CENTRE DE RECERCA MATEMÀTICA

Preprint núm. 1136
January 2013

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WEIGHTED D-T MODULI REVISITED AND APPLIED

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Abstract. We introduce weighted moduli of smoothness for functions $f \in L_p[-1,1] \cap C^{r-1}(-1,1)$, $r \ge 1$, that have an $(r-1)$st absolutely continuous derivative in $(-1,1)$ and such that $\varphi^r f^{(r)}$ is in $L_p[-1,1]$, where $\varphi(x) = (1-x^2)^{1/2}$. These moduli are equivalent to certain weighted D-T moduli, but our definition is more transparent and simpler. In addition, instead of applying these weighted moduli to weighted approximation, which was the purpose of the original D-T moduli, we apply these moduli to obtain Jackson-type estimates on the approximation of functions in $L_p[-1,1]$ (no weight), by means of algebraic polynomials. We also have some inverse theorems that yield characterization of the behavior of the derivatives of the function by means of its degree of approximation.

1. Definitions

Let $\|\cdot\|_p := \|\cdot\|_{\mathbb{L}_p[-1,1]}$, $1 \le p < \infty$, and $\varphi(x) = \sqrt{1-x^2}$.
For $k \in \mathbb{N}_0$, let

$$\Delta_h^k(f,x,J) := \begin{cases} \displaystyle\sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f(x + (i-k/2)h), & \text{if } x \pm kh/2 \in J\,, \\ 0, & \text{otherwise,} \end{cases}$$

be the kth symmetric difference, and let $\Delta_h^k(f,x) := \Delta_h^k(f,x,[-1,1])$.

Definition 1.1. *Let $1 \le p < \infty$ and $r \in \mathbf{N_0}$. Then for $r \ge 1$, let*

$$\mathbf{B}_p^r := \{\, f \mid f^{(r-1)} \in AC_{loc}(-1,1) \quad \text{and} \quad \|f^{(r)}\varphi^r\|_p < +\infty\},$$

and set $\mathbf{B}_p^0 := \mathbf{L}_p[-1,1]$.

For $f \in \mathbf{B}_p^r$ define

$$\omega_{k,r}^{\varphi}(f^{(r)},t)_p := \sup_{0<h\le t} \left\| \mathcal{W}_{kh}(\cdot) \Delta_{h\varphi(\cdot)}^k (f^{(r)}, \cdot) \right\|_p ,$$

2010 *Mathematics Subject Classification.* 41A10, 41A17, 41A25.

Key words and phrases. Approximation by polynomials in the L_p-norm, Degree of approximation, Jackson-type estimates, moduli of smoothness.

*Supported by NSERC of Canada

Part of this work was done while the first two authors were at the Centre de Recerca Matemàtica, Barcelona.

where

$$\mathcal{W}_\delta(x) := ((1-x-\delta\varphi(x)/2)(1+x-\delta\varphi(x)/2))^{1/2}$$

Denote

$$\begin{aligned}\mathfrak{D}_\delta &:= \left\{x \;\middle|\; 1-\delta\varphi(x)/2 \ge |x|\right\} \setminus \{\pm 1\} \\ &= \left\{x \;\middle|\; |x| \le \frac{4-\delta^2}{4+\delta^2}\right\} = [-1+\mu(\delta), 1-\mu(\delta)],\end{aligned}$$

where

$$\mu(\delta) := 2\delta^2/(4+\delta^2).$$

Hence,

$$\omega^\varphi_{k,r}(f,t)_p = \sup_{0<h\le t} \left\| \mathcal{W}^r_{kh}(\cdot)\Delta^k_{h\varphi(\cdot)}(f^{(r)},\cdot)\right\|_{\mathbb{L}_p(\mathfrak{D}_{kh})}.$$

Observe that $\mathfrak{D}_\delta = \emptyset$ if $\delta > 2$, and note that $\Delta^k_{h\varphi(x)}(f,x)$ is defined to be identically 0 if $x \notin \mathfrak{D}_{kh}$ and that $\mathcal{W}_\delta$ is well defined on $\mathfrak{D}_\delta$.

Remark. *When $r=0$, $\omega^\varphi_{k,0}(f,t)_p$ reduces to the well known kth D-T modulus of smoothness* (*see,* e.g., [1]).

A useful observation is that, if $x \in \mathfrak{D}_\delta$, then

$$\mathcal{W}_\delta(x) \le \varphi(u), \quad \text{for all } u \in [-|x|-\delta\varphi(x)/2, |x|+\delta\varphi(x)/2].$$

Indeed, $x \in \mathfrak{D}_\delta$ implies $\mathcal{W}_\delta(x) \ge 0$, so

$$\varphi^2(u) - \mathcal{W}^2_\delta(x) \ge \varphi^2(|x|+\delta\varphi(x)/2) - \mathcal{W}^2_\delta(x) = (1-|x|-\delta\varphi(x)/2)\delta\varphi(x) \ge 0.$$

In particular,

$$\mathcal{W}_\delta(x) \le \varphi(x), \quad x \in \mathfrak{D}_\delta. \tag{1.1}$$

Also, if $\delta \le 1$, then

$$\varphi(x) \le 2\mathcal{W}_\delta(x), \quad x \in \mathfrak{D}_{2\delta}. \tag{1.2}$$

Indeed, $\delta \le 1$, implies $1+|x| \le 2(1+|x|-\delta\varphi(x)/2)$, and $x \in \mathfrak{D}_{2\delta}$, that is, $\delta\varphi(x) \le 1-|x|$, yields $1-|x| \le 2(1-|x|-\delta\varphi(x)/2)$. Hence,

$$\begin{aligned}\varphi^2(x) &= (1-|x|)(1+|x|) \\ &\le 4(1-|x|-\delta\varphi(x)/2)(1+|x|-\delta\varphi(x)/2) = 4\mathcal{W}^2_\delta(x), \quad x \in \mathfrak{D}_{2\delta}.\end{aligned}$$

We also note that

$$\mathfrak{D}_{\delta_1} \subset \mathfrak{D}_{\delta_2} \quad \text{if } \delta_1 > \delta_2.$$

The first important property of the new moduli is stated in the following lemma.

Lemma 1.2. *If $f \in \mathbf{B}^r_p$, then*

$$\omega_{k,r}(f^{(r)},t)_p \to 0, \quad \textit{if} \quad t \to 0+.$$

Proof. Let $\epsilon > 0$. Then there is $\delta > 0$, such that

$$\int_{[-1,1]\setminus\mathfrak{D}_\delta} |\varphi^r(x) f^{(r)}(x)|^p dx < \Big(\frac{\epsilon}{2^{k+2}}\Big)^p.$$

Set

$$g^{(r)}(x) := \begin{cases} f^{(r)}(x), & \text{if} \quad x \in \mathfrak{D}_\delta, \\ 0, & \text{otherwise.} \end{cases}$$

Since $g^{(r)} \in L_p[-1,1]$, a corresponding property of the ordinary D-T modulus of smoothness implies the existence of $t_0 > 0$, such that

$$\omega_k^\varphi(g^{(r)}, t)_p < \frac{\epsilon}{2}, \quad 0 < t \le t_0.$$

For each $h > 0$, we have

$$\begin{aligned} \left\|\mathcal{W}_{kh}^r(\cdot)\Delta_{h\varphi(\cdot)}^k(f^{(r)},\cdot)\right\|_p &\le \left\|\mathcal{W}_{kh}^r(\cdot)\Delta_{h\varphi(\cdot)}^k(g^{(r)},\cdot)\right\|_p + \left\|\mathcal{W}_{kh}^r(\cdot)\Delta_{h\varphi(\cdot)}^k(f^{(r)}-g^{(r)},\cdot)\right\|_p \\ &\le \left\|\Delta_{h\varphi(\cdot)}^k(g^{(r)},\cdot)\right\|_p + \left\|\mathcal{W}_{kh}^r(\cdot)\Delta_{h\varphi(\cdot)}^k(f^{(r)}-g^{(r)},\cdot)\right\|_p \\ &=: I_1 + I_2. \end{aligned}$$

Now if $h \le t_0$, then

$$I_1 < \frac{\epsilon}{2},$$

and

$$\begin{aligned} I_2 &\le \sum_{i=0}^k \binom{k}{i} \Big(\int_{\mathfrak{D}_{kh}} \Big(\mathcal{W}_{kh}^r(x) |f^{(r)}(x+(i-k/2)h\varphi(x)) \\ &\qquad\qquad - g^{(r)}(x+(i-k/2)h\varphi(x))|\Big)^p dx)^{1/p} \\ &\le \sum_{i=0}^k \binom{k}{i} \Big(\int_{\mathfrak{D}_{kh}} (\varphi^r(x+(i-k/2)h\varphi(x)) |f^{(r)}(x+(i-k/2)h\varphi(x)) \\ &\qquad\qquad - g^{(r)}(x+(i-k/2)h\varphi(x))|)^p dx\Big)^{1/p} \\ &\le 2\sum_{i=0}^k \binom{k}{i} \left(\int_{-1}^1 (\varphi^r(u)|f^{(r)}(u) - g^{(r)}(u)|)^p du\right)^{1/p} \\ &\le 2\sum_{i=0}^k \binom{k}{i} \left(\int_{[-1,1]\setminus\mathfrak{D}_\delta} |\varphi^r(u) f^{(r)}(u)|^p\right)^{1/p} \le \frac{\epsilon}{2}, \end{aligned}$$

where for the second inequality we used, for $x \in \mathfrak{D}_{kh}$, the inequality $\mathcal{W}_{kh}(x) \le \varphi(u(x))$, where $u = u(x) := x + (i-k/2)h\varphi(x))$, and the third inequality follows because $u'(x) > \frac{1}{2}$ when $x \in \mathfrak{D}_{kh}$. This completes the proof. □

A similar modulus of smoothness was defined by the third author (see, *e.g.*, [2]) for $p = \infty$ and functions $f \in C[-1,1]$ which are r times continuously differentiable in $(-1,1)$, and such that $\|f^{(r)}\varphi^r\|_\infty < +\infty$ (this collection is known as the Babenko class $\mathbf{B}^r_\infty$). It was proved that in order to guarantee that that modulus tended to zero as $t \to 0$, it is necessary and sufficient that $\lim_{x\to\pm1} f^{(r)}(x)\varphi^r(x) = 0$. Thus, the space C^r_φ, the collection of all such functions has been introduced. Jackson-type estimates were proved for the approximation in the sup-norm of functions $f \in C^r_\varphi$, by algebraic polynomials. Earlier Jackson-type estimates involving the D-T moduli of smoothness [1], required that $f \in C^r[-1,1]$. The Jackson-type estimates for the bigger space have made it possible to characterize those functions whose shape preserving approximation by polynomials is of the rate $O(n^{-\alpha})$, for $\alpha \geq 2$ (see, *e.g.*, [3]-[5]).

In this paper we extend the definitions to $L_p[-1,1]$, $1 \leq p < \infty$. Note that Lemma 1.2 guarantees the convergence to zero of the modulus as $t \to 0$, for all functions in $\mathbf{B}^r_p$.

Our moduli of smoothness are certain type of weighted D-T moduli (see Section 3 for details). However, we give a more transparent and simpler definition of the moduli, which, in particular, makes their monotonicity in t, self-evident. Moreover, we are not interested in weighted approximation, rather we are interested in applying these moduli to estimates on the non-weighted approximation of $f \in \mathbf{B}^r_p$ (see Section 5 for details). We conclude the paper with some comments and new results on more general weighted D-T moduli.

It turns out that our moduli are equivalent to the following K-functionals.

Definition 1.3. *For $k \in \mathbb{N}$, $r \geq 0$ and $f \in \mathbf{B}^r_p$, denote by*

$$K^\varphi_{k,r}(f^{(r)}, t^k)_p := \inf_{g\in\mathbf{B}^{k+r}_p} (\|(f^{(r)} - g^{(r)})\varphi^r\|_p + t^k\|g^{(k+r)}\varphi^{k+r}\|_p),$$

the K-functional.

Indeed, we will prove the following.

Theorem 1.4. *If $f \in \mathbf{B}^r_p$, then*

$$\omega^\varphi_{k,r}(f^{(r)}, t)_p \sim K^\varphi_{k,r}(f^{(r)}, t^k)_p.$$

The proof requires inequalities in two directions, one direction is given in Lemma 2.2 in Section 2 below, and the other is (3.5) in Section 3.

2. Auxiliary lemmas

Lemma 2.1. *If $g \in \mathbf{B}^{r+1}_p$, then $g \in \mathbf{B}^r_p$.*

Proof. Suppose that we are given $g \in \mathbf{B}^{r+1}_p$. Without loss of generality we can assume that $g^{(r)}(0) = 0$.

First, we consider the case $p = 1$. Since $\varphi(u) \geq \varphi(x)$ for $|u| \leq |x|$, we have

$$\begin{aligned}\left\|\varphi^r g^{(r)}\right\|_1 &= \int_{-1}^{1} \varphi^r(x) \left| \int_0^x g^{(r+1)}(u)\, du \right| dx \\ &\leq \int_{-1}^{1} \frac{1}{\varphi(x)} \left| \int_0^x \varphi^{r+1}(u) |g^{(r+1)}(u)|\, du \right| dx \\ &\leq \left\|\varphi^{r+1} g^{(r+1)}\right\|_1 \int_{-1}^{1} \frac{dx}{\varphi(x)} = \pi \left\|\varphi^{r+1} g^{(r+1)}\right\|_1 .\end{aligned}$$

Suppose now that $p > 1$ and either $r \geq 1$, or $r = 0$ and $p \neq 2$. Then using Hölder inequality we have

$$\begin{aligned}\left\|\varphi^r g^{(r)}\right\|_p^p &= \int_{-1}^{1} \varphi^{rp}(x) \left| \int_0^x g^{(r+1)}(u)\, du \right|^p dx \\ &\leq \int_{-1}^{1} \varphi^{rp}(x) \left| \left(\int_0^x \varphi^{-(r+1)q}(u)\, du \right)^{1/q} \left(\int_0^x |\varphi^{r+1}(u) g^{(r+1)}(u)|^p\, du \right)^{1/p} \right|^p dx \\ &\leq \left\|\varphi^{r+1} g^{(r+1)}\right\|_p^p \int_{-1}^{1} \varphi^{rp}(x) \left| \int_0^x \varphi^{-(r+1)q}(u)\, du \right|^{p/q} dx \\ &= 2 \left\|\varphi^{r+1} g^{(r+1)}\right\|_p^p \int_0^1 \varphi^{rp}(x) \left(\int_0^x \varphi^{-(r+1)q}(u)\, du \right)^{p/q} dx \\ &\leq 2^{1+rp/2} \left\|\varphi^{r+1} g^{(r+1)}\right\|_p^p \int_0^1 (1-x)^{rp/2} \left(\int_0^x (1-u)^{-(r+1)q/2}\, du \right)^{p/q} dx \\ &\leq c \left\|\varphi^{r+1} g^{(r+1)}\right\|_p^p \int_0^1 (1-x)^{p/2-1}\, dx \\ &= c \left\|\varphi^{r+1} g^{(r+1)}\right\|_p^p .\end{aligned}$$

Finally, if $p = 2$ and $r = 0$, then

$$\|g\|_2^2 \leq 2 \left\|\varphi g'\right\|_2^2 \int_0^1 \int_0^x \frac{du}{(1-u)}\, dx = 2 \left\|\varphi g'\right\|_2^2 . \qquad \square$$

Remark. *The above proof actually yields that if* $g \in \mathbf{B}_p^{r+1}$*, then*

$$\left\|\varphi^\gamma g^{(r)}\right\|_p < \infty,$$

for any $\gamma > r - 1$*.*

In order to prove the next lemma we note that,

$$\delta|\varphi'(x)| \leq 1, \quad \text{if} \quad x \in \mathfrak{D}_\delta.$$

Indeed, $x \in \mathfrak{D}_\delta$ means that $\delta\varphi(x)/2 \le 1 - |x| = (1+|x|)^{-1}\varphi^2(x)$, that is $\varphi(x) \ge \frac{\delta}{2}(1+|x|)$. Hence

$$\delta|\varphi'(x)| = \delta\frac{|x|}{\varphi(x)} \le \frac{2|x|}{1+|x|} \le 1.$$

If we put $y(x) := x + \delta\varphi(x)/2$ or $y(x) := x - \delta\varphi(x)/2$, then the above implies

$$y'(x) \ge \frac{1}{2}, \quad x \in \mathfrak{D}_\delta. \tag{2.1}$$

Lemma 2.2. *If* $f \in \mathbf{B}_p^r$, *then*

$$\omega_{k,r}^\varphi(f^{(r)}, t)_p \le cK_{k,r}^\varphi(f^{(r)}, t^k)_p.$$

Proof. Suppose that $r \in \mathbf{N}$ (if $r = 0$ this is the well known DT result). Take any $g \in \mathbf{B}_p^{k+r}$. The previous lemma implies $g \in \mathbf{B}_p^r$, whence

$$\omega_{k,r}^\varphi(f, t)_p \le \omega_{k,r}^\varphi(f^{(r)} - g^{(r)}, t)_p + \omega_{k,r}^\varphi(g^{(r)}, t)_p.$$

Take $h \in [0, t]$. For each $i = 0, \dots, k$ put $y_i(x) := x + (i - k/2)h\varphi(x)$. Then (3.1) yields

$$\begin{aligned}\|\varphi^r(y_i)(f^{(r)}(y_i) - g^{(r)}(y_i))\|_{L_p(\mathfrak{D}_{kh})}^p &= \int_{\mathfrak{D}_{kh}} \varphi^r(y_i(x))|f^{(r)}(y_i(x)) - g^{(r)}(y_i(x))|^p\,dx \\ &\le 2\int_{-1}^1 \varphi^r(y)|f^{(r)}(y) - g^{(r)}(y)|^p\,dy \\ &= 2\|\varphi^r(f^{(r)} - g^{(r)})\|_p.\end{aligned}$$

Now, taking into account that $\mathcal{W}_\delta(x) \le \varphi(y)$ for all $y \in [x - \delta\varphi(x)/2, x + \delta\varphi(x)/2]$ and $0 < \delta \le 2$, we get,

$$\begin{aligned}\omega_{k,r}^\varphi(f^{(r)} - g^{(r)}, t)_p &\le \sup_{0<h\le t}\left\|\sum_{i=0}^k \binom{k}{i}\varphi^r(y_i)\,|f(y_i) - g(y_i)|\right\|_{\mathbb{L}_p(\mathfrak{D}_{kh})} \\ &\le 2^{k+1/p}\left\|\varphi^r(f^{(r)} - g^{(r)})\right\|_p.\end{aligned}$$

To estimate the second term $\omega_{k,r}^\varphi(g^{(r)}, t)_p$ we use the identity

$$\Delta_h^k(f, x) = \int_{-h/2}^{h/2}\cdots\int_{-h/2}^{h/2} f^{(k)}(x + u_1 + \cdots + u_k)du_1\cdots du_k.$$

We have

$$\begin{aligned}\omega_{k,r}^\varphi(g^{(r)}, t)_p^p &= \sup_{0<h\le t}\left\|\mathcal{W}_{kh}^r\Delta_{h\varphi}^k(g^{(r)}, \cdot)\right\|_{\mathbb{L}_p(\mathfrak{D}_{kh})}^p \\ &= \sup_{0<h\le t}\left\|\mathcal{W}_{kh}^r\int_{-h\varphi/2}^{h\varphi/2}\cdots\int_{-h\varphi/2}^{h\varphi/2} g^{(k+r)}(\cdot + u_1 + \cdots + u_k)du_1\cdots du_k\right\|_{\mathbb{L}_p(\mathfrak{D}_{kh})}^p.\end{aligned}$$

By Hölder's inequality (with $1/p+1/q=1$), for each u satisfying $-1 < x+u-h\varphi(x)/2 < x+u+h\varphi(x)/2 < 1$, we have

$$\begin{aligned}\left|\int_{-h\varphi(x)/2}^{h\varphi(x)/2} g^{(k+r)}(x+u+u_k)du_k\right| &= \left|\int_{x+u-h\varphi(x)/2}^{x+u+h\varphi(x)/2} g^{(k+r)}(v)dv\right| \\ &\le \int_{x+u-h\varphi(x)/2}^{x+u+h\varphi(x)/2} \frac{\varphi^{k+r}(v)|g^{(k+r)}(v)|}{\varphi^{k+r}(v)}dv \\ &\le \|\varphi^{k+r}g^{(k+r)}\|_{L_p(\mathcal{A}(x,u))}\|\varphi^{-k-r}\|_{L_q(\mathcal{A}(x,u))}.\end{aligned}$$

where

$$\mathcal{A}(x,u) := \left[x+u-\frac{h}{2}\varphi(x), x+u+\frac{h}{2}\varphi(x)\right].$$

Thus, in order to complete the proof, it suffices to prove the inequality

$$\begin{aligned}&\int_{\mathfrak{D}_{kh}}\Bigg(\mathcal{W}_{kh}^r(x)\int_{-h\varphi(x)/2}^{h\varphi(x)/2}\cdots\int_{-h\varphi(x)/2}^{h\varphi(x)/2}\|\varphi^{-k-r}\|_{\mathbb{L}_q(\mathcal{A}(x,u_1+\cdots+u_{k-1}))} \\ &\qquad\times\|\varphi^{k+r}g^{(k+r)}\|_{L_p(\mathcal{A}(x,u_1+\cdots+u_{k-1}))}du_1\cdots du_{k-1}\Bigg)^p dx \\ &\le ch^{kp}\|g^{(k+r)}\varphi^{k+r}\|_p^p.\end{aligned}$$

To this end we write,

$$\int_{\mathfrak{D}_{kh}} = \int_{\mathfrak{D}_{2kh}} + \int_{(\mathfrak{D}_{kh}\setminus\mathfrak{D}_{2kh})\cap[0,1]} + \int_{(\mathfrak{D}_{kh}\setminus\mathfrak{D}_{2kh})\cap[-1,0]} =: I_1+I_2+I_3.$$

Note that for $x\in\mathfrak{D}_{2\delta}$ and $u\in[x-\delta\varphi(x)/2, x+\delta\varphi(x)/2]$, we have

$$\frac{1}{2}(1-|u|) \le \frac{1}{2}(1-|x|+\delta\varphi(x)/2) \le 1-|x| \le 2(1-|x|-\delta\varphi(x)/2) \le 2(1-|u|),$$

where for the second and third inequalities we applied the fact that $\delta\varphi(x) \le 1-|x|$. similarly

$$\frac{1}{2}(1+|u|) \le 1+|x| \le 2(1+|u|).$$

Hence, for $x\in\mathfrak{D}_{2\delta}$ and $u\in[x-\delta\varphi(x)/2, x+\delta\varphi(x)/2]$,

$$\frac{1}{2}\varphi(u) \le \varphi(x) \le 2\varphi(u). \tag{2.2}$$

Also note, that $\delta\varphi(x)\le 1-|x|$ implies

$$\delta \le \varphi(x). \tag{2.3}$$

So, if $x \in \mathfrak{D}_{2kh}$, then

$$\begin{aligned}
&\mathcal{W}_{kh}^r(x) \int_{-h\varphi(x)/2}^{h\varphi(x)/2} \cdots \int_{-h\varphi(x)/2}^{h\varphi(x)/2} \|\varphi^{-k-r}\|_{\mathbb{L}_q(\mathcal{A}(x,u_1+\cdots+u_{k-1}))}\, du_1 \cdots du_{k-1}\\
&\le \varphi^r(x)(h\varphi(x))^{k-1}\frac{2^{k+r}}{\varphi^{k+r}(x)}(h\varphi(x))^{1/q}\\
&= 2^{k+r}h^{k-1+1/q}\varphi^{1/q-1}(x) = 2^{k+r}h^{k-1/p}\varphi^{-1/p}(x),
\end{aligned}$$

where we applied (1.1). Note that the last computations are also valid for $q = \infty$.

Therefore by (2.2) and (2.3),

$$\begin{aligned}
I_1 &\le \int_{\mathfrak{D}_{2kh}} \left(2^{k+r}h^{k-1/p}\varphi^{-1/p}(x)\|\varphi^{k+r}g^{(k+r)}\|_{L_p([x-kh\varphi(x)/2,x+kh\varphi(x)/2])}\right)^p dx\\
&= 2^{p(k+r)}h^{kp-1}\int_{\mathfrak{D}_{2kh}} \frac{1}{\varphi(x)}\int_{x-kh\varphi(x)/2}^{x+kh\varphi(x)/2} \left|\varphi^{k+r}(u)g^{(k+r)}(u)\right|^p dudx\\
&\le ch^{kp-1}\int_{\mathfrak{D}_{2kh}}\int_{x-kh\varphi(x)/2}^{x+kh\varphi(x)/2} \frac{1}{\varphi(u)+kh/2}\left|\varphi^{k+r}(u)g^{(k+r)}(u)\right|^p dudx\\
&= ch^{kp-1}\int_{a_1}^{a_2}\int_{b_1(u)}^{b_2(u)} \frac{1}{\varphi(u)+kh/2}\left|\varphi^{k+r}(u)g^{(k+r)}(u)\right|^p dxdu\\
&= ch^{kp-1}\int_{a_1}^{a_2}\left|\varphi^{k+r}(u)g^{(k+r)}(u)\right|^p \frac{b_2(u)-b_1(u)}{\varphi(u)+kh/2}du,
\end{aligned}$$

where $-1 < a_1 < a_2 < 1$ and

$$b_2(u) - b_1(u) \le \frac{kh\sqrt{1-u^2+(kh/2)^2}}{1+(kh/2)^2} \le kh(\varphi(u) + (kh/2)).$$

Hence,

$$J_1 \le ch^{kp}\|g^{(k+r)}\varphi^{k+r}\|_p^p.$$

Next we estimate I_2, the estimate of I_3 being similar.

Since $k \ge 1$ and $r \ge 1$, we have $(k+r)q > 2$. First, let $p > 1$, so that $q < \infty$. Then,

$$\begin{aligned}
\|\varphi^{-k-r}\|_{\mathbb{L}_q(\mathcal{A}(x,u_1+\cdots+u_{k-1}))} &= \left(\int_{x+u_1+\cdots+u_{k-1}-\frac{h}{2}\varphi(x)}^{x+u_1+\cdots+u_{k-1}+\frac{h}{2}\varphi(x)} \varphi^{-q(k+r)}(v)dv\right)^{\frac{1}{q}}\\
&\le \left(\int_{-\infty}^{x+u_1+\cdots+u_{k-1}+\frac{h}{2}\varphi(x)} (1-v)^{-q(k+r)/2}(v)dv\right)^{\frac{1}{q}}\\
&= c\left(1 - x - u_1 - \cdots - u_{k-1} - \frac{h}{2}\varphi(x)\right)^{-(k+r)/2+1/q},
\end{aligned}$$

where $c = (q(k+r)/2-1)^{-1/q}$.

For $p=1$, so that $q=\infty$, it follows that

$$\|\varphi^{-k-r}\|_{\mathbb{L}_\infty(\mathcal{A}(x,u_1+\cdots+u_{k-1}))} \le c\left(1-x-u_1-\cdots-u_{k-1}-\frac{h}{2}\varphi(x)\right)^{-(k+r)/2}.$$

Now, when $q\neq 2$ or if k is even, then

$$\begin{aligned}
I_2 &\le c\|\varphi^{k+r}g^{(k+r)}\|_p^p \int_{(\mathfrak{D}_{kh}\setminus\mathfrak{D}_{2kh})\cap[0,1]} \Bigg(\mathcal{W}^r_{kh}(x)\int_{-h\varphi(x)/2}^{h\varphi(x)/2}\cdots\int_{-h\varphi(x)/2}^{h\varphi(x)/2} \\
&\quad (1-x-u_1-\cdots-u_{k-1}-h\varphi(x)/2)^{-(k+r)/2+1/q}du_1\cdots du_{k-1}\Bigg)^p dx \\
&\le c2^{pr/2}\|\varphi^{k+r}g^{(k+r)}\|_p^p \int_{(\mathfrak{D}_{kh}\setminus\mathfrak{D}_{2kh})\cap[0,1]} \Bigg((1-x-kh\varphi(x)/2)^{r/2}\int_{-h\varphi(x)/2}^{h\varphi(x)/2}\cdots \\
&\quad \int_{-h\varphi(x)/2}^{h\varphi(x)/2}(1-x-u_1-\cdots-u_{k-1}-h\varphi(x)/2)^{-k/2-r/2+1/q}du_1\cdots du_{k-1}\Bigg)^p dx \\
&\le c2^{pr/2}\|\varphi^{k+r}g^{(k+r)}\|_p^p \int_{(\mathfrak{D}_{kh}\setminus\mathfrak{D}_{2kh})\cap[0,1]} \Bigg(\int_{-h\varphi(x)/2}^{h\varphi(x)/2}\cdots\int_{-h\varphi(x)/2}^{h\varphi(x)/2} \\
&\quad (1-x-u_1-\cdots-u_{k-1}-h\varphi(x)/2)^{-k/2+1/q}du_1\cdots du_{k-1}\Bigg)^p dx \\
&\le ch^{(k/2-1+1/q)2p}\|\varphi^{k+r}g^{(k+r)}\|_p^p \int_{(\mathfrak{D}_{kh}\setminus\mathfrak{D}_{2kh})\cap[0,1]} dx \\
&\le ch^{kp-2}\|\varphi^{k+r}g^{(k+r)}\|_p^p \int_{(\mathfrak{D}_{kh}\setminus\mathfrak{D}_{2kh})\cap[0,1]} dx \le ch^{kp}\|\varphi^{k+r}g^{(k+r)}\|_p^p,
\end{aligned}$$

where the fourth inequality is obtained by straightforward computations. We integrate sufficiently many times to remove the denominator and we observe that no integration gives a "log" term. Also, for $x\in(\mathfrak{D}_{kh}\setminus\mathfrak{D}_{2kh})\cap[0,1]$, we have $1-x\le ch^2$.

If $q=2$ and k is odd, then in order to avoid an integration that gives a "log" term, we ensure the power $(k+1)/2$, in the denominator, instead of $k/2$, and $k+1$ is even. Namely,

$$\begin{aligned}
I_2 &\le c\|\varphi^{k+r}g^{(k+r)}\|_p^p \int_{(\mathfrak{D}_{kh}\setminus\mathfrak{D}_{2kh})\cap[0,1]} \Bigg((1-x-kh\varphi(x)/2)^{1/2}\int_{-h\varphi(x)/2}^{h\varphi(x)/2}\cdots \\
&\quad \int_{-h\varphi(x)/2}^{h\varphi(x)/2}(1-x-u_1-\cdots-u_{k-1}-h\varphi(x)/2)^{-(k+1)/2+1/q}du_1\cdots du_{k-1}\Bigg)^p dx,
\end{aligned}$$

and we proceed in the same way. This completes the proof. □

3. Weighted D-T moduli

Given a function f and weight w defined on $[0,1]$, such that w satisfies certain growth restrictions near the endpoints of the interval (essentially, w behaves like Jacobi weights near the endpoints, see [1, Definition 8.1.1]), and $wf \in L_p[0,1]$. We will refer to such a weight w as legitimate weight. The following weighted K-functionals of f and weight w are defined in [1, (6.1.1)].

$$K_{k,\psi}(f,t^k)_{w,p} := \inf_{g^{(k-1)}\in \mathrm{AC_{loc}}} \left(\|(f-g)w\|_p + t^k \|w\psi^k g^{(k)}\|_p \right).$$

Also defined are the following moduli can be found in [1, p. 218] (with $D=(0,1)$).

$$\begin{aligned}\omega^k_\psi(f,t)_{w,p} := &\sup_{0<h\le t} \left\| w\Delta^k_{h\psi} f\right\|_{\mathbb{L}_p[t_0^*,1-t_1^*]} \\ &+ \sup_{0<h\le t_0^*} \left\| w\overrightarrow{\Delta}^k_h f\right\|_{\mathbb{L}_p[0,12t_0^*]} + \sup_{0<h\le t_1^*} \left\| w\overleftarrow{\Delta}^k_h f\right\|_{\mathbb{L}_p[1-12t_1^*,1]},\end{aligned}$$

where if $\psi(x)=\sqrt{x(1-x)}$, then $t_0^* = t_1^* = k^2t^2$.

It was shown in [1, Theorem 6.1.1] that

$$M^{-1}\omega^k_\psi(f,t)_{w,p} \le K_{k,\psi}(f,t^k)_{w,p} \le M\omega^k_\psi(f,t)_{w,p}, \tag{3.1}$$

for some $M>1$ and all $0<t\le t_0$.

With an obvious modification for $D=(-1,1)$, namely, $\psi := \varphi$, and the weight $w := \varphi^r$, we have

$$\begin{aligned}\tilde\omega^\varphi_{k,r}(f,t)_p &:= \omega^k_\varphi(f,t)_{\varphi^r,p} \\ &= \sup_{0<h\le t} \left\|\varphi^r \Delta^k_{h\varphi}\right\|_{\mathbb{L}_p[-1+t^*,1-t^*]} \\ &\quad + \sup_{0<h\le t^*} \left\|\varphi^r \overrightarrow{\Delta}^k_h\right\|_{\mathbb{L}_p[-1,-1+At_0^*]} + \sup_{0<h\le t^*} \left\|\varphi^r \overleftarrow{\Delta}^k_h\right\|_{\mathbb{L}_p[1-At^*,1]},\end{aligned}$$

where $t^* := 2k^2t^2$ and A is an absolute constant (for example, $A=12$ as in [1]).

In addition in [1, (8.1.2)] the weighted main part moduli of a function f and a legitimate weight w, such that $wf\in L_p[-1,1]$, are defined by

$$\Omega^r_\varphi(f,t)_{w,p} := \sup_{0<h\le t} \left\| w\Delta^k_{h\varphi} f\right\|_{\mathbb{L}_p[t_0^*,1-t_1^*]}. \tag{3.2}$$

Now, it is readily seen that the K-functional defined in Definition 1.3, satisfies

$$K^\varphi_{k,r}(f,t^k)_p = K_{k,\varphi}(f,t^k)_{\varphi^r,p} = \inf_{g^{(k-1)}\in \mathrm{AC_{loc}}} \left(\|(f-g)\varphi^r\|_p + t^k\|\varphi^{k+r} g^{(k)}\|_p \right).$$

Hence, by virtue of (3.1), we obtain

$$M^{-1}\tilde\omega^\varphi_{k,r}(f,t)_p \le K^\varphi_{k,r}(f,t^k)_p \le M\tilde\omega^\varphi_{k,r}(f,t)_p, \tag{3.3}$$

for some $M>1$ and all $0<t\le t_0$.

A similar quantity to the following averaged modulus was considered in [1, (6.1.9)] (recall that $t^* := 2k^2t^2$):

$$\omega_\varphi^{*k}(f,t)_{w,p} = \left(\frac{1}{t}\int_0^t \int_{-1+t^*}^{1-t^*} |w(x)\Delta_{\tau\varphi(x)}^k(f,x)|^p \, dx \, d\tau\right)^{1/p} + \left(\frac{1}{t^*}\int_0^{t^*} \int_{-1}^{-1+At^*} |w(x)\overrightarrow{\Delta}_u^k(f,x)|^p \, dx \, du\right)^{1/p} + \left(\frac{1}{t^*}\int_0^{t^*} \int_{1-At^*}^{1} |w(x)\overleftarrow{\Delta}_u^k(f,x)|^p \, dx \, du\right)^{1/p}. \tag{3.4}$$

From the statement in [1, p. 57], we conclude that

$$K_{k,r}^\varphi(f,t^k)_p \le M_1 \omega_\varphi^{*k}(f,t)_{\varphi^r,p}.$$

We will apply this inequality to complete the proof of Theorem 1.4. Namely, we still have to show, for $f \in \mathbb{B}_p^r$, that

$$K_{k,r}^\varphi(f^{(r)},t^k)_p \le c\omega_{k,r}^\varphi(f^{(r)},t)_p. \tag{3.5}$$

Indeed, it follows from the following.

Lemma 3.1.

$$\omega_\varphi^{*k}(f^{(r)},t)_{\varphi^r,p} \le c\omega_{k,r}^\varphi(f^{(r)},c(k)t)_p, \quad t < c(k).$$

Proof. First, note that $[-1+t^*, 1-t^*] \subset \mathfrak{D}_{2kt}$ and so using (1.2) we have

$$\begin{aligned} &\frac{1}{t}\int_0^t \int_{-1+t^*}^{1-t^*} |\varphi^r(x)\Delta_{\tau\varphi(x)}^k(f^{(r)},x)|^p \, dx \, d\tau \\ &\le \frac{2^{rp}}{t}\int_0^t \int_{-1+t^*}^{1-t^*} |\mathcal{W}_{k\tau}^r(x)\Delta_{\tau\varphi(x)}^k(f^{(r)},x)|^p \, dx \, d\tau \\ &\le \frac{2^{rp}}{t}\int_0^t \int_{\mathfrak{D}_{2kt}} |\mathcal{W}_{k\tau}^r(x)\Delta_{\tau\varphi(x)}^k(f^{(r)},x)|^p \, dx \, d\tau \\ &\le 2^{rp}\omega_{k,r}^\varphi(f^{(r)},t)_p^p. \end{aligned} \tag{3.6}$$

We now estimate the second term (dealing with the function near -1), the third term being similar.

$$
\begin{aligned}
&\frac{1}{t^*}\int_0^{t^*}\int_{-1}^{-1+At^*}|\varphi^r(x)\overrightarrow{\Delta}_u^k(f^{(r)},x)|^p\,dx\,du\\
&=\frac{1}{t^*}\int_0^{t^*}\int_{-1}^{-1+At^*}|\varphi^r(x)\Delta_u^k(f^{(r)},x+ku/2)|^p\,dx\,du\\
&\le\frac{1}{t^*}\int_0^{t^*}\int_{-1+ku/2}^{-1+(A+k/2)t^*}|\varphi^r(y-ku/2)\Delta_u^k(f^{(r)},y)|^p\,dy\,du\\
&\le\frac{1}{t^*}\int_{-1}^{-1+(A+k/2)t^*}\int_0^{2(y+1)/k}|\varphi^r(y-ku/2)\Delta_u^k(f^{(r)},y)|^p\,du\,dy\\
&=\frac{1}{t^*}\int_{-1}^{-1+(A+k/2)t^*}\int_0^{2(y+1)/(k\varphi(y))}\varphi(y)|\varphi^r(y-kh\varphi(y)/2)\Delta_{h\varphi(y)}^k(f^{(r)},y)|^p\,dh\,dy\\
&\le c\frac{1}{t^*}\int_{-1}^{-1+(A+k/2)t^*}\int_0^{2(y+1)/(k\varphi(y))}\varphi(y)|\mathcal{W}_{kh}^r(y)\Delta_{h\varphi(y)}^k(f^{(r)},y)|^p\,dh\,dy\\
&\le c\frac{1}{\sqrt{t^*}}\int_{-1}^{-1+(A+k/2)t^*}\int_0^{2(y+1)/(k\varphi(y))}|\mathcal{W}_{kh}^r(y)\Delta_{h\varphi(y)}^k(f^{(r)},y)|^p\,dh\,dy\\
&\le c\frac{1}{\sqrt{t^*}}\int_0^{c\sqrt{t^*}}\int_{\mathfrak{D}_{kh}\cap[-1,-1+(A+k/2)t^*]}|\mathcal{W}_{kh}^r(y)\Delta_{h\varphi(y)}^k(f^{(r)},y)|^p\,dy\,dh\\
&\le c\omega_{k,r}^{\varphi}(f^{(r)},ct)_p^p.
\end{aligned}
$$

This completes the proof and establishes (3.5). □

4. Averaged moduli

Definition 4.1. *Let $f\in\mathbb{B}_p^r$, $1\le p<\infty$. We define the averaged modulus as follows:*

$$
\omega_{k,r}^{*\varphi}(f^{(r)},t)_p:=\left(\frac{1}{t}\int_0^t\int_{\mathfrak{D}_{k\tau}}|\mathcal{W}_{k\tau}^r(x)\Delta_{\tau\varphi(x)}^k(f^{(r)},x)|^p\,dx\,d\tau\right)^{1/p}.
$$

It is obvious that

$$
\omega_{k,r}^{*\varphi}(f^{(r)},t)_p\le\omega_{k,r}^{\varphi}(f^{(r)},t)_p.
$$

We will prove the opposite inequality.

Theorem 4.2.

$$
\omega_{k,r}^{\varphi}(f^{(r)},t)_p\le c\omega_{k,r}^{*\varphi}(f^{(r)},t)_p.
$$

Corollary 4.3.

$$
\omega_{k,r}^{\varphi}(f^{(r)},t)_p\sim\omega_{k,r}^{*\varphi}(f^{(r)},t)_p.
$$

Theorem 4.2 is a consequence of the following lemma.

Lemma 4.4.

$$\omega_\varphi^{*k}(f^{(r)},t)_{\varphi^r,p} \le c\omega_{k,r}^{*\varphi}(f^{(r)},c(k)t)_p, \quad t < c(k).$$

Proof. We use the definition (3.4) and estimate each of the three terms. Evidently, $[-1+t^*,1-t^*] \subset \mathfrak{D}_{2kt}$ (recall that $t^* = 2k^2t^2$) and

$$\begin{aligned}\frac{1}{t}\int_0^t \int_{-1+t^*}^{1-t^*} |\varphi^r(x)\Delta_{\tau\varphi(x)}^k(f^{(r)},x)|^p\,dx\,d\tau &\le \frac{c}{t}\int_0^t \int_{\mathfrak{D}_{2kt}} |\mathcal{W}_{k\tau}^r(x)\Delta_{\tau\varphi(x)}^k(f^{(r)},x)|^p\,dx\,d\tau \\ &\le c\omega_{k,r}^{*\varphi}(f^{(r)},t)_p^p.\end{aligned}$$

Also, it was shown in our proof of Lemma 3.1 that the second term can be estimated as follows (one can deal with the third term in similar fashion).

$$\begin{aligned}&\frac{1}{t^*}\int_0^{t^*}\int_{-1}^{-1+At^*} |\varphi^r(x)\overrightarrow{\Delta}_u^k(f^{(r)},x)|^p\,dx\,du \\ &\le c\frac{1}{\sqrt{t^*}}\int_{-1}^{-1+(A+k/2)t^*}\int_0^{2(y+1)/(k\varphi(y))} |\mathcal{W}_{kh}^r(y)\Delta_{h\varphi(y)}^k(f^{(r)},y)|^p\,dh\,dy \\ &\le c\frac{1}{\sqrt{t^*}}\int_0^{c\sqrt{t^*}}\int_{\mathfrak{D}_{kh}} |\mathcal{W}_{kh}^r(y)\Delta_{h\varphi(y)}^k(f^{(r)},y)|^p\,dy\,dh \\ &\le c\omega_{k,r}^{*\varphi}(f^{(r)},c(k)t)_p^p.\end{aligned}$$

Note that $\mathfrak{D}_{kh}$ in the above inequality cannot be replaced with $\mathfrak{D}_\delta$ with $\delta > kh$. □

5. Polynomial approximation in L_p

We begin with a theorem that illustrates the hierarchy between the moduli of smoothness.

Theorem 5.1. *If $f \in \mathbf{B}_p^{r+1}$, $r \in \mathbb{N}_0$ and $1 \le p < \infty$, and $k \ge 2$, then*

$$\omega_{k,r}^\varphi(f^{(r)},t)_p \le ct\omega_{k-1,r+1}^\varphi(f^{(r+1)},t)_p.$$

Proof. First, we observe that it follows by Theorem 1.4 that $\omega_{k,r}^\varphi(f^{(r)},t)_p \sim \omega_\varphi^k(f^{(r)},t)_{\varphi^r,p}$. Thus, by virtue of [1, (6.2.9)], we have

$$\omega_{k,r}^\varphi(f^{(r)},t)_p \le c\omega_\varphi^k(f^{(r)},t)_{\varphi^r,p} \le c\int_0^t (\Omega_\varphi^k(f^{(r)},\tau)_{\varphi^r,p}/\tau)\,d\tau,$$

where $\Omega_\varphi^k(f^{(r)},\tau)_{\varphi^r,p}$ was defined in (3.2), with the weight $w = \varphi^r$. Also, by [1, (6.3.2)], we obtain

$$\Omega_\varphi^k(f^{(r)},t)_{\varphi^r,p} \le ct\Omega_\varphi^{k-1}(f^{(r+1)},t)_{\varphi^{r+1},p}.$$

Hence,

$$\omega_{k,r}^{\varphi}(f^{(r)},t)_p \le c\int_0^t \Omega_{\varphi}^{k-1}(f^{(r+1)},\tau)_{\varphi^{r+1},p}\,d\tau$$
$$\le ct\Omega_{\varphi}^{k-1}(f^{(r+1)},t)_{\varphi^{r+1},p} \le \omega_{\varphi}^{k-1}(f^{(r+1)},t)_{\varphi^{r+1},p},$$

where for the second inequality we used the monotonicity of $\Omega_{\varphi}^{k-1}(f^{(r+1)},t)_{\varphi^{r+1},p}$, and for the third we applied [1, (6.2.9)].

In view of the equivalence between $\omega_{k-1,r+1}^{\varphi}(f^{(r+1)},t)_p$ and $\omega_{\varphi}^{k-1}(f^{(r+1)},t)_{\varphi^{r+1},p}$, our proof is complete. □

The rest of this section is devoted to the approximation of functions $f \in L_p[-1,1]$, by polynomials of degree $< n$. Let $\mathcal{P}_n$ be the set of polynomials of degree $< n$ and denote by

$$E_n(f)_p = \inf_{p_n\in\mathcal{P}_n} \|f-p_n\|_p,$$

the degree of approximation of $f \in L_p[-1,1]$ by elements of $\mathcal{P}_n$.

An immediate application of Theorem 5.1, together with [1, Theorem 7.2.1], is the following.

Theorem 5.2. *If $f \in \mathbf{B}_p^r$, then*

$$E_n(f)_p \le \frac{c}{n^r}\omega_{k,r}^{\varphi}(f^{(r)},1/n)_p, \quad n \ge k+r. \tag{5.1}$$

Proof. It follows from [1, Theorem 7.2.1] that

$$E_n(f)_p \le c\omega_{k+r}^{\varphi}(f,1/n)_p, \quad n \ge k+r.$$

Since $f \in \mathbf{B}_p^r$, we apply Theorem 5.1 r times and (5.1) follows. □

Corollary 5.3. *If $f \in \mathbf{B}_p^r$, and if for some $r \in \mathbb{N}_0$, $k \in \mathbb{N}$, and $\alpha > r$, $\omega_{k,r}^{\varphi}(f^{(r)},t)_p = O(t^{\alpha-r})$, then*

$$E_n(f)_p \le cn^{-\alpha}, \quad n \ge k+r. \tag{5.2}$$

Furthermore, we have an inverse to (5.2), see Corollary 5.5 below. We begin with a general inverse result.

Theorem 5.4. *Let $r \in \mathbb{N}_0$, $k \in \mathbb{N}$ and $\alpha > 0$, be such that $r < \alpha < r+k$, and let $f \in L_p[-1,1]$. If*

$$E_n(f)_p \le Mn^{-\alpha}, \quad n \ge 1, \tag{5.3}$$

then $f \in \mathbf{B}_p^r$ and

$$\omega_{k,r}^{\varphi}(f^{(r)},t)_p \le c(M,\alpha,r)t^{\alpha-r}, \quad t > 0.$$

Proof. The proof is based on various results from [1]. First it follows by [1, Theorem 8.2.1] that

$$\Omega_\varphi^{r+k}(f,h)_p := \Omega_\varphi^{r+k}(f,h)_{1,p} \le ch^{r+k} \sum_{0<n<1/h} n^{r+k-1} E_n(f)_p, \quad h>0, \tag{5.4}$$

where we note that $E_n(f)_p$ denotes the degree of approximation by polynomials of degree $< n$, while in [1], it denotes that degree but for the approximation by polynomials of degree $\le n$.

It follows by (5.3) that

$$\Omega_\varphi^{r+k}(f,h)_p \le Mh^\alpha,$$

whence

$$\int_0^1 (\Omega_\varphi^{r+k}(f,\tau)_p/\tau^{r+1})\,d\tau \le c(M,\alpha,r)\int_0^1 \tau^{\alpha-r-1}\,d\tau < \infty.$$

By [1, Theorem 6.3.1], this implies that $f \in C^{r-1}(-1,1)$, that $f^{(r-1)}$ is locally absolutely continuous in $(-1,1)$, and that

$$\Omega_\varphi^k(f^{(r)},t)_{\varphi^r,p} \le c\int_0^t \tau^{\alpha-r-1}\,d\tau = ct^{\alpha-r}.$$

Finally, we apply [1, (6.2.9)], to obtain by Theorem 1.4 and (3.3),

$$\omega_{k,r}^\varphi(f^{(r)},t)_p \le c\omega_\varphi^k(f^{(r)},t)_{\varphi^r,p} \le c\int_0^t (\Omega_\varphi^k(f^{(r)},\tau)_{\varphi^r,p}/\tau)\,d\tau \le ct^{\alpha-r}.$$

This completes the proof. □

Let $P_{k+r} \in \mathcal{P}_{k+r}$, be the best approximation to $f \in L_p[-1,1]$, and set $F := f - P_{k+r}$. Since $\omega_{k,r}^\varphi(p_{k+r}^{(r)},t)_p \equiv 0$ for $p_{k+r} \in \mathcal{P}_{k+r}$, it follows that $\omega_{k,r}^\varphi(f^{(r)},t)_p = \omega_{k,r}^\varphi(F^{(r)},t)_p$, $t>0$, that $E_n(F)_p = \|F\|_p = E_{k+r}(f)_p$, $n \le k+r$, and that we have $E_n(f)_p = E_n(F)_p$, for all $n \ge k+r$. Therefore, an immediate consequence of Theorem 5.4 is the following.

Corollary 5.5. *Let $r \in \mathbb{N}_0$, $k \in \mathbb{N}$ and $\alpha > 0$, be such that $r < \alpha < r+k$, and let $f \in L_p[-1,1]$. If*

$$E_n(f)_p \le n^{-\alpha}, \quad n \ge k+r,$$

then $f \in \mathbf{B}_p^r$ and

$$\omega_{k,r}^\varphi(f^{(r)},t)_p \le c(\alpha,k,r)t^{\alpha-r}, \quad t>0.$$

In [1, p. 72] there is a discussion on characterizing $\omega_\varphi^r(F,t)_p = O(t^\beta)$ for $r \ge 2$ and $0 < r-\beta \le 1$. We may easily derive the conclusions that appear there from the original Jackson type estimates involving the D-T moduli of smoothness and our Corollary 5.5. Indeed, if (in [1]'s notation),

$$\omega_\varphi^r(F,t)_p = O(t^\beta), \quad t \to 0,$$

Then, by the Jackson-type estimates, we have

$$E_n(F)_p \le c\omega_\varphi^r(F,1/n)_p \le cn^{-\beta}, \quad n \ge r = r-2+2.$$

If $0 \le r-2 < \beta < r$ (the above mentioned constraints on β and r (see [1, p. 72, l. 9]) clearly have a typo in the right hand side inequality), then we apply Corollary 5.5, with $k=2$ and r replaced by $r-2$, and obtain that $F^{(r-2)}$ is exists a.e. in $(-1,1)$, and

$$\omega_{2,r-2}^\varphi(F^{(r-2)},t)_p = O(t^{\beta-(r-2)}),$$

or in [1]'s notation

$$\omega_\varphi^2(F^{(r-2)},t)_{\varphi^{r-2},p} = O(t^{\beta-(r-2)}).$$

In fact, we may readily extend the result in the following way.

Theorem 5.6. *Let $k \in \mathbb{N}$ and $0 \le r-k < \beta < r$, and assume $F \in L_p[-1,1]$. If*

$$\omega_\varphi^r(F,t)_p = O(t^\beta), \quad t>0,$$

then $F^{(r-k)}$ is exists a.e. in $(-1,1)$, and

$$\omega_{k,r-k}^\varphi(F^{(r-k)},t)_p = O(t^{\beta-(r-k)}), \quad t>0.$$

We can prove the following extension of Corollary 5.5.

Corollary 5.7. *Let $r \in \mathbb{N}_0$, $k \in \mathbb{N}$ and $\alpha > 0$, be such that $r < \alpha < r+k$, and let $f \in L_p[-1,1]$. If*

$$E_n(f)_p \le n^{-\alpha}, \quad n \ge N, \tag{5.5}$$

for some $N \ge k+r$, then $f \in \mathbf{B}_p^r$ and

$$\omega_{k,r}^\varphi(f^{(r)},t)_p \le c(\alpha,k,r)t^{\alpha-r} + c(N,k,r)t^k E_{k+r}(f)_p, \quad t>0.$$

Proof. As in the explanation above Corollary 5.5, we may prove this corollary with $F = f - P_{k+r}$ instead of f.

Without loss of generality we may assume that $0 < h < N^{-1}$. By (5.4) and (5.5),

$$\begin{aligned}\Omega_\varphi^{r+k}(F,h)_p &\le ch^{r+k}\sum_{0<n<1/h} n^{r+k-1}E_n(F)_p \\ &\le ch^{k+r}N^{k+r}E_{k+r}(F)_p + \sum_{n=N}^{h^{-1}} n^{k+r-1}E_n(F)_p \\ &\le c_1(N,k,r)h^{k+r}E_{k+r}(f)_p + c_2(\alpha,k,r)h^\alpha.\end{aligned}$$

Hence

$$\int_0^1 (\Omega_\varphi^{r+k}(F,\tau)_p/\tau^{r+1})\,d\tau < \infty,$$

which implies that F^{r-1} is locally absolutely continuous in $(-1,1)$, and that

$$\Omega_\varphi^k(F^{(r)},t)_{\varphi^r,p} \le \int_0^t (c_1\tau^{k-1}E_{k+r}(f)_p + c_2\tau^{\alpha-r-1})\,d\tau = c_1t^kE_{k+r}(f)_p + c_2t^{\alpha-r}.$$

So, finally,

$$\begin{aligned}\omega_{k,r}^\varphi(f^{(r)},t)_p = \omega_{k,r}^\varphi(F^{(r)},t)_p &\le c\omega_\varphi^k(F^{(r)},t)_{\varphi^r,p}\\ &\le c\int_0^t (\Omega_\varphi^k(F^{(r)},\tau)_{\varphi^r,p}/\tau)\,d\tau \le c_1t^kE_{k+r}(f)_p + c_2t^{\alpha-r}.\end{aligned}$$

This completes the proof. □

6. Weighted D-T moduli: concluding remarks

The proofs (and therefore the results) of Section 5 may easily be extended to the weighted D-T moduli with weight w which satisfies the conditions of [1, Section 6.1]). So, in particular, we have the hierarchy relations between the weighted moduli of the function (of course, provided its derivative exists), extending [1, Corollary 6.3.3(b)], in the following generalization of Theorem 5.1.

Theorem 6.1. *Let w and ϕ be such $w,\phi\sim 1$ in compacta of $(-1,1)$, and $w(x)\sim(1\mp x)^{\gamma(\pm1)}$ and $\phi(x)\sim(1\mp x)^{\beta(\pm1)}$, as $x\to\pm1$, where $\gamma(\pm1),\beta(\pm1)\ge 0$. Let $0<l<r$, and assume that f is such that $f^{(l-1)}$ is locally absolutely continuous in $(-1,1)$ and $w\phi^lf^{(l)}\in L_p[-1,1]$. Then*

$$\omega_\phi^r(f,t)_{w,p} \le ct^l\omega_\phi^{r-l}(f^{(l)},t)_{w\phi^l,p}, \quad t>0. \tag{6.1}$$

Remark. *The inequality* (6.1) *extends* [1, Corollary 6.3.3(b)], *as we do not require the condition of $\beta(c)\ge 1$, for $c=\pm1$, that appears there.*

Denote by $E_n(f)_{w,p}$, the best weighted L_p approximation of f by polynomials of degree $<n$. Then we have the following Jackson-type estimates.

Theorem 6.2. *Let $0<l<r$ and assume that $f^{(l-1)}$ is locally absolutely continuous in $(-1,1)$ and $w\phi^lf^{(l)}\in L_p[-1,1]$. Then*

$$E_n(f)_{w,p} \le cn^{-l}\omega_\phi^{r-l}(f^{(l)},1/n)_{w\phi^l,p}, \quad n\ge r. \tag{6.2}$$

Proof. It was proved in [1, Theorem 8.2.1] that

$$E_n(f)_{w,p} \le c\int_0^{1/n} (\Omega_\phi^r(f,\tau)_{w,p}/\tau)\,d\tau, \quad n\ge r.$$

By virtue of [1, (6.3.2)],

$$\Omega_\phi^r(f,\tau)_{w,p} \le ct^l\Omega_\phi^{r-l}(f^{(l)},t)_{w\phi^l,p}, \quad 0<t<1.$$

Hence

$$\begin{aligned} E_n(f)_{w,p} &\le c\int_0^{1/n} \tau^{l-1}\Omega_\phi^{r-l}(f^{(l)},\tau)_{w\phi^l,p}\,d\tau \\ &\le cn^{-l}\Omega_\phi^{r-l}(f^{(l)},1/n)_{w\phi^l,p}, \quad n\ge r, \end{aligned}$$

where we used the fact that $\Omega_\phi^{r-l}(f^{(l)},t)_{w\phi^l,p}$ is nondecreasing in t. This, in turn, implies (6.2), by [1, (6.2.9)]. □

We can also obtain an inverse result generalizing Theorem 5.4.

Theorem 6.3. *Let $0<l<r<\alpha$, and let f be such that $wf\in L_p[-1,1]$. If*

$$E_n(f)_{w,p}\le Mn^{-\alpha}, \tag{6.3}$$

then $f^{(l-1)}$ is locally absolutely continuous in $(-1,1)$ and

$$\omega_\phi^{r-l}(f^{(l)},t)_{w\phi^l,p}\le c(M,l,\alpha)t^{\alpha-l}, \quad t>0. \tag{6.4}$$

Remark. *In view of* [1, Theorem 8.2.4], *one may readily obtain from* (6.3) *that*

$$\omega_\phi^{r}(f,t)_{w,p}\le c(M,\alpha)t^{\alpha}, \quad t>0. \tag{6.5}$$

(*Note that* [1, Theorem 8.2.4] *is stated for $w(x)=(1+x)^{\gamma_1}(1-x)^{\gamma_2}$, where $\gamma_i\ge 0$, but can easily be extended to all the above mentioned weights.*) *However, in view of Theorem* 6.1, *one cannot expect to derive* (6.4) *from* (6.5).

Proof. Combining [1, Theorem 8.2.1] and (5.5), we obtain

$$\Omega_\phi^{r}(f,h)_{w,p}\le ch^r\sum_{0<n<1/h} n^{r-1}E_n(f)_{w,p}\le c(M)h^\alpha.$$

Hence,

$$\int_0^1 (\Omega_\phi^r(f,\tau)_{w,p}/\tau^{l+1})\,d\tau\le c\int_0^1 \tau^{\alpha-l-1}\,d\tau<\infty,$$

which, by [1, Theorem 6.3.1(a)], implies that $f^{(l-1)}$ is locally absolutely continuous in $(-1,1)$ and

$$\Omega_\phi^{r-l}(f^{(l)},t)_{w\phi^l,p}\le c\int_0^t (\Omega_\phi^r(f,\tau)_{w,p}/\tau^{l+1})\,d\tau\le c(M,l,\alpha)t^{\alpha-l}, \quad t>0.$$

Finally, we apply [1, (6.2.9)], to get

$$\omega_\phi^{r-l}(f^{(l)},t)_{w\phi^l,p}\le c\int_0^t (\Omega_\phi^{r-l}(f^{(l)},\tau)_{w\phi^l,p}/\tau)\,d\tau\le c(M,l,\alpha)t^{\alpha-l}.$$

This completes the proof. □

The same arguments above Corollary 5.5 yield the following consequence of Theorem 6.3 (analogue of Corollary 5.5 and Corollary 5.7).

Corollary 6.4. *Let $r \in \mathbb{N}_0$, $k \in \mathbb{N}$ and $\alpha > 0$, be such that $r < \alpha < r + k$. Assume w is a weight as above, and let $wf \in L_p[-1, 1]$. If*

$$E_n(f)_{w,p} \le n^{-\alpha}, \quad n \ge N, \tag{6.6}$$

for some $N \ge k + r$, then

$$\omega_\phi^k(f^{(r)}, t)_{\phi^r w,p} \le c(w, \alpha, k, r)t^{\alpha-r} + c(w, N, k, r)t^k E_{k+r}(f)_{w,p}, \quad t > 0.$$

Moreover, if $N = k + r$, then

$$\omega_\phi^k(f^{(r)}, t)_{\phi^r w,p} \le c(w, \alpha, k, r)t^{\alpha-r}, \quad t > 0.$$

Finally, we may state the analogue of Theorem 5.6.

Theorem 6.5. *Let $k \in \mathbb{N}$ and $0 \le r - k < \beta < r$. Assume that the weight w is as above, and F is such that $wF \in L_p[-1, 1]$ and*

$$\omega_\phi^r(F, t)_{w,p} = O(t^\beta), \quad t > 0. \tag{6.7}$$

Then $F^{(r-k)}$ exists a.e. in $(-1, 1)$, and

$$\omega_\phi^k(F^{(r-k)}, t)_{\phi^{r-k}w,p} = O(t^{\beta-(r-k)}), \quad t > 0.$$

Proof. The only difference in the proof is that we do not have the analogues of the Jackson-type estimates for the weighted moduli of smoothness. However, by virtue of [1, Theorem 8.2.1] we have

$$E_n(F)_{w,p} \le c \int_0^{1/n} \Omega_\phi^r(F, \tau)_{w,p} / \tau \, d\tau \le c \int_0^{1/n} \omega_\phi^r(F, \tau)_{w,p} / \tau \, d\tau, \quad n \ge r.$$

It follows by (6.7) that

$$E_n(F)_{w,p} \le cn^{-\beta}, \quad n \ge r,$$

and we conclude the proof as in the discussion following Corollary 5.5. □

Remark. *It is interesting to compare Theorem* 6.3 *to* [1, Corollary 8.2.2].

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