p = \frac{2\log_2 N}{\log_2 N - 2}$, i.e. $\frac{p-2}{2p} = \frac{1}{\log_2 N}$, then $p \leqslant 6$.

Note also that $N^{\frac{1}{2}-\frac{1}{p}} = N^{\frac{1}{\log_2 N}} \leqslant 2$,

Using Lemma 3, we get

$$\frac{1}{|A|^{1/2}}\left|\sum_{m\in A}\hat f(m)\right| \leqslant 24B\log_2(1+\langle A\rangle_M)^{\left(\frac{1}{2}-\frac{1}{\log_2 N}\right)\left(1-\frac{2}{r}\right)}\|f\|_{L_{2,r}}$$
$$\leqslant 24B\log_2(1+\langle A\rangle_M)^{\frac{1}{2}-\frac{1}{r}}\|f\|_{L_{2,r}}.$$

Taking the top exact bound over all finite sets A of $\mathbb{Z}$ we obtain

$$\sup_{A\subset\mathbb{Z}}\frac{1}{|A|^{1/2}\log_2(1+\langle A\rangle_M)^{1/2-1/r}}\left|\sum_{m\in A}\hat f(m)\right| \leqslant 24B\|f\|_{L_{2,r}}$$

Theorem 1 is proved.

Theorem 2 can be proved similarly.

Definition. *Let f be a measurable function. We define $\xi = \{\xi_k\}_{k\in\mathbb{Z}}$ by*

$$\xi_k = \left(\int_{2^k}^{2^{k+1}} f^{*2}(t)dt\right)^{1/2}, k \in \mathbb{Z}.$$

We say that $f \in L_{2,r,q}$, if $\|f\|_{L_{2,r,q}} = \|\xi\|_{l_{r,q}}$, where $l_{r,q}$ is the discrete Lorentz space.

Theorem 3. *Let $2 < r < \infty$, $0 < q \leqslant \infty$, $f \sim \sum_{k=1}^{\infty} a_k\varphi_k$, $f \in L_{2,r,q}$. Then we have*

$$\left(\sum\left(n^{1/r}\left(\sup_{A\in G_n}\frac{1}{|A|^{1/2}\log_2\langle A\rangle_M^{1/2}}\left|\sum\hat f(m)\right|\right)\right)^q\frac{1}{n}\right)^{1/q} \leqslant C\|f\|_{L_{2,r,q}}.$$

with $q = r$ this becomes

$$\left(\sum\left(\sup_{A\in G_n}\frac{1}{|A|^{1/2}\log_2\langle A\rangle_M^{1/2}}\left|\sum\hat f(m)\right|\right)^r\right)^{1/r} \leqslant C\|f\|_{L_{2,r}}.$$

Proof. We define the semi-additive operator T by $Tf \sim \widetilde{a_n}$, where $\widetilde{a_n} = \sup_{A\in G_n}\frac{1}{|A|^{\frac{1}{2}}(\log_2\langle A\rangle_M)^{1/2}}\left|\sum_{m\in A}\hat f(m)\right|$ and $G_n = \{A \subset \mathbb{Z} : \langle A\rangle_M \geqslant 2^n\}$.

Let $2 < r_0 < r < r_1 < \infty$. Using Theorem 1, we obtain

$$\|Tf\|_{l_{r_0,\infty}} = \sup_n n^{\frac{1}{r_0}}\widetilde{a}_n$$

$$= \sup_n n^{1/r_0} \sup_{A\in G_n} \frac{1}{|A|^{1/2}\log_2\langle A\rangle_M^{1/2}} \left|\sum_{m\in A}\hat f(m)\right|$$

$$\leqslant \sup_n \sup_{A\in G_n} \frac{\log_2\langle A\rangle_M^{1/r_0}}{|A|^{1/2}\log_2\langle A\rangle_M^{1/2}} \left|\sum_{m\in A}\hat f(m)\right|$$

$$\leqslant \sup_n \sup_{A\in G_n} \frac{1}{|A|(\log_2\langle A\rangle)^{1/2-1/r_0}} \left|\sum_{m\in A}\hat f(m)\right| \leqslant 20B\|f\|_{L_{2,r_0}}.$$

Similarly,

$$\|Tf\|_{l_{r_1,\infty}} = \sup_n n^{\frac{1}{r_1}}\widetilde{a}_n \leqslant 20B\|f\|_{L_{2,r_1}}.$$

Using the interpolation theorem [see [6]] we obtain $(L_{2,r_0}, L_{2,r_1})_{\theta q} = L_{2,r,q}$, where $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$, $\theta \in (0,1)$. Hence we have

$$\|Tf\|_{l_{r,q}} \leqslant C\|f\|_{L_{2,r,q}}$$

and

$$\left(\sum\left(n^{1/r}\left(\sup_{A\in G_n}\frac{1}{|A|^{1/2}\log_2\langle A\rangle_M^{1/2}}\left|\sum\hat f(m)\right|\right)\right)^q\frac{1}{n}\right)^{1/q} \leqslant C\|f\|_{L_{2,r,q}}$$

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