THE HARDY-LITTLEWOOD-STEIN INEQUALITY FOR GENERALIZED HARDY AND BELLMAN TYPE AVERAGES

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ABSTRACT. In this paper we get estimates for Hardy-Littlwood-Stein inequality with the averages of Hardy and Bellman type, a kind of generalized mean of Fourier coefficients of functions from Lorentz space.

1. Introduction. Let $0 < p < \infty$ and $0 < q \leq \infty$. A measurable 1-periodic function f on $[0,1]$ is said to belong to the Lorentz space $L_{p,q}[0,1]$ if following values are finite:

$$\|f\|_{L_{p,q}[0,1]} = \left(\int_0^1 \left(t^{\frac{1}{p}} f^*(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \quad \text{for } 0 < q < \infty,$$

and in the case $q = \infty$,

$$\|f\|_{L_{p,\infty}[0,1]} = \sup_{0 \leq t \leq 1} t^{\frac{1}{p}} f^*(t).$$

Here $f^*(t) = \inf\{\sigma : \mu\{x : |f(x)| > \sigma\} < t\}$ is the nonincreasing rearrangement of function $f(t)$.

We denote by C constants which depend only of the parameters p, q, p_i, q_i, where $i = 0, 1$, and they may be different everywhere.

In this paper we consider the following problem: Let the function from $L[0;1]$ $f \sim \sum_{k=1}^{\infty} a_k e^{2\pi i k x}$ be the Fourier series by the trigonometric system. If $f \in L_{pq}[0;1]$, then one asks what properties do its Fourier coefficients have?

The Hardy-Littlewood inequality is well known. If $1 < p < 2$ and $f \in L_p[0;1]$ then we have

$$\left(\sum_{k=1}^{\infty} k^{p-2} |a_k|^p \right)^{\frac{1}{p}} \leq C \|f\|_{L_p[0,1]}. \tag{1}$$

This inequality shows the necessary condition for a function to belong to the space $L_p[0;1]$.

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The inequality of Hardy-Littlewood-Stein is also well-known and it is true for the Lorentz space. If $1 < p < 2$, $0 < q \leq \infty$, $p' = \frac{p}{p-1}$ and $f \in L_{p,q}[0,1]$, then we have

$$\left(\sum_{k=1}^{\infty} k^{\frac{q}{p'}-1} (a_k^*)^q \right)^{\frac{1}{q}} \leq C \|f\|_{L_{p,q}[0,1]}. \tag{2}$$

Note that for $2 < p < \infty$ the inequalities (1) and (2) do not hold. The following result was obtained in [1].

Theorem. *Let $2 \leq p < \infty$, $0 < q \leq \infty$, $p' = \frac{p}{p-1}$. If $f \in L_{p,q}[0,1]$ then we have*

$$\left(\sum_{k=1}^{\infty} k^{\frac{q}{p'}-1} \left(\bar{a}_k\right)^q \right)^{\frac{1}{q}} \leq C \|f\|_{L_{p,q}[0,1]} \text{ for } 0 < q < \infty. \tag{3}$$

In the case $q = \infty$, we have

$$\sup_{k \in N} k^{\frac{1}{p'}} \bar{a}_k \leq C \|f\|_{L_{p,\infty}[0,1]}, \tag{4}$$

where $\bar{a}_k = \frac{1}{k} \left| \sum\limits_{m=1}^{k} a_m \right|$.

Here, instead of a_k^* one takes the arithmetic mean, the so-called Hardy mean. Note that for $1 < p \leq 2$ the inequalities (3) and (4) also hold.

We will prove the theorem which shows the inequality with the average of Bellman and Hardy type, a kind of a generalized mean. For example, if the $\bar{a}_k$ is taking averages of the form $\tilde{a}_k = \left| \sum\limits_{k=m}^{\infty} \frac{a_k}{k} \right|$, the inequalities of type (3) and (4) hold for $1 < p < \infty$. Some results were announced in [2]. Also note that in [3] was proved inequalities for Fourier transforms.

2. $\mathbf{n_{p,q,\alpha}}$ and $\mathbf{n_{p,q}}(\lambda)$ spaces. We give auxiliary definitions and lemmas to prove the main theorems.

Definition 1. *Let $0 < p < \infty,\ 0 < q \leq \infty,\ \ \alpha > \frac{1}{p}$, define the space*

$$n_{p,q,\alpha} = \left\{ a = \{a_n\}_{n=1}^{\infty} : \ \|a\|_{n_{p,q,\alpha}} = \left(\sum_{k=1}^{\infty} \left(k^{\frac{1}{p}} \widetilde{a}_k(\alpha) \right)^q \frac{1}{k} \right)^{\frac{1}{q}} < \infty \right\}$$

and for $q = \infty$,

$$\|a\|_{n_{p,\infty,\alpha}} = \sup_{k \in N} k^{\frac{1}{p}} \widetilde{a}_k(\alpha) < \infty,$$

where $\widetilde{a}_k(\alpha) = \sup\limits_{m \geq k} \frac{1}{m^{1-\alpha}} \left| \sum\limits_{s=m}^{\infty} \frac{a_s}{s^{\alpha}} \right|$, *forall* $k \in N$.

Lemma 1. *If $0 < p < \infty$ and $0 < q \leq q_1 \leq \infty$ then $n_{p,q,\alpha} \hookrightarrow n_{p,q_1,\alpha}$.*

Proof. Let $q_1 = \infty$, then from the monotonicity of $\widetilde{a}_k(\alpha)$, we obtain

$$\|a\|_{n_{p,\infty,\alpha}} = \sup_{k\in N} k^{\frac{1}{p}}\widetilde{a}_k(\alpha) = \sup_{k\in N} \left(k^{\frac{q}{p}}\widetilde{a}_k^q(\alpha)\right)^{\frac{1}{q}} \sim \sup_{k\in N} \left(\sum_{m=1}^{k} m^{\frac{q}{p}-1}\widetilde{a}_k^q(\alpha)\right)^{\frac{1}{q}} \leq$$

$$\leq C \sup_{k\in N} \left(\sum_{m=1}^{k} m^{\frac{q}{p}-1}\widetilde{a}_m^q(\alpha)\right)^{\frac{1}{q}} \leq C\|a\|_{n_{p,q,\alpha}}.$$

Now suppose that $0 < q \leq q_1 < \infty$. Then

$$\|a\|_{n_{p,q_1,\alpha}} = \left(\sum_{k=1}^{\infty} \left(k^{\frac{1}{p}+\varepsilon-\varepsilon}\widetilde{a}_k(\alpha)\right)^{q_1} \frac{1}{k}\right)^{\frac{1}{q_1}}.$$

Substituting

$$k^{\frac{1}{p}+\varepsilon} = c\left(\sum_{m=1}^{k} m^{(\frac{1}{p}+\varepsilon)q}\frac{1}{m}\right)^{\frac{1}{q}},$$

and from $\widetilde{a}_k(\alpha) \leq \widetilde{a}_m(\alpha)$, with $m \leq k$, we obtain

$$\|a\|_{n_{p,q_1,\alpha}} \leq c\left(\sum_{k=1}^{\infty} k^{-\varepsilon q_1}\left(\sum_{m=1}^{k} \widetilde{a}_m^q m^{(\frac{1}{p}+\varepsilon)q}\frac{1}{m}\right)^{\frac{q_1}{q}} \frac{1}{k}\right)^{\frac{1}{q_1}}.$$

Using the generalized Minkowski inequality we obtain

$$\|a\|_{n_{p,q_1,\alpha}} \leq C\left(\sum_{m=1}^{\infty} m^{\frac{q}{p}}\widetilde{a}_m^q \frac{1}{m} m^{\varepsilon q}\left(\sum_{k=m}^{\infty} k^{-\varepsilon q_1}\frac{1}{k}\right)^{\frac{q}{q_1}}\right)^{\frac{1}{q}} = C\|a\|_{n_{p,q,\alpha}}.$$

The lemma is proved. □

Lemma 2. *Let $0 < p < \infty$ and $0 < q \leq \infty$, then*

$$\|a\|_{n_{p,q,\alpha}} \sim \left(\sum_{k=0}^{\infty} \left(2^{\frac{k}{p}}\widetilde{a}_{2^k}(\alpha)\right)^q\right)^{\frac{1}{q}}.$$

Proof. From the monotonicity of $\widetilde{a}_k(\alpha)$ we have

$$(\ln 2)^{\frac{1}{q}} 2^{-\frac{1}{p}} 2^{\frac{k+1}{p}} \widetilde{a}_{2^{k+1}}(\alpha) \leq \left(\sum_{m=2^k}^{2^{k+1}-1} \left(m^{\frac{1}{p}}\widetilde{a}_m(\alpha)\right)^q \frac{1}{m}\right)^{\frac{1}{q}} \leq (\ln 2)^{\frac{1}{q}} 2^{\frac{1}{p}} 2^{\frac{k}{p}} \widetilde{a}_{2^k}(\alpha).$$

Taking the q-th power of this inequality, and summing over k we obtain the desired result.

Let (A_0, A_1) be a compatible pair of Banach spaces [4]. Let

$$K(t, a; A_0, A_1) = \inf_{a=a_0+a_1} \left(\|a_0\|_{A_0} + t\|a_1\|_{A_1}\right),\ a \in A_0 + A_1,$$

be the Peter functional, $0 < t < \infty$.

When $0 < q < \infty$, $0 < \theta < 1$, we have

$$(A_0, A_1)_{\theta,q} = \left\{ a \in A_0 + A_1 : \;\; \|a\|_{(A_0,A_1)_{\theta,q}} = \left(\int_0^\infty (t^{-\theta} K(t,a))^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\},$$

and if $q = \infty$, we have

$$(A_0, A_1)_{\theta,\infty} = \left\{ a \in A_0 + A_1 : \;\; \|a\|_{(A_0,A_1)_{\theta,\infty}} = \sup_{0<t<\infty} t^{-\theta} K(t,a)) < \infty \right\}.$$

The lemma is proved. □

Lemma 3. *If* $1 < p_0 < p_1 < \infty$ *and* $0 < q \leq \infty, \;\; 0 < q, q_0, q_1 \leq \infty, \; 0 < \theta < 1$ *, then*

$$(n_{p_0,q_0,\alpha}, \;\; n_{p_1,q_1,\alpha})_{\theta,q} \hookrightarrow n_{p,q,\alpha},$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Proof. By the embedding $n_{p_i,q_i,\alpha} \hookrightarrow n_{p_i,\infty,\alpha}, \;\; i = 0, 1$, we have

$$\|a_i\|_{n_{p_i,\infty,\alpha}} \leq C_i \|a_i\|_{n_{p_i,q_i,\alpha}}, \;\; i = 0, 1.$$

Then for $0 < t < \infty$, we have

$$K(t, a; n_{p_0,\infty,\alpha}, n_{p_1,\infty,\alpha}) = \inf_{a=a_0+a_1} \left(\|a_0\|_{n_{p_0,\infty,\alpha}} + t\|a_1\|_{n_{p_1,\infty,\alpha}} \right) \leq$$

$$\leq C \inf_{a=a_0+a_1} \left(\|a_0\|_{n_{p_0,q_0,\alpha}} + t\|a_1\|_{n_{p_1,q_1,\alpha}} \right) =$$

$$= K\left(t, a; n_{p_0,q_0,\alpha}, n_{p_1,q_1,\alpha}\right), \quad a \in n_{p_0,q_0,\alpha} + n_{p_1,q_1,\alpha}.$$

Hence

$$\|a\|_{\left(n_{p_0,\infty,\alpha}, \;\; n_{p_1,\infty,\alpha}\right)_{\theta,q}} = \left(\int_0^\infty \left(t^{-\theta} K(t, a; n_{p_0,\infty,\alpha}, n_{p_1,\infty,\alpha} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \leq$$

$$\leq C \left(\int_0^\infty \left(t^{-\theta} K(t, a; n_{p_0,q_0,\alpha}, n_{p_1,q_1,\alpha} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} =$$

$$= C\|a\|_{\left(n_{p_0,q_0,\alpha}, \;\; n_{p_1,q_1,\alpha}\right)_{\theta,q}}.$$

Therefore, we obtain

$$(n_{p_0,q_0,\alpha}, \;\; n_{p_1,q_1,\alpha})_{\theta,q} \hookrightarrow (n_{p_0,\infty,\alpha}, \;\; n_{p_1,\infty,\alpha})_{\theta,q}.$$

Therefore, it suffices to prove that

$$(n_{p_0,\infty,\alpha}, \;\; n_{p_1,\infty,\alpha})_{\theta,q} \hookrightarrow n_{p,q,\alpha}.$$

Let $k \in N$. Let $a_k = a_k^0 + a_k^1$ be a representation of a_k the sequence $a = \{a_k\}_{k\in N}$ from $(n_{p_0,\infty,\alpha}, \;\; n_{p_1,\infty,\alpha})_{\theta,q}$, where $a_0 = \{a_k^0\}_{k\in N} \in n_{p_0,\infty,\alpha}$ and $\{a_1\} = \{a_k^1\}_{k\in N} \in n_{p_1,\infty,\alpha}$. As

$$\widetilde{a}_k(\alpha) = \sup_{m\geq k} \frac{1}{m^{1-\alpha}} \left| \sum_{s=m}^{\infty} \frac{a_s}{s^\alpha} \right| \leq$$

$$\leq \sup_{m\geq k} \frac{1}{m^{1-\alpha}} \left| \sum_{s=m}^{\infty} \frac{a_s^0}{s^\alpha} \right| + \sup_{m\geq k} \frac{1}{m^{1-\alpha}} \left| \sum_{s=m}^{\infty} \frac{a_s^1}{s^\alpha} \right| = \widetilde{a}_k^0(\alpha) + \widetilde{a}_k^1(\alpha),$$

we denoting $v(t) = t^{\frac{p_0 p_1}{p_1 - p_0}}$, we obtain

$$\sup_{v(t)\geq s\geq 1} s^{\frac{1}{p_0}} \widetilde{a}_s \leq \sup_{s\geq 1} s^{\frac{1}{p_0}} \widetilde{a}_s^0 + \sup_{v(t)\geq s\geq 1} s^{\frac{1}{p_0} - \frac{1}{p_1} + \frac{1}{p_1}} \widetilde{a}_s^1 \leq$$

$$\leq \sup_{s\in N} s^{\frac{1}{p_0}} \widetilde{a}_s^0 + t \sup_{s\in N} s^{\frac{1}{p_1}} \widetilde{a}_s^1 = \|a_0\|_{n_{p_0,\infty,\alpha}} + t\|a_1\|_{n_{p_1,\infty,\alpha}}.$$

Given the arbitrariness of the representation of $a = a_0 + a_1$, we have

$$K\left(t, a; n_{p_0,\infty,\alpha}, n_{p_1,\infty,\alpha}\right) \geq \sup_{v(t)\geq s\geq 1} s^{\frac{1}{p_0}} \widetilde{a}_s(\alpha).$$

Therefore, when $0 < q \leq \infty$, we have

$$\|a\|_{\left(n_{p_0,\infty,\alpha},\ n_{p_1,\infty,\alpha}\right)_{\theta,q}} = \left(\int_0^\infty \left(t^{-\theta} K\left(t, a; n_{p_0,\infty,\alpha}, n_{p_1,\infty,\alpha}\right)\right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \geq$$

$$\geq \left(\int_0^\infty \left(t^{-\theta} \sup_{v(t)\geq s\geq 1} s^{\frac{1}{p_0}} \widetilde{a}_s(\alpha) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}.$$

We make the substitution $t = u^{\frac{p_1 - p_0}{p_0 p_1}}$ and since $p_0 < p_1$, we obtain

$$\left(\int_0^\infty \left(t^{-\theta} \sup_{v(t)\geq s\geq 1} s^{\frac{1}{p_0}} \widetilde{a}_s(\alpha) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} = \left(\int_0^\infty \left(u^{-\theta(\frac{1}{p_0} - \frac{1}{p_1})} \sup_{u\geq s\geq 1} s^{\frac{1}{p_0}} \widetilde{a}_s(\alpha) \right)^q \frac{du}{u} \right)^{\frac{1}{q}} =$$

$$= \left(\sum_{r=1}^\infty \int_{2^{r-1}-1}^{2^r-1} \left(u^{-\theta(\frac{1}{p_0} - \frac{1}{p_1})} \sup_{u\geq s\geq 1} s^{\frac{1}{p_0}} \widetilde{a}_s(\alpha) \right)^q \frac{du}{u} \right)^{\frac{1}{q}} \geq$$

$$\geq C \left(\sum_{r=1}^\infty \left(2^{-\theta r(\frac{1}{p_0} - \frac{1}{p_1})} \sup_{2^r\geq s\geq 1} s^{\frac{1}{p_0}} \widetilde{a}_s(\alpha) \right)^q \right)^{\frac{1}{q}} \geq$$

$$\geq C \left(\sum_{r=1}^\infty \left(2^{-\theta r(\frac{1}{p_0} - \frac{1}{p_1})} 2^{\frac{r}{p_0}} \widetilde{a}_{2^r}(\alpha) \right)^q \right)^{\frac{1}{q}} = \|a\|_{n_{p,q,\alpha}}.$$

So, we get

$$\|a\|_{n_{p,q,\alpha}} \leq C\|a\|_{\left(n_{p_0,\infty,\alpha},\ n_{p_1,\infty,\alpha}\right)_{\theta,q}}.$$

The lemma is proved. □

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Definition 2. *Let* $1 < p < \infty,\ 0 < q \le \infty$. *For sequence* $\lambda = \{\lambda_k\}_{k=1}^{\infty}$ *such that forall* $m \in N$ $\sum_{k=1}^{m} \lambda_k \ne 0$ *define the space*

$$n_{p,q}(\lambda) = \left\{ a = \{a_n\}_{n=1}^{\infty}: \ \|a\|_{n_{p,q}(\lambda)} = \left(\sum_{k=1}^{\infty} \left(k^{\frac{1}{p}} \overline{a}_k(\lambda) \right)^q \frac{1}{k} \right)^{\frac{1}{q}} < \infty \right\}$$

and for $q = \infty$,

$$\|a\|_{n_{p,\infty}(\lambda)} = \sup_{k \in N} k^{\frac{1}{p}} \overline{a}_k(\lambda) < \infty,$$

where

$$\overline{a}_k(\lambda) = \sup_{m \ge k} \frac{1}{\left| \sum_{s=1}^{m} \lambda_s \right|} \left| \sum_{s=1}^{m} \lambda_s a_s \right|.$$

Similarly, we prove the following lemmas.

Lemma 4. *If* $q_1 > q$, *then* $n_{p,q}(\lambda) \hookrightarrow n_{p,q_1}(\lambda)$.

Lemma 5. *Let* $0 < p < \infty$ *and* $0 < q \le \infty$, *then*

$$\|a\|_{n_{p,q}(\lambda)} \sim \left(\sum_{k=1}^{\infty} \left(2^{\frac{k}{p}} \overline{a}_{2^k}(\lambda) \right)^q \right)^{\frac{1}{q}}.$$

Lemma 6. *If* $1 < p_0 < p_1 < \infty$ *and* $0 < q \le \infty$, $0 < q, q_0, q_1 \le \infty$, $0 < \theta < 1$, *we have*

$$(n_{p_0,q_0}(\lambda),\ \ n_{p_1,q_1}(\lambda))_{\theta q} \hookrightarrow n_{p,q}(\lambda),$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

3. On the summability of Fourier coefficients with the generalized average of Hardy type.

Lemma 7. *Let* $1 < p < \infty,\ \ p' = \frac{p}{p-1}$. *If* $f \in L_1[0,1]$ *and* $f \sim \sum_{k=1}^{\infty} a_k e^{2\pi i k x}$, $x \in [0,1]$, *then*

$$\|f\|_{L_{p',\infty}} \le c \sup_{m \in \mathbb{N}} m^{\frac{1}{p}} |m \Delta a_m|. \tag{5}$$

In this paper inequalities are understand as that if the right-hand side makes sense then the left-hand side also makes sense.

Proof. By the duality relation we have

$$\|f\|_{L_{p',\infty}} = \sup_{\|g\|_{p,1}=1} \int_0^1 \sum_{k=1}^{\infty} a_k e^{2\pi i k x} g(x) dx = \sup_{\|g\|_{p,1}=1} \sum_{k=1}^{\infty} a_k \widehat{g(k)}.$$

Consider the sum $\sum\limits_{k=1}^{\infty} a_k \widehat{g(k)}$. Given that the Fourier coefficients $a_k \to 0$, we have $a_k = \sum\limits_{m=k}^{\infty} \Delta a_m$. Then

$$\sum_{k=1}^{\infty} a_k \widehat{g(k)} = \sum_{k=1}^{\infty} \left(\sum_{m=k}^{\infty} \Delta a_m \right) \widehat{g(k)} = \sum_{m=1}^{\infty} \Delta a_m \sum_{k=1}^{m} \widehat{g(k)} \leq$$

$$\leq \sum_{m=1}^{\infty} |m \Delta a_m| \left| \frac{1}{m} \sum_{k=1}^{m} \widehat{g(k)} \right| \leq \sup_{m \in \mathbb{N}} m^{\frac{1}{p}} |m \Delta a_m| \sum_{m=1}^{\infty} m^{\frac{1}{p'}-1} \left| \frac{1}{m} \sum_{k=1}^{m} \widehat{g(k)} \right|.$$

Using (3) for $q = 1$, we obtain

$$\|f\|_{L_{p',\infty}} \leq C \sup_{\|g\|_{p,1}=1} \left\{ \sup_{m \in \mathbb{N}} m^{\frac{1}{p}} |m \Delta a_m| \|g\|_{L_{p,1}} \right\} = C \sup_{m \in \mathbb{N}} m^{\frac{1}{p}} |m \Delta a_m|.$$

The lemma is proved. □

Theorem 1. *Let* $1 < p < \infty$, $p' = \frac{p}{p-1}$, $0 < q \leq \infty$ *and* $f \in L_{p,q}$, $f \sim \sum\limits_{k=1}^{\infty} a_k e^{2\pi i k x}$.

Let the sequence $\lambda = \{\lambda_k\}$ *satisfy for* $\alpha > \frac{1}{p'}$

$$\sup_{1 \leq m \leq k} m^{2-\alpha} |\lambda_m - \lambda_{m+1}| \leq D \frac{1}{k^{\alpha}} \left| \sum_{m=1}^{k} \lambda_m \right|,$$

where $D > 0$ *is a constant independents of the index* k.
Then we have inequality

$$\left(\sum_{k=1}^{\infty} \left(k^{\frac{1}{p'}} \overline{a}_k(\lambda) \right)^q \frac{1}{k} \right)^{\frac{1}{q}} \leq C \|f\|_{L_{p,q}[0,1]},$$

where $\overline{a}_k(\lambda) = \frac{1}{\left| \sum\limits_{m=1}^{k} \lambda_m \right|} \left| \sum\limits_{m=1}^{k} \lambda_m a_m \right|$, $k \in N$. *For* $q = \infty$, *we have*

$$\sup_{k \in N} k^{\frac{1}{p'}} \overline{a}_k(\lambda) \leq c \|f\|_{L_{p,\infty}}$$

Proof. We estimate

$$\frac{n^{\frac{1}{p'}}}{\left| \sum\limits_{k=1}^{n} \lambda_k \right|} \left| \sum_{m=1}^{n} \lambda_m a_m \right| = \frac{n^{\frac{1}{p'}}}{\left| \sum\limits_{m=1}^{n} \lambda_m \right|} \left| \sum_{m=1}^{n} \lambda_m \int_0^1 f(x) e^{-2\pi i m x} dx \right| \leq$$

$$\leq \int_0^1 |f(x)| \left| \frac{n^{\frac{1}{p'}}}{\left| \sum\limits_{m=1}^{n} \lambda_m \right|} \sum_{m=1}^{n} \lambda_m e^{-2\pi i m x} \right| dx.$$

We set $\frac{n^{\frac{1}{p'}}}{\left|\sum\limits_{r=1}^{n}\lambda_r\right|}\sum\limits_{m=1}^{n}\lambda_m e^{-2\pi imx}=\Phi_n(x)$ and by the Hölder inequality with $\frac{1}{p}+\frac{1}{p'}=1$, we have

$$\|a(\lambda)\|_{l_{p',\infty}}\le\sup_{n\in\mathbb{N}}\int\limits_0^1|f(x)|\,|\Phi_n(x)|\,dx\le\sup_{n\in\mathbb{N}}\|f\|_{L_{p,1}}\|\Phi_n\|_{L_{p',\infty}}.$$

We estimate the norm $\|\Phi_n\|_{L_{p',\infty}}$. The Fourier coefficients of this function are equal to $b_m(\Phi_n)=\frac{n^{\frac{1}{p'}}\lambda_m}{\left|\sum\limits_{r=1}^{n}\lambda_r\right|}$, if $m\le n$; and $b_m(\Phi_n)=0$, if $m>n$.

By (5), we get

$$\|\Phi_n\|_{L_{p',\infty}}\le c\sup_{m\in\mathbb{N}}m^{\frac{1}{p}}|m\Delta b_m|.$$

Consider $I=m^{\frac{1}{p}}|m\Delta b_m|$.

Let $m\le n$, then using the condition of the theorem, we have

$$I=m^{\frac{1}{p}}(m\Delta b_m)=\frac{m^{\frac{1}{p}}mn^{\frac{1}{p'}}|\lambda_m-\lambda_{m+1}|}{\left|\sum\limits_{r=1}^{n}\lambda_r\right|}\le$$

$$\le\frac{n^{\frac{1}{p'}}m^{\alpha-\frac{1}{p'}}}{\left|\sum\limits_{r=1}^{n}\lambda_r\right|}\sup_{1\le k\le n}k^{2-\alpha}|\lambda_k-\lambda_{k+1}|\le$$

$$\le\frac{n^{\frac{1}{p'}}m^{\alpha-\frac{1}{p'}}}{\left|\sum\limits_{r=1}^{n}\lambda_r\right|}\frac{D}{n^{\alpha}}\left|\sum_{m=1}^{n}\lambda_m\right|=D\left(\frac{m}{n}\right)^{\alpha-\frac{1}{p'}}\le D.$$

When $m>n$, we have $I=m^{\frac{1}{p}}|m\Delta b_m|=0$.

Thus, we have

$$\sup_{m\in\mathbb{N}}m^{\frac{1}{p}}|m\Delta b_m|\le D.$$

And consequently, we get

$$\|\Phi_n\|_{p',\infty}\le cD=C.$$

Thus, we have

$$\sup_{n\in\mathbb{N}}\frac{n^{\frac{1}{p'}}}{\left|\sum\limits_{k=1}^{n}\lambda_k\right|}\left|\sum_{m=1}^{n}\lambda_m a_m\right|\le C\|f\|_{L_{p,1}}.$$

Then we obtain

$$\|a\|_{n_{p'\infty}(\lambda)}\le C\|f\|_{L_{p,1}}.$$

Finally, we use Lemma 6 and the interpolation theorem of Marcinkiewicz-Calderon [4]. Since $\alpha>\frac{1}{p'}$ there exist p_0,p_1 such that $1-\alpha<\frac{1}{p_1}<\frac{1}{p}<\frac{1}{p_0}$, i.e. then also satisfy the condition of the theorem. Hence there is $\theta \in (0,1)$ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{p'} = \frac{1-\theta}{p_0'} + \frac{\theta}{p_1'}$. From the above we have

$$\|a\|_{n_{p_0'\infty}(\lambda)} \le C_0 \|f\|_{L_{p_0,1}}$$

and

$$\|a\|_{n_{p_1'\infty}(\lambda)} \le C_1 \|f\|_{L_{p_1,1}}.$$

Then

$$\|a\|_{n_{p'q}(\lambda)} \le C_0^{1-\theta} C_1^{\theta} \|f\|_{L_{p,q}},$$

for any $0 < q \le \infty$. The theorem is proved. □

Corollary 1. *Let $1 < p < \infty$, $p' = \frac{p}{p-1}$, $\beta < \frac{1}{p}$, $0 < q \le \infty$, $f \in L_{p,q}[0,1]$. Then we have*

$$\left(\sum_{k=1}^{\infty}\left(k^{\beta-\frac{1}{p}}\left|\sum_{m=1}^{k}\frac{a_m}{m^{\beta}}\right|\right)^{q}\frac{1}{k}\right)^{\frac{1}{q}} \le c\|f\|_{L_{p,q}[0,1]}.$$

For $q = \infty$, we have

$$\sup_{n\in N} n^{\beta-\frac{1}{p}}\left|\sum_{s=1}^{n}\frac{a_s}{s^{\beta}}\right| \le c\|f\|_{L_{p,\infty}[0,1]}.$$

Proof. Let $\beta < 1/p$, then $1/p' < 1-\beta$ and hence there exists α, such that $1/p' < \alpha < 1-\beta$. We show that $\lambda = \{\frac{1}{m^{\beta}}\}_{m=1}^{\infty}$ satisfies the condition of the theorem 1 with parameter α. Really, we get

$$\sup_{1\le m\le n} m^{2-\alpha}\left(\frac{1}{m^{\beta}} - \frac{1}{(m+1)^{\beta}}\right) \sim \sup_{1\le m\le n} m^{1-\alpha-\beta} = n^{1-\alpha-\beta}.$$

It took into account that $1-\alpha-\beta > 0$. On the other hand

$$\frac{1}{n^{\alpha}}\left|\sum_{m=1}^{n}\frac{1}{m^{\beta}}\right| \sim n^{1-\alpha-\beta}.$$

Considering

$$\left(\sum_{k=1}^{\infty}\left(k^{\beta-\frac{1}{p}}\sum_{m=1}^{k}\frac{a_m}{m^{\beta}}\right)^{q}\frac{1}{k}\right)^{\frac{1}{q}} \sim \left(\sum_{k=1}^{\infty}\left(k^{\frac{1}{p'}}\overline{a}_k(\lambda)\right)^{q}\frac{1}{k}\right)^{\frac{1}{q}},$$

we obtain the required inequality. □

4. On the summability of Fourier coefficients with the average of Bellman type.

Remark 1. *The aim of this section is to obtain a more general inequality for the Bellman transform. In particular, from (3) and (4) we can obtain the following result: Let $1 < p < \infty$, $p' = \frac{p}{p-1}$. If $f \in L_{p,q}[0,1]$ and $f \sim \sum_{k=1}^{\infty} a_k e^{2\pi i k x}$, then we have*

$$\left(\sum_{m=1}^{\infty}\left|\sum_{k=m}^{\infty}\frac{a_k}{k}\right|^{p'}\right)^{\frac{1}{p'}} \le c\|f\|_{L_{p,p'}[0,1]}.$$

Proof. We use the duality theorem (equivalent to the normalization for the space l_p) and changing the order of summation, we obtain

$$\left\| \sum_{k=m}^{\infty} \frac{a_k}{k} \right\|_{l_{p'}} = \sup_{\|b\|_{l_p}=1} \left| \sum_{m=1}^{\infty} b_m \sum_{k=m}^{\infty} \frac{a_k}{k} \right| = \sup_{\|b_m\|_{l_p}=1} \left| \sum_{k=1}^{\infty} \frac{a_k}{k} \sum_{m=1}^{k} b_m \right|. \tag{6}$$

Consider $\sum\limits_{k=1}^{n} \frac{a_k}{k} \sum\limits_{m=1}^{k} b_m$. Applying the Abel transformation:

$$\sum_{k=1}^{n} \frac{a_k}{k} \sum_{m=1}^{k} b_m = \sum_{k=1}^{n-1} \left(\frac{1}{k} \sum_{m=1}^{k} b_m - \frac{1}{k+1} \sum_{m=1}^{k+1} b_m \right) \sum_{r=1}^{k} a_r + \frac{1}{n} \sum_{m=1}^{n} b_m \sum_{r=1}^{n} a_r,$$

We transform it to the following form

$$\left| \sum_{k=1}^{n} \frac{a_k}{k} \sum_{m=1}^{k} b_m \right| \leq \sum_{k=1}^{n-1} \frac{1}{k} \left| \sum_{m=1}^{k} a_m \right| \frac{1}{k+1} \sum_{m=1}^{k+1} |b_m| + \sum_{k=1}^{n-1} \frac{|b_{k+1}|}{k} \left| \sum_{m=1}^{k} a_m \right| +$$
$$+ \frac{1}{n} \left| \sum_{m=1}^{n} b_m \right| \left| \sum_{r=1}^{n} a_r \right| = I_1 + I_2 + I_3.$$

We estimate each term separately. Using Hölder's inequality, if $\frac{1}{p} + \frac{1}{p'} = 1$, and Hardy's [5] inequality and (3) with $q = p'$, we get

$$I_1 \leq \left(\sum_{k=1}^{n} \left| \frac{1}{k} \sum_{m=1}^{k} a_m \right|^{p'} \right)^{\frac{1}{p'}} \left(\sum_{k=1}^{n} \left(\frac{1}{k+1} \sum_{m=1}^{k+1} |b_m| \right)^{p} \right)^{\frac{1}{p}} \leq C \|f\|_{L_{p,p'}} \|b\|_{l_p}.$$

Also we estimate I_2

$$I_2 \leq \left(\sum_{k=1}^{n-1} \left(\frac{1}{k} \left| \sum_{m=1}^{k} a_m \right| \right)^{p'} \right)^{\frac{1}{p'}} \left(\sum_{k=0}^{n} |b_k|^p \right)^{\frac{1}{p}} \leq C \|b\|_{l_p} \|f\|_{L_{p,p'}}.$$

We estimate now I_3. We use Hölder's inequality and taking supremum, we obtain

$$I_3 \leq \|b\|_{l_p} \sup_{n \geq 1} n^{\frac{1}{p'}} \frac{1}{n} \left| \sum_{r=1}^{n} a_r \right|.$$

By (4) and using that the space $L_{p,q}$ is decreasing in the second parameter, we get

$$I_3 \leq C \|b\|_{l_p} \|f\|_{L_{p,\infty}} \leq C \|b\|_{l_p} \|f\|_{L_{p,p'}}.$$

Combining all these estimates for I_1, I_2, I_3, passing to the limit and substituting back in to (6), we obtain the desired inequality. □

A more generalized inequality is proved in the following theorem.

Theorem 2. *Let $1 < p < \infty$, $p' = \frac{p}{p-1}$, $0 < q \le \infty$. If $f \in L_{p,q}[0,1]$ and $f \sim \sum\limits_{k=1}^{\infty} a_k e^{2\pi i k x}$, $\alpha > \frac{1}{p}$, then we have*

$$\left(\sum_{k=1}^{\infty}\left(k^{\alpha-\frac{1}{p}}\left|\sum_{m=k}^{\infty}\frac{a_m}{m^{\alpha}}\right|\right)^{q}\frac{1}{k}\right)^{\frac{1}{q}} \le c\|f\|_{L_{p,q}[0,1]} \qquad \textit{where} \qquad 0<q<\infty,$$

in the case $q=\infty$

$$\sup_{k\in N} k^{\alpha-\frac{1}{p}}\left|\sum_{m=k}^{\infty}\frac{a_m}{m^{\alpha}}\right| \le c\|f\|_{L_{p,\infty}[0,1]}.$$

Proof. We estimate

$$A=\sup_{k\in N} k^{\alpha-\frac{1}{p}}\left|\sum_{m=k}^{\infty}\frac{a_m}{m^{\alpha}}\right|.$$

Substituting the value of the coefficients of Fourier series and interchanging the integral and sum, we obtain

$$A=\sup_{k\in N} k^{\alpha-\frac{1}{p}}\left|\int_0^1 f(x)\Phi_k(x)dx\right|,$$

$\Phi_k(x) = \sum\limits_{m=k}^{\infty}\frac{e^{-2\pi imx}}{m^{\alpha}}$. Using Hölder's inequality for Lorentz space, if $\frac{1}{p}+\frac{1}{p'}=1$, we have

$$A \le \sup_{k\in N} k^{\alpha-\frac{1}{p}}\|f\|_{L_{p,1}}\|\Phi_k\|_{L_{p',\infty}}. \tag{7}$$

Consider $\Phi_k(x) = \sum\limits_{m=k}^{\infty}\frac{e^{-2\pi imx}}{m^{\alpha}}$. Fourier coefficients of this function are monotone decreasing, using Hardy-Littlewood theorem [6], we have

$$\|\Phi_k\|_{L_{p',\infty}} \le C\left\|\frac{1}{m^{\alpha}}\right\|_{l_{p\infty}} = C\sup_{1\le r\le\infty} r^{\frac{1}{p}}\frac{1}{(k+r)^{\alpha}}$$

We prove that

$$B=k^{\alpha-\frac{1}{p}}\sup_{1\le r\le\infty} r^{\frac{1}{p}}\frac{1}{(k+r)^{\alpha}}<\infty$$

We find the maximum of function $g(x)=\frac{x^{\frac{1}{p}}}{(k+x)^{\alpha}}$, and it reaches the maximum in point $x=\frac{k}{\alpha p-1}$, as $\alpha>\frac{1}{p}$, then $x>0$. Hence

$$B=k^{\alpha-\frac{1}{p}}\left(\frac{k}{\alpha p-1}\right)^{\frac{1}{p}}\frac{1}{\left(k+\frac{k}{\alpha p-1}\right)^{\alpha}}=c(\alpha,p).$$

Substituting it to (7), we get $A\le C\|f\|_{L_{p,1}}$, therefore, we obtain

$$\|a\|_{n_{p',\infty,\alpha}} \le C\|f\|_{L_{p,1}[0;1]}.$$

Using also the interpolation theorem of Marcinkiewicz-Calderon [4] and Lemma 3 we obtain stronger inequality

$$\|a\|_{n_{p',q,\alpha}} \leq C\|f\|_{L_{p,q}[0;1]},$$

which implies the desired inequality. □

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