Topology/ Algebraic Geometry/Complex Variables

BRAIDS, CONFORMAL MODULE AND ENTROPY

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ABSTRACT. The conformal module of conjugacy classes of braids implicitly appeared in a paper of Lin and Gorin in connection with their interest in the 13. Hilbert Problem. This invariant is the supremum of conformal modules (in the sense of Ahlfors) of certain annuli related to the conjugacy class. This note states that the conformal module is inverse proportional to a popular dynamical braid invariant, the entropy. The entropy appeared in connection with Thurston's theory of surface homeomorphisms. An application of the concept of conformal module to algebraic geometry is given.

Braids occur in several mathematical fields, sometimes unexpectedly, and one can think of them in several different ways. Braids on n strands can be interpreted as algebraic objects, namely, as elements of the Artin group $\mathcal{B}_n$, or as isotopy classes of geometric braids, or as elements of the mapping class group of the n-punctured disc ([4]). In connection with his interest in the Thirteen's Hilbert Problem Arnol'd gave the following interpretation of the braid group $\mathcal{B}_n$. Denote by $\mathfrak{P}_n$ the space of monic polynomials of degree n without multiple zeros. This space can be parametrized either by the coefficients or by the unordered tuple of zeros of polynomials. This makes $\mathfrak{P}_n$ a complex manifold, in fact, the complement of an algebraic hypersurface in complex Euclidean space $\mathbb{C}^n$. Arnol'd studied its topological invariants ([2]). Choose a base point $p \in \mathfrak{P}_n$. Using the second parametrization Arnol'd interpreted the group $\mathcal{B}_n$ of n-braids as elements of the fundamental group $\pi_1(\mathfrak{P}_n, p)$ with base point p.

The conjugacy classes $\widehat{\mathcal{B}}_n$ of the braid group, equivalently, of the fundamental group $\pi_1(\mathfrak{P}_n, p)$, can be interpreted as free isotopy classes of loops in $\mathfrak{P}_n$. We define the following collection of conformal invariants of the complex manifold $\mathfrak{P}_n$. Consider an element $\hat{b}$ of $\widehat{\mathcal{B}}_n$. We say that a continuous mapping f of an annulus $A = \{z \in \mathbb{C} : r < |z| < R\}$, $0 \leq r \leq R$, into $\mathfrak{P}_n$ represents $\hat{b}$ if for some (and hence for any) circle $\{|z| = \rho\} \subset A$ the loop $f \colon \{|z| = \rho\} \to \mathfrak{P}_n$ represents $\hat{b}$.

Recall that any domain in the complex plane with fundamental group isomorphic to the group of integer numbers $\mathbb{Z}$ is conformally equivalent to an annulus. Ahlfors defined the conformal module of an annulus $A = \{z \in \mathbb{C} : r < |z| < R\}$, as $m(A) = \frac{1}{2\pi} \log(\frac{R}{r})$. Two annuli of finite conformal module are conformally equivalent iff they have equal conformal module.

We call the aforementioned conformal invariants of $\mathfrak{P}_n$ the conformal modules of the conjugacy classes of its fundamental group, or, equivalently, the conformal modules of conjugacy classes of n-braids. They are defined as follows.

Definition *Let $\hat{b}$ be a conjugacy class of n-braids, $n \geq 2$. The conformal module $M(\hat{b})$ of b is defined as $M(\hat{b}) = sup_{\mathcal{A}} m(A)$, where $\mathcal{A}$ denotes the set of all annuli which admit a holomorphic mapping into $\mathfrak{P}_n$ which represents $\hat{b}$.*

For any complex manifold the conformal module of conjugacy classes of its fundamental group can be defined. The collection of conformal modules of all conjugacy classes is a biholomorphic invariant of the manifold. This concept seems to be especially useful for locally symmetric spaces, for instance, for quotients of the n-dimensional round complex ball by a discrete subgroup of its automorphism group. In this case the universal covering is the ball and the fundamental group of the quotient manifold can be identified with the group of covering translations. For each covering translation the problem is to consider the quotient of the ball by the action of the group generated by this single covering translation and to maximize the conformal module of annuli which admit holomorphic mappings into this quotient. The latter concept can be generalized to general mapping class groups. The generalization has relations to symplectic fibrations.

Runge's approximation theorem shows that the conformal module is positive for any conjugacy class of braids. The concept of the conformal module of conjugacy classes of braids appeared (without name) in the paper [6] which was motivated by the interest of the authors in Hilbert's Thirteen's Problem for algebraic functions.

The following objects related to $\mathfrak{P}_n$ have been considered in this connection. A continuous mapping of a Riemann surface X into the set of monic polynomials of fixed degree (maybe, with multiple zeros) is a quasipolynomial. It can be written as $P(x, z) = a_0(x) + a_1(x)z + a_{n-1}(x)z^{n-1} + z^n$, $x \in X$, $\zeta \in \mathbb{C}$, for continuous functions a_j, $j = 1...n$, on X. If the mapping is holomorphic it is called an algebroid function. If the image of the map is contained in $\mathfrak{P}_n$ it is called separable. A separable quasipolynomial is called solvable if it can be globally written as a product of quasipolynomials of degree 1, and is called irreducible if it can not be written as product of two quasipolynomials of positive degree. Two separable quasipolynomials are isotopic if there is a continuous family of separable quasipolynomials joining them. An algebroid function on the complex line $\mathbb{C}$ whose coefficients are polynomials is called an algebraic function. A quasipolynomial P can be considered as a function on $X \times \mathbb{C}$. Its zero set $\mathfrak{S}_P = \{(x, \zeta) \in X \times \mathbb{C}, \ P(x, \zeta) = 0\}$ is a symplectic surface (in the standard symplectic structure), called braided surface due to its relation to braids.

Gorin and Lin considered separable monic polynomials of degree n with coefficients in commutative Banach algebras and asked whether solvability of all

such polynomials implies solvability of all separable quasipolynomials of degree n on the space of maximal ideals of the algebra. They gave a negative answer by finding conjugacy classes of braids with finite conformal module.

The conformal module of conjugacy classes of braids serves as obstruction for the existence of isotopies of quasipolynomials (respectively, of braided surfaces) to algebroid functions (respectively, to complex curves). Indeed, let X be an open Riemann surface. Suppose f is a quasipolynomial of degree n on X. Consider any domain $A \subset X$ which is conformally equivalent to an annulus. The restriction of f to A defines a mapping of the domain A into the space of polynomials $\mathfrak{P}_n$, hence it defines a conjugacy class of n-braids $\hat{b}_{f,A}$.

Lemma 1. *If f is algebroid then $mA \leq M(\hat{b}_{f,A})$.*

Before giving an example of applications of the concept of conformal module of conjugacy classes of braids we compare this concept with a dynamical concept related to braids. Let $\mathbb{D}$ be the unit disc in the complex plane. Denote by E_n^0 the set consisting of the n points $0, \frac{1}{n}, ..., \frac{n-1}{n}$. Consider homeomorphisms of the n-punctured disc $\overline{\mathbb{D}} \setminus E_n^0$, which fix the boundary $\partial \mathbb{D}$ pointwise. Equivalently, these are homeomorphisms of the closed disc $\overline{\mathbb{D}}$ which fix the boundary pointwise and the set E_n^0 setwise. Equip this set of homeomorphisms with compact open topology. The connected components of this space from a group, called mapping class group of the n-punctured disc. This group is isomorphic to $\mathcal{B}_n$. Denote by $\mathcal{H}_b$ the connected component which corresponds to the braid b.

For a homeomorphisms of a compact topological space its topological entropy is an invariant which measures the complexity of its behaviour in terms of iterations. It is defined in terms of the action of the homeomorphism on open covers of the compact space. For a precise definition of topological entropy we refer to the papers [1] or [5]. For a braid b we define its entropy as $h(b) = \inf\{h(\varphi) : \varphi \in \mathcal{H}_b\}$. The value is invariant under conjugation with self-homeomorphisms of the closed disc $\overline{\mathbb{D}}$ that fix the boundary pointwise, hence it does not depend on the choice of the set of punctures and on the choice of the representative of the conjugacy class $\hat{b}$. We write $h(\hat{b}) = h(b)$.

Entropy is a dynamical invariant. It has been considered in connection with Thurston's theory of surface homeomorphisms. Thurston himself used dynamical methods (Markov partitions) to show that many mapping class tori have hyperbolic metric. Detailed proofs of Thurston's theorems are given in [5] where also the entropy of homeomorphisms of closed Riemann surfaces is studied. The study has been extended to Riemann surfaces with punctures. The common definition for braids uses mapping classes of the n-punctured complex plane rather than of the n-punctured disc. One can show that this definition is equivalent to the definition given above. Entropy has been studied intensively. E.g., the lowest non-vanishing entropy among braids on n strands has been computed for small n. There is an algorithm for computing entropy for irreducible braids (respectively,

for irreducible mapping classes) ([3]). Topological methods of fluid mechanics use entropy as a measure of complexity of the arising braids.

It turns out that the dynamical aspect and the conformal aspect are related. The following theorem holds.

Theorem 1. *For each* $\hat{b} \in \widehat{\mathcal{B}}_n$ *($n \geq 2$)*

$$M(\hat{b}) = \frac{\pi}{2} \frac{1}{h(\hat{b})} \,.$$

The proof of the theorem deeply relies on Teichmüller theory, including Royden's theorem on equality of the Teichmüller metric and the Kobayashi metric on Teichmüller space, and Bers' theory of reducible surface homeomorphisms.

Corollary 1. *For each* $\hat{b} \in \widehat{\mathcal{B}}_n$ *($n \geq 2$) and each nonzero integer l*

$$M(\widehat{b^l}) = \frac{1}{l} M(\hat{b}) \,.$$

Indeed, for the entropy the equality $h(\widehat{b^l}) = l\, h(\hat{b})$ holds ([1]).

There is also the concept of the conformal module of braids rather than of conjugacy classes of braids. This notion is based on the conformal module of rectangles which admit holomorphic mappings into $\mathfrak{P}_n$ with suitable boundary conditions on a pair of opposite sides. Recall that the conformal module of a rectangle with sides parallel to the coordinate axes is the ratio of the sidelengths of horizontal and vertical sides. The conformal module of braids is a finer invariant than entropy. If suitably defined it is more appropriate for application, in particular, to real algebraic geometry. In the case of three-braids there are two versions differing by the boundary conditions on horizontal sides of rectangles. (Both should be used.) For the first version one requires that horizontal sides are mapped to polynomials with all zeros on a real line. In the second version one requires that two of the zeros have equal distance from the third. These are the cases appearing on the real axis for polynomials with real coefficients. The invariant can be studied by quasiconformal mappings and elliptic functions. The situation for braids on more than 3 strands is more complex. We intend to come back to this concept in a later paper.

Using the observation of Lemma 1 we give the following example of application of the concept of the conformal module of conjugacy classes of braids.

Let X and Y be open Riemann surfaces and let f be a quasipolynomial on X. Let $w\colon X \to Y$ be a homeomorphism. The homeomorphism w can be interpreted as a new complex structure on X. If w is quasiconformal it represents an element of the Teichmüller space modeled on X. (For Teichmüller theory of open Riemann surfaces see e.g. [7].) Here we do not make any requirement on the homeomorphism w. Denote by f_w the quasipolynomial $f_w(y,z) = f(w^{-1}(y), \zeta)$, $y \in Y$, $\zeta \in \mathbb{C}$. We say that f is isotopic to an algebroid function for the complex structure w if f_w is isotopic to an algebroid function on Y. If two quasiconformal

homeomorphism $w_1 \colon X \to Y_1$ and $w_2 \colon X \to Y_2$ are Teichmüller equivalent then f is isotopic to an algebroid function for the complex structure w_1 iff it is so for w_2. In this case we will say that f is isotopic to an algebroid function for the respective element of the Teichmüller space $\mathcal{T}(X)$ modeled on X.

Let X be an open Riemann surface of finite genus with at most countably many ends. By [8] X is conformally equivalent to a domain Ω on a closed Riemann surface R such that the connected components of $R \setminus \Omega$ are all points or closed geometric discs. A geometric disc is a topological disc whose lift to the universal covering is a round disc (in the standard metric of the covering).

Theorem 2. *Let X be a torus with a geometric disc removed. There exist eight elements of the Teichmüller space $\mathcal{T}(X)$ modeled on X with the following property.*

Let f be an irreducible separable quasipolynomial of degree 3 on X which is isotopic to an algebroid function for each of the eight elements of the Teichmüller space. Then f is isotopic to an algebroid function for each complex structure on X including complex structures of first kind (determining punctured tori X_p).

Moreover, denote by X_p a punctured torus and by f_p an algebroid function on X_p which is isotopic to f. Then $\mathfrak{S}_{f_p}$ is a leaf of a non-singular holomorphic foliation on $X_p \times \mathbb{C}$ and the foliation extends to a non-singular holomorphic foliation on a compact complex space, namely the total space of a degree zero holomorphic $\mathbb{P}^1$-bundle on the closed torus.

Let X be as in Theorem 2. Note that the fundamental group $\pi_1(X, x)$ of X with base point $x \in X$ is a free group on two generators. Consider a quasipolynomial of degree n and all its isotopies which fix the value at the base point x. For each element a of the fundamental group of $\pi_1(X, x)$ this defines a homotopy class of loops $\varphi(a)$ with a base point p in $\mathfrak{P}_n$, hence an n-braid . The mapping φ is a homomorphism from $\pi_1(X, x)$ to $\mathcal{B}_n$. There is a one-to-one correspondence between free isotopy classes of quasipolynomials (without fixing the value at a point) and conjugacy classes of homomorphisms from $\pi_1(X)$ to $\mathcal{B}_n$. The following proposition holds.

Proposition 1. *Let X and f be as in the theorem. Then f corresponds to the conjugacy class of a homomorphism $\varphi \colon \pi_1(X) \to \mathcal{B}_n$ with the following property. There exist $l, q \in \mathbb{Z}$ such that for one of the standard generators of $\pi_1(X)$, say for a_1, we have $\varphi(a_1) = (\sigma_1 \sigma_2)^{3l+1}$, and for the other generator we have $\varphi(a_2) = (\sigma_1 \sigma_2)^q$.*

Here σ_1 and σ_2 are the standard generators of $\mathcal{B}_3$. Theorem 2 is based on Proposition 1 and the following Lemma 2.

Lemma 2. *Let X be a closed Riemann surface with a geometric disc removed. Suppose f is an irreducible separable algebroid function of degree 3 on X. Suppose X contains a domain A one of whose boundary components coincides with the boundary circle of X, such that A is conformally equivalent to an annulus of conformal module strictly larger than $\frac{\pi}{2} \left(\log\left(\frac{3+\sqrt{5}}{2}\right)\right)^{-1}$. Then f is solvable over A.*

Note that $\log(\frac{3+\sqrt{5}}{2})$ is the smallest non-vanishing entropy among 3-braids.

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