# EXTRAPOLATION OF ANALYTIC FUNCTIONS BY SUMS OF THE FORM $\sum_k \lambda_k h(\lambda_k z)$

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ABSTRACT. We consider a method for extrapolating analytic functions $h$ by their respective sums of the form $\sum_k \lambda_k h(\lambda_k z)$ and show advantages of this method.

## 1. Introduction

We study extrapolation of functions $h$ which are analytic in a neighbourhood of the point $z = 0$ by finite sums (we call them *h-sums*) of the following form

$$H_n(z) = H_n(h, \{\lambda_k\}; z) = \sum_{k=1}^{n} \lambda_k h(\lambda_k z), \qquad z, \lambda_k \in \mathbb{C}, \qquad n \in \mathbb{N}.$$

The possibility of uniform approximation of analytic functions by $h$-sums has been proved in [1]; there has been provided an effective method for constructing such sums using Taylor coefficients of an approximating function.

The $h$-sums and their applications were studied then in [2, 3, 4].

Note that $h$-sums naturally generalize the so-called *simple partial fractions*, i.e., rational functions of the form

$$\rho_n(z) = \sum_{k=1}^{n} \frac{1}{z - z_k}, \qquad z, z_k \in \mathbb{C},$$

which are actively studied at present in approximation theory (see [2],[5]—[8]). Indeed, if we take, for example, $h(z) = (z-1)^{-1}$, then for $z_k := \lambda_k^{-1}$ we obtain

$$\rho_n(z) \equiv H_n(z).$$

Let

$$\lambda_1, \dots, \lambda_n, \qquad n \in \mathbb{N},$$

be a set of complex numbers. We use the Newton recurrence formulas (see, for example, [9]):

$$(1) \qquad S_1 = \sigma_1,\ S_m = (-1)^{m+1}\, m\, \sigma_m - \sum_{j=1}^{m-1} (-1)^{m-j} S_j \sigma_{m-j},\ m = 2, 3, \dots,$$

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which express the power sums

$$S_m = S_m(\lambda_1, \dots, \lambda_n) := \sum_{k=1}^{n} \lambda_k^m$$

through the symmetric polynomials

$$\sigma_m = \sigma_m(\lambda_1, \dots, \lambda_n) := \sum\nolimits_{1 \le j_1 < \dots < j_m \le n} \lambda_{j_1} \cdots \lambda_{j_m}, \quad m = 1, 2 \dots, \tag{2}$$

where

$$\sigma_m = 0, \qquad m > n.$$

Let $a > 1$. Choose the numbers $\lambda_k$ in such a way that

$$\sigma_1 = 1, \quad \sigma_m = \frac{(-1)^{m+1}}{m!} \prod_{k=1}^{m-1} (a\,k - 1), \quad m = 2, \dots, n. \tag{3}$$

Then, as it has been shown in [2],

$$\max_{k=1,\dots,n} |\lambda_k| \le a - \frac{a-1}{n}, \qquad S_m = a^{m-1}, \quad m = 1, \dots, n. \tag{4}$$

Let $h$ be a function analytic in the domain

$$D_r := \{z : \ |z| < r\}$$

(where $0 < r \le \infty$ and $D_\infty = \mathbb{C}$), and have the following Taylor expansion

$$h(z) = h_0 + h_1 z + h_2 z^2 + \dots$$

If choose the numbers $\lambda_k$ as it has been shown in (3), then we obtain:

$$H_n^{(1)}(h, a, \{\lambda_k\}; z) := H_n\left(\frac{z}{a}\right) = \sum_{m=0}^{n-1} h_m z^m + \sum_{m=n}^{\infty} h_m S_{m+1} \left(\frac{z}{a}\right)^m,$$

and the sums $H_n$ as functions of $z$ are defined and analytic in $D_r$ and

$$h(z) - H_n\left(\frac{z}{a}\right) = \delta_n(h, a, z) := \sum_{m=n}^{\infty} h_m \left(1 - \frac{S_{m+1}}{a^m}\right) z^m, \qquad z \in D_r. \tag{5}$$

From this, the following extrapolation formula in [2] was obtained (e.i. values of a function $h$ are found using its values in points with less modulus):

$$\begin{gathered} h(z) \approx H_n\left(\frac{z}{a}\right) = \sum_{k=1}^{n} \lambda_k h\left(\frac{\lambda_k}{a} z\right), \\ \left|\frac{\lambda_k}{a} z\right| \le \beta_n |z| < |z| < r, \quad \beta_n := 1 - \frac{a-1}{an}, \end{gathered} \tag{6}$$

with the remainder $\delta_n(h, a, z)$.

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When we use the $\mu$-multiple extrapolation ($\mu \in \mathbb{N}$) we obtain a $\mu$-multiple $h$-sum

$$H_n^{(\mu)}(z) = H_n^{(\mu)}(h, \{\lambda_k\}; a; z)$$

of the form (see [1, 2])

$$H_n^{(\mu)}(z) = \sum_{k_1,\dots,k_\mu=1}^{n} \lambda_{k_1}\cdots\lambda_{k_\mu} h\left(\frac{\lambda_{k_1}\cdots\lambda_{k_\mu}}{a^\mu} z\right), \qquad \left|\frac{\lambda_{k_1}\cdots\lambda_{k_\mu}}{a^\mu} z\right| \le \beta_n^\mu\,|z|. \tag{7}$$

The remainder has been estimated in [2] as

$$\left|h(z) - H_n^{(\mu)}(z)\right| \le n^\mu (1 + a\,n) \sum_{m=n}^{\infty} |h_m||z|^m, \qquad |z| < r. \tag{8}$$

As we can see from (8), if $n$ increases for a fixed $\mu$ the estimate of the remainder improves, but the estimate (7) of the radius of the disc, where all the extrapolation nodes $\lambda_{k_1}\cdots\lambda_{k_\mu} a^{-\mu} z$ lie, becomes worse. On the contrary, if $\mu$ increases for a fixed $n$ the estimate (7) for the nodes improves, but the estimate (8) becomes worse. Therefore, in [2] the following question has been put.

*Whether it is possible to choose $n$ and $\mu$ in such a way that we can extrapolate the values of $h$ with any accuracy on the circle $|z| = r_1 < r$ provided that all the extrapolation nodes $\lambda_{k_1}\cdots\lambda_{k_\mu} a^{-\mu} z$ lie in some fixed disc $|z| \le r_0 < r_1$ ( we suppose $r_0 < r_1 = 1 < r$)?*

In the paper [2], a positive decision has been found for this question in the case of entire functions of a finite order.

**Proposition** [2]**.** *Let $h$ be an entire function of a finite order $\rho$ and $b > \rho$, $a = 2$. Let the numbers $r_0 \in \mathbb{R}$ $\mu_n \in \mathbb{N}$ satisfy the conditions:*

$$\exp\left(-\frac{1}{2\,b}\right) < r_0 < 1, \qquad \mu_n = \left[\frac{\ln r_0}{\ln(1 - \frac{1}{2\,n})}\right] + 1. \tag{9}$$

*Then while $n \to \infty$ the sums $H_n^{(\mu_n)}(z)$ (of the form (7)) converge uniformly to $h(z)$ on the circle $|z| = 1$. Moreover, all the extrapolation nodes lie in the disc $|z| < r_0$.*

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## 2. Main results

In the present paper an explicit form of the remainder of the $\mu$-multiple extrapolation is obtained and the estimate (8) is improved considerably. The question is solved positively for any functions analytic in $D_r$. Moreover, it is shown that if the multiplicity increases, the error of extrapolation does not increase, but the nodes, where we have to know the values of the extrapolated functions, converge to the origin. Let us formulate the main result.

**Theorem.** *Let $h$ be a function analytic in $D_r$ $(0 < r \leq \infty)$, $a > 1$. Then the remainder of the $\mu$-multiple extrapolation has the form:*

$$h(z) - H_n^{(\mu)}(z) = \sum_{m=n}^{\infty} h_m \left(1 - \left(\frac{S_{m+1}}{a^m}\right)^{\mu}\right) z^m, \quad z \in D_r. \tag{10}$$

*and the error estimate (independent of $\mu$) holds:*

$$\left|h(z) - H_n^{(\mu)}(z)\right| \leq \sum_{m=n}^{\infty} |h_m||z|^m, \qquad z \in D_r. \tag{11}$$

*If $r > 1$, $0 < r_0 < 1$, and the numbers $\mu_n \in \mathbb{N}$ satisfy the equality*

$$\mu_n = \left[\frac{\ln r_0}{\ln\left(1 - \frac{a-1}{an}\right)}\right] + 1, \tag{12}$$

*where $[\cdot]$ is an integer part, then for $n \to \infty$ the sums $H_n^{(\mu_n)}(z)$ converge uniformly to $h(z)$ in the circle $|z| = 1$ and all the extrapolation nodes lie in the disc $|z| < r_0$.*

## 3. Proof of main results

**Lemma.** *For $m \geq 1$ and $a > 1$ for the power series $S_m$ of the numbers $\lambda_k$ the inequality holds:*

$$1 \leq S_m \leq a^{m-1}, \qquad m \in \mathbb{N}.$$

**Proof.** Considering (3) and (4), for $m \geq n+1$ from the formula (1) we find:

$$S_m = \sum_{j=m-n}^{m-2} S_j \frac{\prod_{k=1}^{m-j-1}(ak-1)}{(m-j)!} + S_{m-1}, \tag{13}$$

which implies that $S_m \geq 1$ for any $m \geq 1$ and the consequence $S_m$ grows.

The right side of the inequality in Lemma we prove by induction. It is known (see (4)) that

$$S_m = a^{m-1}, \qquad m = 1, \dots, n.$$

Assume that inequalities

$$S_m \leq a^{m-1}$$

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are valid for any $m = 1, \ldots, p-1$ and prove that it is also valid for $m = p$:

$$\begin{aligned} S_p &= \sum_{j=p-n}^{p-2} S_j \frac{\prod_{k=1}^{p-j-1}(ak-1)}{(p-j)!} + S_{p-1} \\ &\le a^{p-2} \sum_{j=p-n}^{p-1} \frac{\Gamma(p-j-a^{-1})}{\Gamma(1-a^{-1})(p-j)!} \\ &= \frac{a^{p-2}}{\Gamma(1-a^{-1})} \sum_{l=1}^{n} \frac{\Gamma(l-a^{-1})}{l!} := a^{p-2} V(a). \end{aligned}$$

In the first inequality we use certain property of the Gamma function

$$\Gamma(z+n) = z(z+1)\cdots(z+n-1)\Gamma(z)$$

for $z = 1 - a^{-1}$.

To complete the proof let show that

$$V(a) < a.$$

Indeed, the sum in $V(a)$ can be found from the obvious inequality

$$\sum_{l=0}^{\infty} \frac{\Gamma(l-a^{-1})}{l!} = \Gamma(-a^{-1}) + \sum_{l=1}^{n} \frac{\Gamma(l-a^{-1})}{l!} + \sum_{l=n+1}^{\infty} \frac{\Gamma(l-a^{-1})}{l!}.$$

It is known (see, e.g., [10]) that

$$\sum_{l=0}^{\infty} \frac{\Gamma(l-a^{-1})}{l!} = 0.$$

We obtain

$$\begin{aligned} V(a) &= -\frac{\Gamma(-a^{-1})}{\Gamma(1-a^{-1})} - \frac{\sum_{l=n+1}^{\infty} \frac{\Gamma(l-a^{-1})}{l!}}{\Gamma(1-a^{-1})} \\ &= a - \frac{\sum_{l=n+1}^{\infty} \frac{\Gamma(l-a^{-1})}{l!}}{\Gamma(1-a^{-1})} < a. \end{aligned}$$

**Proof of Theorem.** We consider the circle

$$|z| = 1.$$

The formula for $\mu_n$ is obtained using inequalities in (6) and (7) from the condition that all the extrapolation nodes must lie in the disc

$$|z| \le r_0 < 1.$$

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Let transfer $H_n^{(\mu)}$ from the formula (7):

$$\begin{aligned} H_n^{(\mu)}(z) &= \sum_{k_1,\dots,k_\mu=1}^{n} \lambda_{k_1}\cdots\lambda_{k_\mu} h\left(\frac{\lambda_{k_1}\cdots\lambda_{k_\mu}}{a^\mu} z\right) \\ &= \sum_{k_1,\dots,k_\mu=1}^{n} \lambda_{k_1}\cdots\lambda_{k_\mu} \sum_{m=0}^{\infty} h_m \left(\frac{\lambda_{k_1}\cdots\lambda_{k_\mu}}{a^\mu} z\right)^m \\ &= \sum_{m=0}^{\infty} h_m \frac{\sum_{k_1,\dots,k_\mu=1}^{n} \left(\lambda_{k_1}\cdots\lambda_{k_\mu}\right)^{m+1}}{a^{m\mu}} z^m \\ &= \sum_{m=0}^{\infty} h_m \left(\frac{S_{m+1}}{a^m}\right)^\mu z^m. \end{aligned}$$

From this we obtain the remainder (which implies the formula (5) for $\mu = 1$)

$$\begin{aligned} h(z) - H_n^{(\mu)}(z) &= \sum_{m=0}^{\infty} h_m z^m - \sum_{m=0}^{\infty} h_m \left(\frac{S_{m+1}}{a^m}\right)^\mu z^m \\ &= \sum_{m=n}^{\infty} h_m \left(1 - \left(\frac{S_{m+1}}{a^m}\right)^\mu\right) z^m, \end{aligned}$$

where

$$0 < \frac{S_{m+1}}{a^m} \le 1$$

by Lemma and it implies the error estimate in Theorem.

## 4. Notes and examples

From statements of the theorem, it follows particularly that in case of the multiple extrapolation the approximation error does not increase (the estimate (11) is independent of $\mu$), but the radius of the circle containing the extrapolation nodes decreases indefinitely (a number of the nodes, of course, increases).

For the $\mu$-multiple extrapolation the sum contains $n^\mu$ terms but we need to know values of the function $h$ only in

$$C_{n+\mu-1}^{\mu} = \frac{(n+\mu-1)!}{\mu!(n-1)!}$$

nodes. For example, for the 2-multiple extrapolation and $n = 6$ it is necessary to know values of the function in 21 nodes but not in 36 points (by a number of the terms in the sum).

Note that extrapolation by $h$-sums in some cases gives more accurate results then traditional interpolation (extrapolation) in roots of unity, i.e. in nodes

$$z_k = e^{2\pi i(k-1)/n}, \qquad k = 1, \dots, n.$$

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Suppose, for example,

$$h(z) = \frac{1}{z-\rho}, \qquad 1 < \rho < 3/2.$$

For $n > 10$ choose $z$ from the conditions $z^n = -1$ and $|z-\rho| < 1$. Then for the respective interpolation polynomial $p$ built using nodes $z_k$ we find (see [11]):

$$\begin{aligned} |h(z) - p(z)| &= \left| \frac{z^n - 1}{(\rho^n - 1)(z-\rho)} \right| \\ &= \frac{2}{\rho^n(1-\rho^{-n})|z-\rho|} \geq \frac{2}{\rho^n |z-\rho|^2}. \end{aligned}$$

From this and (11), it follows that

$$|h(z) - H_n^{(1)}(z)| \leq \frac{1}{\rho^n |z-\rho|} < |z-\rho||h(z)-p(z)|.$$

Now we give some numerical examples. We choose $r_0 = 0.8$, $a = 1.5$ and $n = 4$. In this case we need $\mu = 3$. If we extrapolate, for example, the function

$$h(z) = \frac{1}{z-2}$$

in $z = 1$ using its 20 values for $r_0$, $a$ $n$ we obtain

$$h(1) - H_4^3(1) = -0.0322\ldots.$$

Note that the error of Taylor approximation equals

$$h(1) - \sum_{m=4}^{\infty} h_m = -1/16 = -0.0625.$$

If we use the 8-multiple extrapolation, the nodes (the number of them is 165) lie in the disc $r_0 = 0.41$ and

$$h(1) - H_4^8(1) = -0.0513\ldots,$$

i.e. the error increases slightly and remains less than the error of the remainder of the Taylor series of the function.

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