A SHARP JACKSON INEQUALITY FOR BEST TRIGONOMETRIC APPROXIMATION

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Abstract. The paper presents a sharp Jackson inequality and a corresponding inverse one for best trigonometric approximation in terms of moduli of smoothness that are equivalent to zero on the trigonometric polynomials up to a certain degree. Sharp relations between such moduli of different order are also considered.

1. Introduction

Let $L_p(\mathbb{T})$, $1 \le p \le \infty$, denote the space of the functions with finite L_p-norm on the circle $\mathbb{T}$. We can consider $C(\mathbb{T})$ – the space of the continuous functions on $\mathbb{T}$, in the place of $L_\infty(\mathbb{T})$. Best trigonometric approximation of a function $f \in L_p(\mathbb{T})$ is given by

$$E_n^T(f)_p = \inf_{\tau \in T_n} \|f - \tau\|_p,$$

where T_n denotes the set of the trigonometric polynomials of degree at most n and $\| \circ \|_p$ denotes the usual L_p-norm.

The error $E_n^T(f)_p$ is estimated by the so-called classical moduli of smoothness. To recall, the modulus of smoothness of order $r \in \mathbb{N}$ of $f \in L_p(\mathbb{T})$ is defined by

$$\omega_r(f,t)_p = \sup_{0<h\le t} \|\Delta_h^r f\|_p,$$

where the centred finite difference of order $r \in \mathbb{N}$ of f is given by

$$\Delta_h^r f(x) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} f(x + (r/2 - k)h).$$

The following relation between $E_n^T(f)_p$ and $\omega_r(f,t)_p$ is a classical result in approximation theory (see for example [4, Ch. 7])

$$E_n^T(f)_p \le c\,\omega_r(f, n^{-1})_p, \tag{1.1}$$

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$$\omega_r(f,t)_p \le c\,t^r \sum_{0\le k\le 1/t} (k+1)^{r-1} E_k^T(f)_p.$$

Above and in what follows we denote by c positive constants, which do not depend on the functions in the relations, nor on $n \in \mathbb{N}$ or $0 < t \le t_0$. For $1 < p < \infty$ these estimates were improved by Timan [13, 14] (or see [4, Ch. 7]) in the stronger forms

$$n^{-r}\left\{\sum_{k=0}^{n}(k+1)^{sr-1}E_k^T(f)_p^s\right\}^{1/s} \le c\,\omega_r(f,n^{-1})_p, \tag{1.2}$$

and

$$\omega_r(f,t)_p \le c\,t^r\left\{\sum_{0\le k\le 1/t}(k+1)^{\sigma r-1}E_k^T(f)_p^\sigma\right\}^{1/\sigma}. \tag{1.3}$$

where $s = \max\{p, 2\}$ and $\sigma = \min\{p, 2\}$. The estimate (1.2) is called a sharp Jackson inequality.

Our main goal is to establish analogues of Timan's inequalities with moduli of smoothness, which are equivalent to zero if the function f is a trigonometric polynomial of a certain degree, that is, moduli which are invariant on such trigonometric polynomials. We call them *trigonometric moduli*. Such estimates look natural especially with regard to the Jackson-type inequalities (1.1) and (1.2) and the invariance of best trigonometric approximation. The method we shall use reduces the new inequalities to the classical. On the one hand, this gives a simpler proof of the Jackson inequality given in [9, Theorem 1.1], but on the other hand, we believe, this approach can lead to estimates of best approximation by other systems as well as of the rate of approximation of linear processes by moduli possessing a corresponding natural invariance.

A sharp Jackson inequality in a very general setting for multivariate functions was established by Dai, Ditzian and Tikhonov [3] in terms of K-functionals. Many concrete sharp Jackson estimates are derived, among which about the univariate best algebraic approximation by the Ditzian-Totik modulus, about the multivariate best trigonometric approximation by the classical modulus, and about best approximation by spherical harmonic polynomials by the Ditzian modulus on the sphere [6].

The following two inequalities between the classical moduli of different order are closely related to (1.2) and (1.3):

$$t^r\left\{\int_t^{t_0}\frac{\omega_{r+1}(f,u)_p^s}{u^{sr+1}}\,du\right\}^{1/s} \le c\,\omega_r(f,t)_p \tag{1.4}$$

and

$$\omega_r(f,t)_p \le c\,t^r \left\{ \int_t^{t_0} \frac{\omega_{r+1}(f,u)_p^\sigma}{u^{\sigma r+1}}\,du \right\}^{1/\sigma}, \tag{1.5}$$

where $0 < t \le t_0$. The former was established in [3, (1.6)]; the latter is due to Zygmund [15] and is referred to as a sharp Marchaud inequality. These inequalities were extended to quite general function spaces in [5, 7, 8]. Also, Dai, Ditzian and Tikhonov [3, Theorems 5.3 and 5.5] (see also [2]) established more general forms in terms of K-functionals. There the multivariate form of (1.4) was proved [3, (2.17)]. In the present note we verify the analogues of (1.4) and (1.5) for the trigonometric moduli. Let us mention that their basic properties were established in [9, 10, 11] — they are just similar to those of the classical moduli.

The contents of the paper are organised as follows. In the next section we define the above-mentioned trigonometric moduli. Then in Section 3 and 4 we establish the analogues of (1.2)-(1.3) and (1.4)-(1.5), respectively, in their terms.

2. Trigonometric moduli of smoothness

We shall define two moduli that are identically zero on the trigonometric polynomials up to a certain degree. The first one is based on a modification of the approximated function, whereas the second on a modification of the finite differences.

To define the first modulus, let the 2π-periodic function $\mathfrak{a}$ be given on $[-\pi,\pi]$ by

$$\mathfrak{a}(x) = \frac{1}{2}\,|x|(2\pi - |x|) \tag{2.1}$$

and let for $j \in \mathbb{N}_0$ the bounded linear operator $\mathfrak{A}_j : L_p(\mathbb{T}) \to L_p(\mathbb{T})$, $1 \le p \le \infty$, be defined by

$$\mathfrak{A}_j f = f + j^2 \mathfrak{a} * f.$$

Above, $*$ denotes the convolution on $L_1(\mathbb{T})$

$$f * g(x) = \frac{1}{2\pi} \int_{\mathbb{T}} f(x-y)g(y)\,dy.$$

Further, for $r \in \mathbb{N}_0$ we set

$$\mathfrak{F}_r = \mathfrak{A}_r \cdots \mathfrak{A}_0.$$

Now, we define the trigonometric modulus of smoothness

$$\omega_r^T(f,t)_p = \sup_{0<h\le t} \|\Delta_h^{2r-1} \mathfrak{F}_{r-1} f\|_p.$$

It has the property that $\omega_r^T(f,t)_p \equiv 0$ iff $f \in T_{r-1}$ as it was established in [9, 11].

Babenko, Chernykh and Shevaldin [1] considered another modulus, which is zero on the trigonometric polynomials of degree $r-1$. It is given by

$$\tilde{\omega}_r^T(f,t)_p = \sup_{0<h\le t} \|\widetilde{\Delta}_{r,h} f\|_p,$$

as the modified finite differences $\widetilde{\Delta}_{r,h}$ are defined by

$$\widetilde{\Delta}_{r,h} f(x) = \Delta_{r-1,h} \cdots \Delta_{1,h} \Delta_h f(x), \tag{2.2}$$

where

$$\Delta_{j,h} f(x) = f(x+h) - 2\cos jh \,.\, f(x) + f(x-h), \quad j = 1, 2, \dots$$

In [9, 10, 11] it was shown that the moduli $\omega_r^T(f,t)_p$ and $\tilde{\omega}_r^T(f,t)_p$ characterize the rate of best trigonometric approximation just similarly as the classical modulus in any homogeneous Banach space of periodic functions and, in particular, in L_p. Babenko, Chernykh and Shevaldin [1] proved the Jackson estimate in the case $p = 2$ for the modulus $\tilde{\omega}_r^T(f,t)_2$. Shevaldin [12] verified it for $p = \infty$ and $r = 2$.

3. A sharp Jackson inequality

We shall establish the analogues of (1.2) and (1.3) for the trigonometric moduli $\omega_r^T(f,t)_p$ and $\tilde{\omega}_r^T(f,t)_p$.

Theorem 3.1. *Let $f \in L_p(\mathbb{T})$, $1 < p < \infty$, $s = \max\{p,2\}$, $\sigma = \min\{p,2\}$ and $r \in \mathbb{N}$. Then*

$$n^{1-2r} \left\{ \sum_{k=r-1}^{n} (k+1)^{s(2r-1)-1} E_k^T(f)_p^s \right\}^{1/s} \le c\, \omega_r^T(f, n^{-1})_p \tag{3.1}$$

and

$$\omega_r^T(f,t)_p \le c\, t^{2r-1} \left\{ \sum_{r-1 \le k \le 1/t} (k+1)^{\sigma(2r-1)-1} E_k^T(f)_p^\sigma \right\}^{1/\sigma}, \quad 0 < t \le 1/r. \tag{3.2}$$

The inequalities remain valid with $\tilde{\omega}_r^T(f,t)_p$ in the place of $\omega_r^T(f,t)_p$.

Proof. The moduli $\omega_1^T(f,t)_p$ and $\tilde{\omega}_1^T(f,t)_p$ coincide with $\omega_1(f,t)_p$. So it remains to prove the theorem for $r \ge 2$.

Let $\mathfrak{b}_j$, $j \in \mathbb{N}_0$, be 2π-periodic and defined on $[-\pi, \pi]$ by

$$\mathfrak{b}_j(x) = j(|x| - \pi) \sin |jx|$$

and let

$$\mathfrak{B}_j F = F + \mathfrak{b}_j * F.$$

In [11, Proposition 2.4] we showed that for the bounded operator $\mathfrak{C}_r = \mathfrak{B}_r \cdots \mathfrak{B}_0 : L_p(\mathbb{T}) \to L_p(\mathbb{T})$ we have

$$\mathfrak{C}_r \mathfrak{F}_r f = f - S_r f, \quad f \in L_p(\mathbb{T}),$$

where $S_r f$ is the rth partial sum of the Fourier series of f. Also, as it follows directly from their definition, both operators $\mathfrak{F}_r$ and $\mathfrak{C}_r$ map the set of trigonometric polynomials T_k into itself for any k. Consequently, we get for $k \ge r-1$ and

τ_k the trigonometric polynomial of degree k of best L_p-approximation of $\mathfrak{F}_{r-1}f$ the relation

$$\begin{aligned} E_k^T(f)_p &= E_k^T(\mathfrak{E}_{r-1}\mathfrak{F}_{r-1}f)_p \le \|\mathfrak{E}_{r-1}\mathfrak{F}_{r-1}f - \mathfrak{E}_{r-1}\tau_k\|_p \\ &\le c\,\|\mathfrak{F}_{r-1}f - \tau_k\|_p = c\,E_k^T(\mathfrak{F}_{r-1}f)_p. \end{aligned}$$

Now, (1.2) with $\mathfrak{F}_{r-1}f$ in the place of f and $2r-1$ in the place of r directly implies (3.1).

Similarly, (1.3) and the estimate $E_k^T(\mathfrak{F}_{r-1}f)_p \le c\,E_k^T(f)_p$, $k \in \mathbb{N}_0$ imply

$$\omega_r^T(f,t)_p \le c\,t^{2r-1}\left\{\sum_{0\le k\le 1/t}(k+1)^{\sigma(2r-1)-1}E_k^T(f)_p^\sigma\right\}^{1/\sigma}.$$

Next, we split the sum on the right hand-side into two parts for $0 \le k \le r-2$ and $r-1 \le k \le 1/t$. We estimate above the summands of the first sum using that $k \le r-1$ and $E_k^T(f)_p \le \|f\|_p$ to get

$$\omega_r^T(f,t)_p \le c\,t^{2r-1}\left\{\sum_{r-1\le k\le 1/t}(k+1)^{\sigma(2r-1)-1}E_k^T(f)_p^\sigma + r^{\sigma(2r-1)-1}\|f\|_p\right\}^{1/\sigma}.$$

Now, we replace above f with $f-\tau_{r-1}$, where τ_{r-1} is the trigonometric polynomial of best L_p-approximation of f of degree $r-1$, and use the invariance of $\omega_r^T(f,t)_p$ and $E_k^T(f)_p$, $k \ge r-1$, under addition of trigonometric polynomials of that degree to arrive at (3.2).

The inequalities for $\tilde{\omega}_r^T(f,t)_p$ are derived immediately from those for $\omega_r^T(f,t)_p$ because both moduli are equivalent to the same K-functional, namely,

$$K_r^T(f,t)_p = \inf_{g\in W_p^{2r-1}(\mathbb{T})}\{\|f-g\|_p + t^{2r-1}\|\widetilde{D}_r g\|_p,$$

where $\widetilde{D}_r$ is the differential operator whose kernel is T_{r-1} (see [10, Theorem 4.2] and [11, Theorem 2.1]). $\square$

4. Sharp relations between trigonometric moduli of different order

As we mentioned in the Introduction, the trigonometric moduli possess properties just similar to those of the classical one. In addition, they satisfy the following sharp forms of the inequality $\omega_{r+1}^T(f,t)_p \le c\,\omega_r^T(f,t)_p$ and of the Marchaud inequality.

Theorem 4.1. *Let $f \in L_p(\mathbb{T})$, $1<p<\infty$, $s=\max\{p,2\}$, $\sigma=\min\{p,2\}$ and $r \in \mathbb{N}$. Then for $0<t\le t_0$ there hold*

$$t^{2r-1}\left\{\int_t^{t_0}\frac{\omega_{r+1}^T(f,u)_p^s}{u^{s(2r-1)+1}}\,du\right\}^{1/s} \le c\,\omega_r^T(f,t)_p \tag{4.1}$$

and

$$\omega_r^T(f,t)_p \le c\,t^{2r-1}\left[\left\{\int_t^{t_0}\frac{\omega_{r+1}^T(f,u)_p^\sigma}{u^{\sigma(2r-1)+1}}\,du\right\}^{1/\sigma}+\|f\|_p\right]. \tag{4.2}$$

The inequalities remain valid with $\tilde\omega_r^T(f,t)_p$ in the place of $\omega_r^T(f,t)_p$.

Proof. Iterating [3, (1.6)] (or see (1.4)), we get the inequality

$$t^{2r-1}\left\{\int_t^{t_0}\frac{\omega_{2r+1}(f,u)_p^s}{u^{s(2r-1)+1}}\,du\right\}^{1/s}\le c\,\omega_{2r-1}(f,t)_p. \tag{4.3}$$

Set $F=\mathfrak{F}_{r-1}f$. Then $\mathfrak{F}_r f = F + r^2\mathfrak{a}*F$. In [11, (3.2)] it was proved that $(\mathfrak{a}*g)''=g+\mathit{const}$ for any $g\in L_p(\mathbb{T})$. Then, using basic properties of the classical modulus, we get

$$\omega_{r+1}^T(f,u)_p^s=\omega_{2r+1}(\mathfrak{F}_r f,u)_p^s\le c\left[\omega_{2r+1}(F,u)_p^s+u^{2s}\omega_{2r-1}(F,u)_p^s\right]. \tag{4.4}$$

For the first term on the right above we get by (4.3) with f replaced by F

$$t^{2r-1}\left\{\int_t^{t_0}\frac{\omega_{2r+1}(F,u)_p^s}{u^{s(2r-1)+1}}\,du\right\}^{1/s}\le c\,\omega_r^T(f,t)_p. \tag{4.5}$$

To estimate the second term on the right of (4.4), we proceed as follows. Let $F_t\in W_p^{2r-1}(\mathbb{T})$ be such that

$$\|F-F_t\|_p\le c\,\omega_{2r-1}(F,t)_p \tag{4.6}$$

and

$$t^{2r-1}\|F_t^{(2r-1)}\|_p\le c\,\omega_{2r-1}(F,t)_p. \tag{4.7}$$

For F_t one can take the Steklov mean of F (see e.g. [4, p. 177]). Then we have by basic properties of the classical modulus

$$u^{2s}\omega_{2r-1}(F,u)_p^s\le c\,\|F-F_t\|_p^s+u^{s(2r+1)}\|F_t^{(2r-1)}\|_p^s$$

where $0<u\le t_0$, and, consequently,

$$\begin{aligned} t^{2r-1}&\left\{\int_t^{t_0}\frac{u^{2s}\omega_{2r-1}(F,u)_p^s}{u^{s(2r-1)+1}}\,du\right\}^{1/s}\\ &\le c\,\|F-F_t\|_p+c\,t^{2r-1}\|F_t^{(2r-1)}\|_p\le c\,\omega_r^T(f,t)_p, \end{aligned} \tag{4.8}$$

as at the last step we have applied (4.6)-(4.7) and $\omega_r^T(f,t)_p=\omega_{2r-1}(F,t)_p$.

Now, (4.4), (4.5) and (4.8) imply (4.1).

We proceed to the proof of (4.2). Iterating the sharp Marchaud inequality (1.5), we arrive at

$$\omega_{2r-1}(f,t)_p\le c\,t^{2r-1}\left\{\int_t^{t_0}\frac{\omega_{2r+1}(f,u)_p^\sigma}{u^{\sigma(2r-1)+1}}\,du\right\}^{1/\sigma}.$$

With $\mathfrak{F}_r f$ in the place of f it yields

$$\omega_{2r-1}(\mathfrak{F}_r f,t)_p \le c\, t^{2r-1}\left\{\int_t^{t_0}\frac{\omega_{r+1}^T(f,u)_p^\sigma}{u^{\sigma(2r-1)+1}}\,du\right\}^{1/\sigma}.$$

Thus it remains to show that

$$\omega_r^T(f,t)_p = \omega_{2r-1}(\mathfrak{F}_{r-1}f,t)_p \le c\left(\omega_{2r-1}(\mathfrak{F}_r f,t)_p + t^{2r-1}\|f\|_p\right). \tag{4.9}$$

To verify the latter, we take into account that $\mathfrak{F}_r f = \mathfrak{A}_r F$ with $F = \mathfrak{F}_{r-1}f$; hence $\mathfrak{B}_r\mathfrak{F}_r f = \mathfrak{B}_r\mathfrak{A}_r F = F + \eta_r * F$ with $\eta_r(x) = -1 - 2\cos rx$ as was established in [11, (2.9)]. Set $G = \mathfrak{F}_r f$ and let $G_t \in W_p^{2r-1}(\mathbb{T})$ satisfy (4.6)-(4.7) for G in the place of F. Then

$$\begin{aligned}\omega_{2r-1}(F,t)_p &\le \omega_{2r-1}(\mathfrak{B}_r G - \mathfrak{B}_r G_t,t)_p + \omega_{2r-1}(G_t,t)_p\\ &\qquad + \omega_{2r-1}(\mathfrak{b}_r * G_t,t)_p + \omega_{2r-1}(\eta_r * F,t)_p\\ &\le c\,\|\mathfrak{B}_r G - \mathfrak{B}_r G_t\|_p + t^{2r-1}\|G_t^{(2r-1)}\|_p\\ &\qquad + t^{2r-1}\|\mathfrak{b}_r * G_t^{(2r-1)}\|_p + t^{2r-1}\|\eta^{(2r-1)} * F\|_p\\ &\le c\big(\|G-G_t\|_p + t^{2r-1}\|G_t^{(2r-1)}\|_p + t^{2r-1}\|F\|_p\big)\\ &\le c\left(\omega_{2r-1}(G,t)_p + t^{2r-1}\|f\|_p\right).\end{aligned}$$

Thus (4.9) is proved. □

Remark 4.2. The inequalities (4.1) and (4.2) can be verified by means of (3.1) and (3.2) (see the proof of [3, Theorem 5.3] and [4,]). Moreover, such a proof maybe considered even simpler and shorter than the one we used here. However, we preferred to give a proof based on the properties of the classical moduli and independent of the relation of the other moduli to an approximation process. This demonstrates the advantages of the connection between the new and the classical moduli in transferring properties between them and can be applied to define and establish properties of moduli appropriate for other approximation operators.

Remark 4.3. Replacing in (4.2) f with $f - \tau_{r-1}$, where τ_{r-1} is the trigonometric polynomial of best L_p-approximation of f of degree $r-1$, we immediately arrive at its slightly stronger form

$$\omega_r^T(f,t)_p \le c\, t^{2r-1}\left[\left\{\int_t^{t_0}\frac{\omega_{r+1}^T(f,u)_p^\sigma}{u^{\sigma(2r-1)+1}}\,du\right\}^{1/\sigma} + E_{r-1}^T(f)_p\right].$$

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