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RIGIDITY OF IMMERSED SUBMANIFOLDS IN A HYPERBOLIC SPACE

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Abstract. Let $M^n, n \geq 5$ be a complete noncompact sub-manifold immersed in $\mathbb{H}^{n+p}$. We will prove that there exist certain positive constants α, C such that if $||H|| \leq \alpha$ and the total scalar curvature $||A||_n < C$ then M does not admit any nonconstant harmonic function u with finite energy. Excepting these two conditions, there is no more additional condition on the curvature. Moreover, in the lower dimensional case, namely, $2 \leq n \leq 5$, we show that there exist two certain positive constants $0 < \delta \leq 1$, and β depending only on δ and the first eigenvalue $\lambda_1(M)$ of Laplacian acting on M such that if M satisfies a (δ-SC) condition and $\lambda_1(M)$ has a lower bound then $H^1(L^{2\beta}(M)) = 0$. Again, we do not need to have any additional condition on the curvature.

1. Introduction

It is well-known that the structures of ends or the number of ends of a non-compact immersed submanifold in a Riemannian manifold is related to the space of bounded harmonic functions with finite energy (see [1], [11], [12]). In fact, Li and Tam, in [11], proved that the number of non-parabolic ends of any complete Riemannian manifold is bounded by the dimension of $H^1(L^2(M))$, here we denote by $H^1(L^2(M))$ the space of bounded harmonic functions with finte energy. Due to their result, if the space $H^1(L^2(M))$ is trivial then the submanifold has at most one non-parabolic end. Therefore, it is very interesting to study vanishing property of $H^1(L^2(M))$. There are several work have been done in this direction. For example, in [14], Lei Ni proved that if $M^n(n \geq 3)$ is a complete minimal immersed hypersurface in $\mathbb{R}^{n+1}$, then M does not admit any non-trivial L^2 harmonic one-form, consequently, M has only one end. Recently, in [13], Li, Xu and Zhou shown that if $M^n, (n \geq 3)$ be a complete sub-manifold immersed in $\mathbb{R}^{n+1}$ with zero scalar curvature, then M has one end, provided that the total mean curvature $||H||_n \leq C$, where C is a certain positive number and H denotes the mean curvature of M. When N is a hyperbolic space, Seo [15] proved that there are non L^2 harmonic one form on a complete super stable minimal hypersurface in a hyperbolic space if the first eigenvalue $\lambda_1(M)$ of Laplacian is bounded from below by a certain positive number depending only on the dimension of M.

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Later, Fu and Yang [7] improved the result of Seo by giving a better lower bound of $\lambda_1(M)$. Recently, in [9], Kim and Yun studied complete oriented noncompact hypersurface M^n in a complete Riemannian manifold of nonnegative sectional curvature. They defined a (SC) condition on M and proved that if M satisfies the (SC) condition and $2 \leq n \leq 4$ then there is no non-trivial L^2 harmonic one forms on M. It is important to note that in [9], [13], the authors do not assume the minimality of such a hypersurface nor the constant mean curvature condition. Finally, in [5], we investigate complete hypersurfaces immersed in $\mathbb{R}^{n+1}$ and improve the results in [9] and [13]. In fact, in [5] we remove the condition on scalar curvature in [13], namely, we do not need to assume the zero scalar curvature condition and obtain the same vanishing theorem as in [13]. Moreover, our results in [5] are more general than Kim and Yun's results.

In this paper, motivated by [5], [9] and [13], we consider a complete noncompact immersed submanifold in a hyperbolic space. We will not require the minimality of such a submanifold nor the constant mean curvature condition in our research. In order to establish our result, first we give a definition. Let M^n be an immersed hypersuface in $\mathbb{H}^{n+1}$. For a constant $0 < \alpha \leq 1$, we say that M satisfies the (δ-SC) condition if for any function $\phi \in \mathcal{C}_0^1(M)$,

$$\delta \int_M (-n + |A|^2)\phi^2 \leq \int_M |\nabla \phi|^2, \tag{1.1}$$

where A is the second fundamental form of M. Note that if $\delta = 1$ then the condition (1.1) means the index of the operator $\Delta + (-n + |A|^2)$ is zero, (see [7]). If $\delta = 1$, we also say that M satisfies the (SC) condition. In general, the condition (1.1) can be considered as the δ-stable condition, when N is hyperbolic space. Now, we state our first main theorem.

Theorem 1.1. *Let $M^n, n \geq 5$ be an oriented complete noncompact immersed submanifold in a hyperbolic space $\mathbb{H}^{n+p}$. Assume that there is a constant $0 \leq \alpha < \frac{\sqrt{n}-2}{\sqrt{n}}(n-1)$ such that $||H|| \leq \alpha$. Then there exists a positive constant $C > 0$ such that if the total scalar curvature satisfies*

$$\left(\int_M |A|^n\right)^{\frac{1}{n}} < C$$

then M does not admit any nonconstant harmonic function with finite energy.

It is worth to note that in this theorem we do not assume the minimality of M. When $2 \leq n \leq 4$, we also can prove a vanishing theorem on M provided that M satisfies a (δ-SC) condition and the first eigenvalue of M is bounded from below.

Theorem 1.2. *Let $2 \leq n \leq 4$. Let M^n be a complete hypersurface immersed in a hyperbolic space $\mathbb{H}^{n+1}$. Suppose that M satisfies (SC) condition and*

$$\lambda_1 \geq \left(\sqrt{n-1}+1\right)^2 \frac{\sqrt{n-1}}{2-\sqrt{n-1}},$$

then $H^1(L^2(M)) = 0$ and M has at most one nonparabolic end.

The paper is organized as follows. In the Section 2, we introduce some usefull auxiliary lemmas. Then in the Section 3, we prove the main Theorem 1.1 and its corollaries. Finally, we consider the immersed submanifolds with positive spectrum in the Section 4 and prove the Theorem 1.2.

2. Some usefull lemmas

In this section, we devire some auxiliary lemmas for which we will use later. First, recall that we have the following Sobolev type inequality proved by Hoffman and Spruck [8].

Lemma 2.1 ([8]). *Let M^n be a sub-manifold immersed in $\mathbb{H}^{n+p}$. Then there exists a positive constant $C_1 > 0$ such that for any function $\phi \in \mathcal{C}^1_0(M)$, we have*

$$\left(\int_M |\phi|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq C_1 \left(\int_M |\nabla\phi| + \int_M |H\phi|\right). \tag{2.2}$$

Proof. See [8], Theorem 2.1. □

From Lemma 2.1, we have the following Sobolev inequality proved by Carron [3] (also see [6]) and rigidity property of complete manifolds with finite total mean curvature.

Lemma 2.2. *Let $M^n, n \geq 3$ be an oriented complete sub-manifold immersed in $\mathbb{H}^{n+p}$. If $||H||_n < \infty$, then for any $\phi \in \mathcal{C}^1_0(M)$, we have*

$$\left(\int_M |\phi|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq C_s \int_M |\nabla\phi|^2, \tag{2.3}$$

where

$$C_s = \left(\frac{4C_1(n-1)}{n-2}\right)^2.$$

where C_1 is the constant in the Lemma 2.1. Moreover, each end of M must be non-parabolic.

Proof. The proof of the Lemma is given in [6] (see also [3]). For the completeness, we include the detail here. By the assumption that $\int_M |H|^n < \infty$, there exists a compact subset $D \subset M$ such that

$$\left(\int_{M\setminus D} |H|^n\right)^{\frac{1}{n}} \leq \frac{1}{2C_1}.$$

Let $h \in \mathcal{C}_0^1(M \setminus D)$, the Hölder inequality implies,

$$C_1 \int_{M\setminus D} |Hh| \le C_1 \left(\int_{M\setminus D}\right)^{\frac{1}{n}} \left(\int_{M\setminus D} |h|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}$$
$$\le \frac{1}{2}\left(\int_{M\setminus D} |h|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}.$$

Hence, by (2.2), we have

$$\left(\int_{M\setminus D} |h|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \le 2C_1 \int_{M\setminus D} |\nabla h|.$$

Now, replacing h by $\phi^{\frac{2(n-1)}{n-2}}, \phi \in \mathcal{C}_0^1(M \setminus D)$, we infer

$$\left(\int_{M\setminus D} |\phi|^{\frac{2n}{n-2}}\right)^{\frac{n-1}{n}} \le \frac{4C_1(n-1)}{n-2} \int_{M\setminus D} \left|\phi^{\frac{n}{n-2}} \nabla\phi\right|$$
$$\le \frac{4C_1(n-1)}{n-2} \left(\int_{M\setminus D} |\phi|^{\frac{2n}{n-2}}\right)^{\frac{1}{2}} \left(\int_{M\setminus D} |\nabla\phi|^2\right)^{\frac{1}{2}}.$$

Therefore,

$$\left(\int_{M\setminus D} |\phi|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \le C_s^2 \int_{M\setminus D} |\nabla\phi|^2,$$

for all $\phi \in \mathcal{C}_0^1(M \setminus D)$. By [2] (also see [3]), we obtain the Sobolev inequality

$$\left(\int_{M} |\phi|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \le C_s^2 \int_{M} |\nabla\phi|^2,$$

for all $\phi \in \mathcal{C}_0^1(M)$.

By the Theorem 2.4 and the Proposition 2.5 in [6], each end of M is non-parabolic. The proof is complete. □

Finally, we note that in the paper [4], the authors proved an estimate of the first eigenvalue of Laplacian on M as follows.

Theorem 2.3 ([4]). *Let M^n be a complete noncompact submanifold in the hyperbolic space H^{n+p}. If $||H|| \le \alpha$ for some constant $\alpha < n-1$, then*

$$\lambda_1(M) \ge \frac{1}{4}(n-1-\alpha)^2 > 0.$$

Proof. See [4], Theorem 2. □

3. Proof of the main theorem

To begin with, we first prove the following lemma.

Lemma 3.1. *Let M^n be a complete immersed submanifold in $\mathbb{H}^{n+p}$. Then*

$$Ric_M \geq -(n-1) - \frac{\sqrt{n-1}}{2}\,|A|^2. \tag{3.1}$$

Proof. By [10], it is well-known that

$$\begin{aligned} Ric_M \geq & -(n-1) - \frac{n-1}{n}|A|^2 \\ & - \frac{1}{n^2}\left\{2(n-1)H^2 - (n-2)\sqrt{n-1}\,|H|\sqrt{n|A|^2 - H^2}\right\}. \end{aligned} \tag{3.2}$$

Claim: If $b = \frac{(n-2)^2\sqrt{n-1}}{2n(\sqrt{n-1}+1)^2}$. Then

$$2(n-1)H^2 - (n-2)\sqrt{n-1}\,|H|\sqrt{n|A|^2 - H^2} \geq -bn^2|A|^2. \tag{3.3}$$

Suppose that the claim is proved, then by (3.2), we have

$$\begin{aligned} Ric_M &\geq -(n-1) - \left(\frac{(n-2)^2\sqrt{n-1}}{2n(\sqrt{n-1}+1)^2} + \frac{n-1}{n}\right)|A|^2 \\ &= -(n-1) - \frac{\sqrt{n-1}}{2}\,|A|^2. \end{aligned}$$

Hence, we have proven the conclusion of Lemma 3.1.

The rest of this part is to verify the above Claim. Indeed, if $|A| = 0$ then $H = 0$, so (3.3) is trivial. Assume $|A| > 0$, the inequality (3.3) is equivalent to

$$\frac{(n-2)\sqrt{n-1}}{n^2}\,\frac{|H|}{|A|}\sqrt{n - \frac{H^2}{|A|^2}} - \frac{2(n-1)}{n^2}\,\frac{H^2}{|A|^2} \leq b. \tag{3.4}$$

We define $f_n(t)$ by

$$f_n(t) = \frac{(n-2)\sqrt{n-1}}{n^2}\,t\sqrt{n-t^2} - \frac{2(n-1)}{n^2}\,t^2.$$

Suppose that there is a constant $B > 0$ such that $B \geq \max_{t\in[0,\sqrt{n}]} f_n(t)$. Then,

$$(n-2)\sqrt{n-1}\,t\sqrt{n-t^2} \leq 2(n-1)\,t^2 + Bn^2, \quad \forall\, t \in [0,\sqrt{n}]$$

or equivalently, with $x := t^2$,

$$(n-2)^2(n-1)x(n-x) \leq 4(n-1)^2x^2 + 2B(n-1)n^2x + B^2n^4, \quad \forall\, x \in [0,n]. \tag{3.5}$$

A simple computation shows that (3.5) holds true if

$$B \geq \frac{(n-2)^2\sqrt{n-1}}{2n(\sqrt{n-1}+1)^2}.$$

Now, choose $b = \frac{(n-2)^2\sqrt{n-1}}{2n(\sqrt{n-1}+1)^2}$. The claim is proved. Thus, the proof is complete. $\square$

Now we give a proof of the main theorem.

Proof of Theorem 1.1. Suppose that M admits a nonconstant bounded harmonic function u with finite energy. Let $f = |\nabla u|$.

By Bochner formula, we have

$$\frac{1}{2}\Delta f^2 = |\nabla\nabla u|^2 + Ric_M(\nabla u, \nabla u).$$

Using the refine Kato's inequality $|\nabla\nabla u|^2 \geq \frac{n}{n-1}|\nabla|\nabla u||^2$, we infer

$$f\Delta f + |\nabla f|^2 \geq \frac{n}{n-1}|\nabla f|^2 - (n-1)f^2 - \frac{\sqrt{n-1}}{2}|A|^2 f^2,$$

where we used (3.1). Consequently, we have

$$f\Delta f + \frac{\sqrt{n-1}}{2}|A|^2 f^2 + (n-1)f^2 \geq \frac{1}{n-1}|\nabla f|^2.$$

Let ϕ be a cut-off function. Multiplying both sides of the above inequality by ϕ^2 then integrating over M, we obtain

$$\int_M \phi^2 f\Delta f + \frac{\sqrt{n-1}}{2}\int_M \phi^2|A|^2 f^2 + (n-1)\int_M \phi^2 f^2 \geq \frac{1}{n-1}\int_M \phi^2|\nabla f|^2. \tag{3.6}$$

Since $||H|| \leq \alpha < \frac{\sqrt{n}-2}{\sqrt{n}}(n-1)$, we have $||H|| < n-1$. Thus, Lemma 2.3 is applied to get

$$\frac{1}{4}(n-1-\alpha)^2 \leq \lambda_1(M) \leq \frac{\int_M |\nabla(f\phi)|^2}{\int_M f^2\phi^2}.$$

Combining this inequality with (3.6), we yield

$$\int_M \phi^2 f\Delta f + \frac{\sqrt{n-1}}{2}\int_M \phi^2|A|^2 f^2 + \frac{4(n-1)}{(n-1-\alpha)^2}\int_M |\nabla(f\phi)|^2 \geq \frac{1}{n-1}\int_M \phi^2|\nabla f|^2.$$

Using integration by parts, this implies

$$\begin{aligned}-\int_M \phi^2|\nabla f|^2 - 2\int_M f\phi\langle\nabla f, \nabla\phi\rangle + \frac{\sqrt{n-1}}{2}\int_M \phi^2|A|^2 f^2 \\ + \frac{4(n-1)}{(n-1-\alpha)^2}\int_M |\nabla(f\phi)|^2 \geq \frac{1}{n-1}\int_M \phi^2|\nabla f|^2.\end{aligned}$$

Hence, for any positive number $a > 0$, applying the Schwarz inequality (2.3), we obtain

$$\frac{\sqrt{n-1}}{2}\int_M \phi^2|A|^2 f^2 + \left\{\frac{4(n-1)}{(n-1-\alpha)^2}\left(1+\frac{1}{a}\right) + \frac{1}{a}\right\}\int_M f^2|\nabla\phi|^2$$

$$(3.7)\qquad \geq \left\{\frac{n}{n-1} - \frac{4(n-1)(1+a)}{(n-1-\alpha)^2} - a\right\}\int_M \phi^2|\nabla f|^2$$

On the other hand, since $||H||_n < \infty$, applying the Sobolev inequality (2.3), we infer

$$\int_M |\nabla(f\phi)|^2 \geq C_s^{-1}\left(\int_M (f\phi)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}.$$

Therefore, using the Schwarz inequality, for any positive number $b > 0$, this implies

$$(3.8)\qquad (1+b)\int_M \phi^2|\nabla f|^2 \geq C_s^{-1}\left(\int_M (f\phi)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} - \left(1+\frac{1}{b}\right)\int_M f^2|\nabla\phi|^2.$$

Since $\alpha < \frac{\sqrt{n-2}}{\sqrt{n}}(n-1)$, we can choose $a > 0$ small enough such that

$$\frac{n}{n-1} - \frac{4(n-1)(1+a)}{(n-1-\alpha)^2} - a > 0.$$

Hence, plugging (3.8) into (3.7), we have

$$\frac{\sqrt{n-1}}{2}\int_M \phi^2|A|^2f^2 + \left\{\frac{4(n-1)}{(n-1-\alpha)^2}\left(1+\frac{1}{a}\right)+\frac{1}{a}\right\}\int_M f^2|\nabla\phi|^2$$
$$\geq \frac{C_s^{-1}}{b+1}\left\{\frac{n}{n-1} - \frac{4(n-1)(1+a)}{(n-1-\alpha)^2} - a\right\}\left(\int_M (f\phi)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}$$
$$-\frac{1}{b}\left\{\frac{n}{n-1} - \frac{4(n-1)(1+a)}{(n-1-\alpha)^2} - a\right\}\int_M f^2|\nabla\phi|^2.$$

Moreover, the Hölder inequality implies

$$\int_M \phi^2|A|^2f^2 \leq \left(\int_M |A|^n\right)^{\frac{2}{n}}\left(\int_M (f\phi)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}.$$

Combining the last two inequalities togather, finally, we obtain

$$\left\{\frac{4(n-1)}{(n-1-\alpha)^2}\left(1+\frac{1}{a}\right)+\frac{1}{a}+\frac{1}{b}\left[\frac{n}{n-1}-\frac{4(n-1)(1+a)}{(n-1-\alpha)^2}-a\right]\right\}\int_M f^2|\nabla\phi|^2$$

(3.9)

$$\geq \left\{\frac{C_s^{-1}}{b+1}\left[\frac{n}{n-1}-\frac{4(n-1)(1+a)}{(n-1-\alpha)^2}-a\right]-\frac{\sqrt{n-1}}{2}\left(\int_M |A|^n\right)^{\frac{2}{n}}\right\}\left(\int_M (f\phi)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}.$$

Now, by the assumption on the total scalar curvature, we can choose a, b small enough such that

$$\frac{C_s^{-1}}{b+1}\left[\frac{n}{n-1} - \frac{4(n-1)(1+a)}{(n-1-\alpha)^2} - a\right] - \frac{\sqrt{n-1}}{2}\left(\int_M |A|^n\right)^{\frac{2}{n}} > 0.$$

For any $R > 0$, let ϕ be a cut-off function satisfying $0 \leq \phi \leq 1, \phi \equiv 1$ on $B(R)$, $\phi \equiv 1$ on $M \setminus B(2R)$ and $|\nabla \phi| \leq \frac{1}{R}$. Then, let $R \to \infty$ in (3.9) we conclude that $f \equiv 0$, namely $|\nabla u| = 0$. Hence, u is constant. This is a contradiction. The proof is complete. $\square$

It is easy to see that by the Theorem 1.1 and the Lemma 2.2, we have

Corollary 3.2. *Let $M^n, n \geq 5$ be an oriented complete noncompact immersed submanifold in a hyperbolic space $\mathbb{H}^{n+p}$. Assume that there is a constant $0 \leq \alpha < \frac{\sqrt{n-2}}{\sqrt{n}}(n-1)$ such that $||H|| \leq \alpha$. Then there exists a positive constant $C > 0$ such that if the total scalar curvature satisfies*

$$\left(\int_M |A|^n \right)^{\frac{1}{n}} < C$$

then $H^1(L^2(M)) = 0$. Consequently, M has only one end.

4. Immersed submanifolds with positive spectrum

In this section, we will consider a complete hypersurface of lower dimension immersed in a hyperbolic space. First we have the following vanishing theorem.

Theorem 4.1. *Let $2 \leq n \leq 5$. Let M^n be a complete hypersurface immersed in a hyperbolic space $\mathbb{H}^{n+1}$. Suppose that M satisfies (δ-SC) condition for some $\frac{n-2}{2\sqrt{n-1}} < \delta \leq 1$, if the first eigenvalue of M has lower bound*

$$\lambda_1 = \lambda_1(M) \geq \left(\sqrt{n-1} + 1\right)^2 \left(\frac{2\sqrt{n-1}}{n-2} - \frac{1}{\delta} \right)^{-1}$$

then any harmonic one-form ω on M is trivial, provided that

$$\int_M |\omega|^{2\beta} = o(R^2),$$

where β is a constant satisfying

$$\frac{1 - \sqrt{1 - D\frac{n-2}{n-1}}}{D} < \beta < \frac{1 + \sqrt{1 - D\frac{n-2}{n-1}}}{D}$$

and

$$D = \frac{\sqrt{n-1}}{2\delta} + \frac{1}{\lambda_1} \left\{ \frac{n\sqrt{n-1}}{2} + (n-1) \right\}.$$

Proof. We use the method in [7]. Let ω be a hormonic 1-form as in Theorem 4.1. The Bochner formula and the refine Kato's identity imply

$$|\omega| \, \Delta \, |\omega| \geq \frac{1}{n-1} |\nabla |\omega||^2 + Ric_M(\omega, \omega).$$

By Lemma 3.1, this shows that

$$|\omega|\Delta|\omega| \geq \frac{1}{n-1}|\nabla|\omega||^2 - (n-1)|\omega|^2 - \frac{\sqrt{n-1}}{2}|A|^2|\omega|^2.$$

Now, for any $\alpha > 0$, we have

$$\begin{aligned}
|\omega|^\alpha\Delta|\omega|^\alpha &= |\omega|^\alpha\left(\alpha(\alpha-1)|\omega|^{\alpha-2}|\nabla|\omega||^2 + \alpha|\omega|^{\alpha-1}\Delta|\omega|\right)\\
&= \frac{\alpha-1}{\alpha}|\nabla|\omega|^\alpha|^2 + \alpha|\omega|^{2\alpha-2}|\omega|\Delta|\omega|\\
&\geq \frac{\alpha-1}{\alpha}|\nabla|\omega|^\alpha|^2\\
&\quad + \alpha|\omega|^{2\alpha-2}\left(\frac{1}{n-1}|\nabla|\omega||^2 - (n-1)|\omega|^2 - \frac{\sqrt{n-1}}{2}|A|^2|\omega|^2\right)\\
&\geq \left(1 - \frac{n-2}{(n-1)\alpha}\right)|\nabla|\omega|^\alpha|^2 - \alpha(n-1)|\omega|^{2\alpha} - \alpha\frac{\sqrt{n-1}}{2}|A|^2|\omega|^{2\alpha}. \qquad (4.1)
\end{aligned}$$

Let $q \geq 0$ and let $\phi \in \mathcal{C}_0^\infty(M)$. Multiplying both sides of (4.1) by $|\omega|^{2q\alpha}\phi^2$ then integrating over M, we obtain

$$\begin{aligned}
&\left(1 - \frac{n-2}{(n-1)\alpha}\right)\int_M |\omega|^{2q\alpha}\phi^2|\nabla|\omega|^\alpha|^2\\
&\leq \int_M |\omega|^{(2q+1)\alpha}\phi^2\Delta|\omega|^\alpha + \alpha(n-1)\int_M |\omega|^{2(1+q)\alpha}\phi^2 + \alpha\frac{\sqrt{n-1}}{2}\int_M |A|^2\phi^2|\omega|^{2(q+1)\alpha}\\
&= \alpha(n-1)\int_M |\omega|^{2(1+q)\alpha}\phi^2 + \alpha\frac{\sqrt{n-1}}{2}\int_M |A|^2\phi^2|\omega|^{2(q+1)\alpha}\\
&\quad - (2q+1)\int_M |\omega|^{2q\alpha}|\nabla|\omega|^\alpha|^2\phi^2 - 2\int_M \phi|\omega|^{(2q+1)\alpha}\langle\nabla\phi, \nabla|\omega|^\alpha\rangle.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left(2(q+1) - \frac{n-2}{(n-1)\alpha}\right)\int_M |\omega|^{2q\alpha}|\nabla|\omega|^\alpha|^2\phi^2\\
&\qquad \leq \alpha(n-1)\int_M |\omega|^{2(1+q)\alpha}\phi^2 + \alpha\frac{\sqrt{n-1}}{2}\int_M |A|^2\phi^2|\omega|^{2(q+1)\alpha}\\
&\qquad\quad - 2\int_M \phi|\omega|^{(2q+1)\alpha}\langle\nabla\phi, \nabla|\omega|^\alpha\rangle. \qquad (4.2)
\end{aligned}$$

On the other hand, since M satisfies the (δ-SC) condition and N has nonnegative sectional curvature, we have, for any $\phi \in \mathcal{C}_0^1(M)$,

$$\int_M |\nabla\phi|^2 \geq \delta\int_M \left(\overline{\mathrm{Ric}}(\nu,\nu) + |A|^2\right)\phi^2 \geq \delta\int_M (-n + |A|^2)\phi^2.$$

Replacing ϕ by $|\omega|^{(1+q)\alpha}\phi$ in the above inequality, we obtain

$$\delta\int_M |\omega|^{2(q+1)\alpha}|A|^2\phi^2 \le \int_M |\nabla(|\omega|^{(q+1)\alpha}\phi)|^2 + n\delta\int_M |\omega|^{2(q+1)\alpha}\phi^2. \tag{4.3}$$

Combining (4.2) and (4.3), we infer

$$\begin{aligned}\left(2(q+1)-\frac{n-2}{(n-1)\alpha}\right)\int_M |\omega|^{2q\alpha}|\nabla|\omega|^\alpha|^2\phi^2 \\ \le \frac{\alpha\sqrt{n-1}}{2\delta}\int_M |\nabla(|\omega|^{(q+1)\alpha}\phi)|^2 - 2\int_M |\omega|^{(2q+1)\alpha}\phi\big\langle\nabla\phi,\nabla|\omega|^\alpha\big\rangle. \\ +\alpha\left\{\frac{n\sqrt{n-1}}{2}+(n-1)\right\}\int_M |\omega|^{2(q+1)\alpha}\phi^2\end{aligned} \tag{4.4}$$

On the other hand, by the variational characterization of λ_1, we have

$$\int_M |\omega|^{2(q+1)\alpha}\phi^2 \le \frac{1}{\lambda_1}\int_M |\nabla(|\omega|^{(q+1)\alpha}\phi)|^2. \tag{4.5}$$

Hence, (4.4) implies

$$\begin{aligned}\left(2(q+1)-\frac{n-2}{(n-1)\alpha}\right)\int_M |\omega|^{2q\alpha}|\nabla|\omega|^\alpha|^2\phi^2 \\ \le \left[\frac{\alpha\sqrt{n-1}}{2\delta}+\frac{\alpha}{\lambda_1}\left\{\frac{n\sqrt{n-1}}{2}+(n-1)\right\}\right]\int_M |\nabla(|\omega|^{(q+1)\alpha}\phi)|^2 \\ -2\int_M |\omega|^{(2q+1)\alpha}\phi\,\langle\nabla\phi,\nabla|\omega|^\alpha\rangle\,;\end{aligned}$$

or equivalently,

$$\begin{aligned}\left(2(q+1)-\frac{n-2}{(n-1)\alpha}\right)\int_M |\omega|^{2q\alpha}|\nabla|\omega|^\alpha|^2\phi^2 \\ \le D\alpha(q+1)^2\int_M |\omega|^{2q\alpha}|\nabla|\omega|^\alpha|^2\phi^2 + D\alpha\int_M |\omega|^{2(q+1)\alpha}|\nabla\phi|^2 \\ +\{D\alpha(q+1)-1\}\int_M 2|\omega|^{(2q+1)\alpha}\phi\,\langle\nabla\phi,\nabla|\omega|^\alpha\rangle\,.\end{aligned} \tag{4.6}$$

where

$$D := \frac{\sqrt{n-1}}{2\delta}+\frac{1}{\lambda_1}\left\{\frac{n\sqrt{n-1}}{2}+(n-1)\right\} > 0.$$

For any $\varepsilon > 0$, the Schwarz inequality implies

$$\begin{aligned}\big\{D\alpha(q+1)-1\big\}\int_M 2|\omega|^{(2q+1)\alpha}\phi\,\langle\nabla\phi,\nabla|\omega|^\alpha\rangle \\ \le \big|1-D\alpha(q+1)\big|\int_M |\omega|^{2(q+1)\alpha}|\phi||\nabla|\omega|^\alpha||\nabla\phi|\end{aligned}$$

$$(4.7)\qquad \le \left|1 - D\alpha(q+1)\right| \left\{ \varepsilon \int_M |\omega|^{2q\alpha} |\nabla|\omega|^\alpha|^2 \phi^2 + \frac{1}{\varepsilon} \int_M |\omega|^{2(q+1)\alpha} |\nabla\phi|^2 \right\}.$$

From (4.6) and (4.7), we conclude that

$$\left[2(q+1) - \frac{n-2}{(n-1)\alpha} - D\alpha(q+1)^2 - \left|1 - D\alpha(q+1)\right|\varepsilon\right] \int_M |\omega|^{2q\alpha} |\nabla|\omega|^\alpha|^2 \phi^2$$

$$(4.8)\qquad \le \left\{ D\alpha + \frac{\left|1 - D\alpha(q+1)\right|}{\varepsilon} \right\} \int_M |\omega|^{2(q+1)\alpha} |\nabla\phi|^2$$

Now, choose α, q such that

$$2(q+1) - \frac{n-2}{(n-1)\alpha} - D\alpha(q+1)^2 > 0.$$

Thus, from (4.8), we see that if $\varepsilon > 0$ is small enough then there exists a positive constant C depending on $\varepsilon, q, \alpha, \delta, \lambda_1$ such that

$$(4.9)\qquad \int_M |\omega|^{2q\alpha} |\nabla|\omega|^\alpha|^2 \phi^2 \le C \int_M |\omega|^{2(q+1)\alpha} |\nabla\phi|^2,$$

provided that

$$(4.10)\qquad 2(q+1) - \frac{n-2}{(n-1)\alpha} - D\alpha(q+1)^2 > 0.$$

Let $\beta = \alpha(q+1)$, it is easy to see that (4.10) is equivalent to

$$2\beta - \frac{n-2}{n-1} - D\beta^2 > 0.$$

This inequation is always satisfied by the assumptions

$$\frac{1 - \sqrt{1 - D\frac{n-2}{n-1}}}{D} < \beta < \frac{1 + \sqrt{1 - D\frac{n-2}{n-1}}}{D} \quad \text{and} \quad \lambda_1 \ge (\sqrt{n-1} + 1)^2 \left(\frac{2\sqrt{n-1}}{n-2} - \frac{1}{\delta} \right)^{-1}.$$

Now, let ϕ be a smooth function on $[0, \infty)$ such that $\phi \ge 0, \phi = 1$ on $[0, R]$ and $\phi = 0$ in $[2R, \infty)$ with $|\phi'| \le \frac{2}{R}$, then considering $\phi \circ r$, where r is the function in the definition of $B(R)$, we obtain from (4.9)

$$\int_M |\omega|^{2q\alpha} |\nabla|\omega|^\alpha|^2 \le \frac{4C}{R^2} \int_M |\omega|^{2\beta}.$$

Let $R \to \infty$, by the assumption $\int_M |\omega|^{2\beta} = o(R^2)$, we have that $|\omega|$ is constant. By (4.5), we obtain

$$|\omega|^{2\beta} \int_M \phi^2 \le \frac{4}{\lambda_1 R^2} \int_M |\omega|^{2\beta}.$$

Let $R \to \infty$ again, we conclude that $|\omega| = 0$. Hence ω is trivial. The proof is finished. □

Now, we will give a proof of Theorem 1.2.

Proof of Theorem 1.2. Since M satisfies the (SC) condition, $\delta = 1$. Hence, we can repeat the proof of Theorem 4.1, to obtain $H^1(L^{2\beta}(M)) = 0$, provided that

$$\frac{1-\sqrt{1-D\frac{n-2}{n-1}}}{D} < \beta < \frac{1+\sqrt{1-D\frac{n-2}{n-1}}}{D},$$

where

$$D = \frac{\sqrt{n-1}}{2} + \frac{1}{\lambda_1}\left(\frac{n\sqrt{n-1}}{2} + n - 1\right).$$

Note that the vanishing property of $H^1(L^2(M))$ can be varified if we can choose $\beta = 1$. In fact, by above inequalities, it is sufficient to show that $D \leq 1$. This is satisfied by the assuption

$$\lambda_1 \geq (\sqrt{n-1}+1)^2 \frac{\sqrt{n-1}}{2-\sqrt{n-1}}.$$

The proof is complete. □

Finally, in this section, we give a sufficient condition for immersed hypersurfaces to be satisfying the (δ-SC) condition.

Theorem 4.2. *Let M^n be an immersed hypersurface in $\mathbb{H}^{n+1}, n \geq 3$. If $||A||_n < \frac{1}{\sqrt{\delta C_s}}$, where C_s is the constant in the Lemma 2.2 then M satisfies the (δ-SC) condition.*

Proof. We only need to show that, for any $\phi \in \mathcal{C}_0^1(M)$,

$$\int_M (|\nabla\phi|^2 - \delta(-n + |A|^2)\phi^2) \geq 0.$$

By the assumption on the total scalar curvature, we have $||H||_n \leq \sqrt{n}\,||A||_n < \infty$, hence we can use the Sobolev inequality in Lemma 2.2 to get

$$\int_M (|\nabla\phi|^2 - \delta(-n + |A|^2)\phi^2) \geq \frac{1}{C_s}\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} - \delta\int_M |A|^2 f^2.$$

Moreover, Hölder inequality inplies

$$\int_M |A|^2 f^2 \leq \left(\int_M |A|^n\right)^{\frac{2}{n}} \left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}.$$

Combining two above inequalites, we obtain

$$\int_M (|\nabla\phi|^2 - \delta(-n + |A|^2)\phi^2) \geq \left\{\frac{1}{C_s} - \delta\left(\int_M |A|^n\right)^{\frac{2}{n}}\right\} \left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \geq 0,$$

here we used $||A||_n < \frac{1}{\sqrt{\delta C_s}}$. The proof is complete. □

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