POLYNOMIAL APPROXIMATION IN REARRANGEMENT INVARIANT QUASI BANACH SPACES ON THE UNIT CIRCLE

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Abstract. In this work work we obtain some Jackson type direct theorem and sharp converse theorem of polynomial approximation with respect to fractional order moduli of smoothness in rearrangement invariant quasi Banach spaces on the unit circle.

1. Preliminaries and the main results

Let $\mathcal{M}$ be the set of all measurable functions defined on $\mathsf{T} := \left\{e^{i\theta} : \theta \in [0, 2\pi)\right\}$ and let $\mathcal{M}^+$ be the subset of functions from $\mathcal{M}$ whose values lie in $[0, \infty]$. By χ_E we denote the characteristic function of a measurable set $E \subset \mathsf{T}$. A mapping $\rho\colon \mathcal{M}^+ \to [0, \infty]$ is called a *function norm* if for all constants $a \geq 0$, for all functions $f, g, f_n \quad (n = 1, 2, 3, \ldots)$, and for all measurable subsets E of T, the following properties hold:

(i) $\rho(f) = 0$ iff $f = 0$ a.e.; $\rho(af) = a\rho(f)$; $\rho(f + g) \leq \rho(f) + \rho(g)$,

(ii) if $0 \leq g \leq f$ a.e., then $\rho(g) \leq \rho(f)$, (iii) if $0 \leq f_n \uparrow f$ a.e., then $\rho(f_n) \uparrow \rho(f)$,

(iv) $\rho(\chi_E) < \infty$ holds for every set $E \subset \mathsf{T}$ having a finite Lebesgue measure $|E| < \infty$,

(v) $\int_E |f(w)|\,|dw| \leq C_E\ \rho(f)$ holds for every set $E \subset \mathsf{T}$ having a finite Lebesgue measure $|E| < \infty$, with a constant $C_E \in (0, \infty)$, depending on E and ρ but independent of f.

If ρ is a function norm, its *associate norm* ρ' is defined on $\mathcal{M}^+$ by

$$\rho'(g) := \sup\left\{\int_{\mathsf{T}} |f(w)\,g(w)|\,|dw| : f \in \mathcal{M}^+,\ \rho(f) \leq 1\right\}, \quad g \in \mathcal{M}^+.$$

If ρ is a function norm, then ρ' is itself a function norm [6, pp. 8, Th. 2.2]. Let ρ be a function norm. The collection of functions

$$X(\mathsf{T}) := X(\mathsf{T}, \rho) := \{f \in \mathcal{M} : \rho(|f|) < \infty\}$$

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is called *Banach function space* (shortly BFS) on T. For each $f \in X(\mathsf{T})$ we define

$$\|f\|_{X(\mathsf{T})} := \rho(|f|).$$

A Banach function space X equipped with norm $\|\cdot\|_{X(\mathsf{T})}$ is a Banach space [6, pp. 3,5, Ths. 1.4 and 1.6]. Let ρ' be the associate norm of a function norm ρ. The Banach function space $X(\mathsf{T},\rho')$ determined by the function norm ρ' is called the *associate space* of $X(\mathsf{T}) = X(\mathsf{T},\rho)$ and is denoted by $X'(\mathsf{T})$. It is well-known [6, pp. 9] that

$$\|f\|_{X(\mathsf{T})} = \sup\left\{\int_{\mathsf{T}} |f(w)\,g(w)|\,|dw| : g \in X'(\mathsf{T}),\ \|g\|_{X'(\mathsf{T})} \le 1\right\} \tag{1.1}$$

hold. The distribution function μ_f of a measurable function f is defined as

$$\mu_f(\lambda) = measure\{w \in \mathsf{T} : |f(w)| > \lambda\},\ \lambda \ge 0.$$

A Banach function norm is *rearrangement invariant* if $\rho(f) = \rho(g)$ for every pair of functions f, g which are equimeasurable, that is $\mu_f(\lambda) = \mu_g(\lambda)$.

For a Banach function space $X(\mathsf{T})$, we define $X_r(\mathsf{T}) := \{f \in \mathcal{M} : f^r \in X(\mathsf{T})\}$, $r \in (0,\infty)$ and r-norm as

$$\|f\|_{Xr(\mathsf{T})} := \||f|^r\|_{X(\mathsf{T})}^{1/r}.$$

The spaces $X_p(\Omega)$ on complete σ-finite mesure spaces have been studied and used in [14], [17], [18], [15]. Hardy type inequalities in $X_p(\Omega)$ are investigated in [10].

Throughout this work by C, c, c_i we denote the constants which are absolute or depend only on the parameters given in their brakets.

A *quasi Banach function norm* is a mapping $\rho\colon \mathcal{M}^+ \to [0,\infty]$ such that it satisfies (ii)-(iv) of above definition of function norm but (i) satisfies as a quasinorm, namely, $\rho(f) = 0$ iff $f = 0$ a.e.; $\rho(af) = a\rho(f)$; $\rho(f+g) \le c(\rho(f) + \rho(g))$. If a quasi Banach function norm ρ is rearrangement invariant then the collection of functions $X(\mathsf{T},\rho) = \{f \in \mathcal{M} : \rho(|f|) < \infty\}$ will be called *rearrangement invariant quasi Banach function space* (shortly RIQBFS). A quasi BFS $X(\mathsf{T})$ is said to be p-convex for some $p \in (0,1]$ if there is a c such that for all $f_1, \ldots, f_N \in X(\mathsf{T})$ we have

$$\left\|\left(\sum_{i=1}^{N} |f_i|^p\right)^{1/p}\right\|_{X(\mathsf{T})} \le c\left(\sum_{i=1}^{N} \|f_i\|_{X(\mathsf{T})}^p\right)^{1/p}. \tag{1.2}$$

In this case the condition (1.2) is equivalent to the fact that $X_{1/p}(\mathsf{T})$ is a rearrangement invariant BFS. From (1.1) one can be see that $\|\cdot\|_{X(\mathsf{T})}$ be equivalently represented [8] as

$$\|f\|_{X(\mathsf{T})} \asymp \sup\left\{\left(\int_{\mathsf{T}} |f(w)|^p\,|g(w)|\,|dw|\right)^{1/p} : \|g\|_{Y'(\mathsf{T})} \le 1\right\} \tag{1.3}$$

where $Y'(\mathsf{T})$ is the associate space of the rearrangement invariant BFS $Y(\mathsf{T}) = X_{1/p}(\mathsf{T})$. There are examples [9] of quasi BFS which are not p-convex for any $p > 0$.

$A(x) \asymp B(x)$ will be mean that there exist constants c and C such that $cA(x) \leq B(x) \leq CA(x)$ holds.

Let $X(\mathsf{T})$ be quasi BFS. A function $f \in X(\mathsf{T})$ is said to have *absolutely continuous norm* if

$$\lim_{n\to\infty} \|f\chi_{A_n}\|_{X(\mathsf{T})} = 0$$

for every decreasing sequence of measurable sets (A_n) with $\chi_{A_n} \to 0$ a.e. If every $f \in X(\mathsf{T})$ has this property we will say $X(\mathsf{T})$ has *absolutely continuous norm.*

Hereafter throughout this work we will assume that $\mathbb{X}(\mathsf{T}) := X(\mathsf{T}, AC, p)$ is a RIQBFS which has absolutely continuous norm and p-convex for some $p \in (0,1]$. These assumptions on the function space are not very restrictive. For example Orlicz spaces on T, classical Lorentz spaces $L^{pq}(\mathsf{T})$, $p,q \in (0,\infty)$ (in particular $L^p(\mathsf{T})$ spaces with $p \in (0,1)$), Zygmund spaces $L^p(\log L)^\alpha(\mathsf{T})$, $p \in (0,\infty)$, $\alpha \in \mathbb{R}$, Lorentz $\Lambda(\mathsf{T})$ spaces and Marcinkiewicz spaces on T satisfy [8] these conditions. For a complete treatise of rearrangement invariant BFS and RIQBFS we refer to [13], [6], [12], [7], [11] and [16].

Remark 1. *Let $X(\mathsf{T})$ be a RIQBFS. The following conditions are equivalent:*

(i) The set $\mathcal{P}_n(\mathsf{T}) := \left\{P : P\left(e^{i\theta}\right) = \sum_{j=-n}^{n} c_j e^{ij\theta},\ c_j \in \mathbb{C}\right\}$ of polynomials is dense in $X(\mathsf{T})$.

(ii) The set of continuous functions on T is dense in $X(\mathsf{T})$.

(iii) Rotation operator $R_h f(w) := f\left(we^{ih}\right)$ is a bounded operator in $X(\mathsf{T})$, namely,

$$\|R_h f\|_{X(\mathsf{T})} \leq c\,\|f\|_{X(\mathsf{T})}$$

for every $f \in X(\mathsf{T})$, $h \in [0,2\pi)$ and $w \in \mathsf{T}$.

(iv) $X(\mathsf{T})$ has absolutely continuous norm.

These properties are proved for rearrangement invariant BFS in [6, pp. 157, Lemma 6.3] and they are hold also for RIQBFS $X(\mathsf{T})$ which has absolutely continuous norm.

Let $w \in \mathsf{T}$, $h \in [0,2\pi)$, $\alpha \in \mathbb{R}^+ := (0,\infty)$, $f \in \mathbb{X}(\mathsf{T})$ and we set

$$\Delta_h^\alpha f(w) := (I - R_h)^\alpha f(w) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f\left(we^{ikh}\right)$$

with Binomial coefficients $\binom{\alpha}{k} := \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}$ for $k \geq 1$ and $\binom{\alpha}{0} := 1$, where I is identity operator.

If $\frac{1}{\alpha+1} < p$, then using [19, pp. 14]

$$\left| \binom{\alpha}{k} \right| \leq \frac{c(\alpha)}{k^{\alpha+1}}, \quad k \in \mathbb{Z}^+$$

and hence we obtain

$$\sum_{k=1}^{\infty} \left| \binom{\alpha}{k} \right|^p \leq c(\alpha, p) \sum_{k=1}^{\infty} \frac{c(\alpha)}{k^{p(\alpha+1)}} < \infty. \tag{1.4}$$

On the other hand if g belongs to $Y'(\mathsf{T})$, the associate space of the rearrangement invariant BFS $Y(\mathsf{T}) = X_{1/p}(\mathsf{T})$, then using Levi Monotone Convergence Theorem and Remark 1 (iii) we have

$$\int_{\mathsf{T}} \left| \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f\left(we^{ikh}\right) \right|^p |g(w)| \, |dw|$$

$$\leq \lim_{j \to \infty} \left(\int_{\mathsf{T}} \sum_{k=0}^{j} \left| \binom{\alpha}{k} f\left(we^{ikh}\right) \right|^p |g(w)| \, |dw| \right)$$

$$\leq c \left(\sum_{k=0}^{\infty} \left| \binom{\alpha}{k} \right|^p \right) \left(\sup_{\|g\|_{Y'(\mathsf{T})} \leq 1} \int_{\mathsf{T}} |f(w)|^p \, |g(w)| \, |dw| \right)$$

and hence from (1.3) and (1.4)

$$\|\Delta_h^\alpha f\|_{X(\mathsf{T})} \leq c \, \|f\|_{X(\mathsf{T})}.$$

This last inequality signifies that if $f \in \mathbb{X}(\mathsf{T})$, $\alpha \in \mathbb{R}^+$, $(\alpha+1)^{-1} < p$ and $h \in [0, 2\pi)$, then $\Delta_h^\alpha f \in X(\mathsf{T})$.

Now, if $\alpha \in \mathbb{R}^+$, $f \in \mathbb{X}(\mathsf{T})$, $(\alpha+1)^{-1} < p$ and $h \in [0, 2\pi)$, then we can define the *α-th modulus of smoothness* of a function f as

$$\omega_\alpha(f, \delta)_{X(\mathsf{T})} := \sup_{0 < h \leq \delta} \|\Delta_h^\alpha f\|_{X(\mathsf{T})}, \quad \delta \geq 0.$$

Remark 2. *The α-th modulus of smoothness $\omega_\alpha(f, \delta)_{X(\mathsf{T})}$, $\alpha \in \mathbb{R}^+$, $(\alpha+1)^{-1} <$ p, of $f \in \mathbb{X}(\mathsf{T}) = X(\mathsf{T}, AC, p)$ has the following properties.*

(i) $\omega_\alpha(f, \delta)_{X(\mathsf{T})}$ is non-negative and non-decreasing function of $\delta \geq 0$.

(ii) $\omega_\alpha^p(f_1 + f_2, \cdot)_{X(\mathsf{T})} \leq \omega_\alpha^p(f_1, \cdot)_{X(\mathsf{T})} + \omega_\alpha^p(f_2, \cdot)_{X(\mathsf{T})}$.

(iii) $\lim_{\delta \to 0^+} \omega_\alpha(f, \delta)_{X(\mathsf{T})} = 0$.

By means of Remark 1 (ii) and (iv) let

$$E_n(f)_{X(\mathsf{T})} := \inf_{P \in \mathcal{P}_n} \|f - P\|_{X(\mathsf{T})}, \quad f \in \mathbb{X}(\mathsf{T}), \, n = 0, 1, 2, \ldots.$$

We denote by $X^\alpha(\mathsf{T})$, $\alpha \in \mathbb{R}^+$, the linear space of functions $f \in \mathbb{X}(\mathsf{T})$ such that $f^{(\alpha)} \in X(\mathsf{T})$.

We say that a function $g = f^{(\alpha)}$, $\alpha \in \mathbb{R}^{+}$, is the *$\alpha$th derivative* of $f \in X^{\alpha}(\mathsf{T})$ if there is a function $g \in X(\mathsf{T})$ such that

$$\lim_{h\to 0^{+}} \left\| \frac{\Delta_h^{\alpha}(f)}{h^{\alpha}} - g \right\|_{X(\mathsf{T})} = 0.$$

If a.e. equal functions are identified, then the last condition determines αth derivative uniquely. Also αth derivative is additive with respect to finite number of functions.

The *αth Weyl's derivative* ($\alpha \in \mathbb{R}^{+}$) of a polynomial

$$T_n(\theta) = \sum\nolimits_{v=-n}^{n} \gamma_v e^{iv\theta}, \ \ n \geq 1, \, \theta \in [0, 2\pi), \, \gamma_v \in \mathbb{C}$$

of class $\mathcal{P}_n(\mathsf{T})$ is defined as

$$T_n^{\{\alpha\}}(\theta) = \sum\nolimits_{v\in\mathbb{Z}_n^{*}} \gamma_v (iv)^{\alpha} e^{iv\theta}$$

a.e. on $[0, 2\pi)$, where $\mathbb{Z}_n^{*} := \{\pm 1, \pm 2, \ldots, \pm n\}$ and $(iv)^{-\alpha} := |v|^{-\alpha} e^{(-1/2)\pi i\alpha sign v}$ as principal value.

Remark 3 ([3]). *Let*

$$T_n(\theta) = \sum\nolimits_{v=-n}^{n} \gamma_v e^{iv\theta}, \quad (n \geq 1)$$

be a polynomial of class $\mathcal{P}_n(\mathsf{T})$. Then for every $\alpha \in \mathbb{R}^{+}$ and $\theta \in [0, 2\pi)$ we have

$$T_n^{\{\alpha\}}(\theta) = T_n^{(\alpha)}(\theta).$$

Direct and converse theorems of trigonometric approximation in $\mathbb{X}([0, 2\pi))$, was obtained via the polynomial K-functional in [3]. For the variable exponent Lebesgue spaces similar problems of approximation theory was considered in [2], [1], [5] and [4].

In this work work we obtain some Jackson type direct theorem and sharp converse theorem of polynomial approximation with respect to fractional order moduli of smoothness in these spaces.

Let us denote by $[x]$ the integer part of a real number x and $\{x\} := x - [x]$. Let $q_{\mathbb{X}(\mathsf{T})}$ be the Boyd's upper index of $\mathbb{X}(\mathsf{T})$.

For $\alpha, t \in \mathbb{R}^{+}$ and $f \in \mathbb{X}(\mathsf{T})$ we define for $n = 1, 2, 3, \ldots$, the Polynomial K-functional

$$K_{\alpha}(f, 1/n, X(\mathsf{T}), X^{\alpha}(\mathsf{T})) := \inf\nolimits_{T\in\mathcal{P}_n(\mathsf{T})} \left\{ \|f - T\|_{X(\mathsf{T})} + n^{-\alpha} \left\|T^{(\alpha)}\right\|_{X(\mathsf{T})} \right\}.$$

Theorem 1. *If $\alpha \in R^{+}$, $f \in \mathbb{X}$, $p^{-1} < \min\{\alpha + 1, 2 - \{\alpha\}\}$ and $q_{\mathbb{X}} < \infty$, then the equivalance*

$$\omega_{\alpha}(f, 1/n)_X \asymp K_{\alpha}(f, 1/n, X, X^{\alpha})$$

holds.

The following Jackson type direct theorems of trigonometric approximation hold.

Corollary 1. *If $\alpha \in \mathbb{R}^{+}$, $f \in \mathbb{X}(\mathsf{T})$, $p^{-1} < \min\{\alpha+1, 2-\{\alpha\}\}$ and $q_{\mathbb{X}(\mathsf{T})} < \infty$, then there exists a constant $c > 0$ dependent only on α and $\mathbb{X}(\mathsf{T})$ such that for $n = 1, 2, 3, \ldots$*

$$E_n(f)_{X(\mathsf{T})} \leq c\omega_\alpha\left(f, \frac{1}{n}\right)_{X(\mathsf{T})}$$

holds.

The following converse estimate of trigonometric approximation holds.

Theorem 2. *If $\alpha \in \mathbb{R}^{+}$, $f \in \mathbb{X}(\mathsf{T})$, $(\alpha+1)^{-1} < p$ and $q_{\mathbb{X}(\mathsf{T})} < \infty$, then for $n = 1, 2, 3, \ldots$*

$$\omega_\alpha\left(f, \frac{\pi}{n}\right)_{X(\mathsf{T})} \leq \frac{c}{n^\alpha}\left(\sum\nolimits_{\nu=0}^{n} (\nu+1)^{p\alpha-1} E_\nu^p(f)_{X(\mathsf{T})}\right)^{1/p} \tag{1.5}$$

hold, where the constant $c > 0$ dependent only on α and $\mathbb{X}(\mathsf{T})$.

Corollary 2. *Under the conditions of Theorem 1 the estimate*

$$E_n(f)_{X(\mathsf{T})} = \mathcal{O}\left(n^{-\sigma}\right), \ 1 > \sigma > 0, \ n = 1, 2, \ldots,$$

holds if and only if

$$\omega_\alpha(f, \delta)_{X(\mathsf{T})} = \mathcal{O}(\delta^\sigma).$$

Corollary 3. *Under the conditions of Theorem 1 the converse inequality* (1.5) *is sharp in the sense that*

$$\sup_{E_n(f)_{L^p(\mathsf{T})} \leq 1/n} \omega_1(f, \delta)_{L^p(\mathsf{T})} \asymp \beta(\ln(1/\delta))^{1/p}, \quad 0 < p < 1. \tag{1.6}$$

Exactness of (1.6) can be seen by 2π periodic function $f(\theta) = |\theta|^{1-(1/p)}$, $|\theta| \leq \pi$.

Theorem 3. *Let $f \in \mathbb{X}(\mathsf{T})$ and $q_{\mathbb{X}(\mathsf{T})} < \infty$. If $\beta \in (0, \infty)$ and*

$$\sum_{\nu=1}^{\infty} \nu^{p\beta-1} E_\nu^p(f)_{X(\mathsf{T})} < \infty \tag{1.7}$$

then the derivative $f^{(\beta)}$ exists. Further, denoting by $T_n \in \mathcal{T}_n$, $n \geq 1$, the best approximating polynomial of f in $\|\cdot\|_{X(\mathsf{T})}$ metric we have

$$\left\|f^{(\beta)} - \mathrm{T}_n^{(\beta)}\right\|_{X(\mathsf{T})} \leq c\left(n^\beta E_n(f)_{X(\mathsf{T})} + \left(\sum_{\nu=n+1}^{\infty} \nu^{p\beta-1} E_\nu^p(f)_{X(\mathsf{T})}\right)^{1/p}\right)$$

where the constant $c > 0$ dependent only on β and $\mathbb{X}(\mathsf{T})$.

As a corollary of Theorems 3 and 2

Corollary 4. *Let* $f \in \mathbb{X}(\mathsf{T})$, $\beta \in (0,\infty)$, $q_{\mathbb{X}(\mathsf{T})} < \infty$ *and*

$$\sum_{\nu=1}^{\infty} \nu^{p\alpha-1} E_{\nu}^{p}(f)_{X(\mathsf{T})} < \infty$$

for some $\alpha > 0$. *In this case for* $n = 1, 2, \ldots$, *there exists a constant* $c > 0$ *dependent only on* α, β *and* $\mathbb{X}(\mathsf{T})$ *such that*

$$\omega_{\beta}\left(f^{(\alpha)}, \frac{1}{n}\right)_{X(\mathsf{T})} \le c\left(\frac{1}{n^{\beta}}\left(\sum_{\nu=0}^{n} (\nu+1)^{p(\alpha+\beta)-1} E_{\nu}^{p}(f)_{X(\mathsf{T})}\right)^{1/p} + \right.$$
$$\left. + \left(\sum_{\nu=n+1}^{\infty} \nu^{p\alpha-1} E_{\nu}^{p}(f)_{X(\mathsf{T})}\right)^{1/p}\right)$$

hold.

In the sequel, we denote by c, c_1, c_2,..., positive constants (possibly different at different occurences) that either are absolute or depend on parameters not essential for the argument.

2. Auxiliary results

Theorem 4 (Extrapolation Theorem). [8, Theorem 2.1] *Let* $0 < p_0 < \infty$ *and let* $\mathcal{F}$ *be a family of couples of nonnegative real valued functions such that*

$$\int_{\mathsf{T}} f(w)^{p_0} |dw| \le C \int_{\mathsf{T}} g(w)^{p_0} |dw|, \quad (f,g) \in \mathcal{F}$$

holds with the left hand side is finite. We suppose that $X(\mathsf{T})$ *is a RIQBFS which is p-convex for some* $p \in (0,1]$ *and* $q_{\mathbb{X}(\mathsf{T})} < \infty$. *Then*

$$\|f\|_{X(\mathsf{T})} \le C\|g\|_{X(\mathsf{T})}, \quad (f,g) \in \mathcal{F}$$

holds when the left hand side is finite.

Lemma 1. *Let* $X(\mathsf{T})$ *be a RIQBFS which is p-convex for some* $p \in (0,1]$. *If* $T_n \in \mathcal{P}_n(\mathsf{T})$, $n \ge 1$, $\alpha \in \mathbb{R}^+$, $q_{\mathbb{X}(\mathsf{T})} < \infty$, $0 < h \le 2\pi/n$, *then there exist constants* $c, C > 0$ *such that*

$$\left\|T_n^{(\alpha)}\right\|_{X(\mathsf{T})} \le c\left(\frac{n}{2\sin(nh/2)}\right)^{\alpha} \|\Delta_h^{\alpha} T_n\|_{X(\mathsf{T})}, \tag{2.1}$$

and

$$\|\Delta_h^{\alpha} T_n\|_{X(\mathsf{T})} \le Ch^{\alpha}\left\|T_n^{(\alpha)}\right\|_{X(\mathsf{T})}. \tag{2.2}$$

Proof. Since the inequalities

$$\left\|T_n^{(\alpha)}\right\|_{L^q(\mathsf{T})} \leq c\left(\frac{n}{2\sin(nh/2)}\right)^{\alpha}\left\|\Delta_h^{\alpha}T_n\right\|_{L^q(\mathsf{T})}$$

and

$$\left\|\Delta_h^{\alpha}T_n\right\|_{L^q(\mathsf{T})} \leq Ch^{\alpha}\left\|T_n^{(\alpha)}\right\|_{L^q(\mathsf{T})}$$

are hold for every $q \in [1,\infty)$ we obtain from the Extrapolation Theorem 4 that

$$\left\|T_n^{(\alpha)}\right\|_{X(\mathsf{T})} \leq c\left(\frac{n}{2\sin(nh/2)}\right)^{\alpha}\left\|\Delta_h^{\alpha}T_n\right\|_{X(\mathsf{T})} \quad ,$$

$$\left\|\Delta_h^{\alpha}T_n\right\|_{X(\mathsf{T})} \leq Ch^{\alpha}\left\|T_n^{(\alpha)}\right\|_{X(\mathsf{T})}. \qquad \square$$

Corollary 5. (i) *Taking* $h=\pi/n$ *in* (2.1) *we have*

$$\left\|T_n^{(\alpha)}\right\|_{X(\mathsf{T})} \leq cn^{\alpha}\left\|\Delta_{\pi/n}^{\alpha}T_n\right\|_{X(\mathsf{T})} \tag{2.3}$$

and hence fractional Bernstein Inequality

$$\left\|T_n^{(\alpha)}\right\|_{X(\mathsf{T})} \leq cn^{\alpha}\left\|T_n\right\|_{X(\mathsf{T})}.$$

(ii) *Combining* (2.2) *and* (2.3) *we have*

$$\omega_{\alpha}(T_n,\pi/n)_{X(\mathsf{T})} \leq c\left\|\Delta_{\pi/n}^{\alpha}T_n\right\|_{X(\mathsf{T})}. \tag{2.4}$$

3. Proofs of Main Results

Proof of Theorem 1. Starting with upper inequality we take a $t \in (0,2\pi)$. Then there exists $n \in \mathbb{Z}^+$ such that $\pi/n < t \leq 2\pi/n$. Let t_n^* be the best approximating polynomial to $f \in \mathbb{X}(\mathsf{T})$. Using Corollary 1 of [3] we get

$$\left\|f-t_n^*\right\|_{X(\mathsf{T})} = E_n(f)_{X(\mathsf{T})} = E_n\left(f\left(e^{i\theta}\right)\right)_{X([0,2\pi))} \leq \tag{3.1}$$

$$\leq c\omega_{\alpha}\left(f\left(e^{i\theta}\right),\frac{\pi}{n}\right)_{X([0,2\pi))} \leq c\omega_{\alpha}\left(f,\frac{\pi}{n}\right)_{X(\mathsf{T})}. \tag{3.2}$$

From (2.3), (2.4) and (3.1) we have

$$\left\|t_n^{*(\alpha)}\right\|_{X(\mathsf{T})} \leq cn^{\alpha}\left\|\Delta_{\pi/n}^{\alpha}t_n^*\right\|_{X(\mathsf{T})} \leq$$
$$\leq c(\pi/t)^{\alpha}\left\{c\left\|f-t_n^*\right\|_{X(\mathsf{T})}+\left\|\Delta_{\pi/n}^{\alpha}f\right\|_{X(\mathsf{T})}\right\} \leq ct^{-\alpha}\omega_{\alpha}\left(f,\frac{\pi}{n}\right)_{X(\mathsf{T})}$$

and therefore

$$K_{\alpha}(f,t,X(\mathsf{T}),X^{\alpha}(\mathsf{T})) \leq \left\|f-t_n^*\right\|_{X(\mathsf{T})}+t^{\alpha}\left\|t_n^{*(\alpha)}\right\|_{X(\mathsf{T})} \leq c\omega_{\alpha}(f,t)_{X(\mathsf{T})}.$$

The lower inequality is easy. $\square$

Proof of Theorem 2. Let $T_n \in \mathcal{P}_n(\mathsf{T})$ be the best approximating polynomial of f and let $m \in \mathbb{Z}^+$. We write

$$U_0 = T_1 - T_0,\ U_\upsilon = T_{2^\upsilon} - T_{2^{\upsilon-1}},\ \upsilon \geq 1. \tag{3.3}$$

Then

$$T_{2^m} = T_0 + \sum_{\nu=0}^{m} U_\upsilon.$$

In this case

$$\omega_\alpha^p(f,\pi/n)_{X(\mathsf{T})} \leq \omega_\alpha^p(f - T_{2^m},\pi/n)_{X(\mathsf{T})} + \omega_\alpha^p(T_{2^m},\pi/n)_{X(\mathsf{T})},$$

$$\omega_\alpha^p(f - T_{2^m},\pi/n)_{X(\mathsf{T})} \leq cE_{2^m}^p(f)_{X(\mathsf{T})}$$

and

$$\begin{aligned}\omega_\alpha^p(T_{2^m},\pi/n)_{X(\mathsf{T})} &\leq \omega_\alpha^p(U_0,\pi/n)_{X(\mathsf{T})} + \sum_{\nu=1}^{m}\omega_\alpha^p(U_\upsilon,\pi/n)_{X(\mathsf{T})}\\ &\leq c\left(\frac{\pi}{n}\right)^{p\alpha}\left(\|U_0\|_{X(\mathsf{T})}^p + \sum_{\nu=1}^{m} 2^{\upsilon p\alpha}\|U_\upsilon\|_{X(\mathsf{T})}^p\right).\end{aligned}$$

On the other hand

$$\|U_0\|_{X(\mathsf{T})}^p \leq cE_0^p(f)_{X(\mathsf{T})}$$

and

$$\|U_\upsilon\|_{X(\mathsf{T})}^p \leq 2E_{2^{\upsilon-1}}^p(f)_{X(\mathsf{T})}.$$

It is easily seen that

$$2^{\upsilon p\alpha}E_{2^{\upsilon-1}}^p(f)_{X(\mathsf{T})} \leq c\sum_{\mu=2^{\upsilon-2}+1}^{2^{\upsilon-1}} \mu^{p\alpha-1}E_\mu^p(f)_{X(\mathsf{T})}, \quad \nu = 2,3,\ldots$$

and therefore

$$\omega_\alpha^p(T_{2^m},\pi/n)_{X(\mathsf{T})} \leq c\left(\frac{\pi}{n}\right)^{p\alpha}\left\{E_0^p(f)_{X(\mathsf{T})} + \sum_{\upsilon=1}^{m} 2^{\upsilon p\alpha}E_{2^{\upsilon-1}}^p(f)_{X(\mathsf{T})}\right\}.$$

If we choose $2^{m-1} \leq n < 2^m$, then

$$\omega_\alpha^p(T_{2^m},\pi/(n+1))_{X(\mathsf{T})} \leq \frac{c(\alpha,\mathbb{X})}{n^{p\alpha}}\sum_{\upsilon=0}^{n}(\nu+1)^{p\alpha-1}E_\upsilon^p(f)_{X(\mathsf{T})}$$

and

$$E_{2^m}^p(f)_{X(\mathsf{T})} \leq E_{2^{m-1}}^p(f)_{X(\mathsf{T})} \leq \frac{c}{n^{\alpha p}}\sum_{\upsilon=0}^{n}(\nu+1)^{p\alpha-1}E_\upsilon^p(f)_{X(\mathsf{T})}.$$

Last two inequalities complete the proof. □

Proof of Theorem 3. By Levi's theorem and (3.3)

$$\left\| T_0(\cdot) + \sum_{\upsilon=0}^{\infty} U_\upsilon(\cdot) \right\|_{X(\mathsf{T})}^p = \lim_{r\to\infty} \left\| T_0(\cdot) + \sum_{\upsilon=0}^{r} U_\upsilon(\cdot) \right\|_{X(\mathsf{T})}^p$$

$$\leq c \left\| T_0(\cdot) \right\|_{X(\mathsf{T})}^p + c \lim_{r\to\infty} \sum_{\upsilon=0}^{r} \left\| U_\upsilon(\cdot) \right\|_{X(\mathsf{T})}^p \leq c E_0^p(f)_{X(\mathsf{T})} + c \sum_{\upsilon=1}^{\infty} E_{2^{\upsilon-1}}^p(f)_{X(\mathsf{T})} < \infty.$$

From (1.7) the last series converges and therefore

$$f(w) = \lim_{r\to\infty} T_{2^r}(w) = T_0(w) + \sum_{\upsilon=0}^{\infty} U_\upsilon(w) \text{ a.e.}$$

Analogously using Levi's Theorem

$$\left\| \sum_{\upsilon=0}^{\infty} U_\upsilon^{(\beta)} \right\|_X^p \leq c \sum_{\upsilon=0}^{\infty} \left\| U_\upsilon^{(\beta)} \right\|_X^p \leq c \sum_{\upsilon=0}^{\infty} 2^{\upsilon p \beta} \left\| U_\upsilon \right\|_X^p$$

$$\leq c \left(E_0^p(f)_{X(\mathsf{T})} + \sum_{\upsilon=1}^{\infty} 2^{\upsilon p \beta} E_{2^{\upsilon-1}}^p(f)_{X(\mathsf{T})} \right) < \infty$$

and the series

$$\sum_{\upsilon=0}^{\infty} U_\upsilon^{(\beta)}(\cdot)$$

converges a.e., its sum g is of class X. Now we prove that $g = f^{(\beta)}$ a.e.

For $0 \neq h \in \mathbb{R}$ we have

$$\left\| \frac{\Delta_h^\beta f}{h^\beta} - g \right\|_{X(\mathsf{T})}^p \leq c \left\| \frac{1}{h^\beta} \sum_{k=0}^{N} (-1)^k \binom{\beta}{k} f\left(we^{ikh}\right) - g(w) \right\|_{X(\mathsf{T})}^p$$

$$+ c \left\| \frac{1}{h^\beta} \sum_{k=N+1}^{\infty} (-1)^k \binom{\beta}{k} f\left(we^{ikh}\right) \right\|_{X(\mathsf{T})}^p := c(I_1 + I_2)$$

In this case

$$I_2 \leq \frac{1}{|h|^{\beta p}} \sum_{k=N+1}^{\infty} \left| \binom{\beta}{k} \right|^p \|f\|_{X(\mathsf{T})}^p$$

and hence

$$\lim_{N\to\infty} I_2 = 0.$$

Now by Levi's theorem

$$I_1 = \left\| \frac{1}{h^\beta} \sum_{\upsilon=0}^{\infty} \sum_{k=0}^{N} (-1)^k \binom{\beta}{k} U_\upsilon\left(we^{ikh}\right) - g(w) \right\|_{X(\mathsf{T})}^p$$

$$\leq c\sum_{\upsilon=0}^{\infty}\left\|\frac{1}{h^{\beta}}\sum_{k=0}^{N}(-1)^{k}\left(\begin{array}{c}\beta\\k\end{array}\right)U_{\upsilon}\left(we^{ikh}\right)-U_{\upsilon}^{(\beta)}\left(w\right)\right\|_{X(\mathsf{T})}^{p}=:Y_{N}.$$

The last series converges uniformly in $N\geq 1$, because its υth term doesn't exceed

$$\frac{1}{|h|^{\beta p}}\sum_{k=0}^{\infty}\left|\left(\begin{array}{c}\beta\\k\end{array}\right)\right|^{p}\left(\|U_{\upsilon}\|_{X(\mathsf{T})}^{p}+\|U_{\upsilon}^{(\beta)}\|_{X(\mathsf{T})}^{p}\right)\leq c\left(\frac{1}{|h|^{\beta p}}+1\right)2^{\upsilon p\beta}E_{2^{\upsilon-1}}^{p}\left(f\right)_{X(\mathsf{T})}.$$

From Lebesgue Dominated convergence theorem we have

$$\lim_{N\to\infty}Y_{N}=\sum_{\upsilon=0}^{\infty}\left\|\frac{\Delta_{h}^{\beta}U_{\upsilon}}{h^{\beta}}-U_{\upsilon}^{(\beta)}\right\|_{X(\mathsf{T})}^{p}$$

and then

$$\begin{aligned}\left\|\frac{\Delta_{h}^{\beta}f}{h^{\beta}}-g\right\|_{X(\mathsf{T})}^{p}&\leq c\sum_{\upsilon=0}^{\infty}\left\|\frac{\Delta_{h}^{\beta}U_{\upsilon}}{h^{\beta}}-U_{\upsilon}^{(\beta)}\right\|_{X(\mathsf{T})}^{p}\\&\leq c\sum_{\upsilon=0}^{s}\left\|\frac{\Delta_{h}^{\beta}U_{\upsilon}}{h^{\beta}}-U_{\upsilon}^{(\beta)}\right\|_{X(\mathsf{T})}^{p}+c\sum_{\upsilon=s+1}^{\infty}\left\|\frac{\Delta_{h}^{\beta}U_{\upsilon}}{h^{\beta}}\right\|_{X(\mathsf{T})}^{p}+c\sum_{\upsilon=s+1}^{\infty}\|U_{\upsilon}^{(\beta)}\|_{X(\mathsf{T})}^{p}\\&\leq c\sum_{\upsilon=0}^{s}\left\|\frac{\Delta_{h}^{\beta}U_{\upsilon}}{h^{\beta}}-U_{\upsilon}^{(\beta)}\right\|_{X(\mathsf{T})}^{p}+c\sum_{\upsilon=s+1}^{\infty}2^{\upsilon p\beta}E_{2^{\upsilon-1}}^{p}\left(f\right)_{X(\mathsf{T})}.\end{aligned}$$

For given positive ε the last term is less than ε for sufficiently large s. By Remark 3 we get

$$\lim_{h\to 0^{+}}\left\|\frac{\Delta_{h}^{\beta}U_{\upsilon}}{h^{\beta}}-U_{\upsilon}^{(\beta)}\right\|_{X(\mathsf{T})}^{p}=0$$

and therefore

$$\lim_{h\to 0^{+}}\left\|\frac{\Delta_{h}^{\beta}f}{h^{\beta}}-g\right\|_{X(\mathsf{T})}^{p}<\varepsilon.$$

This implies that $g=f^{(\beta)}$ a.e.

Let $m\in\mathbb{Z}^{+}$ be such that $2^{m-1}\leq n<2^{m}$. We have

$$\begin{aligned}\|T_{n}^{(\beta)}-f^{(\beta)}\|_{X(\mathsf{T})}^{p}&=\left\|T_{n}^{(\beta)}-\sum_{\upsilon=0}^{\infty}U_{\upsilon}^{(\beta)}\right\|_{X(\mathsf{T})}^{p}\\&\leq c\left\|T_{n}^{(\beta)}-T_{2^{m}}^{(\beta)}\right\|_{X(\mathsf{T})}^{p}+c\sum_{\upsilon=m+1}^{\infty}\|U_{\upsilon}^{(\beta)}\|_{X(\mathsf{T})}^{p}\\&\leq c\left(2^{mp\beta}E_{n}^{p}\left(f\right)_{X(\mathsf{T})}+\sum_{\upsilon=m+1}^{\infty}2^{\upsilon p\beta}E_{2^{\upsilon-1}}^{p}\left(f\right)_{X(\mathsf{T})}\right)\end{aligned}$$

$$\leq c\left(n^{\beta p}E_{n}^{p}\left(f\right)_{X(\mathsf{T})}+\sum_{\mu=n+1}^{\infty}\mu^{p\beta-1}E_{\mu}^{p}\left(f\right)_{X(\mathsf{T})}\right)$$

and the result is proved. □

References

[1] Ramazan Akgün, Polynomial approximation of functions in weighted Lebesgue and Smirnov spaces with nonstandard growth, Georgian Math. J. 18 (2011), no. 2, 203-235.

[2] —, Trigonometric approximation of functions in generalized Lebesgue spaces with variable exponent, Ukrainian Math. J., 63 (2011) no.1, 1-26.

[3] —, Approximation by polynomials in rearrangement invariant quasi Banach spaces, Banach J. Math. Anal. 6 (2012) No.2 , pp.113-131.

[4] Ramazan Akgün and Vakhtang Kokilashvili, On converse theorems of trigonometric approximation in weighted variable exponent Lebesgue spaces, Banach J. Math. Anal. 5 (2011), no. 1, 70-82.

[5] —,—, The refined direct and converse inequalities of trigonometric approximation in weighted variable exponent Lebesgue spaces, Georgian Math. J. 18 (2011), no. 3, 399-423.

[6] C. Benneth and R. Sharpley, Interpolation of operators, Pure and applied mathematics, 129, Academic Press, Boston, 1988.

[7] L. Grafakos and N. J. Kalton, Some remarks on multilinear maps and interpolation, Math. Ann. 319 (2001), 151-180.

[8] G. P. Gurbea, J. G-Cuerva, J. M. Martell and C. Perez, Extrapolation with weights, rearrangement-invariant function spaces, modular inequalities and applications to singular integrals, Adv. Math. 203 (2006), 256-318.

[9] W. B. Jahnson and G. Schechtman, Sums of independent random variables in rearrangement invariant function spaces, Ann. Probab. 17 (1987), 789-808.

[10] Pankaj Jain, Babita Gupta and Daulti Verma, Mean inequalities in certain Banach function spaces, J. Math. Anal. Appl. 334 (2007), no. 1, 358-367.

[11] N. J. Kalton, Convexity conditions on non-locally convex lattices, Glasgow J. Math. 25 (1984), 141-152.

[12] S. G. Krein, Yu. I. Petunin and E. M. Semenov, Interpolations of linear operators, Transl. Math. Monogr. 54, AMS, Providence, 1982.

[13] J. Lindenstrauss and L. Tzafriri, Classical Banach spaces I and II, Classics in mathematics, Springer-Verlag, 1996.

[14] G. Ju. Lozanovskii, On reflexive KB spaces, Dokl. Akad. Nauk. SSSR, 158 (1964), 516-519 (In Russian).

[15] Lech Maligranda and Lars Erik Persson, Generalized duality of some Banach function spaces, Nederl. Akad. Wetensch. Indag. Math. 51 (1989), no. 3, 323-338.

[16] S. J. Montgomery-Smith, The Hardy operator and Boyd Indices, Lecture Notes in Pure and Applied Mathematics, vol: 175, Dekker, New York, 1996, pp. 359-364.

[17] Lars Erik Persson, Some elementary inequalities in connection with Xp-spaces, Constructive theory of functions (Varna, 1987), 367–376, Publ. House Bulgar. Acad. Sci., Sofia, 1988.

[18] Lars Erik Persson, On some generalized Orlicz classes and spaces, Research Report 1988-3, Department of Mathematics, Luleå University of Technology, 1988.

[19] S. G. Samko, A. A.. Kilbas and O. I. Marichev, Fractional integrals and derivatives, Theory and applications, Gordon and Breach Science Publishers, 1993.

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