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THE NUMBER OF MEDIUM AMPLITUDE LIMIT CYCLES OF SOME GENERALIZED LIÉNARD SYSTEMS

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Abstract. We will consider two special families of polynomial perturbations of the linear center. For the resulting perturbed systems, which are generalized Liénard systems, we provide the exact upper bound for the number of limit cycles that bifurcate from the periodic orbits of the linear center.

1. Introduction and statement of the results

The bifurcation of limit cycles by perturbing a planar system which has a continuous family of *cycles*, i.e. periodic orbits, has been an intensively studied phenomenon; see for instance [3] and references therein. The simplest planar system having a continuous family of cycles is the linear center, and a special family of its perturbations is given by the generalized polynomial Liénard systems:

$$(1_\varepsilon) \qquad \dot{x} = y + \sum_{i=1}^{\mu} \varepsilon^i F_i(x), \qquad \dot{y} = \sum_{i=0}^{\nu} \varepsilon^i g_i(x),$$

where $\mu \in \mathbb{N}$, $\nu \in \mathbb{N} \cup \{0\}$, $g_0(x) = -x$, $g_i(x)$ and $F_i(x)$ are polynomials for $i \geq 1$, and ε is a small parameter.

The classical and generalized Liénard systems appear very often in several branches of science and engineering, as biology, chemistry, mechanics, electronics, etc., see for instance [14] and references therein. In particular Liénard systems are frequent specially in physiological processes, see for instance [6]. Further, some planar systems can be transformed into (generalized) Liénard systems, see for example [4, 10]. In addition, the generalized polynomial Liénard systems is one of the most considered families in the study of limit cycles, see [13].

We assume that $F_\mu(x) \not\equiv 0$, $g_\nu(x) \not\equiv 0$, $m = \max_{1 \leq i \leq \mu}\{\deg F_i(x)\}$, and $n = \max_{1 \leq i \leq \nu}\{\deg g_i(x)\}$. For a small enough ε, let $\mathcal{H}^\mu_\nu(m,n)$ be the maximum number of limit cycles of (1_ε) that bifurcate from cycles of the *linear center* (1_0), i.e. the maximum number of *medium amplitude limit cycles* which can bifurcate from (1_0) under the perturbation (1_ε). If $\nu = 0$, then $\mathcal{H}^\mu_0(m,n)$ does not depend on n; hence we only write $\mathcal{H}^\mu_0(m)$. The main problem concerning $\mathcal{H}^\mu_\nu(m,n)$ is finding its exact value.

We know from [11] that $\mathcal{H}^1_0(m) \geq [(m-1)/2]$, where $[\cdot]$ denotes the integer part function. Moreover, by following [5, Theorem 3.1] we can prove that $\mathcal{H}^\mu_0(m) = [(m-1)/2]$ for $\mu \geq 1$; Theorem 1 (below) is a generalization of this result. Also, we know from [12] that $\mathcal{H}_1^1(m,n) \geq [(m-1)/2]$, $\mathcal{H}_2^2(m,n) \geq \max\{[(m-1)/2], [m/2]+[n/2]-1\}$, and $\mathcal{H}_3^3(m,n) \geq [(m+n)/2]-1$. However, the exact values of $\mathcal{H}_1^1(m,n)$, $\mathcal{H}_2^2(m,n)$, and $\mathcal{H}_3^3(m,n)$ were not reported there.

In this paper we give the exact value of $\mathcal{H}_\nu^\mu(m,n)$ for two subfamilies of (1_ε). More precisely, we will give the exact value of $\tilde{\mathcal{H}}_\nu^\mu(m,n)$ and $\bar{\mathcal{H}}_\nu^\mu(m,n)$, where $\tilde{\mathcal{H}}_\nu^\mu(m,n)$ is the value of $\mathcal{H}_\nu^\mu(m,n)$ by assuming that $g_i(x)$ is odd for $1 \leq i \leq \nu$, and $\bar{\mathcal{H}}_\nu^\mu(m,n)$ is the value of $\mathcal{H}_\nu^\mu(m,n)$ by assuming that $F_i(x)$ is even for $\mu_0 < i \leq \mu$, where μ_0 with $1 \leq \mu_0 \leq \mu$ is the smallest integer such that $F_{\mu_0}(x) \not\equiv 0$. Of course, if $\mu_0 = \mu$, then $\bar{\mathcal{H}}_\nu^\mu(m,n) = \mathcal{H}_\nu^\mu(m,n)$.

Our main result is the following:

Theorem 1. *(a)* $\tilde{\mathcal{H}}_\nu^\mu(m,n) = \left[\frac{m-1}{2}\right]$. *(b)* $\bar{\mathcal{H}}_\nu^\mu(m,n)$ *is either* $\left[\frac{m-1}{2}\right]$ *if* m *is odd or* $\left[\frac{m}{2}\right]+\left[\frac{n}{2}\right]-1$ *if* m *is even.*

The assumptions on $g_i(x)$ and $F_i(x)$ in definitions of $\tilde{\mathcal{H}}_\nu^\mu(m,n)$ and $\bar{\mathcal{H}}_\nu^\mu(m,n)$, respectively, are necessary. Otherwise, we can construct systems (1_ε) having more medium amplitude limit cycles, see Remark 1 in Section 3.

Theorem 1(b) is a generalization of Theorem 1.1 in [15], where the case $\mu = \nu = 1$ was considered. We note that in such a case $\bar{\mathcal{H}}_1^1(m,n) = \mathcal{H}_1^1(m,n)$. Hence Theorem 1$(b)$ ([15, Theorem 1.1]) gives the exact value of $\mathcal{H}_1^1(m,n)$.

The proof of Theorem 1 is based on computing the maximum number of isolated zeros of the first non-vanishing Poincaré–Pontryagin–Melnikov function of the displacement function of (1_ε), by taking into account the restrictions: $g_i(x)$ odd for $1 \leq i \leq \nu$ and $F_i(x)$ even for $\mu_0 < i \leq \mu$, respectively.

The paper is organized as follows. In Section 2 we recall the definition of the displacement function of (1_ε), as well as the algorithm to compute the Poincaré–Pontryagin–Melnikov functions. Preliminary results that allow us to provide elementary proofs of the main results are given in Section 3. Finally, in Section 4 we will prove Theorem 1.

2. Poincaré–Pontryagin–Melnikov functions

The linear center (1_0) is the Hamiltonian system associated to the polynomial $H = (x^2+y^2)/2$; hence its cycles are the circles $\gamma_c = \{H - c = 0\}$ with $c > 0$. By using c as a parameter, the first return map of (1_ε) can be expressed in terms of ε and c: $\mathcal{P}(\varepsilon, c)$. Therefore the corresponding *displacement function* $L(\varepsilon,c) = \mathcal{P}(\varepsilon,c) - c$ is analytic for small enough ε and can be written as the power series in ε

$$L(\varepsilon,c) = \varepsilon L_1(c) + \varepsilon^2 L_2(c) + O(\varepsilon^3), \quad (2)$$

where $L_i(c)$ with $i \geq 1$ is the *Poincaré–Pontryagin–Melnikov function* of order i, which is defined for $c \geq 0$.

Let $L_k(c)$ with $k \geq 1$ be the first non-vanishing coefficient in (2). The zeros of $L_k(c)$ are important in the study of medium amplitude limit cycles of (1_ε) because of the *Poincaré–Pontryagin–Andronov criterion*: The maximum number of isolated zeros, counting multiplicities, of $L_k(c)$ is an upper bound for $\mathcal{H}_\nu^\mu(m,n)$. Furthermore each simple zero c_0 of $L_k(c)$ corresponds to one and only one limit cycle of (1_ε) with ε small enough bifurcating from the cycle γ_{c_0}.

We know from [7] that $L_k(c)$ has at most $[k(\max\{n,m\}-1)/2]$ positive zeros, counting multiplicities. However, this result does not give the value of $\mathcal{H}_\nu^\mu(m,n)$ because the upper bound for k depending on μ, ν, m, and n is unknown.

Now, we will recall the algorithm to compute the functions $L_i(c)$. System (1_ε) can be written as

$$\dot{x} = y, \qquad \dot{y} = -x + \varepsilon\left(g_1(x) + f_1(x)y\right) + \varepsilon^2\left(g_2(x) + f_2(x)y\right) + \cdots,$$

where $f_i(x) = F_i'(x)$, or equivalently as

$$(3_\varepsilon) \qquad dH - \varepsilon\omega_1 - \varepsilon^2\omega_2 - \cdots = 0,$$

where $\omega_i = (g_i(x) + f_i(x)y)\,dx$, and $\omega_i \equiv 0$ for $i > \max\{\mu,\nu\}$.

As we know, $L_1(c)$ is given by the classical Poincaré–Pontryagin formula

$$L_1(c) = \int_{\gamma_c} \omega_1.$$

The result for computing the higher order Poincaré–Pontryagin–Melnikov functions is the following:

Theorem 2. (Françoise–Iliev–Yakovenko algorithm [2], [7], [16]**).** *If $k \geq 2$ and $L_1(c) \equiv \cdots \equiv L_{k-1}(c) \equiv 0$, then there are polynomials $q_1,\ldots,q_{k-1}$ and $Q_1,\ldots,Q_{k-1}$ such that $\Omega_1 = dQ_1 + q_1 dH,\ldots,\Omega_{k-1} = dQ_{k-1} + q_{k-1}dH$, and*

$$L_k(c) = \int_{\gamma_c} \Omega_k,$$

where

$$\Omega_1 = \omega_1, \text{ and } \Omega_l = \omega_l + \sum_{i+j=l} q_i\omega_j \text{ with } i,j \geq 1 \text{ for } 2 \leq l \leq k.$$

The proof of this result easily follows from the Poincaré–Pontryagin formula, and the Ilyashenko–Gavrilov theorem ([8], [1]): If $\int_{\gamma_c}\omega = 0$ for all $c \geq 0$, then $\omega = dQ + qdH$, where Q and q are polynomials, and by applying an induction argument. For a detailed proof, see for instance [7], [9].

To simplify the computation of the Poincaré–Pontryagin–Melnikov functions, we will give some properties of ω_i.

3. Preliminary results

For computing $L_k(c)$ for (1_ε) we will use the following two elementary lemmas whose proof is omitted.

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Lemma 3. *Let P be a polynomial in the ring $\mathbb{R}[x^2,H]$. We define $\deg_2 P$ to be the degree of P in $\mathbb{R}[x^2,H]$.*

(*a*) *For $i,j\geq 0$ there are homogeneous polynomials Q_{ij}, $q_{ij}\in\mathbb{R}[x^2,H]$ with $\deg_2 Q_{ij}=i+j$ and $\deg_2 q_{ij}=i+j-1$, such that*

$$H^i x^{2j}dx = d\left(xQ_{ij}\right)+\left(xq_{ij}\right)dH \quad \text{or} \quad H^i x^{2j+1}dx = d\left(x^2Q_{ij}\right)+\left(x^2q_{ij}\right)dH.$$

If $i=0$, then $q_{ij}\equiv 0$.

(*b*) *For $i,j\geq 0$ there are homogeneous polynomials Q_{ij}, $q_{ij}\in\mathbb{R}[x^2,H]$ with $\deg_2 Q_{ij}=i+j+1$ and $\deg_2 q_{ij}=i+j$, such that*

$$H^i x^{2j+1}ydx = d\left(yQ_{ij}\right)+\left(yq_{ij}\right)dH.$$

(*c*) *For $i,j\geq 0$ we have* $\displaystyle\int_{\gamma_c} H^i x^{2j}ydx = \frac{-\pi c}{2^j(2j+1)}\binom{2(j+1)}{j+1}c^{i+j}.$

Lemma 4. *If $\omega\in\mathcal{A}:=\left\{(xA+xyB)\,dx\,\middle|\,A,B\in\mathbb{R}[x^2,H]\right\}$ and $q\in\mathcal{S}:=\left\{x^2q_1+yq_2\,\middle|\,q_1,q_2\in\mathbb{R}[x^2,H]\right\}$, then $q\omega\in\mathcal{A}$.*

The next two results are straightforward consequences of these two previous lemmas.

Corollary 5. *If $\omega\in\mathcal{A}$, then $\int_{\gamma_c}\omega\equiv 0$, $\omega=dQ+qdH$ with $q\in\mathcal{S}$, and $q\omega\in\mathcal{A}$.*

Corollary 6. *If* $P\left(x^2\right)=\displaystyle\sum_{r=0}^{d} p_r x^{2r}\in\mathbb{R}\left[x^2\right]$, *then*

$$\int_{\gamma_c} P\left(x^2\right)ydx = -\pi c\left(\sum_{r=0}^{d}\binom{2(r+1)}{r+1}\frac{p_r}{2^r(2r+1)}c^r\right).$$

The following two lemmas will be important in the proof of Theorem 1.

Lemma 7. *Suppose Theorem 2. If $\Omega_l\in\mathcal{A}$ for $1\leq l\leq k-1$, then $\omega_l\in\mathcal{A}$ for $1\leq l\leq k-1$, and $L_k(c)=\int_{\gamma_c}\omega_k$.*

Proof. We proceed by induction on k. If $k=2$, then $\Omega_1=\omega_1\in\mathcal{A}$. Hence $\omega_1=dQ_1+q_1dH$, $q_1\omega_1\in\mathcal{A}$, and $\int_{\gamma_c}q_1\omega_1\equiv 0$ by Corollary 5. Since $L_2(c)=\int_{\gamma_c}\Omega_2=\int_{\gamma_c}\omega_2+\int_{\gamma_c}q_1\omega_1$ because of Theorem 2, $L_2(c)=\int_{\gamma_c}\omega_2$.

We assume that the lemma is true for $k-1$, and we will prove it for k. By assumption, $\Omega_l\in\mathcal{A}$ for $1\leq l\leq k-1$. Then, by Corollary 5, $\Omega_l=dQ_l+q_ldH$ with $q_l\in\mathcal{S}$ for all $1\leq l\leq k-1$. In addition, by the induction hypothesis, $\omega_l\in\mathcal{A}$ for $1\leq l\leq k-2$. Thus, $\overline{\Omega}_{k-1}:=\sum_{i+j=k-1}q_i\omega_j$ with $i,j\geq 1$ is an element of $\mathcal{A}$ following Lemma 4. Since $\omega_{k-1}=\Omega_{k-1}-\overline{\Omega}_{k-1}$, $\omega_{k-1}\in\mathcal{A}$. Hence it is clear that $\overline{\Omega}_k:=\sum_{i+j=k}q_i\omega_j$ with $i,j\geq 1$ is an element of $\mathcal{A}$, which implies that $\int_{\gamma_c}\overline{\Omega}_k\equiv 0$ by Corollary 5. Finally, from Theorem 2 we have $L_k(c)=\int_{\gamma_c}\omega_k+\int_{\gamma_c}\overline{\Omega}_k$. Therefore $L_k(c)=\int_{\gamma_c}\omega_k$. □

Before announce next lemma, we note that each polynomial $h(x) = \sum_{r=0}^{m-1} a_r x^r$ of degree $m-1$ can be written as

$$h(x) = \hat{h}\big(x^2\big) + x\tilde{h}\big(x^2\big), \tag{4}$$

where

$$\hat{h}\big(x^2\big) = \sum_{r=0}^{\left[\frac{m-1}{2}\right]} a_{2r+1} x^{2r}, \qquad \text{and} \qquad \tilde{h}\big(x^2\big) = \sum_{r=0}^{\left[\frac{m-2}{2}\right]} a_{2r+2} x^{2r}.$$

Lemma 8. *Let* $\omega = (g(x) + f(x)y)\,dx$*, where* $f(x) = \sum_{r=0}^{m-1} a_r x^r$ *and* $g(x) = \sum_{s=0}^{n} b_s x^s$.

(*a*) $$\int_{\gamma_c} \omega = \int_{\gamma_c} \hat{f}\big(x^2\big) y dx = -\pi c \left(\sum_{r=0}^{\left[\frac{m-1}{2}\right]} \binom{2(r+1)}{r+1} \frac{a_{2r+1}}{2^r(2r+1)} c^r \right).$$

(*b*) *If* $\int_{\gamma_c} \omega \equiv 0$*, then* $\omega = dQ + (y\bar{q})dH$ *with* $\bar{q} \in \mathbb{R}\left[x^2, H\right]$ *of degree* $\deg_2 \bar{q} = [(m-2)/2]$*, and*

$$\int_{\gamma_c} (y\bar{q})\,\omega = \int_{\gamma_c} \bar{q}\hat{g}\big(x^2\big) y dx = -\pi c \sum_{s=0}^{\left[\frac{n}{2}\right]} \left(\sum_{r=0}^{\left[\frac{m-2}{2}\right]} \binom{2(s+r+1)}{s+r+1} \frac{(b_{2s})(a_{2r+2})}{2^{s+r}(2s+1)} c^{s+r} \right).$$

(*c*) $\int_{\gamma_c} (y\bar{q})\,\omega \equiv 0$ *if and only if* $\bar{q} \equiv 0$*, or* $\hat{g}(x^2) \equiv 0$.

Proof. (*a*). By (*a*) and (*b*) of Lemma 3, $\int_{\gamma_c} \omega = \int_{\gamma_c} \hat{f}(x^2) y dx$. Thus, the statement follows from Corollary 6.

(*b*). If $\int_{\gamma_c} \omega \equiv 0$, then $\hat{f}(x^2) \equiv 0$ by statement (*a*). This property implies that $\omega = g(x)dx + x\tilde{f}(x^2)ydx = d\left(\int g(x)dx\right) + \sum_{r=0}^{\left[\frac{m-2}{2}\right]} a_{2r+2} x^{2r+1} y dx$ by (4). From Lemma 3(*b*) we obtain $x^{2r+1} y dx = d\left(y\overline{Q}_r\right) + (y\bar{q}_r)\,dH$, thus

$$\omega = d\left(\int g(x)dx + y\left(\sum_{r=0}^{\left[\frac{m-2}{2}\right]} a_{2r+2}\overline{Q}_r \right) \right) + \left(y\left(\sum_{r=0}^{\left[\frac{m-2}{2}\right]} a_{2r+2}\bar{q}_r \right) \right) dH = dQ + (y\bar{q})\,dH,$$

where $\overline{Q}_r, \bar{q}_r, \bar{q} \in \mathbb{R}[x^2, H]$ are homogeneous and $\deg_2 \bar{q} = \left[\frac{m-2}{2}\right]$. Moreover, a simple computation shows that

$$\bar{q}_r = 2\sum_{i=0}^{r} \binom{r+1}{i} \left(\frac{r+1-i}{2i+1} \right) (2H)^{r-i} \left(x^2 - 2H\right)^i. \tag{5}$$

As $(y\bar{q})\,\omega = \bar{q}\hat{g}(x^2) y dx + \bar{q}\tilde{g}(x^2) x y dx + \bar{q}\tilde{f}(x^2) x y^2 dx$ and

$$\bar{q}\tilde{f}\big(x^2\big) x y^2 dx = \bar{q}\tilde{f}\big(x^2\big) x \left(2H - x^2\right) dx,$$

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it follows that $(y\overline{q})\,\omega = \overline{q}\hat{g}(x^2)ydx + dQ_2 + q_2dH$ because of statements (a) and (b) of Lemma 3. Hence we obtain

$$\int_{\gamma_c} (y\overline{q})\,\omega = \int_{\gamma_c} \overline{q}\hat{g}\big(x^2\big)ydx = \int_{\gamma_c} \left(\sum_{r=0}^{[(m-2)/2]} a_{2r+2}\overline{q}_r\right)\left(\sum_{s=0}^{[n/2]} b_{2s}x^{2s}\right) ydx.$$

By using expression (5) of $\overline{q}_r$, a straightforward computation, and Lemma 3(c) we obtain the formula given in the statement. Finally, statement (c) follows from the formula given in statement (b). □

Remark 1. System (1_ε) with $\mu = \nu = 1$, $F_1(x) = -x^2$, and $g_1(x) = 1 - x^2$ does not satisfy the hypothesis in definition of $\tilde{\mathcal{H}}^\mu_\nu(m, n)$ because $g_1(x)$ is not an odd function. Here $m = n = 2$ and from Theorem 1(a) it follows that $\tilde{\mathcal{H}}^1_1(2, 2) = 0$; however, for ε small enough, this system has one medium amplitude limit cycle. Indeed, we need only to prove that the first non-vanishing coefficient of the displacement function (2), associated to the system, has a simple positive zero. The system can be written in the form $dH - \varepsilon\omega = 0$ with $\omega = (1 - x^2 - 2xy)dx$. By Lemma 8$(a)$, $L_1(c) \equiv 0$, and by Theorem 2 and Lemma 8(b), $L_2(c) = -\pi c(4 - 2c)$. Now, system (1_ε) with $\mu = \nu = 2$, $F_1(x) = -3x^2$, $F_2(x) = -2x^3$, $g_1(x) = x^2 + x^3$, and $g_2(x) = \left(-5 + 25x^2\right)/6$ does not satisfy the hypothesis in definition of $\bar{\mathcal{H}}^\mu_\nu(m, n)$ because $F_2(x)$ is not an even function. In this case $m = n = 3$ and by Theorem 1(b), $\bar{\mathcal{H}}^2_2(3, 3) = 1$; however, for ε small enough, the resulting system has two medium amplitude limit cycles. Indeed, following previous ideas, and using Theorem 2 and Lemma 8 it is easy to see that $L_1(c) \equiv 0$, $L_2(c) \equiv 0$, and $L_3(c) = -\pi c(c - 1)(c - 2)$.

4. Proof of the main results

We can assume, after a linear change of variables if necessary, that $F_i(0) = 0$ for all $1 \le i \le \mu$. Suppose that $F_i(x) = \sum_{r=1}^{m}(a_{i(r-1)}/r)x^r$ and $g_i(x) = \sum_{s=0}^{n} b_{is}x^s$. Thus, $f_i(x) = F_i'(x) = \sum_{r=0}^{m-1} a_{ir}x^r$ and $g_i(x)$ can be written as $f_i(x) = \hat{f}_i(x^2) + x\tilde{f}_i(x^2)$ and $g_i(x) = \hat{g}_i(x^2) + x\tilde{g}_i(x^2)$, respectively, according to (4).

Proof of Theorem 1. (a). By hypothesis, $g_i(x)$ is odd for $1 \le i \le \nu$, which means that $g_i(x) = x\tilde{g}_i(x^2)$ for $1 \le i \le \nu$. Let $L_k(c)$ be the first non-vanishing Poincaré–Pontryagin–Melnikov function in (2). If $k = 1$, then the theorem is true. Indeed, we have $L_1(c) = \int_{\gamma_c} \omega_1 = \int_{\gamma_c} x\tilde{g}_i(x^2)dx + \int_{\gamma_c} \hat{f}_1(x^2)ydx + \int_{\gamma_c} \tilde{f}_1(x^2)xydx$, and as $\int_{\gamma_c} x\tilde{g}_i(x^2)dx \equiv 0$, and $\int_{\gamma_c} \tilde{f}_1(x^2)xydx \equiv 0$ by Corollary 5, we obtain $L_1(c) = \int_{\gamma_c} \hat{f}_1(x^2)ydx$. From (4) we have $\deg_2 \hat{f}_1(x^2) = [(m-1)/2]$; hence $L_1(c)$ has at most $[(m-1)/2]$ positive zeros because of Corollary 6. Moreover, we can choose suitable coefficients of $F_1(x)$ in such a way that $L_1(c)$ has exactly $[(m-1)/2]$ simple positive zeros. Therefore, by applying the Poincaré–Pontryagin–Andronov criterion it follows that $\tilde{\mathcal{H}}^\mu_\nu(m, n) = [(m-1)/2]$.

Suppose then that $k \geq 2$ and we are therefore in the hypothesis of Theorem 2. If $\Omega_l \in \mathcal{A}$ for $1 \leq l \leq k-1$, then $L_k(c) = \int_{\gamma_c} \omega_k$ by Lemma 7, and by applying the same idea as in previous paragraph, we obtain $\tilde{\mathcal{H}}_\nu^\mu(m,n) = [(m-1)/2]$. Accordingly, it remains to prove that $\Omega_l \in \mathcal{A}$ for $1 \leq l \leq k-1$.

We proceed by induction on k. If $k = 2$, then $L_1(c) \equiv 0$, which implies that $\Omega_1 = \left(x\tilde{g}_1(x^2) + xy\tilde{f}_1(x^2)\right) dx \in \mathcal{A}$. We now assume that the assertion is true for $k-2$, and we will prove it for $k-1$. By induction hypothesis, $\Omega_i \in \mathcal{A}$ for $1 \leq i \leq k-2$, which implies that $\Omega_i = dQ_i + q_i dH$ with $q_i \in \mathcal{S}$ for $1 \leq i \leq k-2$ by Corollary 5. Furthermore, by Lemma 7, $\omega_j \in \mathcal{A}$ for $1 \leq j \leq k-2$. Hence $\overline{\Omega}_{k-1} := \sum_{i+j=k-1} q_i\omega_j$ with $1 \leq i,\ j \leq k-2$ is an element of $\mathcal{A}$ because of Lemma 4. Since $\Omega_{k-1} = \omega_{k-1} + \overline{\Omega}_{k-1}$, $L_{k-1}(c) = \int_{\gamma_c} \Omega_{k-1} = \int_{\gamma_c} \omega_{k-1} = \int_{\gamma_c} \hat{f}_{k-1}(x^2)ydx \equiv 0$, whence $\omega_{k-1} = \left(x\tilde{g}_{k-1}(x^2) + xy\tilde{f}_{k-1}(x^2)\right) dx \in \mathcal{A}$. Therefore $\Omega_{k-1} \in \mathcal{A}$, which completes the proof of statement (a).

(b). First, we will note two properties concerning ω_i and $\int_{\gamma_c} \omega_i$ which we will use along the proof. Second, we will split the proof into two cases: m odd and m even.

For $1 \leq i < \mu_0$ the 1-form $\omega_i = g_i(x)dx$ is exact, that is, $\omega_i = dQ_i + q_i dH$ with $q_i \equiv 0$. Hence, by Theorem 2, $\Omega_i = \omega_i$ and $L_i(c) = \int_{\gamma_c} \Omega_i \equiv 0$ for $1 \leq i < \mu_0$, and $L_{\mu_0}(c) = \int_{\gamma_c} \Omega_{\mu_0} = \int_{\gamma_c} \omega_{\mu_0}$. On the other hand, since $F_i(x)$ is even for $\mu_0 < i \leq \mu$, $f_i(x) = x\tilde{f}_i(x^2)$ for $\mu_0 < i \leq \mu$. Thus, $\omega_i = d\left(\int g_i(x)dx\right) + x\tilde{f}_i(x^2)ydx$ for $i > \mu_0$, and as $x^{2r+1}ydx = d\left(y\overline{Q}_{0r}\right) + (y\bar{q}_{0r})\, dH$ because of Lemma 3(b), we conclude that $\omega_i = d\left(\bar{Q}_i\right) + (y\bar{q}_i)\, dH$; of course $\bar{q}_i \equiv 0$ for $i > \mu$. Therefore $\int_{\gamma_c} \omega_i \equiv 0$ for all $i > \mu_0$.

Case m odd. If m is odd, then $\deg F_{\mu_0}(x) = m$ because $F_i(x)$ is an even polynomial for $\mu_0 < i \leq \mu$. Since $F'_{\mu_0}(x) = f_{\mu_0}(x) = \hat{f}_{\mu_0}(x^2) + x\tilde{f}_{\mu_0}(x^2)$ has an even degree, $\hat{f}_{\mu_0}(x^2) \not\equiv 0$. Hence, from Lemma 8$(a)$ it follows that $L_{\mu_0}(c) = \int_{\gamma_c} \omega_{\mu_0} = \int_{\gamma_c} \hat{f}_{\mu_0}(x^2)ydx \not\equiv 0$, and it has at most $[(m-1)/2]$ positive zeros, counting multiplicities; moreover, we can choose suitable coefficients of $F_{\mu_0}(x)$ in such a way that $L_{\mu_0}(c)$ has exactly $[(m-1)/2]$ simple positive zeros. Therefore by the Poincaré–Pontryagin–Andronov criterion, $\bar{\mathcal{H}}_\nu^\mu(m,n) = [(m-1)/2]$.

Case m even. Let $L_k(c)$ be the first non-vanishing Poincaré–Pontryagin–Melnikov function of (2). If $k = \mu_0$, then $L_{\mu_0}(c)$ has at most $[(m-1)/2]$ positive zeros, counting multiplicities, because of Lemma 8(a). Since m is even, $[(m-1)/2] \leq [m/2] + [n/2] - 1$. Hence $L_{\mu_0}(c)$ has at most $[m/2] + [n/2] - 1$ positive zeros, counting multiplicities.

We claim that if $k \geq \mu_0 + 1$, then $\omega_1, \ldots, \omega_{k-1-\mu_0} \in \mathcal{A}$, $\Omega_i = dQ_i + q_i dH$ with $q_i \in \mathcal{S}$ for $\mu_0 \leq i \leq k-1$, and $L_k(c) = \int_{\gamma_c} (y\bar{q}_{\mu_0})\, \omega_{k-\mu_0} = \int_{\gamma_c} \bar{q}_{\mu_0}\hat{g}_{k-\mu_0}(x^2)ydx$. By assuming that this assertion is true and by applying Lemma 8(b) we conclude that $L_k(c)$ has at most $[m/2] + [n/2] - 1$ positive zeros, counting multiplicities;

moreover, we can choose suitable coefficients of $\bar{q}_{\mu_0}$ and $\hat{g}_{k-\mu_0}(x^2)$ in such a way that $L_k(c)$ has exactly $[m/2]+[n/2]-1$ simple positive zeros. Thus, by the Poincaré–Pontryagin–Andronov criterion, $\bar{\mathcal{H}}^{\mu}_{\nu}(m,n)=[m/2]+[n/2]-1$. Therefore, to finish the proof of statement (b) we need only to confirm the assertion, which we prove next by proceeding by induction on k.

If $k=\mu_0+1$, then we will prove that $\Omega_{\mu_0}=dQ_{\mu_0}+q_{\mu_0}dH$ with $q_{\mu_0}\in\mathcal{S}$, and that $L_{\mu_0+1}(c)=\int_{\gamma_c}\bar{q}_{\mu_0}\hat{g}_1(x^2)ydx$. We know that $\Omega_{\mu_0}=\omega_{\mu_0}$, and from Lemma 8$(b)$ it follows that $\Omega_{\mu_0}=\omega_{\mu_0}=dQ_{\mu_0}+q_{\mu_0}dH$, where $q_{\mu_0}=y\bar{q}_{\mu_0}\not\equiv 0\in\mathcal{S}$. On the other hand, by Theorem 2, $L_{\mu_0+1}(c)=\int_{\gamma_c}\Omega_{\mu_0+1}$, where $\Omega_{\mu_0+1}=\omega_{\mu_0+1}+q_1\omega_{\mu_0}+\cdots+q_{\mu_0-1}\omega_2+q_{\mu_0}\omega_1$. Since $q_i\equiv 0$ for $1\le i<\mu_0$, $\Omega_{\mu_0+1}=\omega_{\mu_0+1}+q_{\mu_0}\omega_1$. Moreover, since $\int_{\gamma_c}\omega_{\mu_0+1}\equiv 0$, $L_{\mu_0+1}(c)=\int_{\gamma_c}(y\bar{q}_{\mu_0})\,\omega_1=\int_{\gamma_c}\bar{q}_{\mu_0}\hat{g}_1(x^2)ydx$.

If $k=\mu_0+2$, then $L_{\mu_0+1}(c)=\int_{\gamma_c}(y\bar{q}_{\mu_0})\,\omega_1=\int_{\gamma_c}\bar{q}_{\mu_0}\hat{g}_1(x^2)ydx\equiv 0$. Since $\bar{q}_{\mu_0}\not\equiv 0$, $\hat{g}_1(x^2)\equiv 0$ by Lemma 8(c). This implies that $\Omega_1=\omega_1\in\mathcal{A}$, and by Corollary 5, $\Omega_1=dQ_1+q_1dH$ with $q_1\in\mathcal{S}$. Moreover, we know that $\omega_{\mu_0}=dQ_{\mu_0}+q_{\mu_0}dH$ with $q_{\mu_0}=y\bar{q}_{\mu_0}\in\mathcal{S}$, and $\omega_{\mu_0+1}=d\left(\bar{Q}_{\mu_0+1}\right)+(y\bar{q}_{\mu_0+1})\,dH$. Thus, $\Omega_{\mu_0+1}=\omega_{\mu_0+1}+q_{\mu_0}\omega_1=dQ_{\mu_0+1}+q_{\mu_0+1}dH$ with $q_{\mu_0+1}\in\mathcal{S}$ because of Corollary 5. On the other hand, from Theorem 2 we have

$$L_{\mu_0+2}(c)=\int_{\gamma_c}\omega_{\mu_0+2}+\int_{\gamma_c}q_1\omega_{\mu_0+1}+\int_{\gamma_c}q_2\omega_{\mu_0}+\cdots+\int_{\gamma_c}q_{\mu_0}\omega_2+\int_{\gamma_c}q_{\mu_0+1}\omega_1.$$

As $\omega_1\in\mathcal{A}$ and $q_{\mu_0+1}\in\mathcal{S}$, then we have $q_{\mu_0+1}\omega_1\in\mathcal{A}$ following Lemma 4 and $\int_{\gamma_c}q_{\mu_0+1}\omega_1\equiv 0$ by Corollary 5. In addition, we know that $q_i\equiv 0$ for $1\le i<\mu_0$ and $\int_{\gamma_c}\omega_{\mu_0+2}\equiv 0$. Hence $L_{\mu_0+2}(c)=\int_{\gamma_c}(y\bar{q}_{\mu_0})\,\omega_2=\int_{\gamma_c}\bar{q}_{\mu_0}\hat{g}_2(x^2)ydx$.

We now assume that the assertion holds for $k-1$, and we will prove it for k. By Theorem 2, $L_k(c)=\int_{\gamma_c}\Omega_k$, where

$$\begin{aligned}\Omega_k=\omega_k+q_1\omega_{k-1}+\cdots+q_{\mu_0-1}\omega_{k+1-\mu_0}+q_{\mu_0}\omega_{k-\mu_0}\\ +q_{\mu_0+1}\omega_{k-1-\mu_0}+\cdots+q_{k-2}\omega_2+q_{k-1}\omega_1.\end{aligned}$$

Since $q_i\equiv 0$ for $1\le i<\mu_0$, $\Omega_k=\omega_k+q_{\mu_0}\omega_{k-\mu_0}+q_{\mu_0+1}\omega_{k-1-\mu_0}+\cdots+q_{k-2}\omega_2+q_{k-1}\omega_1$.

On the other hand, from the induction hypothesis we have $\omega_1,\ldots,\omega_{k-2-\mu_0}\in\mathcal{A}$, $\Omega_i=dQ_i+q_idH$ with $q_i\in\mathcal{S}$ for $\mu_0\le i\le k-2$, and $L_{k-1}(c)=\int_{\gamma_c}\bar{q}_{\mu_0}\hat{g}_{k-1-\mu_0}(x^2)ydx$. Since $L_{k-1}(c)\equiv 0$, $\hat{g}_{k-1-\mu_0}(x^2)\equiv 0$ because of Lemma 8(c), which implies that $\omega_{k-1-\mu_0}\in\mathcal{A}$. Therefore, $q_{\mu_0}\omega_{k-1-\mu_0}+\cdots+q_{k-3}\omega_2+q_{k-2}\omega_1\in\mathcal{A}$ by Lemma 4. Moreover, we have $\omega_{k-1}=d\left(\bar{Q}_{k-1}\right)+(y\bar{q}_{k-1})\,dH$, and by applying Corollary 5 we obtain

$$\Omega_{k-1}=\omega_{k-1}+q_{\mu_0}\omega_{k-1-\mu_0}+\cdots+q_{k-3}\omega_2+q_{k-2}\omega_1=dQ_{k-1}+q_{k-1}dH,\ \text{with } q_{k-1}\in\mathcal{S}.$$

Hence $q_{\mu_0+1}\omega_{k-1-\mu_0}+\cdots+q_{k-2}\omega_2+q_{k-1}\omega_1\in\mathcal{A}$ by Lemma 4. In addition, $\omega_k=d\left(\bar{Q}_k\right)+(y\bar{q}_k)\,dH$. Thus, we obtain $L_k(c)=\int_{\gamma_c}q_{\mu_0}\omega_{k-\mu_0}=\int_{\gamma_c}(y\bar{q}_{\mu_0})\,\omega_{k-\mu_0}=\int_{\gamma_c}\bar{q}_{\mu_0}\hat{g}_{k-\mu_0}(x^2)ydx$. □

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References

[1] L. Gavrilov, *Petrov modules and zeros of Abelian integrals.* Bull. Sci. Math. **122** (1998), no. 8, 571–584.

[2] J. P. Françoise, *Successive derivatives of a first return map, application to the study of quadratic vector fields.* Ergodic Theory Dynam. Systems **16** (1996), no. 1, 87–96.

[3] C. Christopher and C. Li, *Limit Cycles of Differential Equations.* Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2007.

[4] A. Gasull, *Differential equations that can be transformed into equations of Liénard type*, in: Actas del XVLL Colloquio Brasileiro de Matematica, 1989.

[5] A. Gasull and J. Torregrosa, *A relation between small amplitude and big limit cycles.* Rocky Mountain J. Math. **31** (2001), no. 4, 1277–1303.

[6] N. Glade, L. Forest, and J. Demongeot, *Liénard systems and potential-Hamiltonian decomposition. III. Applications.* C. R. Math. Acad. Sci. Paris **344** (2007), no. 4, 253–258.

[7] I. D. Iliev, *The number of limit cycles due to polynomial perturbations of the harmonic oscillator.* Math. Proc. Cambridge Phil. Soc. **127** (1999), 317–322.

[8] Y. Ilyashenko, *The origin of limit cycles under perturbations of the equations* $\frac{dw}{dz} = -\frac{R_z}{R_w}$ *where* $R(z,w)$ *is a polynomial.* USSR Math. Sbornik **78 (120)** (1969), no. 3, 353–364.

[9] Y. Ilyashenko and S. Yakovenko, *Lectures on Analytic Differential Equations.* Graduate Studies in Mathematics, **86**. American Mathematical Society, Providence, RI, 2008.

[10] G. Leonov, *Hilbert's 16th problem for quadratic systems. New methods based on a transformation to the Liénard equation.* Internat. J. Bifur. Chaos Appl. Sci. Engrg. **18** (2008), no. 3, 877–884.

[11] A. Lins, W. de Melo and C. C. Pugh, *On Liénard's equation.* Lecture Notes in Math., Vol. **597**, Springer, Berlin, (1977) 335–357.

[12] J. Llibre, A. C. Mereu and M. A. Teixeira, *Limit cycles of the generalized polynomial Liénard differential equations.* Math. Proc. Cambridge Philos. Soc. **148** (2010), no. 2, 363–383.

[13] J. Llibre, *A survey on the limit cycles of the generalization polynomial Liénard differential equations.* Mathematical models in engineering, biology and medicine, 224–233. AIP Conf. Proc., 1124, Amer. Inst. Phys., Melville, NY, 2009.

[14] H. N. Moreira, *Liénard-type equations and the epidemiology of malaria.* Ecological modelling **60** (1992), no. 2, 139–150.

[15] S. Rebollo-Perdomo, *The infinitesimal Hilbert's 16th problem in the real and complex planes.* Qual. Theory Dyn. Syst., **7** (2009), 467–500.

[16] S. Yakovenko, *A geometric proof of the Bautin theorem.* Concerning the Hilbert 16th problem. Amer. Math. Soc. Transl. Ser. 2, **165**. Amer. Math. Soc., Providence, RI, (1995) 203–219.

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