QUANTUM COHOMOLOGY OF TORIC ORBIFOLDS

EDUARDO GONZALEZ AND CHRIS WOODWARD

Abstract. For any compact toric orbifold (smooth proper Deligne-Mumford toric stack) Y with projective coarse moduli space, we show that the quantum cohomology $QH(Y)$ is canonically isomorphic (in a formal neighborhood of a canonical bulk deformation) to a formal polynomial ring modulo the quantum Stanley-Reisner ideal introduced by Batyrev [4]. This generalizes results of Givental [20], Iritani [31] and Fukaya-Oh-Ohta-Ono [18] for toric manifolds and Coates-Lee-Corti-Tseng [11] for weighted projective spaces. In the language of Landau-Ginzburg potentials, we identify $QH(Y)$ with the ring of functions on the subset $\mathrm{Crit}_+(W)$ of the critical locus $\mathrm{Crit}(W)$ of an explicit potential W consisting of critical points mapping to the interior of the moment polytope, as in [22], [31], [18] in the manifold case. Our proof uses algebro-geometric virtual fundamental classes, a quantum version of Kirwan surjectivity, and an equality of dimensions deduced using a toric minimal model program (tmmp). The previous cases treated by Givental [20], Iritani [31], and Fukaya et al [18] used localization arguments for the residual torus action or open-closed Gromov Witten invariants.

The existence of a Batyrev presentation implies that the quantum cohomology of Y is generically semisimple. This is related by a conjecture of Dubrovin, see [5], to the existence of a full exceptional collection in the derived category of Y proved by Kawamata [34], also using tmmps.

Finally we discuss a connection with Hamiltonian non-displaceability. Any tmmp for Y with generic symplectic class defines a splitting of the quantum cohomology $QH(Y)$ with summands indexed by flips, contractions or fibration in the tmmp, and each summand corresponds a collection of Hamiltonian non-displaceable Lagrangian tori in Y. In particular the existence of infinitely many tmmps can produce open families of Hamiltonian non-displaceable Lagrangians, such as in the examples in Wilson-Woodward [45].

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1. Introduction

According to results of Danilov and Jurkiewicz [14, 32, 33], the rational cohomology ring of a complete rationally-smooth toric variety is the quotient of a polynomial ring generated by prime invariant divisors by the Stanley-Reisner ideal. In addition to relations corresponding to linear equivalence of invariant divisors, there are higher degree relations corresponding to collections of divisor classes whose intersection is empty.

One can reformulate this presentation of the cohomology ring in terms of equivariant cohomology as follows. Let G be a complex reductive group acting on a smooth polarized projective variety X. If the action on the semistable locus X^{ss} is locally free then the geometric invariant theory (git) quotient $X/\!\!/G = X^{\mathrm{ss}}/G$, by which we mean the stack-theoretic quotient of the semistable locus by the group action, is a smooth proper Deligne-Mumford stack with projective coarse moduli space. Under suitable properness assumptions the same holds for quasi-projective X, in particular, for tori G acting on finite-dimensional vector spaces X with weights contained in an open half-space. A result of Kirwan [35] says that the natural map $H_G(X,\mathbb{Q}) \to H(X/\!\!/G,\mathbb{Q})$ is surjective.

Now let G be a complex torus acting on a finite-dimensional complex vector space X equipped with a polarization so that $X/\!\!/G$ is a smooth proper Deligne-Mumford toric stack as in Borisov-Chen-Smith [7]; any such toric stack with projective coarse moduli space arises in this way. In this case $H_G(X)$ may be identified with the ring of polynomial functions on $\mathfrak{g}$, each weight maps to a divisor class in $H(X/\!\!/G)$ under the Kirwan map, and the Stanley-Reisner ideal SR_X^G is precisely the kernel of the Kirwan map. For example, if $G = \mathbb{C}^\times$ acts by scalar multiplication on $X = \mathbb{C}^k$, then $H_G(X) = \mathbb{Q}[\xi]$ is a polynomial ring in a single generator ξ, the git quotient is $X/\!\!/G = \mathbb{P}^{k-1}$, the intersection of the k prime invariant divisors is empty, and the Stanley-Reisner ideal is the ideal $\langle \xi^k \rangle$ generated by ξ^k. This gives the standard description of the cohomology ring of projective space $H(\mathbb{P}^{k-1}) = H_G(X)/SR_X^G = \mathbb{Q}[\xi]/\langle \xi^k \rangle$.

In this paper we give a similar presentation of the quantum cohomology of compact toric orbifolds with projective coarse moduli spaces, via the quantum version of the Kirwan map introduced in [46]. The results here generalize those of Batyrev [4], Givental [20], Lian-Liu-Yau [37], Iritani [29, 30, 31], and Fukaya-Oh-Ohta-Ono [18], who use results of McDuff-Tolman [39]. In particular, Iritani [31] computed the quantum cohomology of toric manifolds using localization arguments, while Fukaya et al [18] gave another proof using open-closed Gromov-Witten invariants. The orbifold quantum cohomology of weighted projective spaces is computed in Coates-Lee-Corti-Tseng [11]. The proof described here has several differences with the previous proofs, even in the case of Fano toric manifolds: it does not use any open Gromov-Witten invariants, proves (as the classical limit) the Danilov-Jurkewicz description, and does not use localization for the residual torus action. Giving a proof for orbifolds also makes the proof for manifolds easier, since one can use toric minimal model programs to prove equality of dimensions of the formal Batyrev ring and quantum cohomology. We introduce the following notations. Let Λ denote the *universal Novikov field* of formal powers of q

$$\Lambda = \left\{ \sum_d c_d q^d, \ c_d \in \mathbb{C}, d \in \mathbb{Q}, \forall e > 0, \#\{d | c_d < e\} < \infty \right\}.$$

We denote by $\Lambda_{\mathbb{Q}} \subset \Lambda$ the subfield with rational coefficients, and by $\Lambda_0 \subset \Lambda$ the subring with only non-negative powers of q. Let

$$QH_G(X) := H_G(X, \mathbb{C}) \otimes_{\mathbb{C}} \Lambda$$

denote the equivariant quantum cohomology of X. We denote by $QH_G(X, \mathbb{Q}) := H_G(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}}$ the subspace with rational coefficients. Equivariant enumeration of stable maps to X defines a family of products

$$\star_\alpha : T_\alpha QH_G(X, \mathbb{Q})^2 \longrightarrow T_\alpha QH_G(X, \mathbb{Q})$$

forming (part of) the structure of a *Frobenius manifold* on $QH_G(X, \mathbb{Q})$ [20] for α in a formal neighborhood of a symplectic class $\omega \in H_2^G(X, \mathbb{Q})$. Explicitly the product $\beta \star_{\alpha+\omega} \gamma$ is defined by

$$(1) \quad \langle \beta \star_{\alpha+\omega} \gamma, \delta \rangle = \sum_{d \in H_2(X,\mathbb{Z}), n \geq 0} (q^{\langle d, \omega \rangle}/n!) \int_{[\overline{\mathcal{M}}_{0,n+3}(X,d)_G \to BG]} \mathrm{ev}_1^* \alpha \cup \ldots \cup \mathrm{ev}_n^* \alpha \cup \mathrm{ev}_{n+1}^* \beta \cup \mathrm{ev}_{n+2}^* \gamma \cup \mathrm{ev}_{n+3}^* \delta^\vee$$

where the integral denotes push-forward using the equivariant virtual fundamental class described in [24]. The *inertia stack* of $X /\!\!/ G$ is

$$I_{X/\!\!/G} = \bigcup_{r>0} \mathrm{Hom}^{\mathrm{rep}}(\mathbb{P}(r), X/\!\!/G) = \bigcup_{[g]} X^{g,\mathrm{ss}}/Z_g$$

where in the first union, $\mathrm{Hom}^{\mathrm{rep}}(\mathbb{P}(r), \cdot)$ denotes representable morphisms from $\mathbb{P}(r) = B\mathbb{Z}_r$ and the second union over conjugacy classes $[g]$ of elements $g \in G$, with $Z_g \subset G$ the centralizer of g. The *rigidified inertia stack* is

$$\overline{I}_{X/\!\!/G} = \bigcup_{r>0} \mathrm{Hom}^{\mathrm{rep}}(\mathbb{P}(r), X/G)/\mathbb{P}(r) = \bigcup_{[g]} X^{g,\mathrm{ss}}/(Z_g/\langle g \rangle)$$

where $\langle g \rangle$ denotes the subgroup generated by g, as in Abramovich-Graber-Vistoli [1]. Let

$$QH(X/\!\!/G) := H(I_{X/\!\!/G}, \mathbb{C}) \otimes \Lambda$$

denote the orbifold quantum cohomology of $X/\!\!/G$, or $QH(X/\!\!/G, \mathbb{Q})$ the version with rational coefficients. Enumeration of twisted stable maps to $X/\!\!/G$ (representable maps from orbifold curves to $X/\!\!/G$) defines a Frobenius manifold structure on $QH(X/\!\!/G)$ [1], [10] given by a family of products defined in a formal neighborhood of an equivariant symplectic class $\omega \in H_2(X/\!\!/G, \mathbb{Q})$ by

$$(2) \quad \star_\alpha : T_\alpha QH(X/\!\!/G, \mathbb{Q})^2 \longrightarrow T_\alpha QH(X/\!\!/G, \mathbb{Q}),$$

$$\langle \beta \star_{\omega+\alpha} \gamma, \delta \rangle := \sum_{d \in H_2(X/\!\!/G,\mathbb{Q}), n \geq 0} (q^{\langle d, \omega \rangle}/n!) \int_{[\overline{\mathcal{M}}_{0,n+3}(X/\!\!/G,d)]} \operatorname{ev}_1^* \alpha \cup \ldots \cup \operatorname{ev}_n^* \alpha \cup \operatorname{ev}_{n+1}^* \beta \cup \operatorname{ev}_{n+2}^* \gamma \cup \operatorname{ev}_{n+3}^* \delta^\vee$$

where the pairing on the left-hand-side is a certain re-scaled Poincaré pairing on the inertia stack $I_{X/\!\!/G}$, see [1].

Some confusion may be caused by the multitude of formal power series rings that one can work over. The equivariant quantum cohomology $QH_G(X)$ can be defined over the larger *equivariant Novikov field* $\Lambda_X^G \subset \operatorname{Hom}(H_2^G(X, \mathbb{Z}), \mathbb{Q})$ consisting of infinite sums $\sum_{i=1}^\infty c_i \delta_{d_i}$ with $\langle d_i, \omega \rangle \to \infty$, where δ_{d_i} is the delta function at $d_i \in H_2^G(X, \mathbb{Z})$. Similarly, $QH(X/\!\!/G)$ can be defined over the equivariant Novikov field $\Lambda_{X/\!\!/G} \subset \operatorname{Hom}(H(X/\!\!/G, \mathbb{Q}), \mathbb{Q})$ consisting of infinite sums $\sum_{i=1}^\infty c_i \delta_{d_i}$ with $\langle d_i, \omega \rangle \to \infty$, where δ_{d_i} is the delta function at $d_i \in H_2(X/\!\!/G, \mathbb{Q})$. However, these more complicated Novikov fields make some of our formulas a bit too long. $QH_G(X)$ is also defined over the universal Novikov ring Λ_0, and if ω is integral, over $\mathbb{Q}[[q]]$. Similarly, $QH(X/\!\!/G)$ is defined over the the Novikov ring Λ_0, and if ω is integral, over $\mathbb{Q}[[q^{1/n}]]$ for n equal to the least common multiple of the orders of the automorphism groups in $X/\!\!/G$. However, it is convenient to work over the field Λ. Also, invariance under Hamiltonian perturbation only holds for Floer/quantum cohomology over the Novikov field Λ, and so working over Λ is more natural for the purposes of symplectic geometry. Unfortunately, Λ and Λ_0 are non-Noetherian over $\mathbb{C}$ and so some care is required when talking about intersection multiplicities etc. In algebraic geometry, one often uses the monoid-algebra of effective curve classes, but we prefer Novikov fields because of the better invariance properties.

In [46] the first author studied the relationship between $QH_G(X)$ and $QH(X/\!\!/G)$ given by virtual enumeration of affine gauged maps, called the *quantum Kirwan map.* An n-marked *affine gauged maps* is a representable morphism from a weighted projective line $\mathbb{P}(1, r)$ for some $r > 0$ to the quotient stack X/G mapping $\mathbb{P}(r) \subset \mathbb{P}(1, r)$ to the semistable locus $X/\!\!/G$. Some of the results of [46] are:

(a) The stack $\mathcal{M}_{n,1}^G(\mathbb{A}, X, d)$ of n-marked affine gauged maps of class $d \in H_2^G(X, \mathbb{Q})$ has a natural compactification $\overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X, d)$ which has evaluation maps

$$\begin{array}{ccccc} & & \overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X) & & \\ & \swarrow^{\operatorname{ev}} & & \searrow^{\operatorname{ev}_\infty} & \\ (X/G)^n & & & & \overline{I}_{X/\!\!/G} \end{array}$$

and a perfect relative obstruction theory over $\overline{\mathcal{M}}_{n,1}(\mathbb{A})$ (the case of X and G trivial) where $\overline{\mathcal{M}}_{n,1}(\mathbb{A})$ is the complexification of Stasheff's multiplihedron.

(b) For any $n \geq 0$, the map defined by virtual enumeration of stable n-marked affine gauged maps

$$(3)\quad \kappa_X^{G,n} : QH_G(X,\mathbb{Q}) \to QH(X/\!\!/G,\mathbb{Q}) \qquad \alpha \longmapsto \sum_{d \in H_2^G(X,\mathbb{Q})} q^{\langle d,\omega\rangle} \operatorname{ev}_{\infty,*} \operatorname{ev}^* \alpha \cup \ldots \cup \alpha$$

is well-defined.

(c) The sum

$$\kappa_X^G ; QH_G(X,\mathbb{Q}) \longrightarrow QH(X/\!\!/G,\mathbb{Q}), \quad \alpha \longmapsto \sum_{n\geq 0} \kappa_X^{G,n}(\alpha)/n!$$

defines a formal map from $QH_G(X,\mathbb{Q})$ to $QH(X/\!\!/G,\mathbb{Q})$ in a neighborhood of the symplectic class ω with the property that each linearization

$$D_\alpha \kappa_X^G \colon T_\alpha QH_G(X,\mathbb{Q}) \longrightarrow T_{\kappa_X^G(\alpha)} QH(X/\!\!/G,\mathbb{Q})$$

is a $\star$-homomorphism with respect to the quantum products.

By analogy with the classical case one hopes to obtain a presentation of the quantum cohomology algebra $T_{\kappa_X^G(\alpha)} QH(X/\!\!/G,\mathbb{Q})$ by showing that $D_\alpha \kappa_X^G$ is surjective and computing its kernel. This hope leads to the following strong and weak quantum version of Kirwan surjectivity. In the strong form, one might hope that κ_X^G has infinite radius of convergence, κ_X^G is surjective, and $D_\alpha\kappa_X^G$ is surjective for any $\alpha \in QH_G(X,\mathbb{Q})$. More modestly, one might hope that $D_\alpha\kappa_X^G$ is surjective for α in a formal neighborhood of a rational symplectic class $\omega \in H_2^G(X,\mathbb{Q})$.

To specialize to the toric case, suppose that G is a complex torus with Lie algebra $\mathfrak{g}$ acting on a finite-dimensional complex vector space X. Let $X_1,\ldots,X_k \subset X$ be the weight spaces of X where $\dim(X_j) = 1$ and G acts on X_j with weight $\mu_j \in \mathfrak{g}^\vee$ in the sense that for $x \in X_j$ and $\xi \in \mathfrak{g}$ we have $\exp(\xi)x = \exp(i\langle \xi,\mu_j\rangle), j = 1,\ldots,k$. We assume that the weights $\mu_j \in \mathfrak{g}^\vee$ are contained in an open half space, that is, for some $\nu \in \mathfrak{g}$ we have $\langle \nu,\mu_i\rangle \in \mathbb{R}_{>0},\ i = 1,\ldots,k$. We also assume that the weights μ_i span $\mathfrak{g}^\vee$, so that G acts generically locally free on X.

We assume that X is equipped with a polarization, that is, an ample G-line bundle $L \to X$, which we may allow to be rational, that is, a integer root of an honest G-line bundle. Let $\omega \in \mathfrak{g}_{\mathbb{Q}}^\vee$ be the vector representing the first Chern class of the polarization $c_1^G(L) \in H_2^G(X,\mathbb{Q})$ under the isomorphism $\mathfrak{g}_{\mathbb{Q}}^\vee \cong H_G^2(X,\mathbb{Q})$. The point ω determines a rational polarization on X with semistable locus

$$X^{\mathrm{ss}} = X \setminus \bigcup_{I \in \mathcal{I}(\omega)} X^I$$

where

$$\mathcal{I}(\omega) = \left\{ I \subset \{1, \ldots, k\} \mid \omega \notin \sum_{i \in I} \mathbb{R}_{\geq 0} \mu_i \right\}$$

is the set of subsets so that ω is not in the span of the corresponding weights and X^I is the intersection of coordinate hyperplanes

$$X^I = \{(x_1, \ldots, x_k) | x_i = 0, \ \forall i \notin I\}.$$

The stable=semistable condition assumption translates to the condition for each $I \in \mathcal{I}(\omega)$ the weights $\mu_i, i \in I$ span $\mathfrak{g}^\vee$. In this case the quotient $X /\!\!/ G = X^{\text{ss}}/G$ is then a smooth (possibly empty) proper Deligne-Mumford stack. We suppose that $X /\!\!/ G$ is non-empty. The *quantum Stanley-Reisner ideal* $QSR_{X,G}(\alpha) \subset QH_G(X, \mathbb{Q})$ is

$$QSR_{X,G}(\alpha) := \langle QSR_{X,G}(d, \alpha), d \in H_2^G(X, \mathbb{Z}) \rangle$$

where

$$QSR_{X,G}(d, \alpha) := \prod_{\langle \mu_j, d \rangle \geq 0} \mu_j^{\langle \mu_j, d \rangle} - q^{\langle d, \alpha \rangle} \prod_{\langle \mu_j, d \rangle \leq 0} \mu_j^{-\langle \mu_j, d \rangle}.$$

If α is the given symplectic class ω, we write $QSR_{X,G} := QSR_{X,G}(\omega)$ The quotient $T_\omega QH_G(X, \mathbb{Q})/QSR_{X,G}$ is the *quantum Stanley-Reisner a.k.a Batyrev ring.* For example, the Batyrev ring for projective space is given as follows. Let $G = \mathbb{C}^\times$ act on $X = \mathbb{C}^k$ by scalar multiplication so that all weights $\mu_1, \ldots, \mu_k$ are equal to $1 \in \mathfrak{g}_\mathbb{Z}^\vee \cong \mathbb{Z}$. Then the stable locus is $X - \{0\}$ and the git quotient is $X /\!\!/ G = \mathbb{P}^{k-1}$. The quantum Stanley-Reisner ideal is generated by the single element $QSR_{X,G}(1) = \xi^k - q$. The Batyrev ring is $T_\omega QH_G(X, \mathbb{Q})/QSR_{X,G} = \Lambda_\mathbb{Q}[\xi]/\langle \xi^k - q \rangle$.

The first formulation of our main result is the following. It says that Batyrev's original suggestion [4] for the quantum cohomology is true, after passing to a suitable formal version of the equivariant cohomology and "quantizing" the divisor classes:

Theorem 1.1. *For a suitable formal version of the equivariant quantum cohomology $QH_G(X)$ (see Section 2) the linearized quantum Kirwan map $D_\alpha \kappa_X^G$ induces an isomorphism $T_\alpha QH_G(X, \mathbb{Q})/QSR_{X,G}(\alpha) \to T_{\kappa_X^G(\alpha)} QH(X /\!\!/ G, \mathbb{Q})$ for α in a formal neighborhood of any rational symplectic class $\omega \in H_G^2(X, \mathbb{Q})$.*

It is important in the above result that the quantum cohomology is defined over the Novikov *field*, or at least, that a suitable rational power of the formal parameter q has been inverted: over a polynomial ring such as $\mathbb{C}[q]$, one does not obtain an surjection because certain elements in twisted sectors are not contained in the image for $q = 0$. Thus one sees a nice presentation of the quantum cohomology only for non-zero q. The necessity of corrections to Batyrev's original conjecture, which involved the divisor classes as generators, was noted in Cox-Katz [12, Example 11.2.5.2] for the second Hirzebruch surface and Spielberg [42] for a toric three-fold. The fact that the change of coordinates restores the original presentation was noted in Guest [26] for semi-Fano toric varieties, and Iritani [31, Section 5], for not-necessarily-Fano toric varieties in general, after passing to a formal completion. See Iritani [31, Example 5.5] and Gonzalez-Iritani [23, Example 3.5] for examples in the toric manifold case.

The presentation of the quantum cohomology in Theorem 1.1 can be re-phrased in terms of Landau-Ginzburg potential as follows, according to suggestions of Givental [20] and the physicists related to mirror symmetry. This formulation will be essential in our proof of the injectivity of the map in Theorem 1.1. The *residual torus*

$$T := (\mathbb{C}^\times)^{\dim(X)}/G$$

has an induced action on $X/\!\!/G$. Let $T_\mathbb{R} \subset T$ denote the unitary part of T

$$T_\mathbb{R} = \exp(\mathfrak{t}_\mathbb{R}), \quad \mathfrak{t}_\mathbb{R} = \operatorname{span}_\mathbb{R} \mathfrak{t}_\mathbb{Z}, \quad \mathfrak{t}_\mathbb{Z} = \exp^{-1}(1)$$

given by exponentiating the real span $\mathfrak{t}_\mathbb{R}$ of the coweights $\mathfrak{t}_\mathbb{Z}$ of T. The Lie algebra $\mathfrak{t}$ of T admits a canonical real splitting $\mathrm{Re} \oplus \mathrm{Im} : \mathfrak{t} \to \mathfrak{t}_\mathbb{R} \oplus i\mathfrak{t}_\mathbb{R}$. The action of $T_\mathbb{R}$ on $X/\!\!/G$ is Hamiltonian, with moment map $\Phi : X/\!\!/G \to \mathfrak{t}^\vee$ induced by the choice of moment map for the action of $U(1)^{\dim(X)}$ on X. Let $\Delta_{X/G} \subset \mathfrak{t}_\mathbb{R}^\vee$ denote its image

$$\Delta_{X/G} := \Phi(X/\!\!/G)$$

the *moment polytope* of $X/\!\!/G$. Let $\nu_j \in \mathfrak{t}_\mathbb{R}, j = 1, \ldots, k$ be the normal vectors to the facets of $\Delta_{X/G}$; these are the images of the minus the standard basis vectors of $\mathbb{R}^{\dim(X)}$ under the projection $\pi_\mathfrak{t}$ to $\mathfrak{t}_\mathbb{R}$:

$$\nu_j = \pi_\mathfrak{t}(-e_j), j = 1, \ldots, k.$$

The moment polytope $\Delta_{X/G}$ is of the form

$$\Delta_{X/G} = \{\mu \in \mathfrak{t}_\mathbb{R}^\vee \mid \langle \mu, \nu_j \rangle \geq \omega_j, j = 1, \ldots, k\}$$

with positions of the facets determined by elements $\omega_j \in \mathbb{Q}$. We say that $\langle \mu, \nu_j \rangle \geq \omega_j$ is a *spurious inequality* if it does not correspond to a facet of $\Delta_{X/G}$. The constants ω_j can be chosen as follows. Given a splitting of the exact sequence

$$0 \longrightarrow \mathfrak{g}_\mathbb{R} \longrightarrow \mathbb{R}^{\dim(X)} \longrightarrow \mathfrak{t}_\mathbb{R},$$

we obtain a dual map $\mathfrak{g}_\mathbb{R}^\vee \to \mathbb{R}^{\dim(X),\vee} \cong \mathbb{R}^{\dim(X)}$. The constants ω_j are the coefficients of the image of the element $\omega \in \mathfrak{g}_\mathbb{R}^\vee$ representing the symplectic class ω under the isomorphism $\mathfrak{g}_\mathbb{R}^\vee \cong H_2^G(X, \mathbb{R})$. The *tropical dual torus* is

$$T^\vee(\Lambda_0) := \left\{ q^\zeta \exp\left(\sum_{d \geq 0} q^d \xi_d \right), \begin{array}{l} \zeta \in \mathfrak{t}_\mathbb{R}^\vee \\ \xi_0 \in \mathfrak{t}^\vee/\mathfrak{t}_\mathbb{Z}^\vee \\ \xi_d \in \mathfrak{t}^\vee, d > 0 \end{array} \right\}$$

where the sum over d satisfies the same finiteness condition as in the definition of Λ, namely for each $e > 0$ the number of non-zero ξ_d with $d < e$ is finite. The Landau-Ginzburg potential associated to the toric stack $X/\!\!/G$ is a function

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on the tropical dual torus given as a sum of monomials whose exponents are the normal vectors to the facets of $\Delta_{X/\!\!/G}$. It was first noticed by Givental [22] that this function is related to the Gromov-Witten theory of $X/\!\!/G$. An explanation from the point of view of mirror symmetry was given in Hori-Vafa [28], and a connection to Floer theory is described in Fukaya et al [17], where the potential appears as a count of holomorphic disks with boundary in a fiber of the moment map. In the later version, the potential receives corrections from nodal holomorphic disks, whereas in Givental [22] and Hori-Vafa [28] there are no corrections; we call the uncorrected version the *naive Landau-Ginzburg potential*, given by

$$W_{X,G} \colon T^\vee(\Lambda_0) \longrightarrow \Lambda, \quad y \longmapsto \sum_{j=1}^k q^{-\omega_j} y^{\nu_j}$$

or more precisely

$$W_{X,G}\left(q^\zeta \exp\left(\sum_{d\geq 0} q^d \xi_d\right)\right) = \sum_{j=1}^k q^{-\omega_j + \langle \zeta, \nu_j\rangle} \exp\left(\sum_{d \geq 0} q^d \langle \xi_d, \nu_j \rangle\right).$$

As it stands, the values of $W_{X,G}$ have negative powers of q. However, later we will always assume that 0 is contained in the moment polytope $\Delta_{X/\!\!/G}$, in which case only positive powers occur. For the many purposes (non-displaceability, Batyrev presentation) it seems that the naive potential is "as good as" the corrected potential. A heuristic argument that the two potentials are related by a geometrically-defined change of coordinates was given in Woodward [47]; for semi-Fano cases it is proved in Chan et al [9] that this coordinate change is the mirror map from Gromov-Witten theory, while Fukaya et al [18, Theorem 11.1] show the existence of some coordinate transformation relating the two. For example, the naive Landau-Ginzburg potential for a product of project lines is given as follows. If $X = \mathbb{C}^4, G = (\mathbb{C}^\times)^2$ with weights $(1,0),(1,0),(0,1),(0,1)$ we have $X/\!\!/G = \mathbb{P}^1 \times \mathbb{P}^1$. With the standard parametrization of the dual torus $T = (\mathbb{C}^\times)^4/G \cong (\mathbb{C}^\times)^2$ the normal vectors to the facets are

$$\nu_1 = (1,0),\ \nu_2 = (-1,0),\ \nu_3 = (0,1),\ \nu_4 = (0,-1)$$

and the moment polytope is $[0,1]^2$. The potential is

$$W_{X,G}(y_1,y_2) = y_1 + q/y_1 + y_2 + q/y_2.$$

The *critical locus* $\operatorname{Crit}(W_{X,G})$ of $W_{X,G}$ is the set of points with vanishing logarithmic derivatives with respect to the coordinates on $T^\vee$,

$$\begin{aligned} \operatorname{Crit}(W_{X,G}) &= \left\{ y \in T^\vee(\Lambda_0) \ \middle|\ \partial_\lambda W_{X,G}(ye^\lambda)|_{\lambda=0} = 0 \quad \forall \lambda \in \mathfrak{t}^\vee_{\mathbb{R}} \right\} \\ &= \left\{ y \in T^\vee(\Lambda_0) \ \middle|\ \sum_{i=1}^k \langle \nu_i, \lambda \rangle q^{-\omega_i} y^{\nu_i} = 0, \quad \forall \lambda \in \mathfrak{t}^\vee_{\mathbb{R}} \right\}. \end{aligned}$$

Assuming that 0 is contained in the moment polytope and ω is integral then the potential $W_{X,G}$ is defined over $\mathbb{C}[q]$ and $\operatorname{Crit}(W_{X,G})$ has a natural scheme structure which is finite over $\operatorname{Spec}\mathbb{C}[q]$. The projection $\mathbb{C}^{\times,\dim(X)} \to T$ induces an injection

$$T^\vee \longrightarrow \mathbb{C}^{\times,\dim(X),\vee}, \ y \longmapsto (y^{\nu_j})_{j=1}^k.$$

Denote by $\operatorname{Crit}_+(W_{X,G}) \subset \operatorname{Crit}(W_{X,G})$ denote the locus of families of critical points that approach $y = 0$ as $q \to 0$, and $\widehat{\operatorname{Jac}}(W_{X,G})$ its *Jacobian ring* of functions, defined first by taking a formal neighborhood of $(y, q) = (0, 0)$, and then inverting powers of q. Continuing the example of a product of projective lines $X/\!\!/G = \mathbb{P}^1 \times \mathbb{P}^1$, the critical locus is defined by

$$0 = y_1\partial_1 W(y_1, y_2) = y_1 - q/y_1, \quad 0 = y_2\partial_2 W(y_1, y_2) = y_2 - q/y_2$$

which has solutions $(y_1, y_2) = (\pm\sqrt{q}, \pm\sqrt{q})$ all of which approach $y = 0$ as $q \to 0$. An interpretation in terms of critical points that lie over the interior of the moment polytope is given in Proposition 3.22. The second formulation of our main result is

Theorem 1.2. *For any rational symplectic class $\omega \in H_2^G(X)$, there is a canonical isomorphism $T_{\kappa_X^G(\omega)} QH(X/\!\!/G) \to \widehat{\operatorname{Jac}}(W_{X,G})$.*

Earlier cases were proved using closed Gromov-Witten invariants in Batyrev [4], Givental [21], Lian-Liu-Yau [37], and, for non-weak-Fano toric manifolds, Iritani [31, 5.11], with another proof given by Brown [8]. A similar result is proved using open-closed Gromov-Witten invariants in Fukaya et al [18], using a potential defined using holomorphic disks, whose leading order terms are the potential above.

The quantum Stanley-Reisner relations were proved in Givental [20] in the Fano case (there were also computations for the semi-Fano case), in the manifold case by Fukaya et al [18] and in general in Woodward [46], and, we understand, in unpublished work by Chung-Ciocan-Fontanine-Kim and Coates, Corti, Iritani, and Tseng. That these relations generate the ideal was expected for some time, see Iritani [30]. To complete the proof, it suffices to show that the rings on both sides have the same dimension; in the Fano case, this follows from Kouchnirenko's theorem [36, 3]. In general, we deduce the dimension equality from the toric minimal model program and an induction. A similar procedure is used by Kawamata [34] to show the existence of an exceptional collection in the derived category of any toric orbifold. The Frobenius manifold structure $QH(Y)$, including the pairing, is expected to be equivalent to Saito's Frobenius structure corresponding to the Landau-Ginzburg potential W, see for example Fukaya et al [18]. However, we do not discuss the pairings in this paper. We give a family version of Theorem 1.2 in Theorem 4.17 below. We end the introduction with examples of the projective plane, written in different ways as a quotient:

Example 1.3. (Projective plane as a quotient by a circle action) The projective plane can be realized as a git quotient of affine space by a circle action as follows. Suppose that $G = \mathbb{C}^\times$ acts on $X = \mathbb{C}^3$ by scalar multiplication. Suppose that the polarization corresponds to a trivial line bundle with a negative weight on the fiber at the origin. The semistable locus is $X^{\rm ss} = X - \{0\}$ and the git quotient is $X/\!\!/G = \mathbb{P}^2$. We take the residual action of $T = (\mathbb{C}^\times)^3/\mathbb{C}^\times$ to have moment polytope in $\mathfrak{t}^\vee \cong \mathbb{R}^2$ equal to

$$\Delta_{X/G} = \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 | \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_1 + \lambda_2 \leq c\}.$$

The corresponding potential is

$$W_{X,G}(y_1, y_2) = y_1 + y_2 + q/y_1 y_2.$$

The critical points are the solutions to

$$y_1 \partial_{y_1} W_{X,G}(y_1, y_2) = y_1 - q/y_1 y_2 = 0, \quad y_2 \partial_{y_2} W_{X,G}(y_1, y_2) = y_2 - q/y_1 y_2 = 0$$

that is, $y_1 = y_2, \quad y_1^3 = y_2^3 = q$ which is the quantum cohomology of $\mathbb{P}^2$.

Example 1.4. (Projective plane as a quotient by a two-torus action) The projective plane $\mathbb{P}^2$ can be realized as a git quotient by a two-dimensional torus as follows. Suppose that $G = (\mathbb{C}^\times)^2$ acts on $X = \mathbb{C}^4$ with weights $(1,0), (1,1), (-1,1), (0,1)$, with polarization corresponding to the weight $(-1,2)$; this is the left-most chamber in Figure 1. The semistable locus consists of points $x = (x_1, x_2, x_3, x_4)$ with $x_1 \neq 0, (x_2, x_3, x_4) \neq 0$, and the git quotient is $X/\!\!/G = \mathbb{P}^2$, with moment polytope for the residual torus $T = (\mathbb{C}^\times)^4/G$ in $\mathfrak{t}^\vee \cong \mathbb{R}^2$. We choose the parametrization

$$(\mathbb{C}^\times)^2 \longrightarrow T, \quad (z_1, z_2) \longmapsto [z_1/z_2, z_2, 1/z_1 z_2, z_1].$$

The moment polytope can be taken to be

$$\Delta_{X/G} = \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_1 + \lambda_2 \leq c_1, -\lambda_1 + \lambda_2 \leq c_2\}$$

for some constants $c_1 < c_2$. In particular, the last equation does not define a facet of $\Delta_{X/G}$, corresponding to the fact that $x_1 = 0$ does not define a divisor in $X/\!\!/G$. To simply notation we take $c_1 = 1$ and $c_2 = 2$. The potential is

$$W_{X,G}(y_1, y_2) = y_1 + y_2 + q/y_1 y_2 + q^2 y_1/y_2.$$

The partial derivatives are

$$\partial_{y_1} W_{X,G}(y_1, y_2) = 1 - q/y_1^2 y_2 + q^2/y_2, \quad \partial_{y_2} W_{X,G}(y_1, y_2) = 1 - q/y_1 y_2^2 - q^2 y_1/y_2^2.$$

The critical values, to leading order, are the three critical points

$$y_1 \sim y_2 \sim \exp(2\pi i k/3) q^{1/3}, k = 0, 1, 2$$

and the two critical points $y_1^2 \sim q^{-1/4}, \quad y_2 \sim -2q^{6/4}$ as shown in Figure 1. The first three resp. second two points resp. do not define elements of $\mathrm{Crit}_+(W_{X,G})$. Hence $\mathrm{Crit}_+(W_{X,G})$ consists of three reduced points, $QH(X/\!\!/G) \cong \mathbb{C}^{\oplus 3}$. The other pictures in Figure 1 show the quotients for the other polarizations; the dotted line represents the span of the equivariant first Chern class $c_1^G(TX)$, for which the quotient $X/\!\!/G$ has a potential with all critical points located at $0 \in \mathfrak{t}^\vee$.

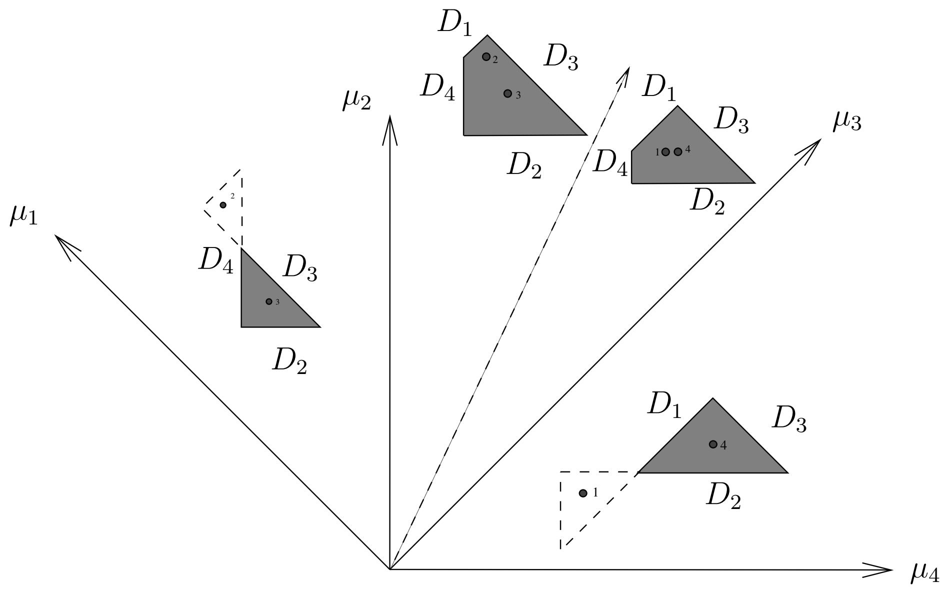

FIGURE 1. Chamber structure for git quotients and the critical locus

2. Quantum Kirwan surjectivity for toric orbifolds

In this section we prove surjectivity for the linearization of the quantum Kirwan map on a formal version of equivariant quantum cohomology; the surjectivity also holds for the usual version but does not lead to an isomorphism. Let X be a smooth polarized projective G-variety, or more generally, a quasiprojective smooth polarized projective G-variety convex at infinity in the sense of [46], such as a finite-dimensional vector space with the action of a torus whose weights are contained in a half-space.

Definition 2.1. (Formal equivariant quantum cohomology ring) Let $\widehat{QH}_G(X)$ be the space of infinite sums

$$\widehat{QH}_G(X) = \left\{ \sum_{i=1}^{\infty} q^{d_i} \alpha_i | \alpha_i \in H_G(X), \ \inf_i d_i > -\infty, \ \lim_{i\to\infty} d_i + \deg(\alpha_i) = \infty \right\}.$$

Proposition 2.2. (Extension of the quantum Kirwan map to the formal equivariant quantum cohomology) *Each Taylor coefficient $\kappa_X^{G,n} : QH_G(X)^n \to QH(X/\!\!/G)$ extends to a map $\widehat{QH}_G(X)^n \to QH(X/\!\!/G)$, still denoted $\kappa_X^{G,n}$.*

Proof. The statement of the proposition follows from the properness result for the energy map: for any $e > 0$, the union of components $\overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X, d)$ for which $\langle d, \omega \rangle < e$ is finite. Hence, the contribution to $\kappa_X^{G,n}$ to any subspace in $QH(X/\!\!/G)$ of bounded q-degree received contributions from only finitely many graded pieces of $QH_G(X)$, which implies that the map extends to the formal version. □

Let X be a finite-dimensional complex vector space with an action of a complex torus G, equipped with a polarization so that the quotient $X/\!\!/G$ is locally free.

Proposition 2.3. (Classification of affine gauged maps in the toric case) *An affine gauged map to X/G of homology class $d \in H_2^G(X,\mathbb{Q})$ is equivalent to a morphism $u = (u_1,\ldots,u_k) : \mathbb{A} \to X$ satisfying*

(a) *the degree of u_j is at most $\langle \mu_j, d\rangle$; and*

(b) *let $u(\infty) = \begin{cases} u_j^{(\langle \mu_j,d\rangle)}/\langle \mu_j,d\rangle! & \langle \mu_j,d\rangle \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$ denote the vector of leading order coefficients with integer exponents. Then $u(\infty) \in X^{\mathrm{ss}}$.*

Proof. By definition, a morphism $\mathbb{P}(1,r) \to X/G$ consists of a G-bundle $P \to \mathbb{P}(1,r)$, given in terms of a clutching function $z \mapsto z^{\lambda/r}$ for some $\lambda \in \mathfrak{g}_{\mathbb{Z}}^\vee$ with $\lambda/r = d$, and a section of the associated X-bundle $u : \mathbb{P}(1,r) \to P \times_G X$. The first condition in the statement of the proposition is the condition that a map $\mathbb{A} \to X$ extend to a global section, while the second is the condition that the extension maps $\mathbb{P}(r)$ to the semistable locus $X/\!\!/G$. The representability condition is that the image of $\mathbb{P}(r)$ is a point $u(\infty)$ in $X/\!\!/G$ with automorphism group containing a group $\mathbb{Z}_r$ generated by $\exp(\lambda/r)$, so that λ/r is the minimal representation of d and each map u above appears once in the classification. □

Theorem 2.4. (Quantum Kirwan surjectivity, toric case) *For any rational symplectic class $\omega \in H_2^G(X,\mathbb{Q})$, the map $D_\omega \kappa_X^G : T_\omega \widehat{QH}_G(X) \to T_{\kappa_X^G(\omega)} QH(X/\!\!/G)$ is surjective.*

If $X/\!\!/G$ is a smooth *variety*, that is, has no orbifold points, then the statement of the theorem follows from Kirwan's surjectivity result from [35], or an explicit description of the classical Kirwan map in the toric case, since the leading order term (setting the Novikov parameter q to zero) is the classical Kirwan map. The novelty of the above theorem is that in the orbifold case, the twisted sectors are also in the image of the quantum Kirwan map, so that $T_{\kappa_X^G(\omega)} QH(X/\!\!/G)$ is a quotient of the usual ring of polynomial invariants $T_\omega QH_G(X) \cong \mathrm{Sym}(\mathfrak{g}^\vee)^G \otimes \Lambda$. The proof of Theorem 2.4 in the orbifold case relies on the following computation which we call a *fractional Batyrev relation* based on similarity with [4]. Recall that the *ceiling* $\lceil x \rceil$ of $x \in \mathbb{R}$ is the smallest integer at least as big as x. We identify $H_2^G(X) \cong \mathfrak{g}$ and $H^2_G(X) \cong \mathfrak{g}^\vee$. Any $d \in H_2^G(X)$ thus defines an element $\exp(d) \in G$, which corresponds to a summand in $H(I_{X/G})$ if it has non-trivial fixed point set in X^{ss}. If so we denote by $1_{\exp(d)} \in H(I_{X/G})$ the degree zero class in the twisted sector (which will have non-zero degree with respect to the grading on $QH(X/\!\!/G)$.) Denote by $\overline{\mathcal{M}}_{1,1}^G(\mathbb{A},X,d)$ the stack of once-marked stable scaled affine gauged maps to X [46]. In general, this compactification allows bubbles in $X/\!\!/G$ and ghost bubbles in X when the points come together. However, since X is affine, there are no non-constant bubbles in X.

Proposition 2.5 (Fractional Batyrev relation)**.** *For any d such that* $\exp(d)$ *has non-empty stabilizer in* $X^{\rm ss}$ *and* $\mathcal{M}_{1,1}(\mathbb{A}, X, d)$ *is non-empty,*

$$D_\omega \kappa_X^G \left(\prod_{\langle \mu_j, d\rangle \geq 0} \mu_j^{\lceil \langle \mu_j, d\rangle \rceil} \right) = 1_{\exp(d)} \prod_{\langle \mu_j, d\rangle \leq 0} \mu_j^{\lceil \langle \mu_j, d\rangle \rceil} q^{\langle \omega, d\rangle} + \text{higher order in } q.$$

Proof of Proposition 2.5. First we compute

$$\int_{[\overline{\mathcal{M}}_{1,1}^G(\mathbb{A},X,d)]} \mathrm{ev}^* \prod_{\langle \mu_j, d\rangle \geq 0} \mu_j^{\lceil \langle \mu_j, d\rangle \rceil} = \int_{[\overline{\mathcal{M}}_{1,1}^G(\mathbb{A},X,d)]} \mathrm{ev}^* \operatorname{Eul} \left(\prod_{\langle \mu_j, d\rangle \geq 0} \mathbb{C}_{\mu_j}^{\lceil \langle \mu_j, d\rangle \rceil} \right).$$

Consider the section

$$\sigma \colon \overline{\mathcal{M}}_{1,1}^G(\mathbb{A}, X, d) \longrightarrow \mathrm{ev}_1^* \prod_{\langle \mu_j, d\rangle \geq 0} \mathbb{C}_{\mu_j}^{\lceil \langle \mu_j, d\rangle \rceil}, \quad u \longmapsto (u_j^{(i)}(z_1))_{j=1,i=0}^{k, \lceil \langle \mu_j, d\rangle \rceil}$$

consisting of the derivatives $u^{(i)}$ of u at the marking z_1; note this is well-defined because of the scaling on the domain, that is, we are modding out by translation on the domain only. On the stratum $\mathcal{M}_{1,1}^G(\mathbb{A}, X)$, σ has zeroes corresponding to maps with all lower-order terms vanishing, and the restriction of ev_∞ to σ^{-1} defines an isomorphism

$$\mathrm{ev}_\infty \colon \sigma^{-1}(0) \longrightarrow I_{X/\!\!/G}(\exp(d))$$

where $I_{X/\!\!/G}(\exp(d))$ is the sector with stabilizer $\exp(d)$. Taking into account the class of the obstruction bundle $\operatorname{Eul}\left(\prod_{\langle \mu_j, d\rangle \leq 0} \mathbb{C}_{\mu_j}^{\lceil \langle \mu_j, d\rangle \rceil - 1}\right)$ this shows that the contribution from the irreducible stratum is $1_{\exp(d)} \prod_{\langle \mu_j, d\rangle \leq 0} \mu_j^{\lceil \langle \mu_j, d\rangle \rceil}$.

Next we examine contributions from the boundary. Any boundary configuration consists of a morphism m $u : \mathbb{A} \to X$ of class d' with $\langle d', \omega\rangle < \langle d, \omega\rangle$. The zero set $\sigma^{-1}(0)$ of σ on such a stratum has $u_j = 0$ if $\langle \mu_j, d\rangle > \langle \mu_j, d'\rangle \in \mathbb{Z}$, and so the components with non-zero leading order correspond to j with $\langle \mu_j, d\rangle \leq \langle \mu_j, d'\rangle$. But these consist of unstable points in X, and so the zero set is empty.

Finally we consider the integral

$$\int_{[\overline{\mathcal{M}}_{1,1}^G(\mathbb{A},X,d')]} \mathrm{ev}^* \prod_{\langle \mu_j, d\rangle \geq 0} \mu_j^{\lceil \langle \mu_j, d\rangle \rceil}$$

for $d \neq d'$. The same argument as in the previous paragraph shows that the integral is zero unless $\langle d, \omega\rangle \leq \langle d', \omega\rangle$. But since ω is generic, this implies strict inequality and so these contributions are of higher energy. □

Corollary 2.6. (Surjectivity onto twisted units) *For any $g \in G$ with non-trivial stabilizer in* $X^{\rm ss}$*, there exists an element $d \in \mathfrak{g}$ with* $\exp(d) = g$ *and* $\langle d, \mu_j\rangle > 0$

for all $j = 1, \dots, k$ *and thus*

$$D_\omega \kappa_X^G \left(\prod_{j=1}^k \mu_j^{\langle \mu_j, d \rangle} \right) = 1_{\exp(d)} q^{\langle d, \omega \rangle} \quad \text{mod higher order in } q.$$

Proof. Since the weights are contained in a half-space, there exists a vector ζ such that $\langle \zeta, \mu_j \rangle > 0$ for $j = 1, \dots, k$. Let $U \subset \mathfrak{g}$ be a compact subset such that $\exp(U) = G$. Then $c\zeta + U$ contains the desired vector d, for $c \gg 0$. □

Notation 2.7. (Cohomology classes in twisted sectors) For any $j \in \{1, \dots, k\}$ such that $\mu_j(d) \in \mathbb{Z}$ denote by $1_{\exp(d)} \delta_J \in H^2(I_{X/G})$ the corresponding divisor class in the twisted sector corresponding to $\exp(d)$, and $1_{\exp(d)} \delta_J = 1_{\exp(d)} \prod_{j \in J} \delta_j \in H(I_{X/G})$ the classical product of divisor classes in the twisted sector for $\exp(d)$. Since each component of $I_{X/G}$ is itself a rationally smooth toric variety, any cohomology class of $I_{X/G}$ arises in this way by the classical description by Danilov-Jurkiewicz.

Corollary 2.8. (Surjectivity onto twisted sectors) *With d as in Corollary 2.6, For any subset J of $\{j, \mu_j(d) \in \mathbb{Z}\}$,*

$$D_\omega \kappa_X^G \left(\prod_{j=1}^k \mu_j^{\langle \mu_j, d \rangle} \prod_{j \in J} \mu_j \right) = 1_{\exp(d)} \delta_J q^{\langle d, \omega \rangle} \quad \text{mod higher order in } q.$$

Proof of Theorem 2.4. For each (g, J) such that the product in classical cohomology $\alpha_{g,J} := 1_g \delta_J \in H(I_{X/G})$, choose an element $\tilde{\alpha}_{g,J}$ such that $D_\omega \kappa_X^G(\tilde{\alpha}_{g,J})$ is equal to $\alpha_{g,J}$ plus terms of higher energy. A recursion produces an inverse. □

We give several examples.

Example 2.9. (Stacky point) Let $X = \mathbb{C}$ with $G = \mathbb{C}^\times$ acting with weight 2, so that $X/G = \mathbb{P}(2) = B\mathbb{Z}_2$. We identify $\mathfrak{g}_{\mathbb{Z}}^\vee = \mathbb{Z}$ in the standard way; then

$$\overline{\mathcal{M}}_{1,1}^G(\mathbb{A}, X, 0) = \{c_0, c_0 \neq 0\}/G \cong \mathbb{P}(2)$$

consists of constant maps while

$$\mathcal{M}_{1,1}^G(\mathbb{A}, X, 1/2) = \{c_1 z + c_0, c_1 \neq 0\}/G \cong \mathbb{C}/\mathbb{Z}_2.$$

It follows that $H_G(X) = \Lambda[\xi]$ and we have $D_\omega \kappa_X^G(1) = 1$ while $D_\omega \kappa_X^G(\xi)$ is half the generator of the twisted sector.

Example 2.10. (The teardrop orbifold, a weighted projective line, reproduced from [46]) If $G = \mathbb{C}^*$ acts on $X = \mathbb{C}^2$ with weights $1, 2$, so that $X /\!\!/ G = \mathbb{P}(1,2)$ is the teardrop orbifold, then $\overline{M}_{1,1}^G(\mathbb{A}, X, 0) = X /\!\!/ G$ implies that $D_\omega \kappa_X^G(1) = 1$ and $D_\omega \kappa_X^G(\xi)$ is the point class. There is a unique map of degree $d = 1/2$, given by $u(z) = (0,1)$, whose derivatives are zero at the marking and leading order term is semistable, and this implies that $D_\omega \kappa_X^G(\xi^2)$ is half the generator of the twisted sector.

Example 2.11. (The second Hirzebruch surface, following [23, Example 3.5]) The second Hirzebruch surface F_2 is a quotient of $X = \mathbb{C}^4$ by the $G = \mathbb{C}^{\times,2}$-action with weights $(0,1), (-2,1), (1,0), (1,0)$. Let $p_1, p_2 \in H_2(X/\!\!/G)$ be the base that corresponds to the zero section and fibre classes. The divisor classes are

$$D_1 = p_2, \quad D_2 = p_2 - 2p_1, \quad D_3 = D_4 = p_1.$$

In the case of nef toric varieties it follows from the adiabatic limit theorem of [46] that the quantum Kirwan map intertwines the I-function and J-function of Givental [20], and so must agree with the mirror transformation for the second Hirzebruch surface, computed in Cox-Katz [12, Example 11.2.5.2]. If $y = (y_1, y_2)$ resp $r = (r_1, r_2)$ is the coordinate on the torus with Lie algebra $H_2^G(X, \mathbb{C})$ resp. $H_2(X/\!\!/G, \mathbb{C})$ (the variables r_1, r_2 were called q_1, q_2 in most of the mirror symmetry literature but we wish to avoid confusion with the univeral Novikov parameter q) then the mirror transformation is

$$y_1 = r_1(1 + r_1)^2, \quad y_2 = r_2(1 + r_1).$$

The image of the divisors classes under the linearized Kirwan map are given by differentiating the mirror transformation. If $\tilde{D}_j = D_\omega \kappa_{X,G}(\mu_j)$, where μ_j is the j-th weight, then

$$\tilde{D}_1 = D_1, \quad \tilde{D}_2 = D_2 + 2\frac{r_1}{1-r_1} D_2, \quad \tilde{D}_3 = D_3 - \frac{r_1}{1-r_1} D_2, \quad \tilde{D}_4 = D_4 - \frac{r_1}{1-r_1} D_2.$$

It was noted in Guest [26] for semi-Fano toric varieties, and Iritani [31, Section 5], Gonzalez-Iritani [23, Example 3.5] these elements satisfy the Batyrev relations with respect to the variables y_1, y_2. It would interesting to derive this directly from the geometric definition of the quantum Kirwan map.

3. Quantum Stanley-Reisner ring and Jacobian ring

In this section we identify the quantum Stanley-Reisner ring with the Jacobian ring, and discuss various extends to formal versions. Consider a git quotient $X/\!\!/G$ of a finite-dimensional complex vector space X by the action of a complex torus G with weights $\mu_1, \ldots, \mu_k$.

Definition 3.1. (a) The *Stanley-Reisner ideal* in $QH_G(X)$ is generated by products of weights $\mu_I = \prod_{i \in I} \mu_i$ where I is a *primitive* collection $I \subset \{1, \ldots, k\}$ with respect to the fan of $X/\!\!/G$: the set $X_I = \{x_i = 0, i \in I\}$ is contained in the unstable locus of X and I is a minimal subset with this property.

(b) (Classical Stanley-Reisner ring) The *Stanley-Reisner ring* is the quotient of $H_G(X)$ by the Stanley-Reisner ideal SR_X^G.

Theorem 3.2. (Rational cohomology of a projective simplicial toric variety) *Suppose that stable=semistable for the G-action on X. Then the Kirwan map $H_G(X, \mathbb{Q}) \to H(X/\!\!/G, \mathbb{Q})$ induces an isomorphism $H_G(X, \mathbb{Q})/SR_X^G \to H(X/\!\!/G, \mathbb{Q})$.*

Proof. This is essentially the Danilov-Jurkiewicz description of the cohomology ring [14, 32, 33], using the fact that each weight function $\mu_i \in \mathfrak{g}_{\mathbb{Q}}^\vee \cong H^2_G(X,\mathbb{Q})$ maps to the corresponding divisor class in $H(X/\!\!/G,\mathbb{Q})$ under the classical Kirwan map. A description from the point of view of equivariant cohomology can be found in Banovero-Brion [6]. □

Note that there are no "linear relations" in the above description; these are in the standard description the kernel of the map $H_{(\mathbb{C}^\times)^{\dim(X)}}(X,\mathbb{Q}) \to H_G(X,\mathbb{Q})$.

Definition 3.3. (a) The *formal quantum Stanley-Reisner ideal* $\widehat{QSR}^G_X \subset \widehat{QH}_G(X,\mathbb{Q})$ is the completion of $QSR_{X,G}$ in $\widehat{QH}_G(X,\mathbb{Q})$. The quotient $\widehat{QH}_G(X,\mathbb{Q})/\widehat{QSR}^G_X$ is the *formal Batyrev ring.*

(b) The *equivariant resp. formal equivariant quantum Stanley-Reisner ideal resp. Batyrev ring* are obtained by replacing the expressions μ_j by their unrestricted versions. Denote by $\epsilon_1,\ldots,\epsilon_k \in \tilde{\mathfrak{g}}^\vee \cong H^2_{\tilde{G}}(X)$ the coordinates (weights) on the big torus

$$\tilde{G} := (\mathbb{C}^\times)^{\dim(X)}$$

acting on X and so that $\mu_1,\ldots,\mu_k \in \mathfrak{g}^\vee \cong QH^2_G(X)$ are their restrictions to $\mathfrak{g}$. The equivariant quantum Stanley-Reisner ideal is generated by

$$QSR^{G,\tilde{G}}_X(d) := \prod_{\langle\mu_j,d\rangle\geq 0} \epsilon_j^{\langle\mu_j,d\rangle} - q^{\langle d,\omega\rangle} \prod_{\langle\mu_j,d\rangle\leq 0} \epsilon_j^{-\langle\mu_j,d\rangle}.$$

Example 3.4. (Batyrev relations for a quotient of affine four-space by a two-torus) Let $G = (\mathbb{C}^\times)^2$ acting on $X = \mathbb{C}^4$ with weights $(-1,1),(0,1),(1,1),(1,0)$. One can think of this as the symplectic cut of $\mathbb{C}^2$ by torus actions with weights $(1,1),(-1,1)$. The corresponding chamber structure and polytopes of the quotients are shown in the Figure 1. The equivariant quantum Stanley-Reisner relations are then

$$\epsilon_1^{-d_1+d_2}\epsilon_2^{d_2}\epsilon_3^{d_1+d_2}\epsilon_4^{d_1} = q^{d_1+d_2}, \quad -d_1+d_2, d_1 \geq 0$$

in particular $\epsilon_1\epsilon_2\epsilon_3 = q, \quad \epsilon_2\epsilon_3^2\epsilon_4 = q^2$. Notice the first relation defines the quantum cohomology for the quotient in the right-most chamber, while the second relation defines quantum cohomology for the quantum cohomology in the left-most chamber. The non-equivariant relations are

$$(-\xi_1+\xi_2)(\xi_2)(\xi_1+\xi_2) = q, \quad \xi_2(\xi_1+\xi_2)^2\xi_1 = q^2.$$

Motivated by considerations from mirror symmetry, Givental [21] and later Hori-Vafa [28], proposed a description of the quantum cohomology in terms of the *Jacobian ring* of functions on the critical locus of a certain function, arising as the *Landau-Ginzburg potential* of the mirror sigma model. In particular, Givental [21] proved an isomorphism of the quantum cohomology of a smooth Fano toric variety with the Jacobian ring.

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Definition 3.5. (Jacobian ring) The *Jacobian ring* $\mathrm{Jac}(W_{X,G})$ of the naive Landau-Ginzburg potential $W_{X,G}$ is the ring of functions on $\mathrm{Crit}(W_{X,G})$. Assuming that ω is integral and 0 lies in the moment polytope, then $\mathrm{Jac}(W_{X,G})$ is defined over $\mathbb{C}[q]$.

Example 3.6. (Jacobian ring for the projective line) If $X/\!\!/G = \mathbb{P}^1$ then $W_{X,G}(y) = y + q/y$ and over $\mathbb{C}[q]$

$$\mathrm{Jac}(W_{X,G}) = \mathbb{C}[y, y^{-1}, q]/\langle y - q/y\rangle = \mathbb{C}[q, y, y^{-1}]/\langle y^2 - q\rangle.$$

Proposition 3.7. (Isomorphism of the Batyrev ring with the Jacobian ring of the naive potential)

$$QH_G(X)/QSR_{X,G} \longrightarrow \mathrm{Jac}(W_{X,G}), \quad [\mu_j] \longmapsto [q^{-\omega_j} y^{\nu_j}], j = 1, \dots, k$$

is well-defined and induces an isomorphism.

Proof. Without the Novikov field, the result is Iritani [30, 3.9]: the linear relations among the weights for $\mathfrak{g}$ on X correspond to the relations on the coordinate ring of $T^\vee$ given by the derivatives of the Landau-Ginzburg potential $W_{X,G}$, since any such relation is of the form

$$\sum_{i=1}^k \mu_i \langle \lambda, \nu_i\rangle = 0.$$

Furthermore the quantum Stanley-Reisner relations $QSR_{X,G}$ correspond to the relations on the various coordinates on the big dual torus $\widetilde{G}^\vee$ restricted to $T^\vee$:

$$\begin{aligned}
\prod_{\langle \mu_j, d\rangle > 0} \mu_j^{\langle \mu_j, d\rangle} \quad &\longmapsto \quad \prod_{\langle \mu_j, d\rangle > 0} q^{-\omega_j \langle \mu_j, d\rangle} y^{\nu_j \langle \mu_j, d\rangle} \\
&= \quad q^{-\sum_{\langle \mu_j, d\rangle > 0} \omega_j \langle \mu_j, d\rangle} y^{\sum_{\langle \mu_j, d\rangle > 0} \nu_j \langle \mu_j, d\rangle} \\
q^{\langle d, \omega\rangle} \prod_{\langle \mu_j, d\rangle < 0} \mu_j^{\langle \mu_j, d\rangle} \quad &\longmapsto \quad q^{\langle d, \omega\rangle} \prod_{\langle \mu_j, d\rangle < 0} q^{-\omega_j \langle \mu_j, d\rangle} y^{\nu_j \langle \mu_j, d\rangle} \\
&= \quad q^{d - \sum_{\langle \mu_j, d\rangle < 0} \omega_j \langle \mu_j, d\rangle} y^{\sum_{\langle \mu_j, d\rangle < 0} \nu_j \langle \mu_j, d\rangle}
\end{aligned}$$

and both lines are equal since

$$\sum_{j=1}^k \nu_j \langle \mu_j, d\rangle = 0, \qquad \sum_{j=1}^k \omega_j \langle \mu_j, d\rangle = \langle \omega, d\rangle.$$

The proof extends over Λ etc. by linearity of both sides over Λ. □

In the following we collect some properties of the Jacobian ring, forgetting the powers of q. For this it is helpful to consider the following family of functions for constant $c \in \mathfrak{g}^\vee$, embedded in $\mathbb{R}^{\dim(X)}$ by the choice of some splitting; we denote by c_j its j-th coordinate.

Notation 3.8. (Landau-Ginzburg potential without q) The *Landau-Ginzburg potential without q and constant c* is

$$W_{X,G,c} \colon T^\vee \longrightarrow \mathbb{C}, \quad y \longmapsto \sum_{j=1}^{k} c_j y^{\nu_j}.$$

Let $\operatorname{Jac}(W_{X,G,c})$ be the coordinate ring of $\operatorname{Crit}(W_{X,G,c})$.

To compute the dimension of $\operatorname{Jac}(W_{X,G,c})$ we recall the following theorem of Kouchnirenko's [36, 3]: Let $W : T^\vee \to \mathbb{C}$ be a function given as a finite sum

$$W(\xi) = \sum_{\lambda \in \mathfrak{t}_\mathbb{Z}} c_\lambda y^\lambda, \quad c_\lambda \in \mathbb{C}$$

as λ ranges over the weights of $T^\vee$, that is, coweights $\mathfrak{t}_\mathbb{Z}$ of T.

Definition 3.9. (Newton polytope) The convex polyhedron

$$\Delta(W) := \operatorname{hull}\{\lambda \in \mathfrak{t}_\mathbb{Z}, c_\lambda \neq 0\}$$

is the *Newton polytope* of W. The function W is *non-degenerate at infinity* if for any face $F \subset \Delta(W)$, the *face polynomial*

$$W_F \colon T^\vee \longrightarrow \mathbb{C}, \quad \xi \longmapsto \sum_{\lambda \in F} c_\lambda y^\lambda$$

has no critical points. By the *multiplicity* of an isolated critical point we mean the intersection multiplicity as in, for example, Fulton's book [19, Lemma 12.1]. By [19, 11.4.4] the intersection is invariant under deformation in the sense that, given a family $W_q(y)$ of functions depending smoothly on a parameter q with isolated critical points, the sum of the multiplicities of critical points y_q converging to y_0 is equal to the multiplicity of y_0.

Theorem 3.10. (Kouchnirenko theorem) *Suppose that W is non-degenerate at infinity, $\{\lambda \in \mathfrak{t}_\mathbb{Z}, c_\lambda \neq 0\}$ generate $\mathfrak{t}_\mathbb{Z}$, and also $\{\lambda - \mu, c_\lambda \neq 0, c_\mu \neq 0\}$ generate $\mathfrak{t}_\mathbb{Z}$. Then $\mathrm{d}W$ is finite in a neighborhood of $\operatorname{Crit}(W)$ and the number $\#\operatorname{Crit}(W)$ of zeroes of $\mathrm{d}W$ counted with multiplicity is equal to $\dim(T)!$ the volume of the Newton polytope,*

$$\#\operatorname{Crit}(W) = \dim(T)! \operatorname{Vol}(\Delta(W)).$$

For a proof we recommend the survey of Atiyah [3], who identifies $\operatorname{Crit}(W)$ with the set of intersection points of a generic torus orbit in the projectivization $\mathbb{P}(\oplus_{c_\lambda \neq 0} \mathbb{C}_\lambda))$ of the sum of weight spaces C_λ with non-zero coefficient c_λ. Thus $\#\operatorname{Crit}(W)$ equals the degree of the toric variety associated to $\Delta(W)$, which is $\dim(T)!$ times the volume of $\Delta(W)$. As noted in Iritani [30, 3.7], if we assume that 0 is contained in the interior of the Newton polytope of $W_{X,G,c}$ then $W_{X,G,c}$ is non-degenerate at infinity.

Proposition 3.11. (Properties of the naive Landau-Ginzburg potential)

(a) *For any non-zero c,* $\dim(\operatorname{Jac}(W_{X,G,c}))$ *is finite and independent of the choice of c.*
(b) *The number* $\#\operatorname{Crit}(W_{X,G,c})$ *of critical points of $W_{X,G,c}$, counted with multiplicity. is equal to* $\dim(T)!$ *times the volume of the dual polytope $\Delta^\vee_{X/G}$ given as the convex hull of the normal vectors ν_j of facets of $\Delta_{X/G}$.*
(c) $\operatorname{Crit}(W_{X,G})$ *is finite over* $\operatorname{Spec}\mathbb{C}[q,q^{-1}]$.
(d) *For generic ω, if $c_j = \omega_j$ for $j = 1,\ldots,k$ then $W_{X,G,c}$ is Morse and* $\operatorname{Jac}(W_{X,G,c})$ *is semisimple.*

Proof. (a)-(c) are consequences of Theorem 3.10. (d) follows from a computation of the Jacobian of $\mathrm{d}W_{X,G,c}$ as in [30, 3.10],

$$\det\left(\sum_{i=1}^{k}\langle\nu_i,\lambda_a\rangle y^{-\nu_i}\langle\nu_i,\lambda_b\rangle\right) > 0$$

for y positive real, that is, of the form $\exp(x), x \in \mathfrak{t}^\vee_{\mathbb{R}}$. □

As noted in Iritani [30, 3.10], [31, 5.10], Kouchnirenko's theorem implies that

Corollary 3.12. *For $c_j = \omega_j, j = 1,\ldots,k$, the order of* $\operatorname{Crit}(W_{X,G,c})$ *is at least* $\dim(QH(X/\!\!/G)$*, with equality if and only if* $c_1(X/\!\!/G) \geq 0$.

Indeed, the volume of the dual polytope is equal to the sum of the volumes of the maximal dimension cones of $X/\!\!/G$ exactly if $c_1(X/\!\!/G) \geq 0$. In general, there must be additional elements in the kernel of $\operatorname{Jac}(W_{X,G}) \to QH(X/\!\!/G)$. See Figure 2.

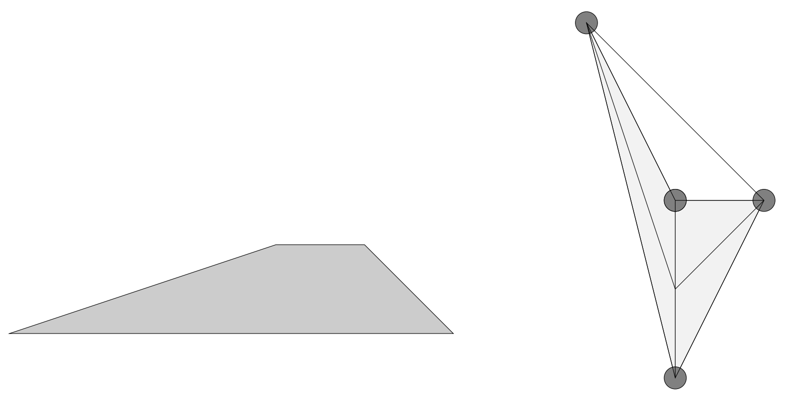

FIGURE 2. A polytope whose dual polytope has too much volume

Proposition 3.13. (Inclusion of the Batyrev relations in the kernel of the quantum Kirwan map) [46] *The kernel of the linearized quantum Kirwan map*

$$D_\alpha\kappa^G_X \colon T_\alpha QH_G(X) \longrightarrow T_{\kappa^G_X(\alpha)}QH(X/\!\!/G)$$

contains the quantum Stanley-Reisner ideal for α in a formal neighborhood of the symplectic class $\omega \in QH_G(X)$, and so induces a map

$$T_\alpha QH_G(X)/QSR_{X,G}(\alpha) \cong \mathrm{Jac}(W_{X,G}) \longrightarrow T_{\kappa_X^G(\alpha)} QH(X/\!\!/G). \tag{4}$$

For the sake of completeness we briefly sketch the proof. The adiabatic limit theorem of [46] relates the genus zero graph potential of $X/\!\!/G$ with the gauged graph potential of X, defined by virtual enumeration of Mundet-stable maps to the quotient stack. Localizing with respect to the $\mathbb{C}^\times$ action on $\mathbb{P}^1$ one obtains a formula involving the the localized graph potential of $X/\!\!/G$, which as in Givental [20] is a fundamental solution for the quantum differential equation on $X/\!\!/G$. It follows that any differential operator which annihilates the localized gauged graph potential, is transformed via $D_\alpha \kappa_X^G$ to a differential operator annihilated the fundamental solution, and so defines a relation. One then checks that the localized gauged potential is the solution to the Gelfand-Kapranov-Zelevinsky hypergeometric system, and the Batyrev relations correspond to differential operators of that system. This completes the sketch. Similar results were obtained for toric manifolds under the name of mirror theorems in, for example, Iritani [31], by writing the toric variety as a complete intersection in another Fano toric variety and applying the Givental formalism.

Because the dimension are wrong (Corollary 3.12) there must be additional relations, that is, generators in the kernel of (4). Several authors suggested, and Iritani [31] and Fukaya et al [18] proved in the case of toric manifolds, that the additional relations correspond to critical points outside of the moment polytope, and can be removed by a suitable formal version of the Jacobian ring of the potential. There seem to be several formal versions which have the same effect: here we adopt the most naive formal version, already introduced in the previous section, which has the effect of introducing the necessary additional relations. In order to relate the formal version $\widehat{QH}_G(X)$ of $QH_G(X)$ discussed in Definition 2.1 to the Jacobian ring $\mathrm{Jac}(W_{X,G})$ we introduce the following notations:

Notation 3.14.

(a) (Fan of the toric stack $X/\!\!/G$) Let $\mathcal{C}_{X/G} = \{C | C \in \mathcal{C}_{X/G}\}$ denote the fan of $X/\!\!/G$. Each cone $C \subset \mathfrak{t}_\mathbb{R}$ in $\mathcal{C}_{X/G}$ is the dual cone of a face of the moment polytope $\Delta_{X/G} \subset \mathfrak{t}_\mathbb{R}^\vee$. By our properness assumption, the interiors of the cones $C \in \mathcal{C}_{X/G}$ give a partition of $\mathfrak{t}_\mathbb{R}$.

(b) (Symplectic class as a piecewise linear function) The symplectic class $\omega \in H_G^2(X)$ defines a convex piecewise linear function $\lambda_{X,G} : \mathcal{C}_{X/G} \to \mathbb{R}$ assigning to each ray the constant in the moment map for the corresponding one-parameter subgroup.

(c) (Order of vanishing at $q = 0$) Denote by $|\ | : \Lambda \to \mathbb{R}$ the map $\sum c_d q^d \to \min(d, c_d \neq 0)$.

We assume for simplicity that 0 is in the interior of the moment polytope, so that $\lambda_{X,G}(\mu)$ are negative. Denote by $R_+(T^\vee(q))$ the *formal coordinate ring* of formal sums

$$R_+(T^\vee(q)) := \left\{ \sum_{\mu \in \mathfrak{t}_\mathbb{Z}} c_\mu y^\mu, \quad c_\mu \in \mathbb{C}[[q]], \quad \inf_{\mu \in \mathfrak{t}_\mathbb{Z}} \lambda_{X,G}(\mu)|c_\mu| > -\infty \right\}. \tag{5}$$

That is, in order for $\sum_\mu c_\mu y^\mu$ to lie in $R_+(T^\vee(q))$, the set of products $|c_\mu|\lambda_{X,G}(\mu)$ should be bounded from below. Using the embedding

$$T^\vee \to \mathbb{C}^{\times,k}, \quad y \mapsto (y^{\nu_j} q^{-\omega_j})_{j=1}^k \tag{6}$$

$R_+(T^\vee(q))$ is the completion with respect to the total degree filtration and so closed under multiplication. The partial derivatives of $W_{X,G}$ generate an ideal in $R_+(T^\vee(q))$ denoted

$$\langle \partial_\xi W_{X,G}, \xi \in \mathfrak{t}^\vee \rangle \subset R_+(T^\vee(q)).$$

The *formal Jacobian ring* is

$$\widehat{\operatorname{Jac}}(W_{X,G}) := \bigcup_{\rho \in \mathbb{R}} \widehat{\operatorname{Jac}}(W_{X,G})_{\geq \rho}, \quad \widehat{\operatorname{Jac}}(W_{X,G})_{\geq \rho} := q^\rho R_+(T^\vee(q)) / \langle \partial_\xi W_{X,G}, \xi \in \mathfrak{t}^\vee \rangle.$$

The formal spectrum of $\widehat{\operatorname{Jac}}(W_{X,G}))_{\geq 0}$ with respect to the ideal generated by q is a formal neighborhood of $(0,0)$ in $\operatorname{Crit}(W_{X,G})$, and can be taken as the ring of functions on $\operatorname{Crit}_+(W_{X,G})$. Working over $\mathbb{C}[[q]]$ this defines a formal scheme structure on $\operatorname{Crit}_+(W_{X,G})$; the same might be said over Λ but the theory of formal schemes usually requires the Noetherian assumption and so we avoid doing so.

Proposition 3.15. (Isomorphism of formal Batyrev and Jacobian rings) *The map $T_\omega QH_G(X)/QSR_{X,G} \to \operatorname{Jac}(W_{X,G})$ extends to an isomorphism*

$$T_\omega \widehat{QH}_G(X)/\widehat{QSR}_X^G \to \widehat{\operatorname{Jac}}(W_{X,G}).$$

Proof. To check that the map is well-defined, it suffices to check that each expression $\sum_{(j_1,\ldots,j_k)} b_{(j_1,\ldots,j_k)} q^{c(j_1,\ldots,j_k)} \prod_{i=1}^k \mu_i^{d_{j_i}}$ lying in $\widehat{QH}_G(X)$ maps to $\widehat{\operatorname{Jac}}(W_{X,G})$. By the definition of $\widehat{QH}_G(X)$, on each subsequence $c_{(j_1,\ldots,j_k)}$ is bounded from below and so this element maps to a formal sum with satisfying the bound (5). Conversely, any element of $\widehat{\operatorname{Jac}}(W_{X,G})$ can be written as the image of an element of $\widehat{QH}_G(X)/\widehat{QSR}_X^G$, by breaking the sum into contributions from the maximal dimensional cones. □

Proposition 3.16. *The following definitions are equivalent:*

(a) (Order of $\operatorname{Crit}_+(W_{X,G})$, algebraically) *The order of* $\operatorname{Crit}_+(W_{X,G})$ *is equal to the the number of points of* $\operatorname{Crit}(W_{X,G})$ *counted with multiplicity whose closure contains $(y,q) = (0,0)$ via the embedding* (6).

(b) (Order of $\operatorname{Crit}_+(W_{X,G})$, analytically) *The order* $\#\operatorname{Crit}_+(W_{X,G})$ *of* $\operatorname{Crit}_+(W_{X,G})$ *is equal to the limit*

$$\lim_{R \to 0} \lim_{q \to 0} \# \operatorname{Crit}(W_{X,G,c}) \cap B_R, \quad c_j = \omega_j, j = 1, \ldots, k$$

where B_R is a ball of radius R centered at 0 *in* $\mathbb{C}^k$.

Proof. Since $\mathrm{Crit}(W_{X,G})$ is flat and finite over $\operatorname{Spec}\mathbb{C}[q,q^{-1}]$, $\mathrm{Crit}_+(W_{X,G})$ is flat and finite over $\operatorname{Spec}\mathbb{C}((q))$ (here we are assuming that 0 is in the interior of the moment polytope). The order of $\mathrm{Crit}_+(W_{X,G})$ is the degree of the morphism, which is invariant under base change to $\operatorname{Spec}\Lambda$, since both $\mathbb{C}((q))$ and Λ are fields. On the other hand, each irreducible components of $\mathrm{Crit}_+(W_{X,G})$ are in bijection with the irreducible components of $\mathrm{Crit}(W_{X,G})$ whose order of vanishing with respect to the divisors $y_1 = 0,\ldots,y_k = 0$ at $q = 0$ is positive, by a simple case of the Grothendieck Existence Theorem [25]; analytically these correspond to critical points in a small ball around $y = 0$ in any particular fiber over $\operatorname{Spec}\mathbb{C}[q,q^{-1}]$. The claim follows. □

We now explore the meaning of the positive part $\mathrm{Crit}_+(W_{X,G})$ more geometrically, in terms of the moment map. Since $\mathrm{Crit}(W_{X,G}) \to \operatorname{Spec}\mathbb{C}[q,q^{-1}]$ is finite, after passing to a finite cover $\operatorname{Spec}\mathbb{C}[[q^{1/n}]]$ we obtain a smooth morphism, and then any irreducible component of $\mathrm{Crit}(W_{X,G})$, considered as a scheme over $\operatorname{Spec}\mathbb{C}[q,q^{-1}]$ defines a point in $T^\vee(\Lambda_0)$.

Definition 3.17. (Tropical moment map) The *tropical moment map*

$$\Psi\colon T^\vee(\Lambda_0) \longrightarrow \mathfrak{t}^\vee, \quad q^\zeta \exp\left(\sum_{d\geq 0}\xi_d q^d\right) \longmapsto \zeta.$$

The following describes the special features of the restriction of the tropical moment map Ψ to the critical locus $\mathrm{Crit}(W_{X,G})$.

Definition 3.18. (Minimal facets) Given $\mu \in \mathfrak{t}^\vee_\mathbb{R}$, a facet of $\Delta_{X/G}$ defined by $\langle \mu, \nu_i = \omega_i$ is *minimal* for μ iff it is among the "closest" among all facets, that is,

$$\langle \mu,\nu_i\rangle - \omega_i = \inf\{\langle\mu,\nu_j\rangle - \omega_j \in \mathbb{R}, \quad j = 1,\ldots,k\}.$$

We denote by $I(\mu)$ the set of facets minimal for μ. More generally, for any subspace $V \subset \mathfrak{t}^\vee_\mathbb{R}$, we denote by $I(\mu, V)$ the set of facets minimal for μ among those with $\nu_j|V \neq 0$.

Remark 3.19. For a generic point $\mu \in \mathfrak{t}^\vee_\mathbb{R}$, there will be a unique minimal facet (the one "closest" to μ) so $I(\mu)$ will have order 1. Each point μ on the boundary of $\Delta_{X/G}$ has minimal facets $I(\mu)$ equal to the set of facets of $\Delta_{X/G}$ containing μ. The same holds in a neighborhood of the boundary, by continuity. For examples of points μ with more than $\dim(T)$ minimal facets, see Figure 3.

Proposition 3.20. (Tropical moment map for critical points) *Given a critical point $y \in \mathrm{Crit}(W_{X,G})$, the tropical moment map $\zeta = \Psi(y) \in \mathfrak{t}^\vee_\mathbb{R}$ has the property that the normal vectors $\nu_j, j \in I(\zeta, V)$ are linearly dependent after restriction to the annihilator V° of V in $\mathfrak{t}^\vee_\mathbb{R}$.*

Proof. For $\lambda \in V$ we have $0 = \partial_\lambda W_{X,G}(y) = \sum_{j=1}^k q^{-\omega_j} y^{\nu_j} \langle \nu_j, \lambda \rangle$. In particular the leading order powers of q must cancel, that is, $\sum_{j \in I(\zeta,V)} y^{\nu_j} \langle \nu_j, \lambda \rangle = 0$. Since this holds for any such λ, $\sum_{j \in I(\zeta,V)} y^{\nu_j} \nu_j = 0$. □

Example 3.21. (a) (Tropical moment map for critical locus for a product of projective lines) Let $X = \mathbb{C}^4$ with $G = (\mathbb{C}^\times)^2$ acting with weights $(1,0),(1,0),(0,1),(0,1)$. Consider the reduction at $\omega = (1,2)$ then $X/\!\!/G = \mathbb{P}^1 \times \mathbb{P}^1$ with moment polytope $[0,2] \times [0,4]$. as above then the critical points are $(y_1, y_2) = (\pm q, \pm q^2)$ which have valuations $(1,2)$, all mapping to the barycenter of the moment polytope. We have $I(\mu) = \{1,2\}$, the facets closest to the critical point, while if $V = \text{span}(1,0)$ then $I(\mu, V) = \{3,4\}$. Note that the vectors $\nu_j, j \in I(\mu)$ or $I(\mu, V)$ are dependent.

(b) (Tropical moment map for critical locus for a family of toric surfaces) Suppose that $X = \mathbb{C}^5$ with $G = (\mathbb{C}^\times)^3$ acting with weight matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

For a suitable choice of ω the quotient $X/\!\!/G$ is the blow-up of projective lines $\mathbb{P}^1 \times \mathbb{P}^1$ at a fixed point, with moment polytope

$$\Delta_{X/G} = \{(\mu_1, \mu_2) \in [0,2] \times [0,4] \mid \mu_1 + \mu_2 \geq \epsilon\}.$$

For $\epsilon \leq 1$, there are two possible values of Ψ on $\text{Crit}(W_{X,G})$: one critical point maps to (ϵ, ϵ), while four other critical points map to $(1,2)$. For $1 \leq \epsilon$, one critical point maps to $(2-\epsilon, \epsilon)$, while the others map to $(1,2)$. See Figure 3. The case $\epsilon = 1$ is special: in this case, one can obtain a line segment of critical points by varying the "bulk deformation", see [16]. This is shown as a dotted line connecting the two critical values in the Figure 3.

Proposition 3.22. *Any point in $y \in \text{Crit}(W_{X,G})$ maps to the interior of the moment polytope $\Delta_{X/G}$ under Ψ iff y represents a point in $\text{Crit}_+(W_{X,G})$, that is, contains $(0,0)$ in its closure.*

Proof. The j-th coordinate of y under the embedding $T^\vee \to \tilde{G}^\vee$ is $y_j = y^{\nu_j} q^{-\omega_j}$ which is asymptotic to a constant times $q^{\langle \nu_j, \Psi(y) \rangle - \omega_j}$. Thus y_j goes to zero as $q \to 0$ iff $\langle \nu_j, \Psi(y) \rangle > \omega_j$. □

By the results of [17], [16], [47], under the assumption that the generic automorphism group is trivial, the image of $\text{Crit}_+(W_{X,G})$ in $\mathfrak{t}_\mathbb{R}^\vee$ consists of moment values such that the corresponding Lagrangian moment fiber is Hamiltonian nondisplaceable, see Section 5.

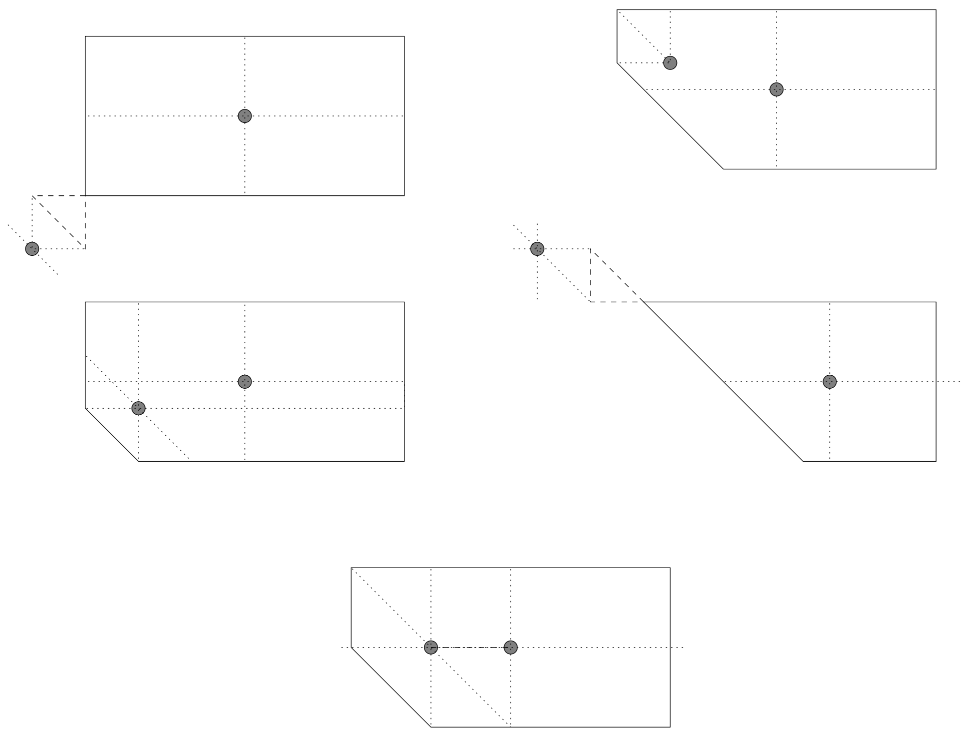

FIGURE 3. Values of the tropical moment map on the critical locus for a family of toric surfaces

4. Dimensional equality via a toric minimal model program

Because of Propositions 3.15 and Theorem 2.4, to complete the description of the quantum cohomology ring, it suffices to show that $D_\alpha \kappa_X^G$ is injective. By the surjectivity result in Theorem 2.4 it suffices to show the equality of dimensions

$$\dim QH(X /\!\!/ G) = \dim \widehat{\operatorname{Jac}}(W_{X,G}). \tag{7}$$

In the case that $X /\!\!/ G$ is Fano and minimally presented as a quotient of X by G (that is, every weight space in X defines a prime divisor of $X /\!\!/ G$) this is a consequence of Kouchnirenko's theorem, see Corollary 3.12.

To reduce to the Fano case, we apply the toric minimal model program introduced by Reid [41]. More precisely, we vary the Kähler class by a multiple of the canonical class until we obtain a Fano fibration, showing that the wall-crossings on both sides of (7) are the same. We wish to emphasize that, although we are using the language of toric minimal models, in fact all of our results are completely *combinatorial*, that is, could be phrased entirely in terms of fans. However, we find the geometric story accompanying the combinatorics rather helpful.

First, recall the general phenomenon of wall-crossing in the context of geometric invariant theory quotients, as in Thaddeus [43]. Let $\omega_t \in H_G^2(X, \mathbb{Q}), t \in [0,1]_{\mathbb{Q}}$ be an affine linear path of Kähler classes, corresponding to a path of rational polarizations (ample G-line bundles) $L_t \to X$. For any $t \in [0,1]_{\mathbb{Q}}$ let

$$X^{t,\mathrm{ss}} = \bigcup_{k>0, s \in H^0(X, L_t^k)^G} \{s \neq 0\}$$

be the semistable locus, and assume that G acts with finite stabilizers on $X^{t,\mathrm{ss}}$ for $t = 0,1$. Then the same is true for generic $t \in [0,1]$ and for such t we denote by

$$X/\!\!/_t G := X^{t,ss}/G$$

the stack-theoretic git quotient with respect to the corresponding polarization. The stack $X/\!\!/_t G$ is a smooth proper Deligne-Mumford stack with projective coarse moduli space.

Proposition 4.1. (Wall-crossing for git quotients) *With G, X, L_t as above.*

(a) (Walls) *there exists a finite collection $t_1, \ldots, t_n \in (0,1)$ of* singular values a.k.a walls *such that there exist semistable points that are not stable;*
(b) (Chambers) *the isomorphism class of the quotient $X/\!\!/_t G$ is independent of t for $t \in (t_j, t_{j+1})$;*
(c) (Wall-crossing) *Suppose that stable=semistable for the G-action on $\mathbb{P}(L_0 \oplus L_1)$. Then as t passes through a singular value, the quotient $X/\!\!/_t G$ goes through a weighted blow-down and blow-up.*

See Figure 4 for an example of the change in moment polytopes under such a variation; the toric case is discussed further in [41], [38, Chapter 14]. A weighted blow-down is a morphism that is an isomorphism except over a single (possibly stacky) point where the fiber is the git quotient of a vector bundle by a circle action. Thus if the point has trivial automorphism group then the fiber is a weighted projective space, but this is not true in general, see for example [38]. First we discuss the invariance of the order of $\mathrm{Crit}_+(W_{X,G})$ in each chamber.

Notation 4.2. For any interval $I \subset \mathbb{R}$, let $\mathrm{Crit}_I(W_{X,G})$ denote the subset of the critical locus $\mathrm{Crit}(W_{X,G})$ consisting of points y with $I \ni \inf_{i=1,\ldots,k} \langle \Psi(y), \nu_i \rangle$.

Corollary 4.3. *Suppose that $\omega_t \in [0,1]$ be an affine linear family of symplectic classes and $s > 0$, such that the family $\omega_t - s c_1^G(X)$ are symplectic and the corresponding symplectic quotients are smooth for all $t \in [0,1]$. Then the order of* $\mathrm{Crit}_{(s,\infty)}(W_{X,G,\omega_t})$ *is independent of t.*

Proof. By Corollary 3.16, the order of $\mathrm{Crit}_{(s,\infty)}(W_{X,G,\omega_t})$ is the number of critical points in a neighborhood of $(q,y) = (0,0)$, for the $-sc_1^G(X)$-shifted potential; after shifting, we may assume $s = 0$. Under the smoothness assumption, all critical points approach either $y_j = 0$ or $y_j = \infty$ in each coordinate y_j on $(C^\times)^k$, and one can make a uniform estimate of the size of the critical points converging to $y = 0$ as $q \to 0$:

$$\exists \epsilon, \delta > 0, \forall y \in \mathrm{Crit}_+ W_{X,G}, |q| < \delta \implies \|y(q)\| < \epsilon.$$

Indeed, since the coefficients of $W_{X,G,c}, c_j = \omega_j$ vary continuously with ω, the space

$$\cup_\omega \mathrm{d}W^{-1}_{X,G,c}(0)|_{c_j=\omega_j}$$

is closed, and so any estimate for a particular ω gives an estimate for ω' in a neighborhood of ω. The statement of the corollary now follows from deformation invariance of the number of these critical points contained in a neighborhood of $y = 0$, counted with multiplicity, with respect to the parameters $c_j = q^{-\omega_j}$. □

In particular,

Proposition 4.4. *If $\omega_{t_-}, \omega_{t_+}$ lie in the same chamber (no singular value is crossed) then* $\dim QH(X/\!\!/_{t_-}G) = \dim QH(X/\!\!/_{t_+}G)$ *and* $\dim \widehat{\mathrm{Jac}}(W_{X,G,t_-}) = \dim \widehat{\mathrm{Jac}}(W_{X,G,t_+})$.

Proof. The first claim follows from 4.1 (b). The second follows from Corollary 4.3. □

We will be particularly interested in the variation of git quotient for toric quotients corresponding to the anti-canonical class $c_1^G(TX)$ in the sense that ω_t is a path of classes satisfying $\frac{d}{dt}\omega_t = -c_1^G(TX)$, see Figure 4. Since $c_1^{\tilde{G}}(TX) = \sum_{i=1}^k \epsilon_i$ the constants defining the moment polytope of $\Delta_{X/G}$ can be taken to be

$$(\omega - tc_1^G(TX))_i = \omega_i - t, \quad i = 1, \dots, k.$$

Hence the facets of the polytope "vary at the same rate":

$$\Delta_{X/_tG} = \{\mu \in \mathfrak{t}^\vee_{\mathbb{R}} \mid \langle \mu, \nu_j \rangle \geq \omega_j + t, j = 1, \dots, k\}.$$

The sequence of a toric varieties obtained in this way is a special case of the minimal model program described in the toric case by Reid [41]; see [13, Chapter 15] for more references. An example of the family of polytopes obtained in this way is shown in Figure 4; the corresponding fans are shown in Figure 5.

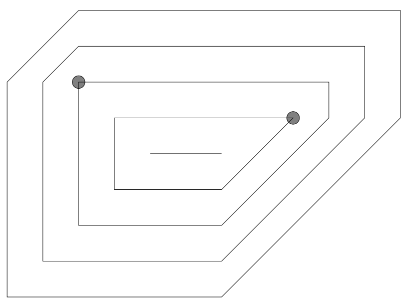

FIGURE 4. Polytopes for a toric minimal model program

We will need the following explicit description of the flips arising from variation in the anticanonical direction.

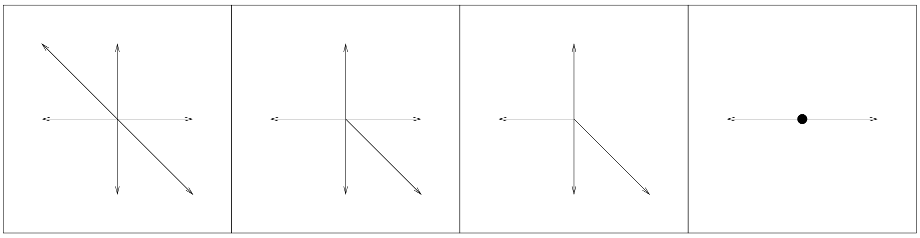

Figure 5. Fans for a toric minimal model program

Proposition 4.5. *For a generic symplectic class ω, at any singular value t, any point in $\mathfrak{t}^\vee$ satisfies at most $\dim(T)+1$ inequalities with strict equality.*

Proof. This is a simple codimension calculation left to the reader. □

Suppose that ω_t is a path of classes as above with $\frac{d}{dt}\omega_t = -c_1^G(X)$.

Definition 4.6. (Flipping simplex) Suppose that $n = \dim(T)$ and $t \in (0,\infty)$ is a singular value, corresponding to an intersection point of facets with normal vectors $\nu_1,\ldots,\nu_{n+1}$, which is a proper subset of the normal vectors, with $(X/\!\!/G)_{t_\pm}$ non-empty for $t_\pm$ in the chamber on either side of t. Let Σ_t denote the *flipping simplex*

$$\Sigma_t := \mathrm{hull}(\nu_1,\ldots,\nu_{n+1}).$$

The toric variety whose fan has the rays $\nu_1,\ldots,\nu_{n+1}$ and no other is projective "stacky weighted projective space" and has moment polytope an n-simplex. It need not be a weighted projective space, see the discussion after Proposition 4.1. However, one may check easily that (as in the case of projective space) the corresponding potential $y \mapsto \sum_{i=1}^{n+1} y^{\nu_i}$ has only non-degenerate critical points.

Proposition 4.7 (Explicit description of flips for the toric minimal model program)**.** *Let $(X/\!\!/G)_t$ be as above, with initial class ω generic so that the condition of Proposition 4.5 is satisfied. For each singular value t, one of the two possibilities holds:*

(a) (Fibration case) *If $0 \in \Sigma_t$ then $X/\!\!/_t G$ undergoes a* fibration *with monotone fibers $(X/\!\!/_t G)'$ over a toric orbifold $(X/\!\!/_t G)''$ of lower dimension. If so*
  (i) *the fan $\mathcal{C}(X/\!\!/_{t\_}G)$ of $\mathcal{C}(X/\!\!/_{t\_}G)$ admits a morphism to the fan $\mathcal{C}((X/\!\!/_t G)')$ of the base, that is, each cone for $X/\!\!/_{t\_}G$ maps to a cone, possibly of lower dimension, of $(X/\!\!/_t G)'$;*
  (ii) *the cones of $\mathcal{C}(X/\!\!/_{t\_}G)$ that map to the origin form the fan of the fiber $(X/\!\!/_t G)''$.*

(b) (Flip or divisorial contraction case) *If $0 \notin \Sigma_t$ then $X/\!\!/_t G$ undergoes a* flip *(resp.* divisorial contraction*, that is, a weighted blow-down followed by weighted blow-up (resp. weighted blow-down only.) If so*

(i) *Denote by* $(\partial\Sigma_t)_\pm$ *the union of facets of* Σ_t *defined by half-spaces that do not resp. do contain* 0*. The corresponding partition of facets determines a partition* $\{I_+, I_-\}$ *of* $\{1, \ldots, n+1\}$*.*

(ii) *The morphism* $(X/\!\!/G)_{t_\pm} \to (X/\!\!/G)_t$ *corresponds to a morphism of fans that is an isomorphism except on the cone generated by* Σ_t*, where it corresponds to the morphism of fans given by the inclusion of the fan defined by* $(\Sigma_t)_\pm$ *to the fan with single cone* $\mathbb{R}_+\Sigma_t$*.*

(iii) *The transition* $X/\!\!/_{t_-}G$ *to* $X/\!\!/_{t_+}G$ *is a weighted blow-down of the orbit that is the intersection of divisors corresponding to* $I_\pm$*.*

Proof. See [13, Lemma 15.3.11], [38, Proof of Proposition 14-2-11] and especially [38, Figure 14-2-12]. The fibration case is straight-forward and left to the reader. For the flip/divisorial contraction case, one can give a proof using variation of git as follows: Let

$$\tilde{\nu}_j := (\nu_j, 1) \in \mathfrak{t}_\mathbb{R} \oplus \mathbb{R}, \ j = 1, \ldots, n$$

and let V denote the affine toric variety with cone generated by $\tilde{\nu}_j, j = 1, \ldots, n$. Locally $X/\!\!/_tG$ is the git quotient of V by the action of $\mathbb{C}^\times$ on the last factor, with polytope for stability parameter t defined by

$$\langle \tilde{\nu}_j, (\mu, t)\rangle = \langle \nu_j, \mu\rangle + t \geq 0, \quad j = 1, \ldots, n+1.$$

The weights of the action are

$$\tilde{\mu}_j = (\mu_j, c_j), \quad \langle \tilde{\mu}_j, \tilde{\nu}_k\rangle = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}.$$

Note that $\langle \tilde{\mu}_j, \tilde{\nu}_k\rangle = \langle \mu_j, \nu_k\rangle + c_j$ and since equality holds for $k \neq j$, the inequality

$$\langle \mu_j, \lambda\rangle + c_j \geq 0$$

defines a facet of Σ_t. The weighted blow-down and blow-up involved from passing to $V/\!\!/_{t_-}\mathbb{C}^\times$ to $V/\!\!/_{t_+}\mathbb{C}^\times$ replaces the projectivization of the sum of the weight spaces with $c_j < 0$ with the projectivization of the sum of the weight spaces with $c_j > 0$; these correspond to the facets of Σ_t such that the corresponding half-space does not resp. does contain the origin. Each orbit in such the projectivizations above has cone given by an intersection of the facets of Σ_t corresponding to weight spaces that it contains, and the claim follows. □

An example is shown in Figure 6, continuing that in Figures 4, 5, where the flipping simplices for each step are shaded.

Lemma 4.8. (Wall-crossing for dimensions) *Let* X, G, ω_t *be as above. In the case that* $X/\!\!/_tG$ *undergoes a flip at a singular value* $t \in (0, \infty)$*, with* $t_\pm = t \pm \epsilon$ *for* ϵ *small, we have*

(a) $\dim(QH(X/\!\!/_{t_+}G)) - \dim(QH(X/\!\!/_{t_-}G)) = \dim(T)!\operatorname{Vol}(\Sigma_t)$; *and*

(b) $\dim \widehat{\operatorname{Jac}}(W_{X,G,t_+}) - \dim \widehat{\operatorname{Jac}}(W_{X,G,t_-}) = \dim(T)!\operatorname{Vol}(\Sigma_t)$.

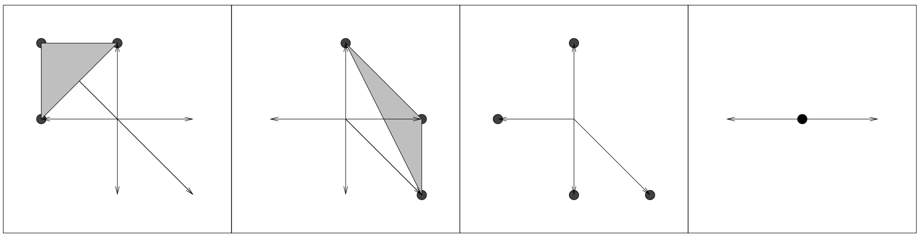

FIGURE 6. Flipping simplices for a toric minimal model program

Proof. (a) To each set $\nu_j, j \in I_\pm$ we add 0 and take the convex hull. The volume of the corresponding polytope is the Euler characteristic of the inertia stack of the corresponding non-compact toric variety, since these are the disjoint union of the maximal dimension cones. It follows that

$$\begin{aligned}\dim QH(X/\!\!/_{t_+}G) - \dim QH(X/\!\!/_{t_-}G) &= \dim(T)!\operatorname{Vol}\operatorname{hull}(\{\nu_j, j \in I_+\} \cup \{0\}) \\ &\quad - \dim(T)!\operatorname{Vol}\operatorname{hull}(\{\nu_j, j \in I_-\} \cup \{0\}) \\ &= \dim(T)!\operatorname{Vol}(\Sigma_t)\end{aligned}$$

which proves the claim. (b) We may suppose that $t_\pm$ are sufficiently close to the critical value t_j so that there exists a number $c > 0$ such that $\Psi(\operatorname{Crit}_+(W_{X,G,t}))$ consists of a single value in $(-c, c)$, which crosses the boundary of $\Delta_{X/\!\!/_tG}$ as t crosses t_j, and the other components of $\Psi(\operatorname{Crit}_+(W_{X,G,t}))$ stay outside of $(-c, c)$ for all $t \in (t_-, t_+)$. The critical value that crosses the boundary corresponds to the intersection of $\dim(T) + 1$-hyperplanes varying linearly in t, and so for each $t \in (t_-, t_+)$ corresponds to a non-degenerate family of critical points by the formal criterion for smoothness. By Kouchnirenko's theorem 3.10 the order of these critical points is $\dim(T)!\operatorname{Vol}(\Sigma_t)$. By this fact and Proposition 3.10 the difference in the number of critical points of the potential before and after the critical value is equal to

$$\begin{aligned}\dim \widehat{\operatorname{Jac}}(W_{X,G,t_+}) - \dim \widehat{\operatorname{Jac}}(W_{X,G,t_-}) &= |\operatorname{Crit}_+(W_{X,G,t_+})| - |\operatorname{Crit}_+(W_{X,G,t_-})| \\ &= |\operatorname{Crit}_{(c,\infty)}(W_{X,G,t_+})| - |\operatorname{Crit}_{(c,\infty)}(W_{X,G,t_-})| \\ &\quad + |\operatorname{Crit}_{(0,c)}(W_{X,G,t_+})| - |\operatorname{Crit}_{(0,c)}(W_{X,G,t_-})| \\ &= 0 + \dim(T)!\operatorname{Vol}(\Sigma_t) = \dim(T)!\operatorname{Vol}(\Sigma_t).\end{aligned}$$

See Figure 7 for an example. □

Remark 4.9. (a) (Termination of a toric minimal model program) In particular, one sees from the above wall-crossing formula that $\dim(QH(X/\!\!/_tG))$ decreases at each wall-crossing. This is one of the proofs of the eventual termination of the toric minimal model program, discussed in [13], [38].

(b) (Dependence of the Jacobian ring on the symplectic class) The location of the critical points also varies with the choice of symplectic class ω, and at

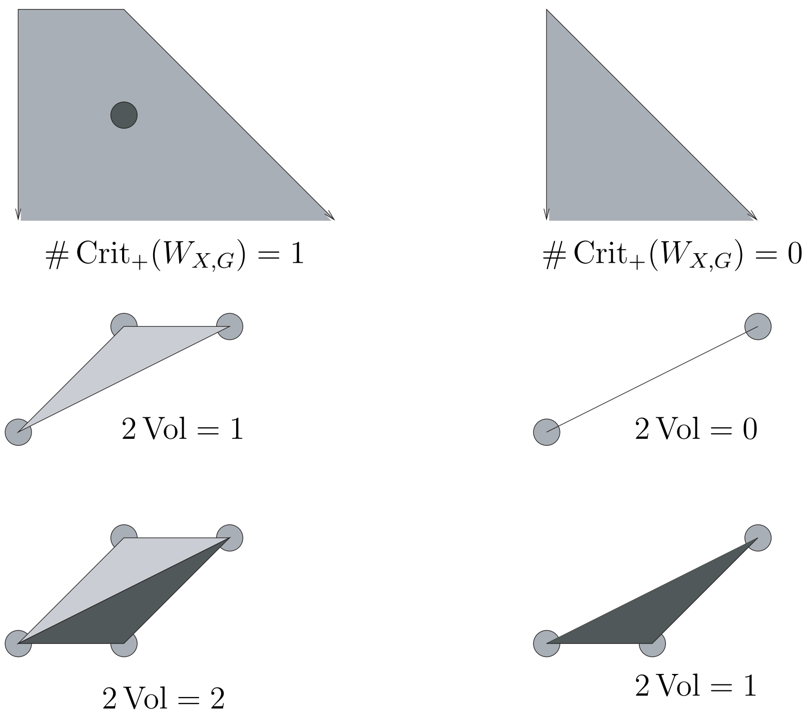

FIGURE 7. Wall-crossing for orbifold Euler characteristics

certain affine linear hyperplanes (occurring when more than $n+2$ normal vectors have a common value) the critical points can "collide" and become degenerate. We call these additional hyperplanes "miniwalls". If $X/\!\!/G$ is monotone then the case that ω is anticanonical is called the *monotone case* and always occurs on (possibly several) miniwalls. In the example in Figure 1, there is a unique miniwall, corresponding to the anticanonical class, shown as a dotted line.

Lemma 4.10. (Critical loci of the Landau-Ginzburg potentials for fibrations) *Let X, G, ω be as above. Suppose that $t \in (0,\infty)$ is a singular value so that $X/\!\!/_t G$ undergoes a fibration over a toric variety $(X/\!\!/_t G)'$ of lower dimension with fiber $(X/\!\!/_t G)''$. Then there is a canonical order-preserving bijection*

$$\mathrm{Crit}_+(W_{X/\!\!/_{t_-}G}) \to \mathrm{Crit}_+(W_{(X/\!\!/_t G)'}) \times \mathrm{Crit}_+(W_{(X/\!\!/_t G)''}).$$

Proof. The fibration of $X/\!\!/_{t_-}G$ induces a fibration of tori and tropical dual tori,

$$1 \to T' \to T \to T'' \to 1, \quad 1 \to T^{\vee,''} \to T^{\vee} \to T^{\vee,'} \to 1.$$

Since the terms in $W_{(X/\!\!/_t G)'}$ vanish on $\hat{T}''$, the second fibration induces a map from $\mathrm{Crit}_+(W_{X/\!\!/_{t_-}G})$ to $\mathrm{Crit}_+(W_{(X/\!\!/_t G)''})$. Each fiber consists of points y such that the induced differential $\mathrm{d}W(y) : \mathfrak{t}^{\vee}/\mathfrak{t}^{\vee,''} \to \mathbb{C}[q]$ vanishes. The leading order terms in this expression arise from the terms of $W_{(X/\!\!/_t G)'}$. Since the critical points for the fiber potential are non-degenerate (because of the genericity assumption on the tmmp) by the formal criterion for smoothness each fiber is in order-preserving bijection with the set of critical points $\mathrm{Crit}_+(W_{(X/\!\!/_t G)'})$. □

Remark 4.11. (The statement only holds for critical points in the interior) The statement of Lemma 4.10 does not hold with $\mathrm{Crit}_+(W_{X/\!\!/G})$ replaced by $\mathrm{Crit}(W_{X/\!\!/G})$, as one can see easily from the case of Hirzebruch surfaces F_n. In this case both the base and fiber are projective lines, so the right hand side has order 4, but the left-hand-side has order $2n$.

Lemma 4.12. (Dimension lemma for fibrations) *With $X/\!\!/_t G$ as above with generic initial symplectic class ω_0, suppose that $t \in (0,\infty)$ is a singular value so that $X/\!\!/_t G$ undergoes a fibration over a toric variety $(X/\!\!/_t G)'$ of lower dimension with fiber $(X/\!\!/_t G)''$. Then*

(a) $\dim(QH(X/\!\!/_{t_-} G)) = \dim(QH((X/\!\!/_t G)'))\dim(QH((X/\!\!/_t G)''))$ *and*
(b) $\dim(\widehat{\mathrm{Jac}}(W_{X/\!\!/_{t_-} G}) = \dim(\widehat{\mathrm{Jac}}(W_{(X/\!\!/_t G)'}))\dim(\widehat{\mathrm{Jac}}(W_{(X/\!\!/_t G)''}))$.

Proof. (a) By Proposition 4.7, the cones of $X/\!\!/_{t_-} G$ of maximal dimension are products of the maximal dimensional cones of $(X/\!\!/_t G)'$ and $(X/\!\!/_t G)''$. It follows the sum of the volumes of the maximal dimensional cones of $X/\!\!/_{t_-} G$ is the product of the corresponding sums for $(X/\!\!/_t G)'$ and $(X/\!\!/_t G)''$. (b) follows from Lemma 4.10. □

Lemma 4.13. (Equality of dimensions in the monotone case) *Suppose that the family $X/\!\!/_t G$ as above has a unique singular point at t_0, and undergoes a fibration* over a point *at t_0. Then $X/\!\!/_t G$ is monotone and* $\dim \widehat{\mathrm{Jac}}(W_{X,G}) = \dim QH(X/\!\!/_t G)$.

Proof. The assumption that $X/\!\!/_t G$ undergoes a fibration over a point means that every facet of $\Delta_{X/\!\!/_t G}$ is equidistant from some point $\mu \in \mathfrak{t}^\vee_{\mathbb{R}}$, so that $X/\!\!/_t G$ is monotone. Without loss of generality we may assume that $\mu = 0$. First suppose that the presentation of X as a symplectic quotient is the minimal one, that is, each weight of X corresponds to a facet of $\Delta_{X/\!\!/G}$ (no spurious facets). In this case all $y \in \mathrm{Crit}_+(W_{X,G})$ have $\Psi(y) = 0$. Thus we may omit the parameters q from the definition of the potential and the number of critical points is equal to $\dim QH(X/\!\!/G)$ by Kouchnirenko's Theorem 3.10.

In the case that the presentation is not minimal, let $W_{X,G,\mathrm{true}} : T^\vee(\Lambda_0) \to \Lambda$ denote the naive Landau-Ginzburg potential associated to the minimal presentation. That is, if

$$\mathcal{T} = \{i, | \operatorname{codim}(\{\langle \nu_i, \mu\rangle = \omega_i\} \cap \Delta_{X/\!\!/G}) = 1\}$$

denotes the indices of the inequalities defining facets of $\Delta_{X/\!\!/G}$ then

$$W_{X,G,\mathrm{true}} = \sum_{i \in \mathcal{T}} q^{-\omega_i} y^{\nu_i}$$

so that each term corresponds to a facet of $\Delta_{X/G}$. Let $W_{X,G,\text{fake}} = W_{X,G} - W_{X,G,\text{true}}$ denote the terms arising from the "fake facets", that is, weights of X that do not define facets of $X/\!\!/G$ so that $W_{X,G} = W_{X,G,\text{true}} + W_{X,G,\text{fake}}$. Let $c_1^G(X)_{\text{true}} \in \mathfrak{g}_{\mathbb{Q}}^\vee \cong H_G^2(X)$ be the sum of the weights corresponding to divisors of $X/\!\!/G$, that is, the true facets. Since the critical locus is non-degenerate, the formal criterion for smoothness implies that there is an an isomorphism $\text{Crit}_+(W_{X,G,\text{true}}) \to \text{Crit}_+(W_{X,G})$, that is, adding in the higher order terms give a deformation of the critical locus lying over the interior of the moment polytope. □

Theorem 4.14. (Equality of Dimensions) *For $X/\!\!/G$ as in the statement of Theorem 1.2,* $\dim \widehat{\text{Jac}}(W_{X,G}) = \dim QH(X/\!\!/G)$.

Proof. By Lemma 4.7 as t varies the toric orbifold $X/\!\!/_t G$ undergoes a finite sequence of weighted blow-ups and blow-downs, the last of which is a fibration to a toric stack of smaller dimension with Fano fibers. The wall-crossing terms are the same, by Lemma 4.8. In the case of a fibration, the equality follows from Lemma 4.12. □

Proof of Theorems 1.2 and 1.1. By [46], the linearization $D_\alpha \kappa_X^G$ of the quantum Kirwan map descends to a map

$$T_\alpha \kappa_X^G / QSR_{X,G} \colon QH_G(X)/QSR_{X,G} \longrightarrow QH(X/\!\!/G)$$

defined in a formal neighborhood of the symplectic class $\omega \in H_2^G(X, \mathbb{Q})$. By Theorem 2.4, $T_\alpha \kappa_X^G / QSR_{X,G}$ is surjective. By Theorem 4.14, the induced map from $\widehat{\text{Jac}}(W_{X,G})$ to $T_{\kappa_G(\alpha)} QH(X/\!\!/G)$ is an isomorphism in a formal neighborhood of $\omega \in QH_G(X)$. Theorem 1.1 follows from the identification with the Jacobian ring in Proposition 3.7. □

Corollary 4.15. *The quantum cohomology of any proper toric orbifold with projective coarse moduli space is semisimple for generic symplectic classes $\omega \in H_G^2(X, \mathbb{Q})$.*

Proof. As explained in [30, Proposition 4.9], this follows from the identification with the Batyrev ring, or rather, the Jacobian ring of the superpotential and the fact proved in [30, Proposition 3.10] that for generic values, the superpotential has only non-degenerate critical values, part (b) of Lemma 3.11. □

For an example of a non-generic symplectic structure that gives non semi-simple quantum cohomology see Ostrover-Tyomkin [40].

Remark 4.16. (Equivariant first Chern class maps to the potential) Under the isomorphism $QH_G(X)/QSR_{X,G} \to \text{Jac}(W_{X,G})$, the first Chern class $c_1^G(X)$ maps to the potential $W_{X,G}$ itself, by definition of the isomorphism. However, $c_1^G(X)$ does not map to $c_1(X/\!\!/G) \in H^2(X/\!\!/G) \subset QH(X/\!\!/G)$ in general, as one can see from the example of $G = \mathbb{C}^\times$ acting on $X = \mathbb{C}$ with weight two, where $X/\!\!/G = \mathbb{C}/\!\!/\mathbb{C}^\times = B\mathbb{Z}_2$ and $c_1^G(X)$ maps to the twisted sector in $X/\!\!/G$.

We now give a family version of Theorem 1.2, which has the same proof as Theorem 1.2 but with slightly more complicated notation. Incorporating the dependence on $\omega \in \mathfrak{g}_{\mathbb{R}}^{\vee}$ we have a family of potentials denoted $W^{\alpha}_{X,G}$ given by

$$(8)\quad W^{\alpha}_{X,G} \colon T^{\vee}(\Lambda_0) \longrightarrow \Lambda,$$
$$q^{\zeta} \exp\left(\sum_{d \geq 0} q^d \xi_d\right) \longmapsto \sum_{j=1}^{k} q^{-\alpha_{2,j} + \langle \zeta, \nu_j \rangle} \exp\left(\sum_{d \geq 0} q^d \langle \xi_d, \nu_j \rangle\right)$$

where $\alpha_{2,j}$ are the $\mathbb{R}^{\dim(X)}$-components of the projection α_2 of $\omega \in H_G(X)$ onto $H^2_G(X)$. The following has the same proof as Theorem 1.2:

Theorem 4.17. *For any rational symplectic class $\omega \in H_2^G(X, \mathbb{Q})$ there is a canonical isomorphism $T_\alpha QH(X/\!\!/G)) \to \widehat{\mathrm{Jac}}(W^{\alpha}_{X,G})$ for α in a formal neighborhood of $\kappa^G_X(\omega)$ in $QH(X/\!\!/G)$.*

The same approach should work for the product structure at any point in the Frobenius manifold for $X/\!\!/G$, not just a formal neighborhood, but so far the quantum Kirwan map is only formal and we lack the necessary convergence results to use it away from the origin. We remark that Iritani [31] has proved several results about convergence of Gromov-Witten invariants on toric varieties. Finally we give the equivariant version.

Corollary 4.18. (Equivariant version of the Batyrev presentation) *There is a canonical isomorphism $T_\alpha \widehat{QH}_{\tilde{G}}(X, \mathbb{Q}) / \widehat{QSR}_X^{G,\tilde{G}}(\alpha) \to T_{\kappa_X^{\tilde{G},G}(\alpha)} QH_{\tilde{G}/G}(X/\!\!/G, \mathbb{Q})$ for α in a formal neighborhood of any rational symplectic class $\omega \in H^2_{\tilde{G}}(X)$.*

Proof. We have already shown in Theorem 1.1 the non-equivariant version of the statement in Corollary 4.18, that is, setting the equivariant parameters for $T = \tilde{G}/G$ to zero. By equivariant formality, $QH_{\tilde{G}/G}(X/\!\!/G, \mathbb{Q})$ is a free $QH_{\tilde{G}/G}(\mathrm{pt}, \mathbb{Q})$-module, and this implies that $\widehat{QH}_{\tilde{G}/G}(X/\!\!/G, \mathbb{Q})$ is a free $\widehat{QH}_{\tilde{G}/G}(\mathrm{pt}, \mathbb{Q})$ module. Since the same is true for the left-hand-side, it follows that the linearization of the equivariant quantum Kirwan map map $T_\omega \widehat{QH}_{\tilde{G}}(X) / \widehat{QSR}_X^{G,\tilde{G}} \to T_{\kappa_X^{\tilde{G},G}(\omega)} QH_{\tilde{G}/G}(X/\!\!/G)$ is also an isomorphism. □

5. Splitting quantum cohomology and non-displaceable Lagrangians

This section is a discussion of how the results here combine with those of [47], [45] on non-displaceable Lagrangian tori. In particular we explain that toric orbifolds can have infinitely many non-displaceable tori because they can have infinitely many tmmps.

Definition 5.1. (Toric minimal model program) Let Y be a smooth proper polarized toric Deligne-Mumford stack with projective coarse moduli space, and G

a torus acting on a vector space X so that $X/\!\!/G$, equipped with its residual torus action, is isomorphic to Y. Suppose that $(X/\!\!/G)_t, t \in [0,\infty)$ denotes the toric varieties in the toric minimal model program obtained by moving the equivariant symplectic class in the direction of $-c_1^G(X)$, with singular values $t_1,\ldots,t_n$. We call the sequence of toric stacks $X/\!\!/_t G$ so obtained (finite except for the variation of symplectic class) a *toric minimal model program* for Y. Let $t_{j,\pm} = t_j \pm \epsilon$ for ϵ sufficiently small so that $t_{j-1} + \epsilon < t_j - \epsilon, j = 2,\ldots,n$. Let

$$d_j = \dim QH(X/\!\!/_{t_{j,-}} G) - \dim QH(X/\!\!/_{t_{j,+}} G)$$

denote the *dimension jump* at t_j. For simplicity, we assume that $(X/\!\!/G)_{t_j}$ has a single singular point, mapping to the *singular moment value*

$$\psi_j \in \partial \Delta_{X/\!\!/_{t_j}G} \subset \Delta_{X/\!\!/_0 G}, \quad j = 1,\ldots,n.$$

Suppose furthermore that for t just before t_n, the quotient $X/\!\!/_t G$ is a fibration over $Y'' = (X/\!\!/_{t_n} G)''$, with monotone fiber $Y' = (X/\!\!/_{t_n} G)'$.

Remark 5.2. (a) (Non-uniqueness of tmmps) Many presentations of $X/\!\!/G$ as a git quotient will give the same toric mmp. However, toric orbifolds can have infinitely many toric mmps, corresponding to different realizations of $X/\!\!/G$ as git quotients. This is because the "fake facets" can "catch up" to the "true facets" under the deformation ω_t. For example, taking the minimal presentation of $\mathbb{P}(1,3,5)$ as a git quotient $\mathbb{C}^3/\!\!/\mathbb{C}^\times$ yields a trivial toric mmp, but introducing a presentation as a quotient $\mathbb{C}^4/\!\!/(C^\times)^2$ yields a toric mmp with a flip to an "orbifold Hirzebruch surface", which is similar to [45]. See Figure 8. The computation Abreu-Borman-McDuff [2, Proposition 4.1.4] show that this never happens in the manifold case.

(b) (Induced tmmp for the fibration at the end of the tmmp) Any presentation of Y as a git quotient $X/\!\!/G$ induces a presentation of the base $(X/\!\!/_{t_n} G)''$ of the final fibration as a git quotient, corresponding to the inequalities that are not defining inequalities for the monotone fiber $(X/\!\!/_{t_n} G)''$, that is, the inequalities which become strict equalities for the final polytope $\Delta_{X/\!\!/_{t_n}G}$. Hence, any presentation of Y induces a tmmp for the base of the final fibration.

Theorem 5.3. *Let Y be a compact toric orbifold with generic symplectic class and $\mathcal{P}$ be a toric mmp for Y with dimension jumps d_j and singular moment values ψ_j, $j = 1,\ldots,n$.*

(a) *There is a canonical decomposition of the quantum cohomology*

$$QH(Y) \cong \bigoplus_{j=1}^{n} QH(Y)_{\mathcal{P},j}, \quad \dim QH(Y)_{\mathcal{P},j} = d_j;$$

(b) *for each $j = 1,\ldots,n-1$, the inverse image L_j of ψ_j in Y is Hamiltonian non-displaceable. The number of local systems making the Lagrangian*

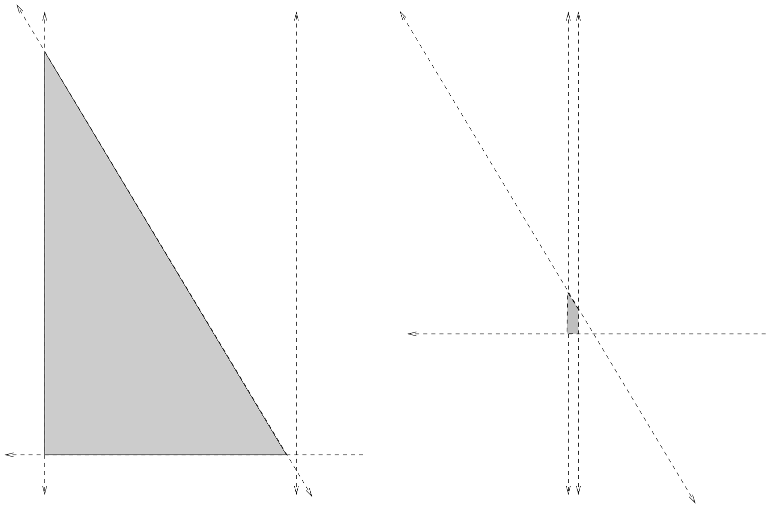

FIGURE 8. Non-trivial toric mmp for $\mathbb{P}(1,3,5)$

Floer homology of L_j non-vanishing, counted with multiplicity, is equal to d_j.

(c) *for $j = n$, the factor $QH(Y)_{\mathcal{P},n}$ further splits*

$$QH(Y)_{\mathcal{P},n} = \bigoplus_{j=1}^{n_1} QH(Y)_{\mathcal{P},n,j}, \quad \dim QH(Y)_{\mathcal{P},n,j} = \dim(QH(Y'))d_{j,1}$$

according to a splitting induced from a tmmp $\mathcal{P}_1$ for the base $Y'' = (X /\!\!/_{t_n} G)''$ with dimension jumps $d_{j,1}$ as in Remark 5.2 (b), and for each singular value $\psi_{j,1}$ in such a tmmp the inverse image in Y is Hamiltonian non-displaceable, etc.

Proof. (a) By the main result Theorem 1.2 $QH(Y) \cong \widehat{\mathrm{Jac}}(W)$, and the latter admits a decomposition into components corresponding to critical points with fixed value of the tropical moment map from Definition 3.17. By Lemma 4.8, each summand has dimension that of the dimension jump in the given tmmp. (b) is a result of [47], with the multiplicity computed using Kouchnirenko's theorem. (c) is a consequence of Lemma 4.12. □

Remark 5.4. (a) The results above are for generic initial symplectic class only. It would be interesting to obtain similar results for non-generic initial classes.

(b) The existence of an open subset of non-displaceable toric fibers in [45] can also be interpreted in terms of the tmmps. The example of $\mathbb{P}(1,3,5)$ shows that tmmps can come in multi-parameter families, with as many parameters as the dimension of the toric orbifolds, causing open subsets of non-displaceable toric moment fibers.

(c) It would be interesting to know whether similar results can be obtained for non-toric minimal model programs. In particular, does the existence of infinitely many minimal model programs cause infinitely many non-displaceable non-Hamiltonian-isotopic Lagrangian tori in non-toric cases as well?

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Eduardo Gonzalez
Department of Mathematics
University of Massachusetts
Boston 100 William T. Morrissey Boulevard
Boston, MA 02125
E-mail address: eduardo@math.umb.edu

Chris Woodward
Mathematics-Hill Center
Rutgers University
110 Frelinghuysen Road, Piscataway
NJ 08854-8019, U.S.A.
E-mail address: ctw@math.rutgers.edu