

# RATIONAL FAMILIES OF INSTANTON BUNDLES ON $\mathbb{P}^{2n+1}$

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ABSTRACT. This paper is devoted to the theory of symplectic instanton bundles on an odd dimensional projective space  $\mathbb{P}^{2n+1}$  with  $n \geq 2$ . The investigation of 't Hooft instanton bundles that were introduced by Ottaviani is continued. Furthermore, the concept of Rao–Skiti instanton bundles is considered. On  $\mathbb{P}^3$ , such instanton bundles were studied independently by Rao and Skiti, but for higher odd-dimensional projective spaces these objects are new. The main results of the article concern the rationality of the moduli spaces of 't Hooft and Rao–Skiti instanton bundles, respectively, and the reducibility of the moduli space of symplectic instanton bundles.

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## 1. INTRODUCTION

The notion of an instanton comes from mathematical physics. It denotes a solution of an equation of motion in quantum theory which describes a (pseudo)particle which exists at only one moment in time. Mathematically, it is a self-dual connection on a principal bundle on a four-dimensional Riemannian manifold. The Penrose–Ward transform turns instantons on the four-dimensional sphere  $S^4$  into holomorphic instanton bundles on the three-dimensional complex

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projective space  $\mathbb{P}^3$  ([3], [4]). This construction turned out to be a major motivation for studying vector bundles on complex projective spaces and similar algebraic varieties. In recent preprints, Jardim, Markushevich, Tikhomirov, and Verbitsky managed to settle fundamental open questions on geometric properties of moduli spaces of instanton bundles on  $\mathbb{P}^3$ , namely smoothness, connectedness and rationality ([20], [21], [35]).

The instanton correspondence exists also on higher odd-dimensional complex projective spaces. In fact, Salamon, Corrigan, Goddard, and Kent introduced a correspondence between self-dual connections on  $\mathrm{Sp}_n(\mathbb{C})$ -bundles on the  $n$ -dimensional quaternionic projective space  $\mathbb{H}\mathbb{P}^n$  and symplectic instanton bundles on the complex projective space  $\mathbb{P}^{2n+1}$ ,  $n \geq 1$  ([11], [13], [29]). For  $n \geq 2$ , the knowledge about moduli spaces of symplectic instanton bundles on  $\mathbb{P}^{2n+1}$  is much less complete, and it is the aim of the present paper to make some progress in this area.

The starting point is the paper by Ottaviani [27]. Ottaviani introduces the notion of symplectic 't Hooft instanton bundles on  $\mathbb{P}^{2n+1}$ ,  $n \geq 2$ , and claims without proof that the closure of the locus of symplectic 't Hooft bundles is an irreducible component of the moduli space of symplectic instanton bundles. So far, we have not been able to prove this claim, but we have checked it for several values of  $n$  and the instanton number  $k$  (Example 4.1). In addition, we construct an irreducible moduli space for symplectic 't Hooft bundles. Our first main result is that this moduli space is stably rational or even rational for many values of  $n$  and  $k$  (Corollary 5.10).

Next, we define the notion of a symplectic Rao–Skiti (RS) instanton bundle on  $\mathbb{P}^{2n+1}$ ,  $n \geq 1$ . Symplectic Rao–Skiti instanton bundles generalize bundles studied by Rao [28] and Skiti [33] on  $\mathbb{P}^3$ . To our knowledge, these instanton bundles haven't been investigated for  $n \geq 2$ , so far. We supply irreducible moduli spaces for symplectic Rao–Skiti instanton bundles and prove their rationality (Corollary 6.10). Another important observation is that there are symplectic Rao–Skiti instanton bundles which are not limits of 't Hooft instanton bundles (Example 3.11). Under the assumption of the claim of Ottaviani (which we have verified in some cases), this implies that the moduli space of symplectic instanton bundles is reducible.

Next, we outline the structure of the paper. In Section 2, we fix notation and briefly recall the definition and basic properties of instanton bundles on projective spaces needed later on. In Section 3, we present the definition and main features of the two irreducible families of symplectic instanton bundles on  $\mathbb{P}^{2n+1}$  that we are going to be interested in, namely, symplectic 't Hooft- and symplectic RS-instanton bundles (see Definition 3.3 and 3.8, respectively). The deformation theory of 't Hooft instanton bundles will be reviewed in Section 4. In that section, we also verify a central claim by Ottaviani with the help of a computer (Example 4.1). Section 5 deals with the construction (Proposition 5.7) and the rationality (Corollary 5.10) of moduli spaces of 't Hooft instanton bundles. Section 6 contains

the determination of the birational type of the moduli stacks (Corollary 6.4) and spaces (Corollary 6.10) of RS-instanton bundles.

**Notation.** Throughout this paper, we will work over the field  $\mathbb{C}$  of complex numbers. Given a vector space  $W$ , we will denote by  $\mathbb{P}(W)$  the projective space of lines in  $W$  and set  $\mathbb{P} := \mathbb{P}^{2n+1} := \mathbb{P}(\mathbb{C}^{2n+2})$ . We will not distinguish between a vector bundle and its locally free sheaf of sections and use the definition of  $\mu$ -(semi)stability due to Mumford and Takemoto [24]. Given an integer  $k \geq 1$ , let  $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$  denote the standard symplectic form on  $\mathbb{C}^{2n+2k}$ .

## 2. MATHEMATICAL INSTANTON BUNDLES

In this section, we recall the definition of (mathematical) instanton bundles and their description in terms of monads.

**Definition 2.1.** Let  $k \geq 1$  be an integer. An *instanton bundle with charge  $k$*  (for short, a  *$k$ -instanton bundle*) is a vector bundle  $E$  on  $\mathbb{P}$  satisfying the following properties:

- i)  $E$  has rank  $2n$ ,
- ii) the Chern polynomial of  $E$  is  $c_t(E) = 1/(1 - t^2)^k$ ,
- iii)  $E$  has natural cohomology in the range  $-(2n + 1) \leq q \leq 0$ , i.e., for any  $q$  in that range, there is at most one integer  $i = i(q)$  such that  $H^i(\mathbb{P}, E(q)) \neq 0$ .
- iv)  $E$  has trivial splitting type, i.e., the restriction of  $E$  to a general line is trivial.

*Remark 2.2.* By [2], Proposition 2.11, any instanton bundle is simple. Nevertheless, it is an open question to determine whether it is  $\mu$ -stable.

**Definition 2.3.** A vector bundle  $E$  on  $\mathbb{P}$  is called *symplectic*, if there exists an isomorphism  $\varphi: E \rightarrow E^*$  such that  $\varphi^* = -\varphi$ . This is equivalent to the existence of a nondegenerate form  $\alpha \in H^0(\mathbb{P}, \bigwedge^2 E)$ .

Notice that, when  $n \geq 2$ , the symplectic structure condition does not follow from the definition. In this paper, we will restrict our attention mainly to symplectic instanton bundles. Using the Beilinson spectral sequence [5], we get the following well-known and useful correspondence between symplectic instanton bundles and self-dual monads:

**Proposition 2.4.** i) *A vector bundle  $E$  over  $\mathbb{P}$  is a symplectic  $k$ -instanton bundle if and only if  $E$  is the cohomology of a symplectic monad*

$$(1) \quad \mathcal{O}_{\mathbb{P}}(-1)^{\oplus k} \xrightarrow{JA^t} \mathcal{O}_{\mathbb{P}}^{2n+2k} \xrightarrow{A} \mathcal{O}_{\mathbb{P}}(1)^{\oplus k}$$

where the matrix

$$A \in \text{Mat}_{k \times (2n+2k)} \left( H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) \right)$$

of linear forms has full rank  $k$  at every point of  $\mathbb{P}$  and satisfies  $AJA^t = 0$ . Conversely, any such matrix  $A$  yields a symplectic monad (1) and hence a symplectic  $k$ -instanton bundle  $E$ .

ii) Abbreviate a monad with cohomology bundle  $E$  as in (1) by  $M_E^\bullet$ . Then, for two such monads  $M_E^\bullet$  and  $N_F^\bullet$ , one has

$$\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}}}(E, F) = \mathrm{Hom}_{\mathrm{cplx}}(M_E^\bullet, N_F^\bullet).$$

*Proof.* i) See [26], Corollary 1.4 and Lemma 1.5, and [18], Proposition 2.4.

ii) See [25], Proposition 1.3, or [18], Proposition 2.4.  $\square$

**Proposition 2.5.** *Let  $E$  be a symplectic  $k$ -instanton bundle on  $\mathbb{P}$  and let  $L \subset \mathbb{P}$  a linear subspace of dimension  $r$ . Then,  $h^0(E|_L) \leq 2n + k - r$ .*

*Proof.* Assume that there is a linear subspace  $L \subset \mathbb{P}$  of dimension  $r$ , such that  $h^0(E|_L) \geq 2n + k - r + 1$  and consider the monad

$$\mathcal{O}_{\mathbb{P}}(-1)^{\oplus k} \xrightarrow{JA^t} \mathcal{O}_{\mathbb{P}}^{2n+2k} \xrightarrow{A} \mathcal{O}_{\mathbb{P}}(1)^{\oplus k}$$

associated with  $E$ . After changing basis, if necessary, we can assume that  $\mathbb{P}$  has homogeneous coordinates  $x_0, x_1, \dots, x_{2n+1}$ ,  $L$  is given by  $x_{r+1} = \dots = x_{2n+1} = 0$  and

$$A|_L = (A_1|A_2) \in \mathrm{Mat}_{k \times (2n+2k)} \left( H^0(\mathbb{P}, \mathcal{O}_L(1)) \right)$$

with  $A_2 = (0) \in \mathrm{Mat}_{k \times (2n+k-r+1)}(H^0(\mathbb{P}, \mathcal{O}_L(1)))$ . By [10], Theorem 2.1, the homogeneous ideal  $I \subset \mathbb{C}[x_0, x_1, \dots, x_r]$  that is generated by the maximal minors of  $A_1 \in \mathrm{Mat}_{k \times (k+r-1)}(H^0(\mathbb{P}, \mathcal{O}_L(1)))$  has height  $\leq r$ . This implies that there exist closed points  $x \in L$ , such that  $\mathrm{rank}(A(x)) < k$ , contradicting the fact that  $A$  is the matrix which defines the symplectic monad associated with  $E$ .  $\square$

From now on, given a symplectic instanton bundle  $E$  on  $\mathbb{P}$  and a linear subspace  $L \subset \mathbb{P}$  of dimension  $r$ , we will say that  $L$  is *unstable with maximal order of instability*, if  $h^0(E|_L) = 2n + k - r$ .

*Remark 2.6.* i) Since any instanton bundle  $E$  on  $\mathbb{P}$  is the cohomology of a linear monad

$$\mathcal{O}_{\mathbb{P}}(-1)^{\oplus k} \xrightarrow{B} \mathcal{O}_{\mathbb{P}}^{2n+2k} \xrightarrow{A} \mathcal{O}_{\mathbb{P}}(1)^{\oplus k},$$

the hypothesis ‘‘symplectic’’ in Proposition 2.5 can be dropped.

ii) The upper bound given in the above proposition is sharp. Indeed, we will see that it is attained by symplectic RS-instanton bundles (see Definition 3.8). More precisely, for any symplectic RS-instanton bundle  $E$  on  $\mathbb{P}$ , we will prove the existence of an  $n$ -dimensional linear subspace  $L \subset \mathbb{P}$  with maximal order of instability (Proposition 3.10).

## 3. SYMPLECTIC 'T HOOFT- AND RS-INSTANTON BUNDLES

This section is devoted to introduce the two kinds of instanton bundles that will be the subject of our research, namely, symplectic 't Hooft- and symplectic RS-instanton bundles on  $\mathbb{P}$ .

Let us start with the definition of symplectic 't Hooft bundles. To this end, we fix integers  $n, k \geq 1$ . Let

$$(2) \quad a \in \text{Mat}_{k \times (n+k)}(\mathbb{C})$$

be a matrix of scalars, and let

$$(3) \quad D := \text{diag}(l_1, \dots, l_{n+k}), \quad l_1, \dots, l_{n+k} \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)),$$

$$(4) \quad D' := \text{diag}(l'_1, \dots, l'_{n+k}), \quad l'_1, \dots, l'_{n+k} \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)),$$

be diagonal matrices with entries in  $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$ . We consider the matrix

$$(5) \quad A := a \cdot (D|D') \in \text{Mat}_{k \times (2n+2k)} \left( H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) \right)$$

as a morphism of vector bundles from  $\mathcal{O}_{\mathbb{P}}^{2n+2k}$  to  $\mathcal{O}_{\mathbb{P}}^{\oplus k}$ .

**Proposition 3.1** (Ottaviani [27], Section 3). *For every choice of  $a, l_j, l'_j$ , we have:*

- i)  $AJA^t = 0$ .
- ii) *The sheaf  $\ker A(1) \subseteq \mathcal{O}_{\mathbb{P}}(1)^{\oplus(2n+2k)}$  has at least  $n+k$  global sections. Moreover, if  $a$  and the  $l_j, l'_j$  are chosen generically, then we have:*
- iii) *The matrix  $A$  has rank  $k$  at every point of  $\mathbb{P}$ .*
- iv) *If  $k \geq 3$ , then  $\ker A(1) \subseteq \mathcal{O}_{\mathbb{P}}(1)^{\oplus(2n+2k)}$  has exactly  $n+k$  global sections.*

*Proof.* i) This follows from the calculation

$$(D|D') \cdot J \cdot (D|D')^t = (D|D') \cdot \begin{pmatrix} D' \\ -D \end{pmatrix} = DD' - D'D = 0.$$

ii) The previous calculation also implies  $A \cdot J \cdot (D|D')^t = 0$ . Consequently, the  $n+k$  columns of the matrix  $J \cdot (D|D')^t$  are elements in  $\ker A(1)$ . We may assume that the linear forms  $l_j$  are all nonzero. Then these columns are linearly independent.

For the rest of the proof, let all  $(k \times k)$ -minors of  $a$  be nonzero, and choose a decomposition

$$H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) = V \oplus W \quad \text{with } \dim V = \dim W = n+1.$$

iii) Choose  $l_1, \dots, l_{n+k} \in V$ , such that any  $n+1$  of them are a basis of  $V$ . Consider a point in  $\mathbb{P}$  where at least one  $v \in V$  does not vanish. Then at most  $n$  of the forms  $l_1, \dots, l_{n+k}$  vanish there, so we can find  $k$  of them that do not. The corresponding  $(k \times k)$ -minor of  $a \cdot \text{diag}(l_1, \dots, l_{n+k})$  is nonzero at this point.

Choosing  $l'_1, \dots, l'_{n+k} \in W$  similarly, we can achieve that  $a \cdot \text{diag}(l'_1, \dots, l'_{n+k})$  has rank  $k$  at all points where at least one  $w \in W$  does not vanish. This covers all points in  $\mathbb{P}$ , thereby proving that iii) holds for some  $a$  and  $l_j, l'_j$ .

iv) We will again take  $l_1, \dots, l_{n+k} \in V$  and  $l'_1, \dots, l'_{n+k} \in W$ . A global section

$$\begin{pmatrix} b \oplus c \\ b' \oplus c' \end{pmatrix} \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}^{2n+2k}(1)), \quad b, b' \in V^{n+k}, \quad c, c' \in W^{n+k},$$

is then in  $\ker A(1)$  if and only if it satisfies the system of linear equations

$$(6) \quad aDb = 0 \in (\text{Sym}^2 V)^k,$$

$$(7) \quad aD'c' = 0 \in (\text{Sym}^2 W)^k,$$

$$(8) \quad a(Dc + D'b') = 0 \in (V \otimes W)^k.$$

Choosing bases  $v_1, \dots, v_{n+1} \in V$  and  $w_1, \dots, w_{n+1} \in W$ , we write

$$D = D_1 v_1 + \dots + D_{n+1} v_{n+1} \quad \text{with } D_1, \dots, D_{n+1} \in \text{Mat}_{(n+k) \times (n+k)}(\mathbb{C}),$$

$$D' = D'_1 w_1 + \dots + D'_{n+1} w_{n+1} \quad \text{with } D'_1, \dots, D'_{n+1} \in \text{Mat}_{(n+k) \times (n+k)}(\mathbb{C}).$$

Specifying the linear forms  $l_1, \dots, l_{n+k} \in V$  and  $l'_1, \dots, l'_{n+k} \in W$  is equivalent to specifying the diagonal matrices  $D_1, \dots, D_{n+1}$  and  $D'_1, \dots, D'_{n+1}$ . We also write

$$b = b_1 v_1 + \dots + b_{n+1} v_{n+1} \quad \text{with } b_i = (b_{i,1}, \dots, b_{i,n+k})^t \in \mathbb{C}^{n+k},$$

$$c = c_1 w_1 + \dots + c_{n+1} w_{n+1} \quad \text{with } c_i = (c_{i,1}, \dots, c_{i,n+k})^t \in \mathbb{C}^{n+k},$$

and similarly for  $b'$  and  $c'$ .

Suppose  $k \geq 2$ . We claim that Equation (6) has only the trivial solution  $b = 0$  if  $l_1, \dots, l_{n+k} \in V$  are generic. It suffices to check this in one special case, say

$$l_j := \begin{cases} v_j, & \text{for } j \leq n+1, \\ v_1 + \dots + v_{n+1}, & \text{for } j \geq n+2. \end{cases}$$

This choice translates into

$$D_i = \text{diag}(\underbrace{0, \dots, 0}_{i-1}, 1, \underbrace{0, \dots, 0}_{n+1-i}, \underbrace{1, \dots, 1}_{k-1}).$$

The coefficient for  $v_i^2$  in Equation (6) reads  $aD_i b_i = 0$ . As the relevant  $(k \times k)$ -minor of  $a$  is nonzero, this implies  $D_i b_i = 0$  and hence

$$(9) \quad b_{i,i} = b_{i,n+2} = b_{i,n+3} = \dots = b_{i,n+k} = 0.$$

Given indices  $1 \leq i_1 < i_2 \leq n+1$ , the coefficient for  $v_{i_1} v_{i_2}$  in Equation (6) reads

$$a(D_{i_1} b_{i_2} + D_{i_2} b_{i_1}) = 0 \in \mathbb{C}^k.$$

Using (9), we see that  $D_{i_1} b_{i_2} + D_{i_2} b_{i_1} \in \mathbb{C}^{n+k}$  has at most  $2 \leq k$  nonzero components; as  $(k \times k)$ -minors of  $a$  are nonzero, we can conclude

$$D_{i_1} b_{i_2} + D_{i_2} b_{i_1} = 0 \in \mathbb{C}^{n+k}$$

and hence  $b_{i_1, i_2} = b_{i_2, i_1} = 0$ . This shows that  $b = 0$  is the only solution of (6) for this particular choice, and consequently for a generic choice, of  $l_1, \dots, l_{n+k} \in V$ . Similarly,  $c' = 0$  is the only solution of (7) for a generic choice of  $l'_1, \dots, l'_{n+k} \in W$ .

Suppose  $k \geq 3$ . It remains to show that Equation (8) has only  $n+k$  linearly independent solutions  $(b', c)$  if  $l_1, \dots, l_{n+k} \in V$  and  $l'_1, \dots, l'_{n+k} \in W$  are generic. Again it suffices to check this in one special case, say

$$l_j := \begin{cases} v_j, & \text{for } j \leq n \\ v_n, & \text{for } j = n+1 \\ v_{n+1}, & \text{for } j \geq n+2 \end{cases} \quad \text{and} \quad l'_j := \begin{cases} w_j, & \text{for } j \leq n+1 \\ w_n, & \text{for } j = n+2 \\ w_{n+1}, & \text{for } j \geq n+3. \end{cases}$$

This choice translates into

$$\begin{aligned} D_i = D'_i &= \text{diag}(\overbrace{0, \dots, 0}^{i-1}, 1, \overbrace{0, \dots, 0, 0, 0, 0, 0, \dots, 0}^{n+k-i}), \quad \text{for } i \leq n-1, \\ D_n &= \text{diag}(0, \dots, 0, 0, 0, \dots, 0, 1, 1, 0, 0, \dots, 0), \\ D_{n+1} &= \text{diag}(0, \dots, 0, 0, 0, \dots, 0, 0, 0, 1, 1, \dots, 1), \\ D'_n &= \text{diag}(0, \dots, 0, 0, 0, \dots, 0, 1, 0, 1, 0, \dots, 0), \\ D'_{n+1} &= \text{diag}(\overbrace{0, \dots, 0, 0, 0, \dots, 0}^{n-1}, 0, 0, 1, 0, \overbrace{1, \dots, 1}^{k-2}); \end{aligned}$$

note that  $D_1 + \dots + D_{n+1}$  and  $D'_1 + \dots + D'_{n+1}$  both equal the identity matrix.

Given indices  $1 \leq i_1, i_2 \leq n+1$ , the coefficient for  $v_{i_1} \otimes w_{i_2}$  in Equation (8) reads

$$a(D_{i_1} c_{i_2} + D'_{i_2} b'_{i_1}) = 0 \in \mathbb{C}^k.$$

Due to the particular choice we have made,  $D_{i_1} c_{i_2} + D'_{i_2} b'_{i_1} \in \mathbb{C}^{n+k}$  has at most  $k$  nonzero components; as  $(k \times k)$ -minors of  $a$  are nonzero, we can conclude

$$D_{i_1} c_{i_2} + D'_{i_2} b'_{i_1} = 0 \in \mathbb{C}^{n+k}.$$

Taking separately the sum over all  $i_2$  and the sum over all  $i_1$ , we get

$$b'_{i_1} = -D_{i_1}(c_1 + \dots + c_{n+1}) \quad \text{and} \quad c_{i_2} = -D'_{i_2}(b'_1 + \dots + b'_{n+1}).$$

Plugging the sum over all  $i_1$  of the former equation into the latter, we conclude

$$c_{i_2} = D'_{i_2}(c_1 + \dots + c_{n+1}).$$

This shows that every solution  $(b', c)$  of Equation (8) satisfies

$$b' = -D(c_1 + \dots + c_{n+1}) \quad \text{and} \quad c = D'(c_1 + \dots + c_{n+1}).$$

As  $c_1 + \dots + c_{n+1} \in \mathbb{C}^{n+k}$ , there are only  $n+k$  linearly independent solutions.  $\square$

*Remark 3.2.* The hypothesis  $k \geq 3$  in Proposition 3.1, iv), cannot be dropped. In fact, it is trivial to check that, if  $k = 1$ , then  $\ker A(1) \subseteq \mathcal{O}_{\mathbb{P}}(1)^{\oplus(2n+2k)}$  has exactly  $2n^2 + 3n + 1$  global sections; and it follows from [2], Theorem 3.14, that if  $k = 2$  then  $h^0(\mathbb{P}, \ker A(1)) = 2n + 2$ .

**Definition 3.3** ([27], Section 3). A symplectic  $k$ -instanton bundle  $E$  on  $\mathbb{P}$  given by a monad

$$\mathcal{O}_{\mathbb{P}}(-1)^{\oplus k} \xrightarrow{JA^t} \mathcal{O}_{\mathbb{P}}^{2n+2k} \xrightarrow{A} \mathcal{O}_{\mathbb{P}}(1)^{\oplus k}$$

is called an 't Hooft instanton bundle, if  $A$  is of the form (5) introduced above.

*Remark 3.4.* i) Proposition 3.1, i) and iii), shows that generic data

$$a \in \text{Mat}_{k \times (n+k)}(\mathbb{C}) \quad \text{and} \quad l_1, \dots, l_{n+k}, l'_1, \dots, l'_{n+k} \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$$

actually define a symplectic 't Hooft  $k$ -instanton bundle  $E$ , via (3), (4), (5) and (1).

ii) It follows from [27], Theorem 3.6, that any 't Hooft instanton bundle on  $\mathbb{P}$  is stable.

iii) The condition on the trivial splitting type follows from the other conditions ([27], Corollary 3.5).

The following statement is also contained in [27], Proposition 3.3 and Theorem 3.7.

**Proposition 3.5.** *Let  $E$  be a symplectic 't Hooft  $k$ -instanton bundle. Then*

$$(10) \quad h^0(\mathbb{P}, E(1)) \geq n$$

with equality if  $k \geq 3$  and  $E$  is generic.

*Proof.* Let  $E$  be a symplectic  $k$ -instanton bundle, given as the cohomology of the monad (1). Then  $E(1)$  is the cohomology of a monad

$$\mathcal{O}_{\mathbb{P}}^{\oplus k} \longrightarrow \mathcal{O}_{\mathbb{P}}(1)^{\oplus (2n+2k)} \xrightarrow{A(1)} \mathcal{O}_{\mathbb{P}}(2)^{\oplus k}.$$

Using  $H^1(\mathbb{P}, \mathcal{O}_{\mathbb{P}}^{\oplus k}) = 0$ , this implies in particular

$$h^0(\mathbb{P}, E(1)) = h^0(\mathbb{P}, \ker A(1)) - k.$$

Hence the claim follows from Proposition 3.1, ii) and iv).  $\square$

*Remark 3.6.* The hypothesis  $k \geq 3$  in the above proposition cannot be dropped. In fact, let  $E$  be a symplectic  $k$ -instanton bundle on  $\mathbb{P}^{2n+1}$ . By Remark 3.2, we know that if  $k = 1$  ( $k = 2$ ) then  $h^0(\mathbb{P}, \ker A(1)) = 2n^2 + 3n + 1$  ( $h^0(\mathbb{P}, \ker A(1)) = 2n + 2$ ) and, hence,  $h^0(\mathbb{P}, E(1)) = 2n^2 + 3n$  ( $h^0(\mathbb{P}, E(1)) = 2n$ , respectively).

We devote the last part of this section to introduce the symplectic RS-instanton bundles. Again, we fix integers  $k, n \geq 1$ . Let

$$(11) \quad H := (h_{ij}) \in \text{Mat}_{k \times (n+k)} \left( H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) \right)$$

be a persymmetric matrix of linear forms  $h_{ij} \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$ , i.e., a matrix such that  $h_{ij} = h_{st}$  if  $i + j = s + t$ . Let

$$(12) \quad L := \{ f_0 = f_1 = \dots = f_n = 0 \}$$



be a linear  $n$ -space which contains no zeroes of the maximal minors of  $H$ . We consider the matrices

$$(13) \quad F := \begin{pmatrix} f_0 & f_1 & \cdots & f_n & 0 & \cdots & \cdots & 0 \\ 0 & f_0 & f_1 & \cdots & f_n & 0 & \cdots & 0 \\ 0 & 0 & f_0 & f_1 & \cdots & f_n & \ddots & \vdots \\ \vdots & \cdots & \ddots & \ddots & \ddots & \cdots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & f_0 & f_1 & \cdots & f_n \end{pmatrix}$$

and

$$(14) \quad A := (F|H) \in \text{Mat}_{k \times (2n+2k)} \left( H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) \right).$$

The matrix  $A$  will be considered as a morphism of vector bundles from  $\mathcal{O}_{\mathbb{P}}^{2n+2k}$  to  $\mathcal{O}_{\mathbb{P}}(1)^{\oplus k}$  and it satisfies the following properties.

**Proposition 3.7.** *For general choices of  $f_s, h_{ij} \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$ ,  $0 \leq s \leq n$ ,  $1 \leq i \leq k$  and  $0 \leq j \leq n+k-1$ , we have:*

- i)  $AJA^t = 0$ .
- ii) *The sheaf  $\ker A_L \subseteq \mathcal{O}_L^{2n+2k}$  has exactly  $n+k$  global sections.*
- iii) *The matrix  $A$  has rank  $k$  at every point  $x$  of  $\mathbb{P}$ .*

*Proof.* i) This part follows from the calculation

$$(F|H) \cdot J \cdot (F|H)^t = (F|H) \cdot \begin{pmatrix} H^t \\ -F^t \end{pmatrix} = FH^t - HF^t = 0.$$

ii) This is due to the fact that

$$A_L = (0|H_L)$$

has exactly  $n+k$  independent syzygies of degree 0. (In general, we denote by  $\text{Syz}_d(A)$  the space of relations  $\sum_{i=1}^q \alpha_i \cdot c_i = 0$  among the column vectors  $c_1, \dots, c_q$  of the matrix  $A$  with  $\deg(\alpha_i) = d$ ,  $i = 1, \dots, q$ ,  $d \geq 0$ .)

iii) We deduce this from the fact that the linear space  $L = \{f_0 = f_1 = \cdots = f_n = 0\}$  contains no zeroes of the maximal minors of  $H$ .  $\square$

**Definition 3.8.** A symplectic  $k$ -instanton bundle  $E$  on  $\mathbb{P}$  given by a monad

$$\mathcal{O}_{\mathbb{P}}(-1)^{\oplus k} \xrightarrow{JA^t} \mathcal{O}_{\mathbb{P}}^{2n+2k} \xrightarrow{A} \mathcal{O}_{\mathbb{P}}(1)^{\oplus k}$$

is called an *RS-instanton bundle*, if  $A$  is of the form (14) introduced above.

*Remark 3.9.* i) For  $n = 1$ , symplectic RS-instanton bundles were studied by Rao in [28] and by Skiti in [33]. They are characterized as rank 2 instanton bundles on  $\mathbb{P}^3$  with a jumping line of maximal order.

ii) Examples of symplectic RS-instanton bundle  $E$  on  $\mathbb{P}$  include the special symplectic instanton bundles introduced by Spindler and Trautmann in [34], Definition 4.1.

iii) Since the family of symplectic RS-instanton bundles is irreducible, it follows from the fact that special symplectic instanton bundles are stable [2] that there exists a non-empty open subset of stable symplectic RS-instanton bundles.

In the next proposition we are going to prove that the bound given in Proposition 2.5 is sharp. Indeed, we have

**Proposition 3.10.** *Let  $E$  be a symplectic RS-instanton bundle on  $\mathbb{P}$ . There exists an unstable  $n$ -dimensional linear subspace  $L \subset \mathbb{P}$  with maximal order of instability. Moreover, if  $E$  is generic, then  $L$  is unique.*

*Proof.* Let  $E$  be a symplectic RS-instanton bundle  $E$  on  $\mathbb{P}$  associated with an  $n$ -space  $L = \{f_0 = f_1 = \dots = f_n = 0\}$ . By Proposition 2.5,  $h^0(E|_L) \leq n + k$ . On the other hand, it is easy to see, using Proposition 3.7 and the display of the monad associated with  $E$ , that  $h^0(E|_L)$  equals the number of linearly independent linear syzygies of the matrix  $A$  evaluated on  $f_0 = f_1 = \dots = f_n = 0$ . Hence  $h^0(E|_L) \geq n + k$  and thus we get the equality.

It remains to prove the unicity. Let  $E$  be a generic symplectic RS-instanton bundle on  $\mathbb{P}$ . After a change of coordinates if necessary we can assume without loss of generality that  $E$  is the cohomology of the monad

$$(15) \quad \mathcal{O}_{\mathbb{P}}(-1)^{\oplus k} \xrightarrow{JA^t} \mathcal{O}_{\mathbb{P}}^{2n+2k} \xrightarrow{A} \mathcal{O}_{\mathbb{P}}(1)^{\oplus k}$$

where  $A = (F|H)$  with  $H = (h_{ij})$  a persymmetric matrix of linear forms and

$$F = \begin{pmatrix} x_0 & x_1 & \cdots & x_n & 0 & \cdots & \cdots & 0 \\ 0 & x_0 & x_1 & \cdots & x_n & 0 & \cdots & 0 \\ 0 & 0 & x_0 & x_1 & \cdots & x_n & \ddots & \vdots \\ \vdots & \cdots & \ddots & \ddots & \ddots & \cdots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & x_0 & x_1 & \cdots & x_n \end{pmatrix},$$

$x_0, \dots, x_{2n+1}$  being homogeneous coordinates on  $\mathbb{P}$ . Let  $L$  be the  $n$ -space  $\{x_0 = x_1 = \dots = x_n = 0\}$ . It follows from the first part of the proof that  $h^0(E|_L) = 2n + k - n = n + k$ . The fact that  $L$  is the unique  $n$ -space with this property results from the following

**Claim.** *For any  $n$ -space  $L' \subset \mathbb{P}$ ,  $L' \neq L$ ,  $h^0(E|_{L'}) < n + k$ .*

We now prove the claim. Since  $L' \neq L$ , there exists  $i \in \{0, \dots, n\}$ , such that  $L' \not\subseteq \{x_i = 0\}$ . Without loss of generality assume that  $i = 0$ . Associated with the monad (15), we have the two short exact sequences

$$0 \longrightarrow K \longrightarrow \mathcal{O}_{\mathbb{P}}^{2n+2k} \xrightarrow{A} \mathcal{O}_{\mathbb{P}}(1)^{\oplus k} \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}}(-1)^{\oplus k} \longrightarrow K \longrightarrow E \longrightarrow 0$$

where  $K$  stands for the kernel of  $A$ . Restricting both sequences to  $L'$  and taking cohomology, we see that  $h^0(E|_{L'}) = h^0(K|_{L'})$ . Since  $L' \not\subseteq \{x_0 = 0\}$  and the

linear forms  $h_{ij}$  are general, we have  $h^0(K|_{L'}) \leq n+k-1$ . This finishes the proof of the claim and the proposition.  $\square$

By means of the following example, we will see that there are symplectic RS-instanton bundles which are not symplectic 't Hooft bundles.

*Example 3.11.* Fix homogeneous coordinates  $x_0, \dots, x_n, y_0, \dots, y_n$  on  $\mathbb{P}$  and consider the linear  $n$ -space  $L \subset \mathbb{P}$  defined by  $x_0 = x_1 = \dots = x_n = 0$ . Consider the persymmetric matrix

$$H := \begin{pmatrix} y_{n-1} + y_n & 0 & \cdots & 0 & y_0 & \cdots & y_{n-1} & y_n \\ & 0 & \cdots & 0 & y_0 & \cdots & y_{n-1} & y_n & 0 \\ & \vdots & \ddots & \ddots & \cdots & \ddots & \ddots & \ddots & \vdots \\ & 0 & y_0 & \cdots & y_{n-1} & y_n & 0 & \cdots & 0 \\ y_0 & \cdots & y_{n-1} & y_n & 0 & \cdots & 0 & y_{n-1} - y_n \end{pmatrix}$$

and the matrix

$$F = \begin{pmatrix} x_0 & x_1 & \cdots & x_n & 0 & \cdots & \cdots & 0 \\ 0 & x_0 & x_1 & \cdots & x_n & 0 & \cdots & 0 \\ 0 & 0 & x_0 & x_1 & \cdots & x_n & \ddots & \vdots \\ \vdots & \cdots & \ddots & \ddots & \ddots & \cdots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & x_0 & x_1 & \cdots & x_n \end{pmatrix}.$$

Denote by  $E$  the RS-instanton bundle defined by the  $((2n+2k) \times k)$ -matrix  $A = (F|H)$ . Assume that  $k > n$ . Then,  $E$  is an RS-instanton bundle which is not of 't Hooft type. Indeed, this will follow from Proposition 3.5 and the following claim.

**Claim.**  $H^0(\mathbb{P}, E(1)) = 0$ .

*Proof.* By definition,  $E$  is the cohomology of the monad

$$\mathcal{O}_{\mathbb{P}}(-1)^{\oplus k} \xrightarrow{JA^t} \mathcal{O}_{\mathbb{P}}^{2n+2k} \xrightarrow{A} \mathcal{O}_{\mathbb{P}}(1)^{\oplus k}.$$

We have the two short exact sequences

$$\begin{aligned} 0 &\longrightarrow K \longrightarrow \mathcal{O}_{\mathbb{P}}^{2n+2k} \xrightarrow{A} \mathcal{O}_{\mathbb{P}}(1)^{\oplus k}, \\ 0 &\longrightarrow \mathcal{O}_{\mathbb{P}}(-1)^{\oplus k} \longrightarrow K \longrightarrow E \longrightarrow 0, \end{aligned}$$

where  $K$  denotes the kernel of  $A$ . Then,

$$\text{Syz}_1(A) = \ker\left(H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)^{\oplus(2n+2k)}) \longrightarrow H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2)^{\oplus k})\right)$$

is the space of the syzygies of  $A$  of degree one, and we define  $\text{syz}_1(A)$  as its dimension. Twisting  $K$  and  $E$  by  $\mathcal{O}_{\mathbb{P}}(1)$  and taking cohomology, we easily deduce that

$$h^0(\mathbb{P}, E(1)) = \text{syz}_1(A) - k.$$

Let us prove that the space of syzygies of degree one of  $A$  is generated by the following independent vectors:

$$\begin{aligned} v_1 &= (-y_{n-1} - y_n, \overbrace{0, \dots, 0}^{k-2}, -y_0, \dots, -y_n, x_0, \dots, x_n, \overbrace{0, \dots, 0}^{k-1})^t, \\ v_i &= (\overbrace{0, \dots, 0}^{k-i}, -y_0, \dots, -y_n, \overbrace{0, \dots, 0}^{2i-2}, x_0, \dots, x_n, \overbrace{0, \dots, 0}^{k-i})^t, \quad 2 \leq i \leq k-1, \\ v_k &= (-y_0, \dots, -y_n, \overbrace{0, \dots, 0}^{k-2}, -y_{n-1} + y_n, \overbrace{0, \dots, 0}^{k-1}, x_0, \dots, x_n)^t. \end{aligned}$$

Since  $A \cdot v_i = 0$ , it is clear that  $v_i$  belongs to the space of syzygies of degree one of  $A$ ,  $1 \leq i \leq k$ . Let us see that the other inclusion also holds. Let

$$v = (\alpha_1, \dots, \alpha_{n+k}, \beta_1, \dots, \beta_{n+k})^t \in \text{Syz}_1(A)$$

with

$$\alpha_i := \alpha_{x_0}^i x_0 + \alpha_{x_1}^i x_1 + \dots + \alpha_{x_n}^i x_n + \alpha_{y_0}^i y_0 + \alpha_{y_1}^i y_1 + \dots + \alpha_{y_n}^i y_n \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$$

$$\beta_i := \beta_{x_0}^i x_0 + \beta_{x_1}^i x_1 + \dots + \beta_{x_n}^i x_n + \beta_{y_0}^i y_0 + \beta_{y_1}^i y_1 + \dots + \beta_{y_n}^i y_n \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)).$$

Since  $v \in \text{Syz}_1(A)$ , we have

(16)

$$\begin{aligned} \alpha_1 x_0 + \alpha_2 x_1 + \dots + \alpha_{n+1} x_n + \beta_1 (y_{n-1} + y_n) + \beta_k y_0 + \dots + \beta_{n+k} y_n &= 0, \\ \alpha_i x_0 + \alpha_{i+1} x_1 + \dots + \alpha_{n+i} x_n + \beta_{k-i+1} y_0 + \dots + \beta_{n+k-i+1} y_n &= 0, \\ \alpha_k x_0 + \alpha_{k+1} x_1 + \dots + \alpha_{n+k} x_n + \beta_1 y_0 + \dots + \beta_{n+1} y_n + \beta_{n+k} (y_{n-1} - y_n) &= 0, \end{aligned}$$

$2 \leq i \leq k-1$ . Restricting (16) to  $y_0 = y_1 = \dots = y_n = 0$ , we get

$$\alpha_{x_j}^q = 0, \quad \text{for } 1 \leq q \leq n+k, \quad 0 \leq j \leq n,$$

and, restricting (16) to  $x_0 = x_1 = \dots = x_n = 0$ , we get

$$\beta_{y_j}^q = 0, \quad \text{for } 1 \leq q \leq n+k, \quad 0 \leq j \leq n.$$

Hence, the linear forms  $\beta_1, \dots, \beta_{n+k}$  only depend on  $x_0, \dots, x_n$  and the linear forms  $\alpha_1, \dots, \alpha_{n+k}$  only depend on  $y_0, \dots, y_n$ . On the other hand, since, in each of the equations in (16), the coefficient of  $x_j y_i$  must be zero,  $0 \leq i, j \leq n$ , we get the following relations:

- For  $0 \leq i \leq n-2$ ,  $0 \leq j \leq n$ ,

$$(17) \quad \alpha_{y_i}^{l+j} + \beta_{x_j}^{k-l+i+1} = 0, \quad 1 \leq l \leq k.$$

- For  $i = n-1$ ,  $0 \leq j \leq n$ ,

$$(18) \quad \begin{aligned} \alpha_{y_{n-1}}^{j+1} + \beta_{x_j}^1 + \beta_{x_j}^{k+n-1} &= 0, \\ \alpha_{y_{n-1}}^{j+l} + \beta_{x_j}^{k+n-l} &= 0, \quad 2 \leq l \leq k-1, \\ \alpha_{y_{n-1}}^{j+k} + \beta_{x_j}^n + \beta_{x_j}^{k+n} &= 0. \end{aligned}$$

- For  $i = n$ ,  $0 \leq j \leq n$ ,

$$(19) \quad \begin{aligned} \alpha_{y_n}^{j+1} + \beta_{x_j}^1 + \beta_{x_j}^{k+n} &= 0, \\ \alpha_{y_n}^{j+l} + \beta_{x_j}^{k+n-l+1} &= 0, \quad 2 \leq l \leq k-1, \\ \alpha_{y_n}^{j+k} + \beta_{x_j}^{n+1} - \beta_{x_j}^{k+n} &= 0. \end{aligned}$$

From these equations, we deduce that

$$(20) \quad \begin{aligned} \beta_{x_i}^i &= \beta_{x_{i+1}}^i = \cdots = \beta_{x_n}^i = 0, \quad 1 \leq i \leq n, \\ \beta_{x_0}^i &= \beta_{x_1}^i = \cdots = \beta_{x_{i-k-1}}^i = 0, \quad k+1 \leq i \leq n+k, \\ \alpha_{y_i}^i &= \alpha_{y_{i+1}}^i = \cdots = \alpha_{y_n}^i = 0, \quad 2 \leq i \leq n, \\ \alpha_{y_0}^i &= \alpha_{y_1}^i = \cdots = \alpha_{y_{i-k-1}}^i = 0, \quad k+1 \leq i \leq n+k-1, \end{aligned}$$

and that

$$\begin{aligned} \alpha_{y_1}^1 &= \alpha_{y_2}^1 = \cdots = \alpha_{y_{n-2}}^1 = 0, \\ \alpha_{y_0}^{n+k} &= \alpha_{y_1}^{n+k} = \alpha_{y_2}^{n+k} = \cdots = \alpha_{y_{n-2}}^{n+k} = 0. \end{aligned}$$

Putting everything together, we get

$$v = \sum_{l=1}^k \beta_{x_n}^{n+l} v_l,$$

i.e.,  $v$  is a linear combination of  $v_1, \dots, v_k$ . This finishes the proof of the claim.  $\square$

*Remark 3.12.* The fact that there are symplectic 't Hooft bundles which are not symplectic RS-instanton bundles follows from Sections 5 and 6 where we prove that the family of symplectic 't Hooft bundles on  $\mathbb{P}$  with charge  $k$  is irreducible of dimension  $5kn + 4n^2$  and the family of symplectic RS-instanton bundles on  $\mathbb{P}$  with charge  $k$  is irreducible of dimension  $(4n+2) \cdot k + 4n^2 + 2n - 4$ .

#### 4. DEFORMATIONS OF GENERIC SYMPLECTIC 'T HOOFT INSTANTON BUNDLES

Let  $E$  be a generic symplectic 't Hooft  $k$ -instanton bundle over  $\mathbb{P}$ . Ottaviani claims in [27] that every infinitesimal deformation of  $E$  as a symplectic bundle comes from a deformation as a symplectic 't Hooft bundle for  $n \geq 2$  and  $k \geq 9$ . This would mean that (the closure of) the locus of 't Hooft bundles is an irreducible component of the moduli space of symplectic instanton bundles and be very interesting in view of our rationality result 5.10. Unfortunately, there is not much of a proof for that in [27], just a reference to a future paper which has not yet appeared.

In this section we will see that, at least for small values of  $k$  and/or  $n$ , this is not the case. Let  $A$  be a matrix of the form (5) above, with generic parameters  $a$  and  $l_j, l'_j$ . Then as we already quoted,  $A$  defines a generic symplectic 't Hooft instanton bundle  $E$ , namely the cohomology of the monad (1). An infinitesimal deformation

$$A + \varepsilon X \quad \text{with} \quad X \in \text{Mat}_{k \times (2n+2k)} \left( H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) \right)$$

of  $A$  over  $\mathbb{C}[\varepsilon^2]$  with  $\varepsilon^2 = 0$  still defines a symplectic instanton bundle if and only if

$$(A + \varepsilon X) \cdot J \cdot (A + \varepsilon X)^t = 0.$$

Since  $\varepsilon^2 = 0$ , this is equivalent to  $AJX^t + XJA^t = 0$ . Using  $J^t = -J$ , is also equivalent to the following condition:

The matrix  $A \cdot J \cdot X^t \in \text{Mat}_{k \times k}(H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2)))$  is symmetric.

The vector space of all matrices  $X$  of this format has dimension  $k \cdot (2n + 2k) \cdot (2n + 2)$ . The symmetry in question amounts to  $k \cdot (k - 1)/2$  linear conditions in  $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2))$ , and hence to  $(k \cdot (k - 1)/2) \cdot ((2n + 2) \cdot (2n + 3)/2)$  linear conditions in  $\mathbb{C}$ ; hence

$$\begin{aligned} \dim_{\mathbb{C}} \left\{ X \in \text{Mat}_{k \times (2n+2k)} \left( H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) \right) \mid AJX^t \text{ is symmetric} \right\} \\ \geq k \cdot (2n + 2k) \cdot (2n + 2) - \frac{k(k-1)}{2} \cdot \frac{(2n+2) \cdot (2n+3)}{2}. \end{aligned}$$

Note that this lower bound becomes negative, and hence trivially satisfied, if  $n$  and  $k$  are large. In particular, this shows that the conditions are then not independent.

Such an infinitesimal deformation of  $A$  gives the trivial deformation of  $E$  if and only if it comes from the infinitesimal action of  $\text{GL}_k(\mathbb{C}) \times \text{Sp}_{2n+2k}(\mathbb{C})$ . This infinitesimal action is free, since the group of symplectic automorphisms of the generic  $E$  is finite. Hence the vector space of deformations of  $E$  as a symplectic instanton bundle has dimension at least

$$\begin{aligned} k(2n+2k)(2n+2) - \frac{k(k-1)}{2} \cdot \frac{(2n+2)(2n+3)}{2} - k^2 - \frac{(2n+2k)(2n+2k+1)}{2} \\ = (n+k) \cdot (2n+1) \cdot (2k-1) - \frac{k \cdot (k-1)}{2} \cdot \frac{(2n+2) \cdot (2n+3)}{2} - k^2 \\ = (n+k) \cdot (2n+1) \cdot (2k-1) - \frac{k \cdot (k-1)}{2} \cdot (2n^2 + 5n + 3) - k^2 \\ = -k^2 \cdot \frac{2n^2 - 3n + 1}{2} + k \cdot \frac{10n^2 + 5n + 1}{2} - 2n^2 - n \\ = -(2n-1) \cdot (n-1) \cdot \frac{k \cdot (k-1)}{2} + 4n \cdot (n+1) \cdot k - (2n+1) \cdot n \\ = -n^2 \cdot (k^2 - 5k + 2) + n \cdot \frac{3k^2 + 5k - 2}{2} - \frac{k^2 - k}{2}. \end{aligned}$$

On the other hand, the moduli space of symplectic 't Hooft instanton bundles is generically smooth of dimension  $5kn + 4n^2$ . Hence, a necessary condition for every deformation of a generic 't Hooft bundle being an 't Hooft bundle, too, is that

$$-n^2 \cdot (k^2 - 5k + 6) + n \cdot \frac{3k^2 - 5k - 2}{2} - \frac{k^2 - k}{2} \leq 0.$$

Because of the leading term, this necessary condition is satisfied for  $n, k \gg 0$ . Concerning smaller values, it

- is false for  $n = 2$  and  $3 \leq k \leq 8$ ,
- holds for  $n = 2$  and  $k \geq 9$ ,
- is false for  $n = 3$  and  $3 \leq k \leq 5$ ,
- holds for  $n = 3$  and  $k \geq 6$ ,
- is false for  $k = 3$  and  $n \gg 0$ ,
- holds for  $k \geq 4$  and  $n \gg 0$ .

In the cases where this inequality is false, every 't Hooft bundle will have deformations which aren't 't Hooft bundles.

*Example 4.1.* The claim of Ottaviani in question is also equivalent to

$$\begin{aligned} & \dim_{\mathbb{C}} \{ X \in \text{Mat}_{k \times (2n+2k)} (H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))) \mid A \cdot J \cdot X^t \text{ is symmetric} \} \\ &= (5kn + 4n^2) + \dim(\text{GL}_k(\mathbb{C}) \times \text{Sp}_{2n+2k}(\mathbb{C})) \\ &= (n+k) \cdot (6n+3k+1) \end{aligned}$$

where  $A$  is still a matrix of the form (5) above, with generic parameters  $a$  and  $l_j, l'_j$ . Using the MAPLE, we checked this claim for the following values of  $(n, k)$ :

$$(2, 9), (2, 10), (3, 6), (3, 7), (3, 8), (3, 9), (4, 5), (4, 6), (4, 7), (5, 5), (5, 6).$$

In these cases, the closure of the locus of 't Hooft bundles is an irreducible component of the moduli space of symplectic instanton bundles.

## 5. THE MODULI SPACE OF 'T HOOFT INSTANTON BUNDLES AND ITS BIRATIONAL TYPE

Consider the symplectic vector space

$$U := \mathbb{C}^2$$

and the vector space

$$V := \mathbb{C}^k.$$

We identify the matrix  $a$  in (2) with an element

$$a \in V^{\oplus(n+k)}.$$

For each  $j$ , we identify the pair  $l_j, l'_j \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$  in (3) and (4) with a linear map

$$U^* \longrightarrow H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)), \quad j \in \{1, \dots, n+k\}.$$

Choosing a basis of  $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$  identifies these maps with elements  $L_j \in U^{\oplus(2n+2)}$ ,  $j = 1, \dots, n+k$ . We put  $L = (L_j)_{j=1, \dots, n+k} \in (U^{\oplus(2n+2)})^{\oplus(n+k)}$ . For each element in the vector space

$$(U^{\oplus(2n+2)} \oplus V)^{\oplus(n+k)},$$

we thus obtain a linear map

$$A: U^* \otimes \mathcal{O}_{\mathbb{P}}^{\oplus(n+k)} \xrightarrow{\text{diag}(L)} \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n+k)} \xrightarrow{a} V \otimes \mathcal{O}_{\mathbb{P}}(1)$$

as in (5). It satisfies  $AJA^t = 0$  for the standard symplectic form  $J$  on  $(U^*)^{\oplus(n+k)}$ .

Consider the linear algebraic group

$$G_{n,k} := (\mathrm{SL}(U) \times \mathbb{G}_m) \wr S_{n+k} \times \mathrm{GL}(V).$$

Recall that the wreath product  $(\mathrm{SL}(U) \times \mathbb{G}_m) \wr S_{n+k}$  is the semidirect product

$$1 \longrightarrow (\mathrm{SL}(U) \times \mathbb{G}_m)^{\times(n+k)} \longrightarrow (\mathrm{SL}(U) \times \mathbb{G}_m) \wr S_{n+k} \xleftarrow{\quad} S_{n+k} \longrightarrow 1$$

where  $S_{n+k}$  acts on  $(\mathrm{SL}(U) \times \mathbb{G}_m)^{\times(n+k)}$  by permuting the factors.

We can write each element  $g \in G_{n,k}$  in the form

$$g = (\sigma \cdot (\beta_j, \gamma_j)_j, \alpha)$$

with  $\alpha \in \mathrm{GL}(V)$ ,  $\beta_j \in \mathrm{SL}(U)$ ,  $\gamma_j \in \mathbb{G}_m$  and  $\sigma \in S_{n+k}$ . It induces an isomorphism

$$\begin{array}{ccccc} U^* \otimes \mathcal{O}_{\mathbb{P}}^{\oplus(n+k)} & \xrightarrow{\mathrm{diag}(g \cdot L)} & \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n+k)} & \xrightarrow{g \cdot a} & V \otimes \mathcal{O}_{\mathbb{P}}(1) \\ \sigma \circ (\beta_j^*)_j \downarrow & & \downarrow \sigma \circ (\gamma_j)_j & & \downarrow \alpha \\ U^* \otimes \mathcal{O}_{\mathbb{P}}^{\oplus(n+k)} & \xrightarrow{\mathrm{diag}(L)} & \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n+k)} & \xrightarrow{a} & V \otimes \mathcal{O}_{\mathbb{P}}(1) \end{array}$$

with

$$(21) \quad (g \cdot a)_j := \alpha^{-1}(\gamma_j a_{\sigma(j)}),$$

$$(22) \quad (g \cdot L)_j := \gamma_j^{-1} \beta_j(L_{\sigma(j)}).$$

These formulas define a linear action of  $G_{n,k}$  on  $(U^{\oplus(2n+2)} \oplus V)^{\oplus(n+k)}$ . The subgroup

$$\mu_2 \hookrightarrow G_{n,k}$$

given by the diagonal embedding  $\mu_2 \hookrightarrow (\mathrm{SL}(U) \times \mathbb{G}_m)^{\times(n+k)} \times \mathrm{GL}(V)$  acts trivially.

Proposition 3.1, iii), states  $R_{n,k} \neq \emptyset$  for the open locus

$$R_{n,k} \subseteq (U^{\oplus(2n+2)} \oplus V)^{\oplus(n+k)}$$

where  $A$  has rank  $k$  at every point in  $\mathbb{P}$ . The subset  $R_{n,k}$  is preserved by  $G_{n,k}$ . Each point in  $R_{n,k}$  defines a symplectic monad (1) whose cohomology is a symplectic  $k$ -instanton bundle  $E$ ; it is constant up to isomorphism on the  $G_{n,k}$ -orbits. Note that  $E$  is stable, by [27], Theorem 3.6. We will call the points of  $R_{n,k}$  *stable 't Hooft data* in the sequel. We leave it to the reader to formulate the moduli problem of stable 't Hooft data; compare for example [18], Remark 3.5. In the following, we will demonstrate that the geometric quotient

$$R_{n,k}/G_{n,k}$$

exists as a smooth quasi-projective variety. For this, we recall some details of the construction of the moduli space  $\mathrm{MI}_{\mathbb{P}^{2n+1}}(k)$  of stable symplectic  $k$ -instanton bundles on  $\mathbb{P}^{2n+1}$  contained in [26].



**Theorem 5.1** (The first fundamental theorem of invariant theory for symplectic groups). *Let  $E$  and  $F$  be finite dimensional complex vector spaces,  $\varphi: F \rightarrow F^\vee$  a symplectic form on  $F$ , and  $S$  the isometry group of  $(F, \varphi)$ . Then, the  $S$ -invariant map*

$$\begin{aligned} \kappa: \operatorname{Hom}(E, F) &\longrightarrow \operatorname{Hom}_{\operatorname{AS}}(E, E^\vee) \\ f &\longmapsto f^\vee \circ \varphi \circ f \end{aligned}$$

*induces a closed embedding of the categorical quotient of the vector space  $\operatorname{Hom}(E, F)$  by the action of  $S$  into  $\operatorname{Hom}_{\operatorname{AS}}(E, E^\vee)$ , the sub vector space of anti-symmetric homomorphisms. The image consists of those homomorphisms whose rank is less than or equal to  $\min\{\dim(E), \dim(F)\}$ .*

*Proof.* [14], Theorem 5.2.2 and Lemma 5.2.4. □

In the study of monads, we look at matrices in the vector space

$$M := \operatorname{Hom}(U^{\oplus(n+k)}, V^{\oplus(2n+2)})$$

and, in the study of symplectic monads, at matrices in the closed subvariety

$$(23) \quad \operatorname{SM} := \{ A \in M \mid AJA^t = 0 \}.$$

According to Theorem 5.1,

$$\begin{aligned} \kappa: \mathbb{P}(M) &\longrightarrow \mathbb{P}\left(\operatorname{Hom}_{\operatorname{AS}}\left((V^{\oplus(2n+2)})^\vee, V^{\oplus(2n+2)}\right)\right) \\ [A] &\longmapsto [K(A): (V^{\oplus(2n+2)})^\vee \xrightarrow{A^\vee} (U^{\oplus(n+k)})^\vee \xrightarrow{J} U^{\oplus(n+k)} \xrightarrow{A} V^{\oplus(2n+2)}] \end{aligned}$$

is a model for the categorical quotient of  $\mathbb{P}(M)$  by the given  $\operatorname{Sp}_{2(n+k)}(\mathbb{C})$ -action, possibly followed by a closed embedding.

*Remark 5.2.* One calls  $K(A)$  the *Kronecker module* of  $A$ . We may view  $K(A)$  as an element of

$$H := \operatorname{Hom}\left(\bigwedge^2(\mathbb{C}^{2n+2}), \operatorname{Hom}(V^\vee, V)\right).$$

If  $A \in \operatorname{SM}$ , then  $K(A)$  takes values in the sub vector space  $\operatorname{Hom}_S(V^\vee, V)$  of symmetric homomorphisms, i.e.,  $K(A)$  is a symmetric Kronecker module (compare [26], p. 39).

Finally, one has to study the  $\operatorname{SL}(V)$ -action on  $\mathbb{P}(H)$ . By work of Hulek [19], a point in  $\mathbb{P}(H)$  which comes from a monad whose cohomology is a stable instanton bundle is  $\operatorname{SL}(V)$ -stable ([26], Lemmas 1.11 and 1.12). Let

$$\operatorname{SI}_k \subset \mathbb{P}(H)$$

be the  $\operatorname{SL}(V)$ -invariant locally closed subset of points coming from monads with stable  $k$ -instanton bundles as cohomology. By the aforementioned result of Hulek, the geometric quotient

$$\operatorname{MI}_{\mathbb{P}^{2n+1}}(k) := \operatorname{SI}_k / \operatorname{SL}(V)$$

exists as quasi-projective variety. It is the moduli space of stable  $k$ -instanton bundles on  $\mathbb{P}^{2n+1}$ .

*Remark 5.3.* Set

$$\tilde{\mathbb{S}}I_k := \mathbb{P}(\text{SM}) \cap \kappa^{-1}(\text{SI}_k)$$

and let  $\mathcal{M}_{\mathbb{P}^{2n+1}}(k)$  be the moduli space of stable vector bundles with Chern character  $1/(1-t^2)^k$  on  $\mathbb{P} = \mathbb{P}^{2n+1}$ . We have the induced morphism

$$\tilde{\mathbb{S}}I_k \longrightarrow \mathcal{M}_{\mathbb{P}^{2n+1}}(k).$$

Since this morphism factorizes over the categorical quotient of  $\tilde{\mathbb{S}}I_k$  by the action of  $\text{Sp}_{2n}(\mathbb{C}) \times \mathbb{P}\text{GL}(V)$ , Proposition 2.4, ii), shows that this quotient is actually a geometric one. Remark 2.2 implies further that the quotient map (compare [23], Proposition 1.3.1, or [31], Theorem 1.5.3.1, i)

$$\tilde{\mathbb{S}}I_k \longrightarrow \tilde{\mathbb{S}}I_k / (\text{Sp}_{2n}(\mathbb{C}) \times \mathbb{P}\text{GL}(V)) = \tilde{\mathbb{S}}I_k / (\text{Sp}_{2n}(\mathbb{C}) \times \text{SL}(V)) = \text{SI}_k / \text{SL}(V)$$

is a principal  $(\text{Sp}_{2n}(\mathbb{C}) \times \mathbb{P}\text{GL}(V))$ -bundle.

We now look at the linear  $G_{n,k}$ -action on

$$W_{n,k} := (U^{\oplus(2n+2)} \oplus V)^{\oplus(n+k)}.$$

The subgroup

$$\mathbb{G}_m^{\times(n+k+1)} \cong \mathbb{G}_m^{n+k} \times \mathcal{L}(\text{GL}(V)) \subset G_{n,k}$$

is normal, and we set

$$H_{n,k} := G_{n,k} / \mathbb{G}_m^{\times(n+k+1)}.$$

Let us first look at the  $\mathbb{G}_m^{\times(n+k+1)}$ -action on  $W_{n,k}$ . We use the isogeny

$$\begin{aligned} \mathbb{G}_m^{\times(n+k)} \times \mathbb{G}_m &\longrightarrow \mathbb{G}_m^{\times(n+k)} \times \mathbb{G}_m \\ ((\gamma_j)_{j=1,\dots,n+k}, z) &\longmapsto ((z^{-1} \cdot \gamma_j)_{j=1,\dots,n+k}, z^2). \end{aligned}$$

Then, the last factor just acts by multiplying everything by  $z$ ,  $z \in \mathbb{G}_m$ . The  $\mathbb{G}_m^{\times(n+k+1)}$ -semistable points in  $W_{n,k}$  correspond to the  $\mathbb{G}_m^{\times(n+k)}$ -semistable points in  $\mathbb{P}(W_{n,k})$ , and these are easy to describe.

**Lemma 5.4.** *For a point  $p = [(L_j)_{j=1,\dots,n+k}, a] \in \mathbb{P}(W_{n,k})$ , the following conditions are equivalent:*

- i) *The point  $p$  is semistable with respect to the action of  $\mathbb{G}_m^{\times(n+k)}$ .*
- ii) *The point  $p$  is stable with respect to the action of  $\mathbb{G}_m^{\times(n+k)}$ .*
- iii)  $\forall j \in \{1, \dots, n+k\} : a_j \neq 0$  and  $L_j \neq 0$ .

As before, we may view a tuple  $(L_j)_{j=1,\dots,n+k} \in (U^{\oplus(2n+2)})^{\oplus(n+k)}$  as a matrix  $(D|D')$  where  $D$  and  $D'$  are diagonal matrices of the format  $(n+k) \times (n+k)$  with entries in  $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$ . We define

$$\begin{aligned} \alpha : W_{n,k} &\longrightarrow \text{M} \\ ((L_j)_{j=1,\dots,n+k}, a) &\longmapsto a \cdot (D|D'). \end{aligned}$$

The map  $\alpha$  is invariant under the  $\mathbb{G}_m^{\times(n+k)}$ -action, and the semistability condition in Lemma 5.4 clearly implies that  $\alpha(p) \neq 0$  holds for every  $\mathbb{G}_m^{\times(n+k)}$ -semistable point  $p = ((L_j)_{j=1, \dots, n+k}, a) \in W_{n,k}$ . Therefore,  $\alpha$  descends to a morphism

$$\bar{\alpha}: \bar{W}_{n,k} := W_{n,k}/\mathbb{G}_m^{\times(n+k+1)} \longrightarrow \mathbb{P}(M)$$

between projective varieties, and the image of  $\bar{\alpha}$  is contained in the closed subvariety  $\mathbb{P}(SM)$ . Using Lemma 5.4, one checks that  $\bar{\alpha}$  is injective, too.

The group  $H_{n,k}$  acts on the quotient  $\bar{W}_{n,k}$ . There is a natural linearization of this action. Note that the vector space  $U$  is equipped with the symplectic form  $U \times U \longrightarrow \mathbb{C}$ ,  $(u, u') \longmapsto \det(u|u')$ . Our convention is also such that the symplectic form on  $U^{\oplus(n+k)}$  is the direct sum of the symplectic forms on the summands. In this way,  $H_{n,k}$  becomes a subgroup of  $\mathrm{Sp}_{2n}(\mathbb{C}) \times \mathbb{P}\mathrm{GL}(V)$ .

*Remark 5.5.* It is straightforward to compute the GIT-semistable and stable points for the  $H_{n,k}$ -action on  $\bar{W}_{n,k}$ . Unfortunately, the condition one finds is weaker than the condition of  $(\mathrm{Sp}_{2n}(\mathbb{C}) \times \mathbb{P}\mathrm{GL}(V))$ -semistability and stability on  $\mathbb{P}(M)$ . For example, using the first fundamental theorem of invariant theory (Theorem 5.1), we can determine the  $\mathrm{SL}(U)^{\times(n+k)}$ - and  $\mathrm{Sp}_{2(n+k)}(\mathbb{C})$ -semistable points. The  $\mathrm{Sp}_{2(n+k)}(\mathbb{C})$ -semistable points in  $\mathbb{P}(M)$  are just those points

$$[A] \in \mathbb{P}(M), \quad A: U^{\oplus(n+k)} \longrightarrow V^{\oplus(2n+2)},$$

for which the image of  $A^\vee$  is not an isotropic subspace of  $(U^{\oplus(n+k)})^\vee$ . Viewing  $A$  as a tuple of linear maps

$$A_i: U \longrightarrow V^{\oplus(2n+2)}, \quad i = 1, \dots, n+k,$$

we see that  $[A]$  is  $\mathrm{SL}(U)^{\times(n+k)}$ -semistable, if and only if the image of  $A_i^\vee$  is not an isotropic subspace of  $U$ ,  $i = 1, \dots, n+k$ . It is now clear that the  $\mathrm{SL}(U)^{\times(n+k)}$ -semistability of  $[A]$  will in general not imply its  $\mathrm{Sp}_{2n}(\mathbb{C})$ -semistability, not even if  $[A]$  lies in the image of  $\bar{\alpha}$ .

Unfortunately, this means that the GIT-quotient  $\bar{W}_{n,k} // H_{n,k}$  will not map to the GIT-quotient  $\mathbb{P}(M) // (\mathrm{Sp}_{2n}(\mathbb{C}) \times \mathbb{P}\mathrm{GL}(V))$ . This makes it harder to understand the map from the moduli space of stable 't Hooft data to the moduli space of stable symplectic instanton bundles (see Problem 5.8).

On the positive side, the facts that  $\bar{\alpha}$  is finite and equivariant and that  $H_{n,k}$  is a subgroup of  $\mathrm{Sp}_{2n}(\mathbb{C}) \times \mathbb{P}\mathrm{GL}(V)$  imply that a point  $p \in W_{n,k}/\mathbb{G}_m^{\times(n+k)}$  whose image  $\bar{\alpha}(p) \in \mathbb{P}(M)$  is  $\mathrm{Sp}_{2n}(\mathbb{C}) \times \mathbb{P}\mathrm{GL}(V)$ -stable is  $H_{n,k}$ -stable. This applies, in particular, to the points of the open subset

$$\bar{R}_{n,k} := R_{n,k}/\mathbb{G}_m^{\times(n+k+1)} \subset W_{n,k}/\mathbb{G}_m^{\times(n+k+1)} = \bar{W}_{n,k}$$

(compare Remark 5.3). Therefore, the geometric quotient

$$\bar{R}_{n,k}/H_{n,k}$$

exists as a quasi-projective variety. Remark 5.3 also implies that the quotient map

$$\bar{R}_{n,k} \longrightarrow \bar{R}_{n,k}/H_{n,k}$$

is a principal  $H_{n,k}$ -bundle. Since  $R_{n,k} \longrightarrow \bar{R}_{n,k}$  is a  $\mathbb{G}_m^{\times(n+k+1)}$ -bundle, the variety  $\bar{R}_{n,k}/H_{n,k}$  is actually smooth. As in [23], Proposition 1.3.1, or [31], Theorem 1.5.3.1, i), one checks

$$\bar{R}_{n,k}/H_{n,k} \cong \mathbb{P}(W_{n,k})/(G_{n,k}/\mathbb{G}_m) \cong W_{n,k}/G_{n,k}.$$

In the study of the birational geometry of this quotient, we will need the following open subset: Let

$$R_{n,k}^0 \subseteq R_{n,k} \subseteq (U^{\oplus(2n+2)} \oplus V)^{\oplus(n+k)}$$

be the open locus where moreover  $l_j$  and  $l'_j$  are linearly independent for each  $j$ , the  $n+k$  linear subspaces  $\{l_j = l'_j = 0\} \subseteq \mathbb{P}$  are distinct, and

$$\ker A(1) \subseteq U^* \otimes \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n+k)}$$

has exactly  $n+k$  global sections. If  $k \geq 3$ , then  $R_{n,k}^0 \neq \emptyset$  due to Proposition 3.1, iv).

**Lemma 5.6.** *Given two points*

$$(a, L), (\tilde{a}, \tilde{L}) \in R_{n,k}^0,$$

*with corresponding symplectic instanton bundles  $E, \tilde{E}$ , every isomorphism*

$$\varphi: \tilde{E} \longrightarrow E$$

*is induced by a unique group element  $g \in G_{n,k}$  with  $g \cdot (a, L) = (\tilde{a}, \tilde{L})$ .*

*Proof.* The isomorphism  $\varphi$  comes from a unique isomorphism of symplectic monads

$$\begin{array}{ccc} U^* \otimes \mathcal{O}_{\mathbb{P}}^{\oplus(n+k)} & \xrightarrow{\tilde{A}} & V \otimes \mathcal{O}_{\mathbb{P}}(1) \\ \beta^* \downarrow & & \downarrow \alpha \\ U^* \otimes \mathcal{O}_{\mathbb{P}}^{\oplus(n+k)} & \xrightarrow{A} & V \otimes \mathcal{O}_{\mathbb{P}}(1) \end{array}$$

with  $\alpha \in \mathrm{GL}(V)$  and  $\beta \in \mathrm{Sp}(U^{\oplus(n+k)})$ ; here  $A := a \circ \mathrm{diag}(L)$  and  $\tilde{A} := \tilde{a} \circ \mathrm{diag}(\tilde{L})$ .

As we have seen in the proof of Proposition 3.1, i) and ii), the  $n+k$  global sections of  $\ker A(1)$  and of  $\ker \tilde{A}(1)$  are given by the two maps

$$J \circ \mathrm{diag}(L)^t \text{ and } J \circ \mathrm{diag}(\tilde{L})^t: \mathcal{O}_{\mathbb{P}}^{\oplus(n+k)} \longrightarrow U^* \otimes \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n+k)}.$$

Since our isomorphism of symplectic monads has to respect these, it has the form

$$\begin{array}{ccccc} U^* \otimes \mathcal{O}_{\mathbb{P}}^{\oplus(n+k)} & \xrightarrow{\text{diag}(\tilde{L})} & \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n+k)} & \xrightarrow{\tilde{a}} & V \otimes \mathcal{O}_{\mathbb{P}}(1) \\ \beta^* \downarrow & & \downarrow \gamma & & \downarrow \alpha \\ U^* \otimes \mathcal{O}_{\mathbb{P}}^{\oplus(n+k)} & \xrightarrow{\text{diag}(L)} & \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n+k)} & \xrightarrow{a} & V \otimes \mathcal{O}_{\mathbb{P}}(1) \end{array}$$

with  $\gamma \in \text{GL}_{n+k}(\mathbb{C})$ . The loci in  $\mathbb{P}$  where  $\text{diag}(L)$  and  $\text{diag}(\tilde{L})$  are not surjective are

$$\bigcup_j \{l_j = l'_j = 0\} \quad \text{and} \quad \bigcup_j \{\tilde{l}_j = \tilde{l}'_j = 0\}.$$

Since our isomorphisms  $\beta$  and  $\gamma$  have to respect these, they are of the form

$$\beta = \sigma \circ (\beta_j)_j \quad \text{and} \quad \gamma = \sigma \circ (\gamma_j)_j$$

with  $\beta_j \in \text{GL}(U)$ ,  $\gamma_j \in \mathbb{G}_m$  and  $\sigma \in S_{n+k}$ . Here,  $\beta_j \in \text{SL}(U)$  as  $\beta$  is symplectic.  $\square$

We summarize our discussion:

**Proposition 5.7.** *Fix positive integers  $n$  and  $k$ . Then, the moduli space*

$$\text{HI}_{\mathbb{P}^{2n+1}}(k) := R_{n,k}/G_{n,k}$$

*of stable 't Hooft data exists as a smooth quasi-projective scheme and comes with a generically injective morphism  $\iota$  to the moduli space  $\text{MI}_{\mathbb{P}^{2n+1}}(k)$  of stable symplectic  $k$ -instanton bundles on  $\mathbb{P}^{2n+1}$ . The moduli space  $\text{HI}_{\mathbb{P}^{2n+1}}(k)$  contains*

$$\text{HI}_{\mathbb{P}^{2n+1}}^0(k) := R_{n,k}^0/G_{n,k}$$

*as a dense open subscheme. The morphism  $\iota$  is injective on  $\text{HI}_{\mathbb{P}^{2n+1}}^0(k)$ .*

*Problem 5.8.* i) Describe the fibers of the morphism  $\text{HI}_{\mathbb{P}^{2n+1}}(k) \longrightarrow \text{MI}_{\mathbb{P}^{2n+1}}(k)$ .

ii) Is the morphism  $\text{HI}_{\mathbb{P}^{2n+1}}(k) \longrightarrow \text{MI}_{\mathbb{P}^{2n+1}}(k)$  proper, or at least its image closed?

We now investigate the birational geometry of the moduli space  $\text{HI}_{\mathbb{P}^{2n+1}}(k)$ . In what follows by the *quotient of a vector space* (here it will be  $\text{Mat}_{2^e \times 2^e}^{\text{sym}}(\mathbb{C})^2$ ) *modulo a group* (here it will be  $\mathbb{P}\text{O}_{2^e}$ ) *that acts generically freely*, we mean the quotient of an open subvariety where the action is free; this is well-defined up to birational equivalence.

**Theorem 5.9.** *Assume  $k \geq 3$ . The coarse moduli scheme  $R_{n,k}^0/G_{n,k}$  of 'generic' 't Hooft data on  $\mathbb{P}^{2n+1}$  is birational to*

$$\mathbb{C}^{5kn+4n^2-2^e(2^e+3)/2} \times \left( \frac{\text{Mat}_{2^e \times 2^e}^{\text{sym}}(\mathbb{C})^2}{\mathbb{P}\text{O}_{2^e}} \right)$$

*where  $2^e$  is the largest power of 2 that divides both  $n$  and  $k$ .*

*Here the projective orthogonal group  $\mathbb{P}\text{O}_{2^e} := \text{O}_{2^e}/\mu_2$  acts on the vector space  $\text{Mat}_{2^e \times 2^e}^{\text{sym}}(\mathbb{C})^2$  of pairs of symmetric matrices by simultaneous conjugation.*

**Corollary 5.10.** *Assume  $k \geq 3$  and let  $R_{n,k}^0/G_{n,k}$  be the coarse moduli scheme of ‘generic’ t Hooft data on  $\mathbb{P}^{2n+1}$ . It holds:*

- i) *If  $\gcd(n, k) \not\equiv 0 \pmod{4}$ , then  $R_{n,k}^0/G_{n,k}$  is rational.*
- ii) *If  $\gcd(n, k) \not\equiv 0 \pmod{16}$ , then  $R_{n,k}^0/G_{n,k}$  is stably rational.*

*Proof.* It follows from Theorem 5.9 and the fact that the quotient  $\text{Mat}_{2^e \times 2^e}^{\text{sym}}(\mathbb{C})^2/\mathbb{P}\text{O}_{2^e}$  is rational for  $2^e = 1$  and for  $2^e = 2$ . It is stably rational for  $2^e = 4$  [30] and for  $2^e = 8$  [6].  $\square$

*Problem 5.11.* It would be nice to determine the level of stable rationality of the variety  $R_{n,k}^0/G_{n,k}$ .

*Proof of Theorem 5.9.* By Lemma 5.6 and Proposition 5.7, we have

$$\frac{R_{n,k}^0}{G_{n,k}} \simeq \frac{(U^{\oplus(2n+2)} \oplus V)^{\oplus(n+k)}}{G_{n,k}/\mu_2}$$

where  $G_{n,k} = (\text{SL}(U) \times \mathbb{G}_m) \wr S_{n+k} \times \text{GL}(V)$  acts via the formulas (21) and (22).

Let  $u_1, u_2$  be a basis of  $U$ , and put  $u = (u_1, u_2) \in U^2$ . Then the  $G_{n,k}$ -orbit of  $(u, \dots, u) \in (U^2)^{n+k}$  is open, with stabilizer  $(\mu_2 \wr S_{n+k}) \times \text{GL}(V) \subseteq G_{n,k}$ . Hence

$$\frac{R_{n,k}^0}{G_{n,k}} \simeq \frac{(U^{2n} \oplus V)^{n+k}}{((\mu_2 \wr S_{n+k}) \times \text{GL}(V))/\mu_2}$$

where each copy of  $\mu_2$  acts trivially on  $U^{2n}$  and via its nontrivial character on  $V$ .

Here the group action is already generically free on the direct summand  $V^{n+k}$ , because the finite group  $(\mu_2 \wr S_{n+k})/\mu_2$  acts effectively, and hence generically freely, on  $V^{n+k}/\text{GL}(V) \simeq \text{Gr}_k(\mathbb{C}^{n+k})$ . Thus the no-name lemma ([12], Corollary 3.8) yields

$$\begin{aligned} \frac{R_{n,k}^0}{G_{n,k}} &\simeq \frac{V^{n+k}}{((\mu_2 \wr S_{n+k}) \times \text{GL}(V))/\mu_2} \times \mathbb{C}^{4n(n+k)} \\ &\simeq \frac{(\mathbb{C} \oplus V)^{n+k}}{((\mu_2 \wr S_{n+k}) \times \text{GL}(V))/\mu_2} \times \mathbb{C}^{(4n-1)(n+k)} \end{aligned}$$

where each  $\mu_2$  acts trivially on the corresponding summand  $\mathbb{C}$ .

Sending  $\lambda_1, \dots, \lambda_{n+k} \in \mathbb{C}$  to the matrix  $\text{diag}(\lambda_1, \dots, \lambda_{n+k})$  defines a morphism

$$\mathbb{C}^{n+k} \longrightarrow \text{Mat}_{(n+k) \times (n+k)}^{\text{sym}}(\mathbb{C}).$$

This morphism is equivariant for the above (permutation) action of  $\mu_2 \wr S_{n+k}$  on  $\mathbb{C}^{n+k}$  and its action as a subgroup of  $\text{O}_{n+k}$  on  $\text{Mat}_{(n+k) \times (n+k)}^{\text{sym}}(\mathbb{C})$  by conjugation.

Given  $v_1, \dots, v_{n+k} \in V$ , they define a linear map  $V^* \longrightarrow \mathbb{C}^{n+k}$ ; sending them to the image of this map whenever it is injective defines a dominant rational map

$$V^{n+k} \dashrightarrow \text{Gr}_k(\mathbb{C}^{n+k})$$

which is invariant under  $\text{GL}(V)$  and equivariant under  $\mu_2 \wr S_{n+k}$  (this time acting as a subgroup of  $\text{O}_{n+k}$ , so each  $\mu_2$  acts nontrivially on one copy of  $\mathbb{C}$ ). Taking

the product of this morphism and this rational map, we thus obtain a rational map

$$(24) \quad \frac{(\mathbb{C} \oplus V)^{n+k}}{((\mu_2 \wr S_{n+k}) \times \mathrm{GL}(V))/\mu_2} \dashrightarrow \frac{\mathrm{Mat}_{(n+k) \times (n+k)}^{\mathrm{sym}}(\mathbb{C}) \times \mathrm{Gr}_k(\mathbb{C}^{n+k})}{\mathbb{P}\mathrm{O}_{n+k}}.$$

The symmetric matrices in question correspond to endomorphisms of  $\mathbb{C}^{n+k}$  which are self-adjoint for the standard bilinear form. It follows that eigenvectors with different eigenvalues are orthogonal. A generic self-adjoint endomorphism has no multiple eigenvalues, and its eigenvectors are not isotropic. Hence it admits an orthonormal basis consisting of eigenvectors, which is unique up to permutation and signs. This means that the rational map (24) is indeed birational, so

$$\frac{R_{n,k}^0}{G_{n,k}} \simeq \frac{\mathrm{Mat}_{(n+k) \times (n+k)}^{\mathrm{sym}}(\mathbb{C}) \times \mathrm{Gr}_k(\mathbb{C}^{n+k})}{\mathbb{P}\mathrm{O}_{n+k}} \times \mathbb{C}^{(4n-1)(n+k)}.$$

Now the theorem is a consequence of the following proposition.  $\square$

**Proposition 5.12.** *Suppose  $d = d_1 + d_2$  with  $d_1, d_2 \geq 1$ , and  $\nu \in \{1, 2\}$ . Then*

$$\frac{\mathrm{Mat}_{d \times d}^{\mathrm{sym}}(\mathbb{C}) \times \mathrm{Gr}_{d_\nu}(\mathbb{C}^d)}{\mathbb{P}\mathrm{O}_d} \simeq \frac{\mathrm{Mat}_{2^e \times 2^e}^{\mathrm{sym}}(\mathbb{C})^2}{\mathbb{P}\mathrm{O}_{2^e}} \times \mathbb{C}^{d+d_1d_2-2^e(2^e+3)/2}$$

where  $2^e$  is the largest power of 2 that divides both  $d_1$  and  $d_2$ .

*Proof.* We endow  $\mathbb{C}^d$  with the standard symmetric bilinear form  $b_d: \mathbb{C}^d \times \mathbb{C}^d \rightarrow \mathbb{C}$ . Fix  $\nu = 1$  or  $\nu = 2$ . Let a generic linear subspace  $U_\nu \subseteq \mathbb{C}^d$  of dimension  $d_\nu$  be given. Then  $U_{3-\nu} := U_\nu^\perp$  has dimension  $d_{3-\nu}$ , and

$$\mathbb{C}^d = U_1 \oplus U_2.$$

Let moreover  $f: \mathbb{C}^d \rightarrow \mathbb{C}^d$  be a generic self-adjoint endomorphism. With respect to the above orthogonal direct sum decomposition, we write

$$f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} : U_1 \oplus U_2 \rightarrow U_1 \oplus U_2.$$

The self-adjointness  $f = f^*$  means  $f_{11} = f_{11}^*$ ,  $f_{22} = f_{22}^*$  and  $f_{21} = f_{12}^*$ . We may assume  $d_1 \geq d_2$  without loss of generality. Since  $f$  is generic,  $f_{12}: U_2 \rightarrow U_1$  is then injective. We choose an orthonormal basis of  $U_1$ .

In the case  $d_1 > d_2$ , we define a dominant rational map

$$(25) \quad \frac{\mathrm{Mat}_{d \times d}^{\mathrm{sym}}(\mathbb{C}) \times \mathrm{Gr}_{d_\nu}(\mathbb{C}^d)}{\mathbb{P}\mathrm{O}_d} \dashrightarrow \frac{\mathrm{Mat}_{d_1 \times d_1}^{\mathrm{sym}}(\mathbb{C}) \times \mathrm{Gr}_{d_2}(\mathbb{C}^{d_1})}{\mathbb{P}\mathrm{O}_{d_1}}$$

by sending the pair  $(f, U_\nu)$  to  $f_{11}: U_1 \rightarrow U_1$  and  $U'_2 := f_{12}(U_2) \subseteq U_1$ , where we identify  $U_1$  with  $\mathbb{C}^{d_1}$  via the chosen basis; this is well-defined modulo  $\mathbb{P}\mathrm{O}_{d_1}$ .

Using the isomorphism  $f_{12}: U_2 \rightarrow U'_2$ , the restriction of  $b_d$  to  $U_2 \times U_2$  and the map  $f_{22}: U_2 \rightarrow U_2$  yield a nondegenerate symmetric bilinear form on  $U'_2$ , and a self-adjoint endomorphism  $f'_{22}$  of  $U'_2$ . From all these, we can reconstruct  $f$  up to  $\mathbb{P}\mathrm{O}_d$  by choosing an orthogonal isomorphism  $U_1 \oplus U'_2 \cong \mathbb{C}^d$ .

Given a generic point  $(f_{11}: U_1 \rightarrow U_1, U'_2 \subseteq U_1)$  in the image of our rational map (25), we have just seen that the fiber over it parameterizes nondegenerate symmetric bilinear forms on  $U'_2$  together with self-adjoint endomorphisms  $f'_{22}$  of  $U'_2$ . Since  $\mu_2 \subseteq \mathcal{O}_{d_1}$  acts by its nontrivial character on  $U'_2 \subseteq U_1$ , it acts trivially on these forms and endomorphisms; hence the rational map (25) is birationally a tower of two vector bundles, both of rank  $d_2 \cdot (d_2 + 1)/2$ . This proves

$$(26) \quad \frac{\mathrm{Mat}_{d \times d}^{\mathrm{sym}}(\mathbb{C}) \times \mathrm{Gr}_{d_\nu}(\mathbb{C}^d)}{\mathbb{P}\mathcal{O}_d} \simeq \frac{\mathrm{Mat}_{d_1 \times d_1}^{\mathrm{sym}}(\mathbb{C}) \times \mathrm{Gr}_{d_2}(\mathbb{C}^{d_1})}{\mathbb{P}\mathcal{O}_{d_1}} \times \mathbb{C}^{d_2(d_2+1)}$$

for  $\nu = 1$  and for  $\nu = 2$ ; recall that we are assuming  $d = d_1 + d_2$  with  $d_1 > d_2$ .

In the case  $d_1 = d_2$ , genericity of  $f$  implies that  $f_{21}: U_1 \rightarrow U_2$  is an isomorphism. We define a dominant rational map

$$(27) \quad \frac{\mathrm{Mat}_{d \times d}^{\mathrm{sym}}(\mathbb{C}) \times \mathrm{Gr}_{d_1}(\mathbb{C}^d)}{\mathbb{P}\mathcal{O}_d} \dashrightarrow \frac{\mathrm{Mat}_{d_1 \times d_1}^{\mathrm{sym}}(\mathbb{C})^2}{\mathbb{P}\mathcal{O}_{d_1}}$$

by sending  $(f, U_1)$  to  $f_{11}: U_1 \rightarrow U_1$  and  $f_{21}^*(b_d): U_1 \times U_1 \rightarrow \mathbb{C}$ , which correspond to symmetric matrices via the chosen basis of  $U_1$ ; this is well-defined modulo  $\mathbb{P}\mathcal{O}_{d_1}$ .

The endomorphism  $f'_{22} := f_{21}^{-1} \circ f_{22} \circ f_{21}$  of  $U_1$  is self-adjoint with respect to  $f_{21}^*(b_d)$ . From all these, we can again reconstruct  $f$  up to  $\mathbb{P}\mathcal{O}_d$ . Since  $\mu_2 \subseteq \mathcal{O}_{d_1}$  acts trivially on these endomorphisms  $f'_{22}$ , the rational map (27) is birationally a vector bundle of rank  $d_1 \cdot (d_1 + 1)/2$ . This proves

$$(28) \quad \frac{\mathrm{Mat}_{d \times d}^{\mathrm{sym}}(\mathbb{C}) \times \mathrm{Gr}_{d_1}(\mathbb{C}^d)}{\mathbb{P}\mathcal{O}_d} \simeq \frac{\mathrm{Mat}_{d_1 \times d_1}^{\mathrm{sym}}(\mathbb{C})^2}{\mathbb{P}\mathcal{O}_{d_1}} \times \mathbb{C}^{d_1(d_1+1)/2}$$

under the assumption  $d = d_1 + d_2$  with  $d_1 = d_2$ .

Now let's return to the general case  $d = d_1 + d_2$  with  $d_1, d_2 \geq 1$ . Following the Euclidean algorithm as it computes  $h := \gcd(d_1, d_2)$ , and composing with the corresponding birational equivalence (26) or (28) in each step, we get

$$(29) \quad \frac{\mathrm{Mat}_{d \times d}^{\mathrm{sym}}(\mathbb{C}) \times \mathrm{Gr}_{d_\nu}(\mathbb{C}^d)}{\mathbb{P}\mathcal{O}_d} \simeq \frac{\mathrm{Mat}_{h \times h}^{\mathrm{sym}}(\mathbb{C})^2}{\mathbb{P}\mathcal{O}_h} \times \mathbb{C}^{d+d_1d_2-h(h+3)/2}.$$

Recall that  $2^e$  is the largest power of 2 dividing  $h$ . If  $h = 2^e$ , then we are done, so we assume  $2^e < h$ . Since the action of  $\mathbb{P}\mathcal{O}_h$  is generically free here, the stack quotient  $[\mathrm{Mat}_{h \times h}^{\mathrm{sym}}(\mathbb{C})^2 / \mathcal{O}_h]$  is generically a  $\mu_2$ -gerbe over our birational quotient modulo  $\mathbb{P}\mathcal{O}_h$ . The standard representation  $\mathbb{C}^h$  of  $\mathcal{O}_h$  yields a vector bundle of rank  $h$  and nontrivial weight on this  $\mu_2$ -gerbe, whose index (at the generic point) therefore divides  $h$ , and hence divides  $2^e$ . It follows that the Grassmannian bundle with fibers  $\mathrm{Gr}_{2^e}(\mathbb{C}^h)$  over this stack quotient has a rational generic fiber, so

$$\frac{\mathrm{Mat}_{h \times h}^{\mathrm{sym}}(\mathbb{C})^2}{\mathbb{P}\mathcal{O}_h} \times \mathbb{C}^{2^e(h-2^e)} \simeq \frac{\mathrm{Mat}_{h \times h}^{\mathrm{sym}}(\mathbb{C})^2 \times \mathrm{Gr}_{2^e}(\mathbb{C}^h)}{\mathbb{P}\mathcal{O}_h}.$$

Applying (29) once more, with  $h$  and  $2^e$  instead of  $d$  and  $d_\nu$ , completes the proof.  $\square$



*Remark 5.13.* The same argument shows that the stack quotient  $[R_{n,k}^0/G_{n,k}]$  is birational to  $\mathbb{C}^{5kn+4n^2-2^e(2^e+3)/2} \times [\text{Mat}_{2^e \times 2^e}^{\text{sym}}(\mathbb{C})^2/O_{2^e}]$ . In particular, there is a Poincaré family on some dense open part if and only if  $2^e = 1$ , which means that  $n$  or  $k$  is odd. Otherwise, the obstruction is a Brauer class of order 2 and index  $2^e$ ; cf. [8] and [1], Theorem 3.

## 6. THE MODULI SPACE OF RS-INSTANTON BUNDLES AND ITS BIRATIONAL TYPE

In this section, we construct the moduli stack of RS-instanton bundles on  $\mathbb{P}$  as well as the moduli space of stable RS-instanton bundles, and we determine the birational type of these objects. Put  $U := \mathbb{C}^2$ , and let  $p, q \geq 1$  be integers. We consider the multiplication map

$$\mu: S^p U \otimes S^q U \longrightarrow S^{p+q} U.$$

**Lemma 6.1.** *For a linear hyperplane  $\wp \subset S^p U$ , the following are equivalent:*

- i) *There is a line  $\ell \subset U$  with  $\wp = \ell \cdot S^{p-1} U$ .*
- ii) *The restriction  $\wp \otimes S^q U \longrightarrow S^{p+q} U$  of  $\mu$  is not surjective.*

*Proof.* The case  $p = 1$  is trivial, so we assume  $p \geq 2$ . It is obvious that i) implies ii).

For the converse, we assume that there is no line  $\ell \subset U$  with  $\wp = \ell \cdot S^{p-1} U$ . We identify  $U$  with  $H^0(\mathbb{P}^1, \mathcal{O}(1))$ ; then

$$\wp \subset S^p U = H^0(\mathbb{P}^1, \mathcal{O}(p)).$$

The assumption on  $\wp$  means that the canonical evaluation map

$$\eta: \mathcal{O}_{\mathbb{P}^1}^p \cong \wp \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(p)$$

is surjective. The kernel of  $\eta$  is a vector bundle of rank  $p - 1$  over  $\mathbb{P}^1$ , so

$$(30) \quad \ker(\eta) \cong \bigoplus_{i=1}^{p-1} \mathcal{O}_{\mathbb{P}^1}(a_i) \quad \text{with } a_1, \dots, a_{p-1} \in \mathbb{Z},$$

due to Grothendieck's splitting theorem. We have a short exact sequence

$$0 \longrightarrow \ker(\eta) \longrightarrow \wp \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1} \xrightarrow{\eta} \mathcal{O}_{\mathbb{P}^1}(p) \longrightarrow 0$$

of vector bundles over  $\mathbb{P}^1$ . The associated long exact cohomology sequence reads

$$0 \longrightarrow H^0(\mathbb{P}^1, \ker(\eta)) \longrightarrow \wp \longrightarrow S^p U \longrightarrow H^1(\mathbb{P}^1, \ker(\eta)) \longrightarrow 0.$$

Hence we conclude  $H^0(\mathbb{P}^1, \ker(\eta)) = 0$  and  $H^1(\mathbb{P}^1, \ker(\eta)) \cong \mathbb{C}$ . Comparing this with the decomposition (30), we see  $\ker(\eta) \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{p-2} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$  and in particular

$$H^1(\mathbb{P}^1, \ker(\eta) \otimes \mathcal{O}_{\mathbb{P}^1}(q)) = 0.$$

This implies that the following map induced by  $\eta$  is surjective:

$$H^0(\mathbb{P}^1, \wp \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1}(q)) \longrightarrow H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(p+q)).$$

But this map is the restriction of the multiplication map  $\mu$ ; therefore this restriction is surjective. This shows that ii) is false if i) is false.  $\square$

**Corollary 6.2.** *Suppose that three linear maps*

$$\alpha \in \mathrm{GL}(S^p U), \quad \beta \in \mathrm{GL}(S^q U) \quad \text{and} \quad \gamma \in \mathrm{GL}(S^{p+q} U)$$

*satisfy  $\mu \circ (\alpha \otimes \beta) = \gamma \circ \mu$ . Then there is a linear map  $g \in \mathrm{GL}(U)$  such that*

$$S^p g \in \mathbb{C}^* \cdot \alpha, \quad S^q g \in \mathbb{C}^* \cdot \beta \quad \text{and} \quad S^{p+q} g \in \mathbb{C}^* \cdot \gamma.$$

*This linear map  $g$  is unique up to multiplication by  $\mathbb{C}^*$ .*

*Proof.* The map

$$\begin{aligned} \mathbb{P}(U) &\longrightarrow \mathbb{P}(S^p U)^* \\ \ell &\longmapsto \ell \cdot S^{p-1} U \end{aligned}$$

is a closed immersion and its image is stable under the automorphism of  $\mathbb{P}(S^p U)^*$  induced by  $\alpha$ , according to the previous lemma. Thus,  $\alpha$  induces an automorphism of  $\mathbb{P}(U)$ , which we lift to an automorphism  $g$  of  $U$ . Modifying  $\alpha$ ,  $\beta$  and  $\gamma$  by  $S^p g$ ,  $S^q g$  and  $S^{p+q} g$ , respectively, we may assume  $g = \mathrm{id}_U$ . This means that the automorphism of  $\mathbb{P}(S^p U)^*$  induced by  $\alpha$  restricts to the identity on  $\mathbb{P}(U) \subseteq \mathbb{P}(S^p U)^*$ .

Given a line  $\ell$  in  $U$ , the hyperplane  $\wp := \ell \cdot S^{q-1} U$  in  $S^q U$  has the property that  $(\ell \cdot S^{p-1} U) \otimes S^q U$  and  $S^p U \otimes \wp$  have the same image under  $\mu$ . According to Corollary 6.2, this characterizes  $\wp$  uniquely. Since  $\alpha(\ell \cdot S^{p-1} U) = \ell \cdot S^{p-1} U$  due to the previous paragraph, we conclude  $\beta(\ell \cdot S^{q-1} U) = \ell \cdot S^{q-1} U$  as well.

The fundamental theorem of algebra states that every line  $\ell$  in  $S^p U$  is a product of  $p$  lines  $\ell_1, \dots, \ell_p$  in  $U$ . If  $\ell$  is generic, then the  $\ell_i$  are all distinct, and  $\ell$  is the intersection of the hyperplanes  $\wp_i = \ell_i \cdot S^{p-1} U$  in  $S^p U$ . Due to the first paragraph, we have  $\alpha(\wp_i) = \wp_i$  for all  $i$ , and hence  $\alpha(\ell) = \ell$  for generic lines  $\ell$ . It follows that  $\alpha$  induces the identity on  $\mathbb{P}(S^p U)$ , and hence  $\alpha \in \mathbb{C}^* \cdot \mathrm{id}$ .

Applying the same arguments to  $\beta$ , we get  $\beta \in \mathbb{C}^* \cdot \mathrm{id}$  as well. Because  $\mu$  is surjective, this implies  $\gamma \in \mathbb{C}^* \cdot \mathrm{id}$ , too.  $\square$

Let

$$f: (S^n U)^* \longrightarrow H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$$

be a linear embedding. We fix an integer  $k \geq 1$  and consider a linear map

$$h: S^{n+2k-2} U \longrightarrow H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)).$$

We endow the trivial algebraic vector bundle of rank  $2n + 2k$  over  $\mathbb{P}$

$$E_{n,k}^0 := ((S^{n+k-1} U)^* \oplus S^{n+k-1} U) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}}$$

with the standard symplectic form  $J$ . We also form the algebraic vector bundles

$$E_{n,k}^{-1} := (S^{k-1} U) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}}(-1) \quad \text{and} \quad E_{n,k}^1 := (S^{k-1} U)^* \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}}(1)$$

of rank  $k$  over  $\mathbb{P}$ . The linear maps  $f$  and  $h$  define a morphism of vector bundles

$$A_{f,h}: E_{n,k}^0 \longrightarrow E_{n,k}^1$$

whose induced map of global sections is the direct sum of the following two maps:

$$(31) \quad (S^{n+k-1}U)^* \xrightarrow{\mu^*} (S^{k-1}U)^* \otimes (S^n U)^* \xrightarrow{\text{id} \otimes f} (S^{k-1}U)^* \otimes H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)),$$

$$(32) \quad S^{n+k-1}U \xrightarrow{\mu^*} (S^{k-1}U)^* \otimes S^{n+2k-2}U \xrightarrow{\text{id} \otimes h} (S^{k-1}U)^* \otimes H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)).$$

We assume that  $A_{f,h}$  is surjective; this is an open condition on  $h$ . The composition

$$(33) \quad E_{n,k}^{-1} \xrightarrow{J \circ A_{f,h}^*} E_{n,k}^0 \xrightarrow{A_{f,h}} E_{n,k}^1$$

vanishes; this follows easily from the observation that the bilinear map

$$S^{k-1}U \otimes S^{k-1}U \xrightarrow{\mu} (S^n U)^* \otimes S^{n+2k-2}U \xrightarrow{f \otimes h} H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2))$$

is symmetric. Thus, for a generic choice of  $h$ , (33) is a monad and defines a symplectic instanton bundle  $E$ .

The special form of the map  $A_{f,h}$  means precisely that  $E$  is an RS-instanton bundle. To see this, choose a basis of  $U$ , and endow the symmetric powers of  $U$  and their duals with the induced bases. Let  $f_i, h_i \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$  denote the images of these basis vectors under  $f, h$ . Then the map (31) corresponds to the special matrix  $F$  with entries  $f_i$ , and the map (32) corresponds to the persymmetric matrix  $H$  with entries  $h_i$ . Moreover, for a generic choice of the  $h_i$ , the  $n$ -space  $\{f_0 = f_1 = \dots = f_n = 0\}$  does not contain any zeroes of the maximal minors of  $H$ , and thus we conclude that  $E$  is an RS-instanton bundle.

Conversely, if  $E$  is an RS-instanton bundle, then the entries of the matrices  $F$  and  $H$  define linear maps  $f$  and  $h$  as above. This shows that the RS-instanton bundles are precisely the instanton bundles arising from monads of the form (33).

Let  $L_f \subseteq \mathbb{P}$  denote the locus where all sections in the image of  $f$  vanish. Then  $L_f$  is a linear subspace of dimension  $n$  in  $\mathbb{P}$ ; it is the linear subspace restricted to which  $E$  has  $n+k$  global sections by construction. If  $E$  is generic, then  $L_f$  is the only linear subspace of dimension  $n$  in  $\mathbb{P}$  with that property; see Proposition 3.10.

The next question is: When are two such RS-instanton bundles isomorphic? Let

$$g \in \text{GL}(U), \quad t \in \mathbb{C}^* \quad \text{and} \quad u \in (S^{2n+2k-2}U)^*$$

be given. They define an isomorphism of symplectic monads

$$(34) \quad \begin{array}{ccccc} E_{n,k}^{-1} & \xrightarrow{J \circ A_{f,h}} & E_{n,k}^0 & \xrightarrow{A_{f,h}} & E_{n,k}^1 \\ \downarrow t^{-1} \cdot S^{k-1}g & & \downarrow \begin{pmatrix} (S^{n+k-1}g^{-1})^* & (\text{id} \otimes u) \circ \mu^* \\ 0 & S^{n+k-1}g \end{pmatrix} & & \downarrow t(S^{k-1}g^{-1})^* \\ E_{n,k}^{-1} & \xrightarrow{J \circ A_{f',h'}} & E_{n,k}^0 & \xrightarrow{A_{f',h'}} & E_{n,k}^1 \end{array}$$

where the vertical map in the middle contains as a matrix entry the composition

$$S^{n+k-1}U \xrightarrow{\mu_*} (S^{n+k-1}U)^* \otimes S^{2n+2k-2}U \xrightarrow{\text{id} \otimes u} (S^{n+k-1}U)^*$$

and the monad in the second row comes from the maps

$$(35) \quad f' := tf \circ (S^n g)^* \quad \text{and} \quad h' := t(h - (u \otimes f) \circ \mu_*) \circ (S^{n+2k-2}g^{-1}).$$

Here the definition of the map  $h'$  involves the composition

$$S^{n+2k-2}U \xrightarrow{\mu_*} S^{2n+2k-2}U \otimes (S^n U)^* \xrightarrow{u \otimes f} H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)).$$

A straightforward computation shows that Diagram (34) commutes indeed and that the induced isomorphism of RS-instanton bundles is symplectic. Note that  $L_f = L_{f'}$ .

The formulas (35) define a linear group action of the semidirect product

$$G := (\text{GL}(U) \times \mathbb{C}^*) \ltimes (S^{2n+2k-2}U)^*$$

on the vector space containing the pairs  $(f, h)$  considered above, which is

$$\text{RS} := \text{RS}_{n,k} := (S^n U \oplus (S^{n+2k-2}U)^*) \otimes H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)),$$

such that each group element sending  $(f, h)$  to  $(f', h')$  induces a symplectic isomorphism (34) from the monad given by  $A_{f,h}$  to the monad given by  $A_{f',h'}$ . This isomorphism is the identity if and only if the triple  $(g, t, u)$  is in the image of the roots of unity  $\mu_{n+k-1} \subseteq \mathbb{C}^*$  under the embedding

$$\begin{aligned} \mu_{n+k-1} &\longrightarrow (\text{GL}(U) \times \mathbb{C}^*) \ltimes (S^{2n+2k-2}U)^* \\ \rho &\longmapsto (\rho \cdot \text{id}_U, \rho^{-n}, 0). \end{aligned}$$

This image is a normal subgroup, and the resulting quotient group

$$G := \frac{(\text{GL}(U) \times \mathbb{C}^*) \ltimes (S^{2n+2k-2}U)^*}{\mu_{n+k-1}}$$

still acts on the vector space RS.

**Proposition 6.3.** *Given two RS-instanton bundles*

$$E = \ker(A_{f,h}) / \text{im}(J \circ A_{f,h}^*) \quad \text{and} \quad E' = \ker(A_{f',h'}) / \text{im}(J \circ A_{f',h'}^*)$$

with  $L_f = L_{f'}$  and a symplectic isomorphism  $\varphi: E \rightarrow E'$ , there is a unique element of  $G$  which sends  $(f, h)$  to  $(f', h')$  and gives back  $\varphi$  via Diagram (34).

*Proof.* The uniqueness follows from the observation that the isomorphism (34) is the identity only if the triple  $(g, t, u)$  is in the image of  $\mu_{n+k-1}$ . We prove the existence.

The given isomorphism  $\varphi$  lifts uniquely to an isomorphism

$$\varphi^\bullet: (E_{n,k}^\bullet, A_{f,h}) \xrightarrow{\sim} (E_{n,k}^\bullet, A_{f',h'})$$

of symplectic monads, which in turn is given by two linear automorphisms

$$\varphi^0 \in \text{Sp}((S^{n+k-1}U)^* \oplus S^{n+k-1}U, J) \quad \text{and} \quad \varphi^1 \in \text{GL}((S^{k-1}U)^*)$$

satisfying the following equation:

$$(36) \quad A_{f',h'} \circ \varphi^0 = \varphi^1 \circ A_{f,h}.$$

Let's first restrict this equation to  $L := L_f = L_{f'} \subset \mathbb{P}$ . The first summand  $(S^{n+k-1}U)^*$  of  $H^0(\mathbb{P}, E_{n,k}^0)$  consists exactly of the sections which vanish on  $L$ . It follows that this summand has to be preserved by  $\varphi^0$ , so

$$\varphi^0 = \begin{pmatrix} (\gamma^{-1})^* & \tau \\ 0 & \gamma \end{pmatrix}$$

for some linear automorphism  $\gamma \in \text{GL}(S^{n+k-1}U)$  and some symmetric bilinear form  $\tau: S^{n+k-1}U \rightarrow (S^{n+k-1}U)^*$ .

The assumption  $L_f = L_{f'}$  means that the linear embeddings  $f$  and  $f'$  have the same image, so there is a linear automorphism  $\beta \in \text{GL}(S^n U)$  with  $f' = f \circ \beta^*$ . We can write  $\varphi^1 = (\alpha^{-1})^*$  for a unique element  $\alpha \in \text{GL}(S^{k-1}U)$ .

Now the first component of Equation (36) simplifies to  $\mu \circ (\alpha \otimes \beta) = \gamma \circ \mu$ . This allows us to apply Corollary 6.2. Multiplying the resulting element  $g \in \text{GL}(U)$  by a nonzero scalar if necessary, we can even achieve  $\gamma = S^{n+k-1}g$  besides  $S^{k-1}g \in \mathbb{C}^* \cdot \alpha$  and  $S^n g \in \mathbb{C}^* \cdot \beta$ . Comparing the scalars, we get more precisely  $\alpha = t^{-1} \cdot S^{k-1}g$  and  $\beta = t \cdot S^n g$  for some  $t \in \mathbb{C}^*$ .

We replace the pair  $(f, h)$  by its image under the group element given by the triple  $(g, t, 0)$ . This reduces us without loss of generality to the case where

$$f = f', \quad \varphi^0 = \begin{pmatrix} \text{id} & \tau \\ 0 & \text{id} \end{pmatrix} \quad \text{and} \quad \varphi^1 = \text{id}$$

for some symmetric bilinear form  $\tau: S^{n+k-1}U \rightarrow (S^{n+k-1}U)^*$ . In this situation, the second component of Equation (36) means that the diagram

$$\begin{array}{ccc} S^{k-1}U \otimes S^{n+k-1}U & \xrightarrow{\mu} & S^{n+2k-2}U \xrightarrow{h-h'} H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) \\ \tau \downarrow & & \parallel \\ S^{k-1}U \otimes (S^{n+k-1}U)^* & \xrightarrow{\mu^*} & (S^n U)^* \xrightarrow{f} H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) \end{array}$$

commutes. Since  $\mu$  is surjective, we conclude that the image of  $h - h'$  is contained in the image of  $f$ . As  $f$  is injective, this implies  $h - h' = f \circ \delta$  for some linear map

$$\delta: S^{n+2k-2}U \rightarrow (S^n U)^*.$$

Using this, the previous commutative diagram yields the commutative diagram

$$\begin{array}{ccc} S^n U \otimes S^{k-1}U \otimes S^{n+k-1}U & \xrightarrow{\text{id} \otimes \mu} & S^n U \otimes S^{n+2k-2}U \\ \mu \otimes \text{id} \downarrow & & \downarrow \delta \\ S^{n+k-1}U \otimes (S^{n+k-1}U) & \xrightarrow{\tau} & \mathbb{C}. \end{array}$$

The composition  $\tau \circ (\mu \otimes \text{id}) = (\text{id} \otimes \mu) \circ \delta$  in this diagram is a multilinear form

$$U^{\otimes(2n+2k-2)} = U^{\otimes n} \otimes U^{\otimes(k-1)} \otimes U^{\otimes(n+k-1)} \longrightarrow \mathbb{C}$$

which is invariant under the two subgroups  $S_{n+k-1} \times S_{n+k-1}$  and  $S_n \times S_{n+2k-2}$  in the symmetric group  $S_{2n+2k-2}$ . But these two subgroups generate the full group  $S_{2n+2k-2}$ , so the multilinear form descends to a linear form

$$u: S^{2n+2k-2}U \longrightarrow \mathbb{C}$$

such that  $\delta = u \circ \mu$  and  $\tau = u \circ \mu$ . It follows that the isomorphism  $\varphi^\bullet$  comes from the element in  $G$  given by the triple  $(\text{id}_U, 1, u)$ .  $\square$

**Corollary 6.4.** *The quotient stack  $[\text{RS}/G]$  is birational to the moduli stack of RS-instanton bundles  $E$  with charge  $k$  over  $\mathbb{P}^{2n+1}$ .*

*Remark 6.5.* Note that the quotient stack  $[\text{RS}/G]$  has dimension

$$\begin{aligned} \dim(\text{RS}) - \dim(G) &= ((n+1) + (n+2k-1)) \cdot (2n+2) - (4+1 + (2n+2k-1)) \\ &= (4n+2) \cdot k + 4n^2 + 2n - 4. \end{aligned}$$

In the special case  $n=1$ , this coincides with the dimension  $6k+2$  of the moduli space introduced by Rao and by Skiti in [28] and [33], respectively.

Our next aim is to determine the birational type of this moduli stack. For that, we choose a basis of  $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$ . This provides a  $G$ -equivariant isomorphism

$$\text{RS} \cong (S^n U)^{\oplus(2n+2)} \oplus ((S^{n+2k-2}U)^*)^{\oplus(2n+2)}.$$

We start with the quotient of the first summand  $(S^n U)^{\oplus(2n+2)}$  modulo the subgroup

$$(\text{GL}(U) \times \mathbb{C}^*) / \iota(\mu_{n+k-1})$$

of  $G$ , whose definition involves the embedding

$$\begin{aligned} \iota: \mathbb{C}^* &\longrightarrow \text{GL}(U) \times \mathbb{C}^* \\ \lambda &\longmapsto (\lambda \cdot \text{id}_U, \lambda^{-n}). \end{aligned}$$

Here the group action is the restriction of the action (35). So the group element represented by  $g \in \text{GL}(U)$  and  $t \in \mathbb{C}^*$  acts on  $(S^n U)^{\oplus(2n+2)}$  as the automorphism

$$(37) \quad \begin{aligned} (S^n U)^{\oplus(2n+2)} &\longrightarrow (S^n U)^{\oplus(2n+2)} \\ f &\longmapsto (S^n g)(tf). \end{aligned}$$

**Proposition 6.6.** *The stack quotient of the vector space  $(S^n U)^{\oplus(2n+2)}$  modulo the above linear group action of  $(\text{GL}(U) \times \mathbb{C}^*) / \iota(\mu_{n+k-1})$  is*

- i) *birational to  $\mathbb{A}^{2n^2+4n-2} \times B\mathbb{C}^*$ , if  $n$  or  $k$  is odd;*
- ii) *birational to  $\mathbb{A}^{2n^2+4n-7} \times [\text{End}(U)^2 / \text{GL}(U)]$ , if  $n$  and  $k$  are even.*

Here  $B\mathbb{C}^*$  is the classifying stack of  $\mathbb{C}^*$ , and  $[\text{End}(U)^2 / \text{GL}(U)]$  is the stack quotient of  $\text{End}(U)^2$  modulo the linear action of  $\text{GL}(U)$  by simultaneous conjugation.

*Proof.* We consider the central extension of linear algebraic groups

$$(38) \quad 1 \longrightarrow \mathbb{C}^* \cong \frac{\mathbb{C}^*}{\mu_{n+k-1}} \xrightarrow{\iota} \frac{\mathrm{GL}(U) \times \mathbb{C}^*}{\iota(\mu_{n+k-1})} \xrightarrow{\pi} \frac{\mathrm{GL}(U)}{\mu_n} \longrightarrow 1$$

where  $\pi$  sends the class of a pair  $(g, t)$  to the class of  $t^{1/n} \cdot g \in \mathrm{GL}(U)$ ; note that the  $n$ th root  $t^{1/n}$  is well-defined modulo  $\mu_n$ . Our group action (37) is the composition of  $\pi$  and the natural action of  $\mathrm{GL}(U)/\mu_n$  on  $(S^n U)^{\oplus(2n+2)}$  via  $n$ th symmetric powers.

Suppose that  $n$  is odd. Then the central extension (38) splits, because the map

$$\begin{aligned} \frac{\mathrm{GL}(U) \times \mathbb{C}^*}{\iota(\mu_{n+k-1})} &\longrightarrow \frac{\mathbb{C}^*}{\mu_{n+k-1}} \\ [g, t] &\longmapsto \left[ t \cdot \det(g)^{\frac{n+1}{2}} \right] \end{aligned}$$

is a left inverse of  $\iota$ . Hence the stack quotient in question is birational to  $B\mathbb{C}^*$  times the stack quotient of  $(S^n U)^{\oplus(2n+2)}$  modulo  $\mathrm{GL}(U)/\mu_n$ . Note that already the action of  $\mathrm{GL}(U)/\mu_n$  on  $(S^n U)^{\oplus 2}$  is generically free. Moreover, the isomorphism

$$\begin{aligned} \mathrm{GL}(U)/\mu_n &\xrightarrow{\sim} \mathrm{GL}(U) \\ [g] &\longmapsto \det(g)^{\frac{n-1}{2}} \cdot g \end{aligned}$$

shows that  $\mathrm{GL}(U)/\mu_n$  also has a generically free representation of dimension 4 containing an open orbit. The no-name lemma ([12], Corollary 3.8) allows us to replace the representation  $(S^n U)^{\oplus(2n+2)}$  by one which contains this 4-dimensional representation as a direct summand, and then to conclude that the quotient is rational, as required.

Now suppose that  $n$  is even and  $k$  is odd. Then (38) also splits, since the map

$$\begin{aligned} \frac{\mathrm{GL}(U) \times \mathbb{C}^*}{\iota(\mu_{n+k-1})} &\longrightarrow \frac{\mathbb{C}^*}{\mu_{n+k-1}} \\ [g, t] &\longmapsto \left[ \det(g)^{\frac{n+k-1}{2}} \right] \end{aligned}$$

is a left inverse of  $\iota$ . Hence the stack quotient in question is again birational to  $B\mathbb{C}^*$  times the stack quotient of  $(S^n U)^{\oplus(2n+2)}$  modulo  $\mathrm{GL}(U)/\mu_n$ . The isomorphism

$$\begin{aligned} \mathrm{GL}(U)/\mu_n &\xrightarrow{\sim} \mathbb{P}\mathrm{GL}(U) \times \mathbb{C}^* \\ [g] &\longmapsto \left[ g, \det(g)^{\frac{n}{2}} \right] \end{aligned}$$

provides a generically free action of  $\mathrm{GL}(U)/\mu_n$  on  $\mathrm{End}(U)^2 \oplus \mathbb{C}$ , such that the quotient is rational. Using the no-name lemma as before, it follows that the quotient of  $(S^n U)^{\oplus(2n+2)}$  modulo  $\mathrm{GL}(U)/\mu_n$  is also rational, as required.

Finally, suppose that  $n$  and  $k$  are both even. We use the group isomorphism

$$\begin{aligned} \frac{\mathrm{GL}(U) \times \mathbb{C}^*}{\iota(\mu_{n+k-1})} &\xrightarrow{\sim} \mathrm{GL}(U) \times \mathbb{C}^* \\ [g, t] &\longmapsto \left( \det(g)^{\frac{n+k-2}{2}} \cdot g, \det(g)^{\frac{n}{2}} \cdot t \right) \end{aligned}$$

whose composition with the embedding  $\iota$  of  $\mathbb{C}^* \cong \mathbb{C}^*/\mu_{n+k-1}$  is the embedding

$$\begin{aligned} \mathbb{C}^* &\longrightarrow \mathrm{GL}(U) \times \mathbb{C}^* \\ \lambda &\longmapsto (\lambda \cdot \mathrm{id}_U, 1). \end{aligned}$$

Thus the cokernel of  $\iota$  is isomorphic to  $\mathbb{P} \mathrm{GL}(U) \times \mathbb{C}^*$ , which again acts generically freely on  $\mathrm{End}(U)^2 \oplus \mathbb{C}$ . The no-name lemma allows us to replace  $(S^n U)^{\oplus(2n+2)}$  by the direct sum of  $\mathrm{End}(U)^2 \oplus \mathbb{C}$  and a trivial representation of dimension

$$(n+1) \cdot (2n+2) - 2 \cdot 4 - 1 = 2n^2 + 4n - 7,$$

whose quotient modulo  $\mathrm{GL}(U) \times \mathbb{C}^*$  is precisely as claimed.  $\square$

**Corollary 6.7.** *The moduli stack of RS-instanton bundles with charge  $k$  over  $\mathbb{P}^{2n+1}$  is*

- i) *birational to  $\mathbb{A}^{(4n+2)k+4n^2+2n-4} \times B\mu_2$ , if  $n$  or  $k$  is odd;*
- ii) *birational to  $\mathbb{A}^{(4n+2)k+4n^2+2n-9} \times [\mathrm{End}(U)^2/\mathrm{SL}(U)]$ , if  $n$  and  $k$  are even.*

*Proof.* The projection onto the first summand

$$\mathrm{RS} \cong (S^n U)^{\oplus(2n+2)} \oplus ((S^{n+2k-2} U)^*)^{\oplus(2n+2)} \longrightarrow (S^n U)^{\oplus(2n+2)}$$

descends to a 1-morphism of stacks

$$\Phi: \left[ \frac{\mathrm{RS}}{G} \right] \longrightarrow \left[ \frac{(S^n U)^{\oplus(2n+2)}}{(\mathrm{GL}(U) \times \mathbb{C}^*)/\iota(\mu_{n+k-1})} \right].$$

The latter stack is generically a gerbe with band  $\mathbb{C}^*$  due to the previous proposition. The 1-morphism  $\Phi$  is a vector bundle  $\mathcal{V}$  with fibers

$$((S^{n+2k-2} U)^*)^{\oplus(2n+2)} / f_*((S^{2n+2k-2} U)^*),$$

on which the automorphism groups  $\mathbb{C}^*$  of the gerbe act with weight 2. Its rank is

$$\mathrm{rank}(\mathcal{V}) = (n+2k-1) \cdot (2n+2) - (2n+2k-1) = (4n+2) \cdot k + 2n^2 - 2n - 1.$$

Now suppose that  $n$  or  $k$  is odd. Let  $\mathcal{L}$  denote the tensor square of the universal line bundle over the classifying stack  $B\mathbb{C}^*$ . Then the automorphism groups  $\mathbb{C}^*$  of this neutral gerbe  $B\mathbb{C}^*$  act with the same weight 2 on the fibers of  $\mathcal{L}$ . But any two vector bundles over a  $\mathbb{C}^*$ -gerbe with the same rank and the same weight are isomorphic over some dense open substack of that  $\mathbb{C}^*$ -gerbe, according to [17], Lemma 4.10. Using Proposition 6.6, we conclude that the total space  $[\mathrm{RS}/G]$  of  $\mathcal{V}$  is birational to  $\mathbb{A}^{2n^2+4n-2}$  times the total space of the vector bundle

$$\mathcal{L}^{\oplus(4n+2)k+2n^2-2n-1} \longrightarrow B\mathbb{C}^*.$$



The projection of the line bundle  $\mathcal{L}$  onto  $B\mathbb{C}^*$  coincides with the natural morphism

$$\varphi: B\mu_2 \longrightarrow B\mathbb{C}^*.$$

Hence the total space of  $\mathcal{L}^{\oplus N+1}$  over  $B\mathbb{C}^*$  is the total space of  $\varphi^* \mathcal{L}^{\oplus N}$  over  $B\mu_2$ . But this pullback vector bundle is trivial, because  $\mu_2 \subset \mathbb{C}^*$  acts trivially on the fibers of  $\mathcal{L}$ . This shows that the total space of  $\mathcal{L}^{\oplus N+1}$  over  $B\mathbb{C}^*$  is isomorphic to  $\mathbb{A}^N \times B\mu_2$ . It follows that the stack  $[\text{RS}/G]$  is birational to

$$\mathbb{A}^{2n^2+4n-2} \times \mathbb{A}^{(4n+2)k+2n^2-2n-2} \times B\mu_2$$

in the case where  $n$  or  $k$  is odd.

Now suppose that  $n$  and  $k$  are even. Let  $\mathcal{U}$  denote the universal vector bundle of rank 2 over the quotient stack  $[\text{End}(U)^2/\text{GL}(U)]$ , or in other words the pullback of the universal vector bundle of rank 2 over the classifying stack  $B\text{GL}(U)$ . On the line bundle  $\det(\mathcal{U})$ , the automorphism groups  $\mathbb{C}^*$  of the dense open  $\mathbb{C}^*$ -gerbe in  $[\text{End}(U)^2/\text{GL}(U)]$  act again with weight 2. Combining [17], Lemma 4.10, and Proposition 6.6 as before, we conclude that the stack  $[\text{RS}/G]$  is birational to  $\mathbb{A}^{2n^2+4n-7}$  times the total space of the vector bundle

$$\det(\mathcal{U})^{\oplus(4n+2)k+2n^2-2n-1} \longrightarrow [\text{End}(U)^2/\text{GL}(U)].$$

The projection of the line bundle  $\det(\mathcal{U})$  onto  $[\text{End}(U)^2/\text{GL}(U)]$  coincides with the natural morphism

$$\psi: [\text{End}(U)^2/\text{SL}(U)] \longrightarrow [\text{End}(U)^2/\text{GL}(U)].$$

Hence the total space of  $\det(\mathcal{U})^{\oplus N+1}$  over  $[\text{End}(U)^2/\text{GL}(U)]$  is the total space of  $\psi^* \det(\mathcal{U})^{\oplus N}$  over  $[\text{End}(U)^2/\text{SL}(U)]$ . But this pullback vector bundle is now trivial over a dense open substack of  $[\text{End}(U)^2/\text{SL}(U)]$ , because the generic stabilizer  $\mu_2 \subset \text{SL}(U)$  acts trivially on the fibers of  $\det(\mathcal{U})$ ; cf. [17], Corollary 4.8. This shows that the total space of  $\det(\mathcal{U})^{\oplus N+1}$  over  $[\text{End}(U)^2/\text{GL}(U)]$  is birational to  $\mathbb{A}^N \times [\text{End}(U)^2/\text{SL}(U)]$ . It follows that the stack  $[\text{RS}/G]$  is birational to

$$\mathbb{A}^{2n^2+4n-7} \times \mathbb{A}^{(4n+2)k+2n^2-2n-2} \times [\text{End}(U)^2/\text{SL}(U)]$$

in the case where  $n$  and  $k$  are both even. □

Next, we would like to construct the moduli space for Rao–Skiti instanton bundles as an algebraic variety. We have already set up the group action for this moduli problem. Since the group  $G$  that acts is not reductive, it seems that the construction of the quotient is not as straightforward as in the setting of 't Hooft instanton bundles. However, we can easily reduce to a quotient problem for a reductive group. That problem can be solved with the construction of [26] and some descent theory. So, let us make all this precise. Denote by

$$\text{RS}_{n,k}^s \subset \text{RS}_{n,k}$$

the  $G$ -invariant open subset of RS-monads as in (33) whose cohomology is a stable symplectic instanton bundle. By Remark 3.9, iii), this set is non-empty.

**Proposition 6.8.** *For given positive integers  $n$  and  $k$ , the moduli space*

$$\text{RSI}_{\mathbb{P}^{2n+1}}(k) := \text{RS}_{n,k}^s/G$$

*of stable RS-instanton bundles with charge  $k$  on  $\mathbb{P}^{2n+1}$  exists as a smooth quasi-projective variety. It is equipped with a generically injective morphism*

$$\iota: \text{RSI}_{\mathbb{P}^{2n+1}}(k) \longrightarrow \text{MI}_{\mathbb{P}^{2n+1}}(k)$$

*to the moduli space of stable symplectic instanton bundles with charge  $k$  on  $\mathbb{P}^{2n+1}$ .*

*Problem 6.9.* i) Study the properties of the map  $\iota$ . Is its image closed or is  $\iota$  even proper? Describe all the fibers of  $\iota$ .

ii) Is every RS-instanton bundle stable?

*Proof of Proposition 6.8.* Formulas (31) and (32) describe an embedding

$$\psi: \text{RS}_{n,k} \longrightarrow \text{SM}$$

of  $\text{RS}_{n,k}$  into the space of all symplectic monads (compare (23)). We also have a homomorphism

$$\alpha: G \longrightarrow H, \quad H := \text{Sp}((S^{n+k-1}U)^* \oplus S^{n+k-1}U, J) \times \text{GL}(V).$$

Note that  $\alpha$  is injective (see Proposition 6.3 and its proof for the details).

The principal  $G$ -bundle  $H \longrightarrow H/G$  and the action of  $G$  on  $\text{RS}_{n,k}^s$  define the associated fiber bundle

$$H \times^G \text{RS}_{n,k}^s \longrightarrow H/G$$

with typical fiber  $\text{RS}_{n,k}^s$  (see [32], §3.2). It is a smooth quasi-projective variety on which the reductive group  $H$  acts from the left. By Proposition 3.10 and Proposition 6.3, we have a generically injective and  $H$ -equivariant morphism

$$a: H \times^G \text{RS}_{n,k}^s \longrightarrow \text{SM}.$$

Since  $H \times^G \text{RS}_{n,k}^s$  is normal, the  $H$ -action on this space can be linearized in some ample line bundle  $A$  ([22], Corollary 1.6).

Let

$$\widehat{\text{SI}}_k \subset \text{SM} \setminus \{0\}$$

be the preimage of  $\widehat{\text{SI}}_k$  under the projection  $\text{SM} \setminus \{0\} \longrightarrow \mathbb{P}(\text{SM})$ . Remark 5.3 shows that

$$\widehat{\text{SI}}_k \longrightarrow \text{MI}_{\mathbb{P}^{2n+1}}(k)$$

is a principal  $H$ -bundle. In particular, it is a faithfully flat morphism. The  $H$ -action on  $H \times^G \text{RS}_{n,k}^s$  and the linearization of this action in  $A$  provide the data which enable us to descend the  $H$ -equivariant morphism

$$H \times^G \text{RS}_{n,k}^s \longrightarrow \text{SM}$$

to a quasi-projective variety  $Q$  and a morphism  $Q \rightarrow \text{MI}_{\mathbb{P}^{2n+1}}(k)$ . For this, one applies a descent theorem by Grothendieck ([15], Exposé VIII, Proposition 7.8; [9], Theorem 7, p. 138). Here,

$$H \times^G \text{RS}_{n,k}^s \rightarrow Q$$

is also a principal  $H$ -bundle. In particular,  $Q$  is the categorical quotient for  $\text{RS}_{n,k}^s$  with respect to the  $G$ -action.  $\square$

Proposition 6.6 implies the following statements:

**Corollary 6.10.** *The coarse moduli space  $\text{RSI}_{\mathbb{P}^{2n+1}}(k)$  of stable RS-instanton bundles with charge  $k$  over  $\mathbb{P}^{2n+1}$  is rational.*

**Corollary 6.11.** *There is a Poincaré family parameterized by some open subscheme of the coarse moduli space  $\text{RSI}_{\mathbb{P}^{2n+1}}(k)$  of stable RS-instanton bundles with charge  $k$  over  $\mathbb{P}^{2n+1}$  if and only if  $n$  is odd or  $k$  is odd.*

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