Approximation Theory and Harmonic Analysis on the Unit Sphere

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CHAPTER 1

Approximation by trigonometric polynomials

The unit sphere on the plane is just a circle, denoted by $\mathbb{S}^1 := \{x_1^2 + x_2^2 = 1\}$. Under the polar coordinates $(x_1, x_2) = r(\cos\theta, \sin\theta)$, the functions on the circle $\mathbb{S}^1$ can be identified with 2π-periodic functions in θ.

We start with a quick review of approximation by trigonometric polynomials, which is at the root of the approximation theory.

For $n = 0, 1, \ldots$, consider the space $\Pi_n(\mathbb{S}^1)$ of trigonometric polynomials of degree at most n, defined by

$$\Pi_n(\mathbb{S}^1) := \operatorname{span}\{1, \cos\theta, \sin\theta, \ldots, \cos n\theta, \sin n\theta\}.$$

Let $L^p(\mathbb{S}^1)$ denote the space of Lebesgue integrable functions such that

$$\|f\|_p := \left(\int_0^{2\pi} |f(\theta)|^p d\theta\right)^{1/p} < \infty, \quad \text{where} \quad 1 \le p < \infty,$$

whereas for $p = \infty$, we consider $C(\mathbb{S}^1)$ instead, with $\|f\|_\infty := \max_{\theta\in[0,2\pi)} |f(\theta)|$. For $f \in L^p(\mathbb{S}^1)$ if $1 \le p < \infty$ or $f \in C(\mathbb{S}^1)$ if $p = \infty$, the error of best approximation by trigonometric polynomials of degree at most n is defined by

$$E_n(f)_p := \inf_{g\in\Pi_n(\mathbb{S}^1)} \|f - g\|_p, \qquad 1 \le p \le \infty.$$

The central problem in trigonometric approximation theory is to characterize $E_n(f)_p$ in terms of the smoothness of the function f. For this purpose we need the notion of modulus of smoothness, usually defined via the forward difference of f.

Let I denote the identity operator and S_h be the translation operator defined by $S_h f(x) := f(x + h)$. For $r = 1, 2, \ldots$, the forward difference operator $\overrightarrow{\triangle}_h^r$ is defined by

$$\overrightarrow{\triangle}_h := S_h - I \quad \text{and} \quad \overrightarrow{\triangle}_h^r := \overrightarrow{\triangle}_h^{r-1}\overrightarrow{\triangle}_h,$$

or $\overrightarrow{\triangle}_h = (S_h - I)^r$. The binomial theorem implies that

$$\overrightarrow{\triangle}_h^r f(x) = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(x + hj).$$

The usual difference operator $\triangle$ is related to $\overrightarrow{\triangle}_h$ by $\triangle = -\overrightarrow{\triangle}_h$.

Definition 1.0.1. *For $f \in L^p(\mathbb{S}^1)$ if $1 \le p < \infty$ or $f \in C(\mathbb{S}^1)$ if $p = \infty$, $r = 1, 2, \ldots$ and $t > 0$,*

$$\omega_r(f, t)_p := \sup_{|\theta|\le t} \left\|\overrightarrow{\triangle}_\theta^r f\right\|_p, \qquad 1 \le p \le \infty.$$

The modulus of smoothness $\omega_r(f, t)_p$ is a continuous and increasing function of t and it satisfies

$$\lim_{t\to 0+} \omega_r(f; t)_p = 0, \qquad \forall f \in L^p,\ 1 \le p < \infty, \quad \text{or } f \in C,\ p = \infty.$$

Furthermore it satisfies the following properties:

(1) $\omega_r(f + g; t)_p \le \omega_r(f; t)_p + \omega_r(g; t)_p$.

(2) $\omega_r(f;\lambda t)_p \le (\lambda+1)^r \omega_r(f;t)_p, \quad \lambda \ge 0.$
(3) $\omega_r(f;t)_p \le ct^r \omega_r(f^{(r)};t)_p, \quad$ if $f^{(r)} \in L^p(\mathbb{S}^1)$.

The main theorem of the trigonometric best approximation is the following:

THEOREM 1.0.2. *For $f \in L^p(\mathbb{S}^1)$ if $1 \le p < \infty$ or $f \in C(\mathbb{S}^1)$ if $p = \infty$,*

$$E_n(f)_p \le c\,\omega_r\left(f;(n+1)^{-1}\right)_p, \quad 1 \le p \le \infty, \quad n = 0,1,2,.... \tag{1.0.1}$$

On the other hand,

$$\omega_r(f;n^{-1})_p \le c\,n^{-r}\sum_{k=1}^{n} k^{r-1}E_{k-1}(f)_p, \qquad 1 \le p \le \infty. \tag{1.0.2}$$

PROOF. The inequality (1.0.1) is usually called the Jackson estimate, its proof requires constructing a trigonometric polynomial whose error of approximation is bounded by the modulus of smoothness. The standard construction is given by

$$J_n(\theta) := \lambda_n \int_0^{2\pi} f(\phi)K_{n,r}(\theta-\phi)d\phi = f * K_{n,r}(\theta),$$

where

$$K_{n,r}(\theta) = \lambda_{n,r}\left(\frac{\sin m\theta/2}{\sin\theta/2}\right)^{2r}, \qquad m = \left\lfloor \frac{n}{r}\right\rfloor + 1,$$

and the constant $\lambda_{n,r}$ is determined by $\int_0^{2\pi} K_{n,r}(\theta)d\theta = 1$. The kernel is an even non-negtive trigonometric polynomial of degree $\le n$. Using the fact that $\sin\theta/2 \sim \theta$ for $0 \le \theta \le 2\pi$, it is easy to see that

$$\lambda_{n,r}^{-1} = \int_0^{2\pi}\left(\frac{\sin m\theta/2}{\sin\theta/2}\right)^{2r} d\theta \sim \int_0^{2\pi}\left(\frac{\sin m\theta/2}{\theta}\right)^{2r} d\theta \le cn^{2r-1}$$

by a change of variable $\phi \mapsto m\theta$ in the last integral, and the same prove shows that

$$\int_0^{2\pi} \theta^k K_{n,r}\theta)d\theta \le cn^{-k}, \quad k = 0,1,\ldots,2r-2.0.$$

The polynomials that give the Jackson estimate is defined by

$$S_n f(\theta) = \int_0^{2\pi}\left[f(\theta) - (-1)^r \overrightarrow{\triangle}_h^r f(\theta)\right] K_{n,r}(\theta)d\theta.$$

It is not difficult to see that S_n is a trigonometric polynomials of degree $\le n$. Now, by Minkowski's theorem,

$$\begin{aligned}\|J_n f - f\|_p &\le \left\|\int_0^{2\pi} \overrightarrow{\triangle}_h^r f(x)K_{n,r}(\theta)d\theta\right\|_p \le \int_0^{2\pi}\omega_r(f;|\theta|)_p K_{n,r}(\theta)d\theta \\ &\le \omega_r(f;n^{-1})\int_0^{2\pi}(n|\theta|+1)^r K_{n,r}(\theta)d\theta \le c\omega_r(f;n^{-1}).\end{aligned}$$

The inequality (1.0.2) is often called the Bernstein estimate as its proof relies on the Bernstein inequality, which states that for any $T_n \in \Pi_n(\mathbb{S}^1)$,

$$\|T_n^{(k)}\| \le n^k\|T_n\|_p, \qquad k = 1,2,..., \quad 1 \le p \le \infty.$$

Let S_n denote the trigonometric polynomial in $\Pi_n(\mathbb{S}^1)$ such that $E_n(f) = \|f - s_n\|_p$ and set $S_{2^{-1}} = S_0$. Then

$$\omega_r(f;h)_p \le \omega_r(f - S_{2^m};h)_p + \omega_r(S_{2^m};h)_p$$

for $m = 1,2,\ldots$. Hence,

$$\omega_r(f - S_{2^m};h)_p \le 2^r\|f - S_{2^m}\|_p = 2^r E_{2^m}(f)_p,$$

whereas by the Bernstein inequality

$$\begin{aligned}\|\overrightarrow{\triangle}_h^r S_{2^m}\|_p &\le c\,h^r \|S_{2^m}^{(r)}\|_p \le c\,h^r \sum_{k=0}^{m} \|S_{2^k}^{(r)} - S_{2^{k-1}}^{(r)}\|_p \\ &\le c\,h^r \sum_{k=0}^{m} 2^{kr} \|S_{2^k} - S_{2^{k-1}}\|_p \le c\,h^r \sum_{k=0}^{m} 2^{kr} E_{2^{k-1}}(f)_p,\end{aligned}$$

since $\|S_{2^k} - S_{2^{k-1}}\|_p \le \|f - S_{2^k}\|_p + \|f - S_{2^{k-1}}\|_p$. Consequently, it follows that

$$\omega_r(f;h)_p \le 2^r E_{2^m}(f)_p + c\,h^r \sum_{k=0}^{m} 2^{kr} E_{2^{k-1}}(f)_p.$$

Directly from its definition, $E_n(f)_p$ is a non-increasing function of n, hence

$$2^{kr} E_{2^{k-1}}(f)_p \le 2^{2r-1} \sum_{j=2^{k-2}}^{2^{k-1}} (j+1)^{r-1} E_j(f)_p.$$

Combining the last two inequalities proves (1.0.2) for $n = 2^m$. For a given positive integer n, choose m so that $2^m < n \le 2^{m+1}$. Then (1.0.2) can be deduced from the special case of $n = 2^m$ by the monotonicity of $\omega_r(f;h)$ in h and $E_n(f)_p$ in n. □

Together the two inequalities in the theorem show that

$$E_n(f)_p \sim n^{-\alpha} \quad \text{iff} \quad \omega_r(f;t)_p \sim t^{-\alpha}$$

whenever $\alpha < r$. The inverse estimate (1.0.2) is called weak type since it gives an estimate that contains an additional $\log n$ factor when $E_n(f)_p \sim n^{-r}$.

Several further properties of the forward difference and the modulus of smoothness are stated below.

LEMMA 1.0.3. *Let $r \in \mathbb{N}$ and $f \in L^p(\mathbb{S}^1)$ if $1 \le p < \infty$ or $f \in C(\mathbb{S}^1)$ if $p = \infty$.*

(i) *For $0 < h < 2\pi$ and $f^{(r)} \in L^p(\mathbb{S}^1)$ when needed,*

$$\left\|\overrightarrow{\triangle}_h^r f\right\|_p \le c\,2^r \|f\|_p, \quad \text{and} \quad \left\|\overrightarrow{\triangle}_h^r f\right\|_p \le c\,h^r \|f^{(r)}\|_p.$$

(ii) *If $T_n \in \Pi_n(\mathbb{S}^1)$ is a trigonometric polynomial of degree at most n, then*

$$\left\|T_n^{(r)}\right\|_p \sim h^{-r} \left\|\overrightarrow{\triangle}_h^r T_n\right\|_p, \quad 0 \le h \le \pi n^{-1},$$

with the constant of equivalence depending only on r.

(iii) *For $0 < t < 2\pi$,*

$$\sup_{|\theta| \le t} \left\|\overrightarrow{\triangle}_\theta^r f\right\|_p \sim \left(\frac{1}{t}\int_0^t \left\|\overrightarrow{\triangle}_\theta^r f\right\|_p^p d\theta\right)^{\frac{1}{p}}.$$

LEMMA 1.0.4. *(Marchaud inequality) Let $r, s \in \mathbb{N}$ and $1 \le r < s$. If $f \in L^p(\mathbb{S}^1)$, $1 \le p < \infty$, or $f \in C(\mathbb{S}^1)$ if $p = \infty$, then for $0 < t < 1/2$,*

$$\omega_r(f;t)_p \le c_s t^r \int_t^1 \frac{\omega_s(f;u)_p}{u^{r+1}}\,du. \tag{1.0.3}$$

The main references for this chapter are the books [**14, 33, 42, 52**].

CHAPTER 2

Spherical Harmonics

For $x \in \mathbb{R}^d$, we write $x = (x_1, \dots, x_d)$. The inner product of $x, y \in \mathbb{R}^d$ is denoted by $\langle x, y\rangle := \sum_{i=1}^d x_i y_i$ and the norm of x is denoted by $\|x\| := \sqrt{\langle x, x\rangle}$. We now consider the unit sphere $\mathbb{S}^{d-1} := \{x : \|x\| = 1\}$ of $\mathbb{R}^d$.

2.1. Space of spherical harmonics and orthogonal bases

Let $\mathbb{N}_0$ denote the set of nonnegative integers. For $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$, a monomial x^α is a product $x^\alpha = x_1^{\alpha_1} \dots x_d^{\alpha_d}$, which has degree $|\alpha| := \alpha_1 + \dots + \alpha_d$.

A homogeneous polynomial P of degree n is a linear combination of monomials of degree n, that is,

$$P(x) = \sum_{|\alpha|=n} c_\alpha x^\alpha, \qquad c_\alpha \in \mathbb{R} \quad \text{or} \quad \mathbb{C}.$$

A polynomial of (total) degree at most n is of the form $P(x) = \sum_{|\alpha|\le n} c_\alpha x^\alpha$. Let $\mathcal{P}_n^d$ denote the space of real homogeneous polynomials of degree n and let Π_n^d denote the space of real polynomials of degree at most n. Counting the cardinalities of $\{\alpha \in \mathbb{N}_0^d : |\alpha| = n\}$ and $\{\alpha \in \mathbb{N}_0^d : |\alpha| \le n\}$ shows that

$$\dim \mathcal{P}_n^d = \binom{n+d-1}{n} \quad \text{and} \quad \dim \Pi_n^d = \binom{n+d}{n}.$$

Let ∂_i be the partial derivative in the i-th variable and Δ the Laplacian operator

$$\Delta = \partial_1^2 + \dots + \partial_d^2.$$

Definition 2.1.1. *For $n = 0, 1, 2, \dots$ let $\mathcal{H}_n^d$ be the linear space of real harmonic polynomials, homogeneous of degree n, on $\mathbb{R}^d$, that is,*

$$\mathcal{H}_n^d = \left\{P \in \mathcal{P}_n^d : \Delta P = 0\right\}.$$

Spherical harmonics are the restrictions of elements in $\mathcal{H}_n^d$ on the unit sphere. If $Y \in \mathcal{H}_n^d$, then $Y(x) = \|x\|^n Y(x')$ where $x = \|x\|x'$ and $x' \in \mathbb{S}^{d-1}$. Strictly speaking, one should make a distinction between $\mathcal{H}_n^d$ and its restriction on the sphere. We will however also call $\mathcal{H}_n^d$ the space of spherical harmonics. When it is necessary to emphasize the restriction on the sphere, we shall use the notation $\mathcal{H}_n^d|_{\mathbb{S}^{d-1}}$.

Spherical harmonics of different degrees are orthogonal with respect to

$$\langle f, g\rangle_{\mathbb{S}^{d-1}} := \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} f(x)g(x)d\sigma(x) \tag{2.1.1}$$

where $d\sigma$ is the Lebesgue measure on $\mathbb{S}^{d-1}$ and ω_d denotes the surface area of $\mathbb{S}^{d-1}$,

$$\omega_d := \int_{\mathbb{S}^{d-1}} d\sigma = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{2.1.2}$$

Theorem 2.1.2. *If $Y_n \in \mathcal{H}_n^d$, $Y_m \in \mathcal{H}_m^d$, and $n \ne m$, then $\langle Y_n, Y_m\rangle_{\mathbb{S}^{d-1}} = 0$.*

Proof. Let $\frac{\partial}{\partial r}$ denote the normal derivative. Since Y_n is homogeneous, $Y_n(x) = r^n Y_n(x')$, where $x = rx'$ and $x' \in \mathbb{S}^{d-1}$, so that $\frac{\partial Y_n}{\partial r}(x') = nY_n(x')$ for $x' \in \mathbb{S}^{d-1}$ and $n \ge 0$. By Green's identity,

$$\begin{aligned}(n-m)\int_{\mathbb{S}^{d-1}} Y_n Y_m d\sigma &= \int_{\mathbb{S}^{d-1}} \left(Y_m \frac{\partial Y_n}{\partial r} - Y_n \frac{\partial Y_m}{\partial r}\right) d\sigma \\ &= \int_{\mathbb{B}^d} (Y_m \Delta Y_n - Y_n \Delta Y_m)\, dx = 0,\end{aligned}$$

since $\Delta Y_n = 0$ and $\Delta Y_m = 0$. □

Theorem 2.1.3. *For $n = 0, 1, 2, \ldots$, there is a decomposition of $\mathcal{P}_n^d$,*

$$\mathcal{P}_n^d = \bigoplus_{0 \le j \le n/2} \|x\|^{2j} \mathcal{H}_{n-2j}^d.$$

In other words, for each $P \in \mathcal{P}_n^d$, there is a unique decomposition

$$P(x) = \sum_{0 \le j \le n/2} \|x\|^{2j} P_{n-2j}(x) \quad \textit{with} \quad P_{n-2j} \in \mathcal{H}_{n-2j}^d.$$

Proof. The proof uses induction. Evidently $\mathcal{P}_0^d = \mathcal{H}_0^d$ and $\mathcal{P}_1^d = \mathcal{H}_1^d$. Since $\Delta \mathcal{P}_n^d \subset \mathcal{P}_{n-2}^d$, $\dim \mathcal{H}_n^d \ge \dim \mathcal{P}_n^d - \dim \mathcal{P}_{n-2}^d$. Suppose the statement holds for $m = 0, 1, \ldots, n-1$. Then $\|x\|^2 \mathcal{P}_{n-2}^d$ is a subspace of $\mathcal{P}_n^d$ and it is isomorphic to $\mathcal{P}_{n-2}^d$. By induction hypothesis, $\|x\|^2 \mathcal{P}_{n-2}^d = \bigoplus_{0 \le j \le n/2-1} \|x\|^{2j+1} \mathcal{H}_{n-2-2j}^d$. Hence, by the previous theorem, $\mathcal{H}_n^d$ is orthogonal to $\|x\|^2 \mathcal{P}_{n-2}^d$, so that $\dim \mathcal{H}_n^d + \dim \mathcal{P}_{n-2}^d \le \dim \mathcal{P}_n^d$. Consequently, $\mathcal{P}_n^d = \mathcal{H}_n^d \oplus \|x\|^2 \mathcal{P}_{n-2}^d$. □

The polar coordinates $(x_1, x_2) = (r\cos\theta, r\sin\theta)$, $r \ge 0$, $0 \le \theta \le 2\pi$, gives a coordinates for $\mathbb{S}^1$ when $r = 1$. The high dimensional analogue is the spherical polar coordinates defined by

$$\begin{cases} x_1 = r\sin\theta_{d-1} \ldots \sin\theta_2 \sin\theta_1, \\ x_2 = r\sin\theta_{d-1} \ldots \sin\theta_2 \cos\theta_1, \\ \cdots \\ x_{d-1} = r\sin\theta_{d-1}\cos\theta_{d-2}, \\ x_d = r\cos\theta_{d-1}, \end{cases} \tag{2.1.3}$$

where $r \ge 0$, $0 \le \theta_1 \le 2\pi$, $0 \le \theta_i \le \pi$ for $i = 2, \ldots, d-1$. When $r = 1$ these are the coordinates for the unit sphere $\mathbb{S}^{d-1}$, which are in fact defined recursively by

$$x = (\xi \sin\theta_{d-1}, \cos\theta_{d-1}) \in \mathbb{S}^{d-1}, \qquad \xi \in \mathbb{S}^{d-2}.$$

Let $d\sigma = d\sigma_d$ be the Lebesgue measure on $\mathbb{S}^{d-1}$. Then it is easy to verify that

$$d\sigma = d\sigma_d = \prod_{j=1}^{d-2} (\sin\theta_{d-j})^{d-j-1}\, d\theta_{d-1} \ldots d\theta_2 d\theta_1 \tag{2.1.4}$$

in the spherical coordinates (2.1.3). Furthermore, we have

$$\int_{\mathbb{S}^{d-1}} f(x) d\sigma_d(x) = \int_0^\pi \int_{\mathbb{S}^{d-2}} f(\xi \sin\theta, \cos\theta) d\sigma_{d-1}(\xi) (\sin\theta)^{d-2} d\theta. \tag{2.1.5}$$

An orthogonal basis of spherical harmonics can be given in terms of the Gegenbauer polynomials $C_n^\lambda(t)$, which are orthogonal with respect to weight function

$$w_\lambda(t) := (1-t^2)^{\lambda - \frac{1}{2}}, \qquad t \in [-1, 1], \quad \lambda > -1/2.$$

More precisely, making a change of variable $t \mapsto \cos\theta$, we have

$$\int_0^\pi C_n^\lambda(\cos\theta) C_m^\lambda(\cos\theta) (\sin\theta)^{d-2} d\theta = \frac{\sqrt{\pi}\,\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d}{2})} \frac{\lambda (2\lambda)_n}{(n+\lambda) n!} \delta_{m,n}, \tag{2.1.6}$$

where $(a)_n$ denote the Pochhammer symbol $(a)_n = a(a+1)\cdots(a+n-1)$. Together with (2.1.5), this allows us to write down a basis of spherical harmonics in terms of the Gegenbauer polynomials in the spherical coordinates.

THEOREM 2.1.4. *For $d > 2$ and $\alpha \in \mathbb{N}_0^d$, define*

$$Y_\alpha(x) := [h_\alpha]^{-1} r^{|\alpha|} g_\alpha(\theta_1) \prod_{j=1}^{d-2} (\sin\theta_{d-j})^{|\alpha^{j+1}|} C_{\alpha_j}^{\lambda_j}(\cos\theta_{d-j}), \tag{2.1.7}$$

where $g_\alpha(\theta_1) = \cos\alpha_{d-1}\theta_1$ for $\alpha_d = 0$, $\sin\alpha_{d-1}\theta_1$ for $\alpha_d = 1$, $|\alpha^j| = \alpha_j + \ldots + \alpha_d$, $\lambda_j = |\alpha^{j+1}| + (d-j-1)/2$, and

$$[h_\alpha]^2 := b_\alpha \prod_{j=1}^{d-2} \frac{\alpha_j!(\frac{d-j+1}{2})_{|\alpha^{j+1}|}(\alpha_j + \lambda_j)}{(2\lambda_j)_{\alpha_j}(\frac{d-j}{2})_{|\alpha^{j+1}|}\lambda_j}$$

in which $b_\alpha = 2$ if $\alpha_{d-1} + \alpha_d > 0$, $= 1$ otherwise. Then $\{Y_\alpha : |\alpha| = n, \alpha_d = 0, 1\}$ is an orthonormal basis of $\mathcal{H}_n^d$; that is, $\langle Y_\alpha, Y_\beta\rangle_{\mathbb{S}^{d-1}} = \delta_{\alpha,\beta}$.

2.2. Projection operators and Zonal harmonics

Let $L^2(\mathbb{S}^{d-1})$ denote the space of square integrable functions on $\mathbb{S}^{d-1}$. Let

$$\mathrm{proj}_n : L^2(\mathbb{S}^{d-1}) \mapsto \mathcal{H}_n^d$$

denote the orthogonal projection from $L^2(\mathbb{S}^{d-1})$ onto $\mathcal{H}_n^d$.

DEFINITION 2.2.1. *The reproducing kernel $Z_n(\cdot,\cdot)$ of $\mathcal{H}_n^d$ is uniquely determined by*

$$\frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} Z_n(x,y)p(y)d\sigma(y) = p(x), \quad \forall p \in \mathcal{H}_n^d, \quad x \in \mathbb{S}^{d-1} \tag{2.2.1}$$

and the requirement that $Z_n(x,\cdot)$ is an element of $\mathcal{H}_n^d$ for each fixed x.

That the kernel is well defined and unique follows from the Riesz representation theorem, applying on the linear functional $L(Y) := Y(x)$, $Y \in \mathcal{H}_n^d$, for a fixed $x \in \mathbb{S}^{d-1}$. The reproducing kernel is also the kernel for the projection operator:

$$\mathrm{proj}_n f(x) = \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} f(y)Z_n(x,y)d\sigma(y). \tag{2.2.2}$$

LEMMA 2.2.2. *The kernel $Z_n(\cdot,\cdot)$ satisfies the following properties:*

1. *In terms of an orthonormal basis $\{Y_j : 1 \le j \le \dim\mathcal{H}_n^d\}$ of $\mathcal{H}_n^d$,*

$$Z_n(x,y) = \sum_{k=1}^{\dim\mathcal{H}_n^d} Y_k(x)Y_k(y), \quad x, y \in \mathbb{S}^{d-1}. \tag{2.2.3}$$

2. *Despite* (2.2.3), *Z_n is independent of the particular choice of bases of $\mathcal{H}_n^d$.*
3. *For any $\xi, \eta \in \mathbb{S}^{d-1}$,*

$$\frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} Z_n(\xi,y)Z_n(\eta,y)d\sigma(y) = Z_n(\xi,\eta). \tag{2.2.4}$$

4. *$Z_n(x,y)$ depends only on $\langle x,y\rangle$.*

PROOF. Since $Z_n(x,\cdot) \in \mathcal{H}_n^d$, it can be expressed as $Z_n(x,y) = \sum_k c_k Y_k(y)$ where the coefficients are determined by (2.2.1) as $c_k = Y_k(x)$. The uniqueness implies that Z_n is independent of the choice of bases. This can also be shown directly as follows. Let $\mathbb{Y}_n = (Y_1, ..., Y_N)$ with $N = \dim\mathcal{H}_n^d$ and regard it as a column vector. Then $Z_n(x,y) = [\mathbb{Y}_n(x)]^{\mathrm{tr}}\mathbb{Y}_n(y)$. If $\{Y_j' : 1 \le j \le N\}$ is another orthonormal basis of $\mathcal{H}_n^d$, then $\mathbb{Y}_n' = Q\mathbb{Y}_n$. Since the orthonormality of $\{Y_j\}$ can be expressed as $\frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} \mathbb{Y}_n(x)[\mathbb{Y}_n(x)]^{\rm tr} d\sigma(x)$ is an identity matrix, it follows readily that Q is an orthogonal matrix. Hence, $Z_n(x,y) = [\mathbb{Y}_n(x)]^{\rm tr} Q^{\rm tr} Q \mathbb{Y}_n(y) = [\mathbb{Y}'_n(x)]^{\rm tr}\mathbb{Y}'_n(y)$.

The uniqueness of $Z_n(x,y)$ shows that $Z_n(Qx,Qy) = Z_n(x,y)$ for all $Q \in O(d)$. Since for $x,y \in \mathbb{S}^{d-1}$ there exists a $Q \in SO(d)$ such that $Qx = (0,...,0,1)$ and $Qy = (0,...,0,\sqrt{1-\langle x,y\rangle^2}, \langle x,y\rangle)$, this shows that $Z_n(x,y)$ depends only on $\langle x,y\rangle$. $\square$

From the third property of the theorem, $Z_n(x,y) = F_n(\langle x,y\rangle)$, which is often called a zonal harmonic as it is harmonic and depends only on $\langle x,y\rangle$. We now derive a closed formula of F_n, which turns out to be a multiple of the Gegenbauer polynomial, C_n^λ, of degree n.

THEOREM 2.2.3. *For $n \in \mathbb{N}_0$ and $x,y \in \mathbb{S}^{d-1}$, $d \geq 3$,*

$$Z_n(x,y) = \frac{n+\lambda}{\lambda} C_n^\lambda(\langle x,y\rangle), \quad \lambda = \frac{d-2}{2}. \tag{2.2.5}$$

PROOF. Let $Z_n(x,y) = F_n(\langle x,y\rangle)$. Since $Z_n(x,\cdot)$ and $Z_m(x,\cdot)$ are orthogonal over $\mathbb{S}^{d-1}$ and their integrals can be written as an integral of one variable, we obtain

$$0 = \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} F_n(\langle x,y\rangle)F_m(\langle x,y\rangle)d\sigma(y) = \frac{\omega_{d-1}}{\omega_d}\int_{-1}^1 F_n(t)F_m(t)(1-t^2)^{\frac{d-3}{2}}dt.$$

Hence, by the uniqueness of the Genbauer polynomials, $F_n(t) = c_n C_n^\lambda(t)$. The constant c_n can be determined by setting $x = y$ in (2.2.4) and integrating over $\mathbb{S}^{d-1}$, we obtain

$$F_n(1) = \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} Z_n(\langle x,x\rangle)d\sigma(x) = \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} \sum_{k=1}^{\dim \mathcal{H}_n^d} Y_k^2(x)d\sigma(x) = \dim \mathcal{H}_n^d,$$

so that

$$c_n^2 C_n^\lambda(1) = \dim \mathcal{H}_n^d.$$

Since $C_n^\lambda(1) = \binom{n+2\lambda-1}{n}$, we can deduce that $c_n = (n+\lambda)/\lambda$. $\square$

Let $\{Y_i : 1 \le i \le \dim \mathcal{H}_n^d\}$ be an orthonormal basis of $\mathcal{H}_n^d$. Then (2.2.5) states

$$\sum_{j=1}^{\dim \mathcal{H}_n^d} Y_j(x)Y_j(y) = \frac{n+\lambda}{\lambda} C_n^\lambda(\langle x,y\rangle), \qquad \lambda = \frac{d-2}{2}. \tag{2.2.6}$$

This identity is usually referred to as the addition formula of spherical harmonics, since for $d = 2$ it is the additional formula of the cosine function.

COROLLARY 2.2.4. *For $n \in \mathbb{N}_0$ and $x,y \in \mathbb{S}^{d-1}$, $d \geq 3$,*

$$|Z_n(x,y)| \le \dim \mathcal{H}_n^d \quad \text{and} \quad Z_n(x,x) = \dim \mathcal{H}_n^d.$$

The functions on $\mathbb{S}^{d-1}$ that depend only on $\langle x,y\rangle$ are analogues of radial functions on $\mathbb{R}^d$. For such functions, there is a Funk-Hecke formula given blow.

THEOREM 2.2.5. *Let f be an integrable function such that $\int_{-1}^1 |f(t)|(1-t^2)^{(d-3)/2}dt$ is finite and $d \ge 2$. Then for any $Y_n \in \mathcal{H}_n^d$,*

$$\int_{\mathbb{S}^{d-1}} f(\langle x,y\rangle)Y_n(y)d\sigma(y) = \lambda_n(f)Y_n(x), \quad x \in \mathbb{S}^{d-1}, \tag{2.2.7}$$

where $\lambda_n(f)$ is a constant defined by

$$\lambda_n(f) = \omega_{d-1}\int_{-1}^1 f(t)\frac{C_n^{\frac{d-2}{2}}(t)}{C_n^{\frac{d-2}{2}}(1)}(1-t^2)^{\frac{d-3}{2}}dt.$$

2.3. Laplace-Beltrami operator

The operator in the section heading is the spherical part of the Laplace operator, which we denote by Δ_0. The operator Δ_0 plays an important role for analysis on the sphere.

LEMMA 2.3.1. *In the spherical-polar coordinates $x = r\xi$, $r > 0$, $\xi \in \mathbb{S}^{d-1}$, the Laplace operator satisfies*

$$\Delta = \frac{d^2}{dr^2} + \frac{d-1}{r}\frac{d}{dr} + \frac{1}{r^2}\Delta_0, \tag{2.3.1}$$

where

$$\Delta_0 = \sum_{i=1}^{d-1} \frac{\partial^2}{\partial \xi_i^2} - \sum_{i=1}^{d-1}\sum_{j=1}^{d-1} \xi_i \xi_j \frac{\partial^2}{\partial \xi_i \partial \xi_j} - (d-1)\sum_{i=1}^{d-1} \xi_i \frac{\partial}{\partial \xi_i}. \tag{2.3.2}$$

COROLLARY 2.3.2. *Let $f \in C^2(\mathbb{S}^{d-1})$. Define $F(y) := f(y/\|y\|)$, $y \in \mathbb{R}^d$. Then*

$$\Delta_0 f(x) = \Delta F(x), \qquad x \in \mathbb{S}^{d-1}. \tag{2.3.3}$$

The corollary follows immediately from (2.3.1), since $x/\|x\|$ is independent of r. The expressions in (2.3.3) show that Δ_0 and ∇_0 are independent of coordinates of $\mathbb{S}^{d-1}$. In fact, we could take (2.3.3) as the definition of Δ_0 and ∇_0.

Our next result shows that the spherical harmonics are eigenfunctions of the Laplace-Beltrami operator.

THEOREM 2.3.3. *The spherical harmonics are eigenfunctions of Δ_0,*

$$\Delta_0 Y(\xi) = -n(n+d-2)Y(\xi), \quad \forall Y \in \mathcal{H}_n^d, \quad \xi \in \mathbb{S}^{d-1}. \tag{2.3.4}$$

PROOF. Let $x = r\xi$, $\xi \in \mathbb{S}^{d-1}$. Since $Y \in \mathcal{H}_n^d$ is homogeneous, $Y(x) = r^n Y(\xi)$ and, by (2.3.1),

$$0 = \Delta Y(x) = n(n-1)r^{n-2}Y(\xi) + (d-1)nr^{n-2}Y(\xi) + r^{n-2}\Delta_0 Y(\xi),$$

which is (2.3.4) upon dividing r^{n-2}. □

The identity (2.3.4) also implies that Δ_0 is self-adjoint, that is,

$$\int_{\mathbb{S}^{d-1}} f(x)\Delta_0 g(x) d\sigma = \int_{\mathbb{S}^{d-1}} \Delta_0 f(x) g(x) d\sigma,$$

which can also be proved directly.

Next we introduce angular derivatives in higher dimensions:

DEFINITION 2.3.4. *For $x \in \mathbb{R}^d$ and $1 \le i \ne j \le n$, define*

$$D_{i,j} := x_i \partial_j - x_j \partial_i = \frac{\partial}{\partial \theta_{i,j}}, \tag{2.3.5}$$

where $\theta_{i,j}$ is the angle of polar coordinates in (x_i, x_j)-plane, defined by $(x_i, x_j) = r_{i,j}(\cos\theta_{i,j}, \sin\theta_{i,j})$, $r_{i,j} \ge 0$ and $0 \le \theta_{i,j} \le 2\pi$.

By its definition with partial derivatives on $\mathbb{R}^d$, $D_{i,j}$ acts on $\mathbb{R}^d$, yet the second equality in (2.3.5) shows that it acts on the sphere $\mathbb{S}^{d-1}$. Thus, for f defined on $\mathbb{R}^d$,

$$(D_{i,j} f)(\xi) = D_{i,j}[f(\xi)], \qquad \xi \in \mathbb{S}^{d-1}, \tag{2.3.6}$$

where the right hand side means that $D_{i,j}$ is acting on $f(\xi)$.

Since $D_{i,j} = -D_{i,j}$, the number of distinct operators $D_{i,j}$ is $\binom{d}{2}$. The operator Δ_0 can be decomposed in terms of them.

THEOREM 2.3.5. *For $x \in \mathbb{S}^{d-1}$, Δ_0 satisfies the decomposition*

$$\Delta_0 = \sum_{1 \le i < j \le d} D_{i,j}^2. \tag{2.3.7}$$

The operators $D_{i,j}$ will play an important role for approximation theory on the sphere. We state two more properties of these operators.

LEMMA 2.3.6. *(i) For $1 \le i < j \le d$, the operators $D_{i,j}$ commutes with Δ_0. In particular, $D_{i,j}$ maps $\mathcal{H}_n^d$ to itself.*

(ii) For $f, g \in C^1(\mathbb{S}^{d-1})$ and $1 \le i \ne j \le d$,

$$\int_{\mathbb{S}^{d-1}} f(x) D_{i,j} g(x) d\sigma(x) = - \int_{\mathbb{S}^{d-1}} D_{i,j} f(x) g(x) d\sigma(x). \tag{2.3.8}$$

References. The books that contain materials on spherical harmonics are [**1, 2, 19, 20, 21, 22, 23, 39, 43**]. The operators $D_{i,j}$ are discussed in [**12**].

CHAPTER 3

Convolution Operator and Spherical Harmonic Expansion

3.1. Convolution and harmonic expansion on the sphere

The fact that the reproducing kernel of $\mathcal{H}_n^d$ depends only on $\langle x, y\rangle$ suggests a definition of a convolution operator on the sphere. Let

$$w_\lambda(x) = (1-x^2)^{\lambda-1/2}, \quad \lambda > -\tfrac{1}{2}, \quad x \in (-1,1).$$

Definition 3.1.1. *For $f \in L^1(\mathbb{S}^{d-1})$ and $g \in L^1(w_\lambda, [-1,1])$ with $\lambda = \frac{d-2}{2}$,*

$$(f * g)(x) := \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} f(y) g(\langle x, y\rangle) d\sigma(y). \tag{3.1.1}$$

Denote the norm of the space $L^p(w_\lambda;[-1,1])$ by $\|\cdot\|_{\lambda,p}$; for $g \in L^p(w_\lambda;[-1,1])$,

$$\|g\|_{\lambda,p} := \left(c_\lambda \int_{-1}^1 |g(x)|^p w_\lambda(x)dx\right)^{1/p}, \quad 1 \le p \le \infty,$$

where c_λ is the normalization constant such that $c_\lambda \int_{-1}^1 w_\lambda(t)dt = 1$, and the norm is taken as the uniform norm when $p=\infty$. The convolution on the sphere satisfies Young's inequality:

Theorem 3.1.2. *Let $p,q,r \ge 1$ and $p^{-1} = r^{-1} + q^{-1} - 1$. For $f \in L^q(\mathbb{S}^{d-1})$ and $g \in L^r(w_\lambda;[-1,1])$ with $\lambda = \frac{d-2}{2}$,*

$$\|f * g\|_p \le \|f\|_q \|g\|_{\lambda,r}. \tag{3.1.2}$$

In particular, for $1 \le p \le \infty$,

$$\|f * g\|_p \le \|f\|_p \|g\|_{\lambda,1} \quad \textit{and} \quad \|f * g\|_p \le \|f\|_1 \|g\|_{\lambda,p}. \tag{3.1.3}$$

In particular, (3.1.3) shows that $f * g$ is well defined. By (2.2.2) and (2.2.5), proj_n is a convolution operator:

$$\operatorname{proj}_n f = f * Z_n, \qquad Z_n(t) := \frac{n+\lambda}{\lambda} C_n^\lambda(t) \quad \text{with} \quad \lambda = \frac{d-2}{2}. \tag{3.1.4}$$

For $g \in L^1(w_\lambda;[-1,1])$, let $\widehat{g}_n^\lambda$ denote the Fourier coefficient of g with respect to the Gegenbauer polynomials,

$$\widehat{g}_n^\lambda := c_\lambda \int_{-1}^1 f(t) \frac{C_n^\lambda(t)}{C_n^\lambda(1)} (1-t^2)^{\lambda-\frac{1}{2}} dt.$$

Theorem 3.1.3. *For $f \in L^1(\mathbb{S}^{d-1})$ and $g \in L^1(w_\lambda;[-1,1])$ with $\lambda = \frac{d-2}{2}$,*

$$\operatorname{proj}_n(f * g) = \widehat{g}_n^\lambda \operatorname{proj}_n f, \quad n = 0,1,2\ldots. \tag{3.1.5}$$

Proof. By (2.2.2) and the Funk-Hecke formula in Theorem 2.2.5

$$\begin{aligned}\operatorname{proj}_n(f*g)(x) &= \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} (f*g)(\xi) Z_n(x,\xi)d\sigma(\xi)\\ &= \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} f(y)\left(\frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} g(\langle \xi, y\rangle) Z_n(x,\xi)d\sigma(\xi)\right)d\sigma(y)\\ &= \widehat{g}_n^\lambda \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} f(y)Z_n(x,y)d\sigma(y) = \widehat{g}_n^\lambda \operatorname{proj}_n g(x),\end{aligned}$$

where we have used the fact that $c_\lambda = \omega_{d-1}/\omega_d$ when $\lambda = \frac{d-2}{2}$. □

The identity (3.1.5) can be viewed as an analogy that the Fourier transform of $f*g$ is equal to the product of the Fourier transforms of f and g. It justifies calling the right hand side of (3.1.1) convolution.

With respect to an orthonormal basis $\{Y_\alpha\}$, say (2.1.7), a function f in $L^2(\mathbb{S}^{d-1})$ can be expanded in a Fourier series

$$f(x) = \sum c_\alpha Y_\alpha(x), \quad \text{where} \quad c_\alpha = \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} f(y)Y_\alpha(y)d\sigma.$$

It is often more convenient to consider the orthogonal expansions in terms of the spaces $\mathcal{H}_n^d$. Collecting terms of spherical harmonics of the same degree, the Fourier series takes the form, by (2.2.2) and (2.2.3),

$$f(x) = \sum_{n=0}^{\infty} \operatorname{proj}_n f(x), \tag{3.1.6}$$

where $\operatorname{proj}_n f$ is the orthogonal projection of f onto the space $\mathcal{H}_n^d$. The formulation of (3.1.6) is independent of a particular choice of orthogonal basis. In particular, the n-th partial sum of (3.1.6) is defined by

$$S_n f := \sum_{k=0}^{n} \operatorname{proj}_k f. \tag{3.1.7}$$

By (2.2.2), $S_n f$ can be written as an integral operator whose kernel enjoys a close form in terms of Jacobi polynomials.

Proposition 3.1.4. *For $n = 0, 1, 2, \ldots$,*

$$S_n f(x) = \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} f(y) K_n(\langle x, y\rangle) d\sigma(y) = (f * K_n)(x), \tag{3.1.8}$$

where the kernel K_n satisfies, with $\lambda = \frac{d-2}{2}$,

$$K_n(t) := \sum_{k=0}^{n} \frac{k+\lambda}{\lambda} C_k^\lambda(t) = \frac{(2\lambda+1)_n}{(\lambda+\frac{1}{2})_n} P_n^{(\lambda+\frac{1}{2},\lambda-\frac{1}{2})}(t). \tag{3.1.9}$$

Since the space of spherical polynomials is dense in $C(\mathbb{S}^{d-1})$ by the Weierstrass theorem and, as a consequence, dense in $L^2(\mathbb{S}^{d-1})$, the following theorem is a standard Hilbert space result for $L^2(\mathbb{S}^{d-1})$:

Theorem 3.1.5. *The family of spherical harmonics are dense in $L^2(\mathbb{S}^{d-1})$ and*

$$L^2(\mathbb{S}^{d-1}) = \sum_{n=0}^{\infty} \mathcal{H}_n^d : \quad f = \sum_{n=0}^{\infty} \operatorname{proj}_n f$$

in the sense that $\lim_{n\to\infty} \|f - S_n f\|_2 = 0$ *for any $f \in L^2(\mathbb{S}^{d-1})$. In particular, for $f \in L^2(\mathbb{S}^{d-1})$ the Parseval identity holds,*

$$\|f\|_2^2 = \sum_{n=0}^{\infty} \|\operatorname{proj}_n f\|_2^2.$$

Just as in the case of classical Fourier series in several variables, $S_n f$ does not converge, in general, pointwisely or in L^p for $p \neq 2$. The operator norm of S_n in $L^p(\mathbb{S}^{d-1})$ is defined by

$$\|S_n\|_p := \sup_{\|f\|_p=1} \|S_n f\|_p.$$

THEOREM 3.1.6. *Let $d > 2$. Then $\|S_n\|_\infty = \|S_n\|_1 = \Lambda_n$, where*

$$\Lambda_n := \max_{x\in\mathbb{S}^{d-1}} \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} |K_n(x,y)| d\sigma(y) \sim n^{\frac{d-2}{2}}.$$

PROOF. That $\|S_n\|_\infty = \|S_n\|_1 = \Lambda_n$ follows from the standard argument for linear integral operators. By the closed form of $K_n(x,y)$ in (3.1.9), we obtain

$$\Lambda_n = \frac{(2\lambda+1)_n}{(\lambda+\frac{1}{2})_n} c_\lambda \int_{-1}^{1} |P_n^{(\lambda+\frac{1}{2},\lambda-\frac{1}{2})}(t)|(1-t^2)^{\lambda-\frac{1}{2}} dt, \quad \lambda = \frac{d-2}{2},$$

from which the asymptotic follows from that of the Jacobi polynomials. □

In the case of $d = 2$, $\|S_n\|_\infty \sim \log n$ as shown in the classical Fourier analysis.

The constant Λ_n is often called the Lebesgue constant. Since it is unbounded as $n \to \infty$, the uniform boundedness principle implies that there is a function $f \in C(\mathbb{S}^{d-1})$ for which $S_n f$ does not converge to f in the uniform norm. We then look for summability methods of the spherical harmonic series that will ensure the convergence. One important class of such methods are the Cesàro means.

DEFINITION 3.1.7. *For $\delta \in \mathbb{R}$, the Cesàro (C,δ) means of the sequence $\{a_k\}_{k=0}^\infty$ are defined by*

$$s_n^\delta := \frac{1}{A_n^\delta} \sum_{k=0}^{n} A_{n-k}^\delta a_k, \quad n = 0, 1, \ldots, \tag{3.1.10}$$

where

$$A_k^\delta = \binom{k+\delta}{k} = \frac{(\delta+k)(\delta+k-1)\ldots(\delta+1)}{k!}. \tag{3.1.11}$$

The sequence is called (C,δ) summable if s_n^δ converges as $n \to \infty$.

A simple exercise shows that if s_n^δ converges to s, then $s_n^{\delta+\tau}$ converges to s for all $\tau > 0$.

Denote by $S_n^\delta f$ the (C,δ) means of the spherical harmonic series,

$$S_n^\delta f := \frac{1}{A_n^\delta} \sum_{k=0}^{n} A_{n-k}^\delta \operatorname{proj}_k f. \tag{3.1.12}$$

If $\delta = 0$, then $S_n^\delta f = S_n f$. By (3.1.4), $S_n^\delta f$ can be written as a convolution operator

$$S_n^\delta f = f * K_n^\delta, \qquad K_n^\delta(t) := \frac{1}{A_n^\delta} \sum_{k=0}^{n} A_{n-k}^\delta \frac{k+\lambda}{\lambda} C_k^\lambda(t), \tag{3.1.13}$$

where $\lambda = (d-2)/2$. This kernel is closely connected to the Cesàro means $s_n^\delta(w_\lambda; f)$ of the Fourier orthogonal series in the Gegenbauer polynomials. Indeed, let $k_n^\delta(w_\lambda; \cdot, \cdot)$ denote the kernel of $s_n^\delta(w_\lambda; f)$,

$$s_n^\delta(w_\lambda; f, x) = c_\lambda \int_{-1}^{1} f(y) k_n^\delta(w_\lambda; x, y) w_\lambda(y) dy,$$

then it is easy to verify that

$$K_n^\delta(t) = k_n^\delta\left(w_\lambda; 1, t\right), \qquad \lambda = \tfrac{d-2}{2}. \tag{3.1.14}$$

As a consequence, some of the convergence results of spherical harmonic series can be deduced from those of the Gegenbauer series. Here is one example:

THEOREM 3.1.8. *For $\delta \ge d-1$, S_n^δ is a nonnegative operator; that is, $S_n^\delta f(x) \ge 0$ if $f(x) \ge 0$ for all $x \in \mathbb{S}^{d-1}$.*

PROOF. By (3.1.13), we only need to show $K_n^{d-1}(t) \ge 0$ for $t \in [-1,1]$, which follows, by (3.1.14), from the classical result of Kogbetiliantz that $k_n^\delta(w_\lambda; 1, t) \ge 0$ if $\delta \ge 2\lambda - 1$ ([**1**, p. 389]). □

Moreover, the relation (3.1.14) shows that the Lebesgue constant of S_n^δ can be deduced from the Lebesgue function of $s_n^\delta(w_\lambda)$ evaluated at the point $x = 1$.

THEOREM 3.1.9. *Let $\lambda = \frac{d-2}{2}$. Then $\|S_n^\delta\|_\infty = \|S_n^\delta\|_1 = \Lambda_n^\delta$, where*

$$\Lambda_n^\delta := \max_{x \in \mathbb{S}^{d-1}} \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} |K_n^\delta(x,y)| d\sigma(y) \sim \begin{cases} n^{\lambda-\delta}, & 0 < \delta < \lambda, \\ \log n, & \delta = \lambda, \\ 1 & \delta > \lambda. \end{cases}$$

In particular, $\|S_n^\delta\|_\infty$ and $\|S_n^\delta\|_1$ are bounded if and only if $\delta > \frac{d-2}{2}$.

PROOF. Again, that $\|S_n^\delta\|_\infty = \|S_n^\delta\|_1 = \Lambda_n^\delta$ follows from the standard arguement. The integral of the zonal function on the sphere can be reduced to one-variable, which gives

$$\Lambda_n^\delta = c_\lambda \int_{-1}^1 \left| k_n^\delta(w_\lambda; 1, t) \right| (1-t^2)^{\lambda - \frac{1}{2}} dt.$$

The asymptotic of this integral is given in [**41**, Section 9.4]. □

COROLLARY 3.1.10. *If $\delta > \frac{d-2}{2}$, then for $f \in L^p(\mathbb{S}^{d-1})$ and $1 \le p \le \infty$, or $f \in C(\mathbb{S}^{d-1})$ and $p = \infty$,*

$$\sup_n \|S_n^\delta f\|_p \le c \|f\|_p \quad \text{and} \quad \lim_{n\to\infty} \|S_n^\delta f - f\|_p = 0.$$

Furthermore, for $p = 1$ or ∞, the convergence fails in general if $\delta = \frac{d-2}{2}$.

The index $\lambda = \frac{d-2}{2}$ is often called the critical index for the (C,δ) means of the spherical harmonic series on $\mathbb{S}^{d-1}$. We will need the following pointwise estimate of the Cesàro kernel.

LEMMA 3.1.11. *Let $\lambda = \frac{d-2}{2} \ge 0$. If $0 \le \delta \le \lambda + 1$,*

$$|K_n^\delta(t)| \le c n^{\lambda-\delta} \left[(1 - t + n^{-2})^{-(\delta+\lambda+1)/2} + (1 + t + n^{-2})^{-\lambda/2} \right].$$

If $\lambda + 1 \le \delta \le 2\lambda + 1$,

$$|K_n^\delta(t)| \le c n^{-1} \left[(1 - t + n^{-2})^{-(\lambda+1)} + (1 + t + n^{-2})^{-(2\lambda+1-\delta)/2} \right].$$

If $\delta \ge 2\lambda + 1$,

$$|K_n^\delta(t)| \le c n^{-1} (1 - t + n^{-2})^{-(\lambda+1)}.$$

3.2. Maximal function and Cesàro summability

For $x \in \mathbb{S}^{d-1}$ and $\theta > 0$ we define a spherical cap, $\mathsf{c}(x,\theta)$, centered at x by

$$\mathsf{c}(x,\theta) := \{ y \in \mathbb{S}^{d-1} : \langle x, y \rangle \ge \cos\theta \}. \tag{3.2.1}$$

Let $|\mathsf{c}(x,\theta)|$ denote the surface area of $\mathsf{c}(x,\theta)$, that is,

$$|\mathsf{c}(x,\theta)| := \int_{\mathsf{c}(x,\theta)} d\sigma(y) = \omega_{d-1} \int_0^\theta (\sin\phi)^{d-2} d\phi. \tag{3.2.2}$$

Then the Hardy-Littlewood maximum function on the sphere is defined as follows:

Definition 3.2.1. *For $f \in L^1(\mathbb{S}^{d-1})$, we define*

$$Mf(x) := \sup_{0<\theta\le\pi} \frac{1}{|\mathsf{c}(x,\theta)|} \int_{\mathsf{c}(x,\theta)} |f(y)|d\sigma(y).$$

Given $E \subset \mathbb{S}^{d-1}$, we denote by meas$(E)$ the Lebesgue measure $\int_E d\sigma(x)$ of E. The maximal function satisfies a weak boundedness for L^1 functions and a strong boundedness for L^p functions.

Theorem 3.2.2. (1) *For $f \in L^1(\mathbb{S}^{d-1})$ and $\alpha > 0$,*

$$\text{meas}\left\{x \in \mathbb{S}^{d-1} : Mf(x) \ge \alpha\right\} \le c\frac{\|f\|_1}{\alpha}.$$

(2) *Let $f \in L^p(\mathbb{S}^{d-1})$, $1 < p < \infty$, or $f \in C(\mathbb{S}^{d-1})$ for $p = \infty$. Then the maximal function is a strong type (p,p) operator for $1 < p \le \infty$; that is,*

$$\|Mf\|_p \le c\|f\|_p, \qquad 1 < p \le \infty.$$

(3) *Assume that $g \in L^1([-1,1], w_\lambda)$ and $k(\theta) := g(\cos\theta)$ is a continuous, non-negative and decreasing function on $[0,\pi]$. Then for $f \in L^1(\mathbb{S}^{d-1})$,*

$$|(f * g)(x)| \le cM(|f|)(x), \quad x \in \mathbb{S}^{d-1},$$

where $c = \int_0^\pi k(\theta)(\sin\theta)^{d-2}d\theta$.

For $\delta \ge 0$, we define the maximal Cesàro (C,δ) operator by

$$S_*^\delta f(x) := \sup_{N\ge 0} |S_N^\delta f(x)|, \qquad x \in \mathbb{S}^{d-1}.$$

It turns out that the maximal Cesàro operator S_*^δ can be controlled pointwisely by the Hardy-Littlewood maximal function whenever $\delta > \frac{d-2}{2}$.

Theorem 3.2.3. *If $\delta > \lambda := \frac{d-2}{2}$ and $f \in L^1(\mathbb{S}^{d-1})$ then for every $x \in \mathbb{S}^{d-1}$,*

$$S_*^\delta f(x) \le c\left[Mf(x) + Mf(-x)\right]. \tag{3.2.3}$$

If, in addition, $\delta \ge d-1$, then the term $Mf(-x)$ in (3.2.3) *can be dropped.*

Using Theorem 3.2.3, Theorem 3.2.2, we deduce the following corollary:

Corollary 3.2.4. *If $\delta > \frac{d-2}{2}$ and $f \in L(\mathbb{S}^{d-1})$, then $\lim_{N\to\infty} S_N^\delta f(x) = f(x)$ for almost every $x \in \mathbb{S}^{d-1}$ and, moreover,*

$$\text{meas}\{x \in \mathbb{S}^{d-1} : S_*^\delta f(x) > \alpha\} \le c\frac{\|f\|_1}{\alpha}, \quad \forall \alpha > 0.$$

3.3. Convergence of Cesàro means: further results

According to Corollary 3.1.10, the (C,δ) means $S_n^\delta f$ converges to f in the $L^1(\mathbb{S}^{d-1})$ norm or in the uniform norm if and only if $\delta > \frac{d-2}{2}$. We also know, since $S_n^0 f = S_n f$, that the convergence holds for $\delta \ge 0$ in the $L^2(\mathbb{S}^{d-1})$ norm. The case $1 < p < \infty$ is more delicate and far more difficult to obtain. We state the main results that have been proven below.

Throughout this section, we set for $1 \le p \le \infty$,

$$\delta(p) := \max\left\{0, (d-1)\left|\frac{1}{p} - \frac{1}{2}\right| - \frac{1}{2}\right\}. \tag{3.3.4}$$

We start with a negative result of Bonami and Clerc [**4**, Theorem 5.1].

Theorem 3.3.1. *If $1 \le p \le \infty$ and $0 \le \delta \le \delta(p)$, then there exists a function $f \in L^p(\mathbb{S}^{d-1})$ such that $S_n^\delta f$ does not converge in $L^p(\mathbb{S}^{d-1})$. In particular, if $\delta = 0$ and $p \ne 2$, then there exists a function $f \in L^p(\mathbb{S}^{d-1})$ such that $S_n f$ does not converge in $L^p(\mathbb{S}^{d-1})$.*

Theorem 3.3.1 also implies that if $1 \le p \le \infty$ and $0 \le \delta \le \delta(p)$, then $\{\|S_n^\delta\|_p\}_{n=1}^\infty$ is unbounded.

In the positive direction, the convergence of $S_n^\delta f$ depends on a sharp bound of the projection operator. Such a bound was established by Sogge [**37**].

THEOREM 3.3.2. *If $1 \le p \le \infty$ and $|\frac{1}{p} - \frac{1}{2}| \ge \frac{1}{d}$, then*

$$\| \operatorname{proj}_n(f)\|_2 \le c_d n^{\delta(p)} \|f\|_p. \tag{3.3.5}$$

The connection between (3.3.5) and the convergence of S_n^δ, revealed in the proof of Theorem 5.2 in [**4**], leads to the following theorem in [**37**].

THEOREM 3.3.3. *If $1 \le p < \infty$ and $|\frac{1}{p} - \frac{1}{2}| \ge \frac{1}{d}$, then $\lim_{n\to\infty} \|S_n^\delta f - f\|_p = 0$ holds for all $f \in L^p(\mathbb{S}^{d-1})$ if and only if $\delta > \delta(p)$.*

In the case of $d = 2$, Sogge [**37**] further proved that the conclusion of Theorem 3.3.3 remains true without the assumption $|1/2 - 1/p| \ge 1/3$.

For the maximal Cesàro operator S_*^δ, the following result was proved, using Stein's interpolation theorem for analytic families of operators, in [**4**].

THEOREM 3.3.4. *If $1 < p \le 2$, $\delta > (d-2)(\frac{1}{p} - \frac{1}{2})$ and $f \in L^p(\mathbb{S}^{d-1})$, then*

$$\|S_*^\delta f\|_p \le C_p \|f\|_p.$$

Together with Corollary 3.2.4, Theorem 3.3.4 implies the following corollary.

COROLLARY 3.3.5. *If $1 \le p \le 2$, $f \in L^p(\mathbb{S}^{d-1})$ and $\delta > (d-2)(\frac{1}{p} - \frac{1}{2})$ then*

$$\lim_{n\to\infty} S_n^\delta(f)(x) = f(x)$$

for almost every $x \in \mathbb{S}^{d-1}$.

3.4. Near best approximation operators and highly localized kernels

For a given function $f \in L^p(\mathbb{S}^{d-1})$, its Cesàro means $S_n^\delta f$ provide a sequence of polynomials that approximate f. These means are useful for our further study, but they are not ideal for quantitative results in approximation theory.

DEFINITION 3.4.1. *Let $f \in L^p(\mathbb{S}^{d-1})$ if $1 \le p < \infty$ and $f \in C(\mathbb{S}^{d-1})$ if $p = \infty$. For $n \ge 0$, the error of the best approximation to f by polynomials of degree at most n is defined by*

$$E_n(f)_p := \inf_{g \in \Pi_n(\mathbb{S}^{d-1})} \|f - g\|_p, \qquad 1 \le p \le \infty. \tag{3.4.1}$$

The best approximation element exists, since $\Pi_n^d(\mathbb{S}^{d-1})$ is a finite dimensional space, by a general theorem in the Banach space ([**14**, p. 59]). Finding such a polynomial, however, is not easy. For most applications, fortunately, it is sufficient to find a polynomial that is near best approximation.

DEFINITION 3.4.2. *Let η be a C^∞-function on $[0,\infty)$ such that $\eta(t) = 1$ for $0 \le t \le 1$ and $\eta(t) = 0$ for $t \ge 2$. Define*

$$L_n f := f * L_n = \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} f(y) L_n(\langle x, y\rangle)\, d\sigma(y), \quad x \in \mathbb{S}^{d-1}, \tag{3.4.2}$$

for $n = 0, 1, 2, \ldots,$ where

$$L_n(t) := \sum_{k=0}^\infty \eta\left(\frac{k}{n}\right) \frac{k+\lambda}{\lambda} C_k^\lambda(t), \qquad \lambda = \frac{d-2}{2}, \quad t \in [-1,1]. \tag{3.4.3}$$

In the following, we shall call functions that satisfy the requirement for the η function in the above theorem a C^∞ cut-off function, or simply a cut-off function.

Since η is supported on $[0, 2]$, the summation in $L_n f$ can be terminated at $k = 2n - 1$, so that $L_n f$ is a polynomial of degree at most $2n - 1$. It approximates f as well as the best approximation polynomial of degree n.

THEOREM 3.4.3. *Let $f \in L^p$ if $1 \le p < \infty$ and $f \in C(\mathbb{S}^{d-1})$ if $p = \infty$. Then*
(1) *$L_n f \in \Pi^d_{2n-1}$ and $L_n f = f$ for $f \in \Pi^d_n$.*
(2) *For $n \in \mathbb{N}$, $\|L_n f\|_p \le c\|f\|_p$.*
(3) *For $n \in \mathbb{N}$,*

$$\|f - L_n f\|_p \le (1 + c)E_n(f)_p. \tag{3.4.4}$$

PROOF. We have already shown that $L_n f$ is a polynomial of degree at most $2n - 1$. Using the projection operator proj_n of $\mathcal{H}^d_n$, we can write

$$L_n f = \sum_{k=0}^{\infty} \eta\left(\frac{k}{n}\right) \operatorname{proj}_k f.$$

Since the definition of η shows that $\eta(\frac{k}{n}) = 1$ for $0 \le k \le n$, it follows readily that $L_n f = \sum_{k=0}^{n} \operatorname{proj}_k f = f$ if $f \in \Pi^d_n$. This proves (1).

By Young's inequality, Theorem 3.1.2, $\|L_n f\|_p \le \|f\|_p \|\eta_n\|_{\lambda,1}$, where $\lambda = \frac{d-2}{2}$. The proof of (2) reduces to showing that $\|\eta_n\|_{\lambda,1}$ is bounded. Let σ be a positive integer such that $\sigma \ge d - 1$ so that, see Theorem 3.1.8, the (C, σ) means $K^\sigma_n(t)$ of the sequence $\frac{k+\lambda}{\lambda} C^\lambda_k(t)$ is nonnegative on $[-1, 1]$. Using summation by parts repeatedly on η_n, we can write

$$\eta_n(t) = \sum_{k=1}^{\infty} \triangle^{\sigma+1} \eta\left(\frac{k}{n}\right) \binom{k+\sigma}{k} K^\sigma_k(t),$$

where $\triangle^m$ acts on the function $t \mapsto \eta(\frac{t}{n})$. Since $\eta \in C^\infty[0, +\infty)$ implies that $|\triangle^{\sigma+1}\eta(k/n)| \le cn^{-\sigma-1}$ and $\binom{k+\sigma}{k} \le ck^\sigma$,

$$\|\eta_n\|_{\lambda,1} \le cn^{-\sigma-1} \sum_{k=1}^{2n} \left|\triangle^{\sigma+1}\eta\Big(\frac{k}{n}\Big)\right| k^\sigma \le c$$

as the support of η is on $[0, 2]$. This completes the proof of (2).

The proof of (3) is an easy consequence of (1) and (2). Indeed, let p_n be the best approximation polynomial of degree n. Then, (1) shows $L_n p_n = p_n$, so that

$$\|f - L_n f\|_p \le \|f - p_n\| + \|L_n(f - p_n)\| \le (1 + c)\|f - p_n\|_p = (1 + c)E_n(f)_p$$

by the triangle inequality and (2). □

The fact that $L_n f$ reproduces polynomials of degree up to n and it is a polynomial of degree at most $2n$ itself makes it a fundamental tool for polynomial approximation. Even more, its kernel, $L_n(t)$, possesses a remarkable property that L_n and its derivatives $L^{(j)}_n$ are highly localized at $t = 1$. More precisely, we state the following theorem.

THEOREM 3.4.4. *Let ℓ be a positive integer. For $n \ge 1$ and $\theta \in [0, \pi]$,*

$$|L^{(j)}_n(\cos\theta)| \le c_{\ell,j} \left\|\eta^{(3\ell-1)}\right\|_\infty n^{d-1+2j}(1 + n\theta)^{-\ell}, \quad j = 0, 1, \cdots. \tag{3.4.5}$$

By choosing ℓ large but fixed, the theorem shows that L_n and its derivatives decay faster than any polynomial of a fixed degree. This desirable property will be useful in many occasions.

References. The convolution operator is often studied in terms of the spherical mean, which were studied in approximation theory in [**3, 32, 35**], see also [**31, 44**].

For early study of the Cesàro summability, see the references in [**20**, Chapt. 12]. Several important references are [**4, 6, 37**]. The operator $L_n f$ was used in [**35**] but the use of such an operator on the sphere already appeared in [**24**]. The fast decaying of the kernel was established in [**5, 27, 30, 34**]. Under additional assumptions on the cut-off function, the rate of decay can be improved to subexponential estimate [**25**].

CHAPTER 4

Approximation on the Sphere

There are several ways to define a modulus of smoothness on the sphere $\mathbb{S}^{d-1}$. The one we choose in this chapter has the advantage that it relies on the modulus of smoothness on $\mathbb{S}^1$, even though $\mathbb{S}^{d-1}$ is not in itself a product space of $\mathbb{S}^1$, which allows us to utilize the classical results of trigonometric approximation. The results in this chapter are established fairly recently in [**12**].

4.1. Modulus of smoothness on the unit sphere

Let $\{e_1, \cdots, e_d\}$ be the standard basis for $\mathbb{R}^d$: the ith coordinate of e_j is 1 if $i = j$, 0 if $i \neq j$. For $1 \le i \neq j \le d$ and $t \in \mathbb{R}$, let $Q_{i,j,t}$ denote a rotation by the angle t in the (x_i, x_j)-plane, oriented such that the rotation from the vector e_i to the vector e_j is assumed to be positive. As an example, for $(i, j) = (1, 2)$ and $(x_1, x_2) = s(\cos\theta, \sin\theta)$, the action of the rotation $Q_{1,2,t} \in SO(d)$ is given by

$$\begin{aligned} Q_{1,2,t}(x_1, \ldots, x_d) &= (x_1 \cos t - x_2 \sin t, x_1 \sin t + x_2 \cos t, x_3, \cdots, x_d) \\ &= (s\cos(\phi + t), s\sin(\phi + t), x_3, \cdots, x_d). \end{aligned} \tag{4.1.1}$$

There are $d(d-1)/2$ distinct angles $\theta_{i,j}$, too many for a basis of $\mathbb{S}^{d-1}$, but they are closely related to the Euler angles that form a basis for $SO(d)$. Let $T(Q_{i,j.\theta})$ be the map that sends f to $T(Q_{i,j,\theta})f(x) = f(Q_{i,j,-\theta}x)$. Written explicitly, for example, for $(i, j) = (1, 2)$,

$$T(Q_{1,2,\theta}f)(x) = f(x_1 \cos\theta + x_2 \sin\theta, -x_1 \sin\theta + x_2 \cos\theta, x_3, \ldots, x_d). \tag{4.1.2}$$

Recall that $D_{i,j} = x_i\partial_j - x_j\partial_i$. Then it is easy to see that

$$\frac{dT(Q_{i,j,\theta})}{d\theta}\Big|_{\theta=0} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} = D_{i,j}, \tag{4.1.3}$$

where the second equality follows from (4.1.2).

For $r = 1, 2, \ldots$, we use $Q_{i,j,t}$ to define the difference operator

$$\triangle^r_{i,j,\theta} := (I - T(Q_{i,j,\theta}))^r, \qquad 1 \le i \neq j \le d,$$

where $T(Q)$ denotes the rotation operator $T(Q)f(x) = f(Q^{-1}x)$. Since $Q_{i,j,\theta} = Q_{j,i,-\theta}$, we have $\triangle^r_{i,j,\theta} = \triangle^r_{j,i,-\theta}$. Because of (4.1.1), $\triangle^r_{i,j,\theta}$ can be expressed in the forward difference. For instance, take $(i, j) = (1, 2)$ as an example,

$$\triangle^r_{1,2,\theta}f(x) = \overrightarrow{\triangle}^r_\theta f\left(x_1\cos(\cdot) - x_2\sin(\cdot), x_1\sin(\cdot) + x_2\cos(\cdot), x_3, \ldots, x_d\right), \tag{4.1.4}$$

where $\overrightarrow{\triangle}^r_\theta$ is acted on the variable $(\cdot)$ and is evaluated at $t = 0$. Our modulus of smoothness on the sphere is defined in terms of these differences.

Definition 4.1.1. *For $r \in \mathbb{N}$, $t > 0$, and $f \in L^p(\mathbb{S}^{d-1})$, $1 \le p < \infty$, or $f \in C(\mathbb{S}^{d-1})$ for $p = \infty$, define*

$$\omega_r(f, t)_p := \max_{1 \le i < j \le d} \sup_{|\theta| \le t} \left\| \triangle^r_{i,j,\theta} f \right\|_p. \tag{4.1.5}$$

For $r = 1$ we write $\omega(f, t)_p := \omega_1(f, t)_p$.

By (4.1.4), the quantity $\sup_{|\theta|\le t}\|\triangle^r_{i,j,\theta}f\|_p$ resembles the moduli of smoothness on the largest circle of the intersection of (x_i,x_j) plane and $\mathbb{S}^{d-1}$ and our modulus of smoothness $\omega_r(f,t)_p$ is the maximum among all possible choices of (i,j).

Just as in the case of $\mathbb{S}^1$, the modulus of smoothness is a continuous and increasing function of t with $\omega_r(f;t)_p \to 0$, $t\to 0+$, and it satisfies the properties

(1) For $s<r$, $\omega_r(f,t)_p \le 2^{r-s}\omega_s(f,t)_p$.
(2) For $\lambda>0$, $\omega_r(f;\lambda t)_p \le (\lambda+1)^r\omega_r(f;t)_p$.

Several further properties of $\omega_r(f;t)$ are more conveniently, and more useful, if stated in terms of $\Delta^r_{i,j,\theta}$.

Lemma 4.1.2. *Let $r\in\mathbb{N}$ and let $f\in L^p(\mathbb{S}^{d-1})$ with $1\le p<\infty$, or $f\in C(\mathbb{S}^{d-1})$ when $p=\infty$.*

(i) *For any $\lambda>0$, $t\in(0,2\pi]$, and $1\le i<j\le d$, we have*
$$\sup_{|\theta|\le\lambda t}\|\triangle^r_{i,j,\theta}f\|_p \le (\lambda+1)^r\sup_{|\theta|\le t}\|\triangle^r_{i,j,\theta}f\|_p.$$
(ii) *For $1\le i\ne j\le d$ and $\theta\in[-\pi,\pi]$,*
$$\|\triangle^r_{i,j,\theta}f\|_p\le 2^r\|f\|_p \quad \text{and} \quad \|\triangle^r_{i,j,\theta}f\|_p\le c\,|\theta|^r\|D^r_{i,j}f\|_p.$$
(iii) *If $f\in\Pi^d_n$ and $1\le i<j\le d$, then*
$$\|\triangle^r_{i,j,n^{-1}}f\|_p\sim n^{-r}\|D^r_{i,j}f\|_p$$
(iv) *For $1\le i<j\le d$ and $t\in(0,2\pi)$,*
$$\sup_{|\theta|\le t}\|\triangle^r_{i,j,\theta}f\|_p^p\sim\frac{1}{t}\int_0^t\|\triangle^r_{i,j,\theta}f\|_p^p\,d\theta$$
with $\|\cdot\|_p^p$ replaced by $\|\cdot\|_\infty$ when $p=\infty$.

Proof. Clearly we only need to consider the case of $(i,j)=(1,2)$. The proof comes down essentially to that of one variable. Indeed, we define
$$g_{s,y}(\phi):=f(s\cos\phi,s\sin\phi,\sqrt{1-s^2}y),\quad y\in\mathbb{S}^{d-3},\quad s\in[0,1],\quad \phi\in[0,2\pi].$$
For $d>3$, the integral over $\mathbb{S}^{d-1}$ can be written as
$$\begin{aligned}\int_{\mathbb{S}^{d-1}}f(y)\,d\sigma(y)&=\int_{\mathbb{B}^2}\int_{\mathbb{S}^{d-3}}f(x_1,x_2,\sqrt{1-\|x\|^2}y)\,d\sigma(y)\,(1-\|x\|^2)^{\frac{d-4}{2}}dx\\&=\int_0^1 s(1-s^2)^{\frac{d-4}{2}}\int_{\mathbb{S}^{d-3}}\int_0^{2\pi}g_{s,y}(\phi)\,d\phi\,d\sigma(y)\,ds,\end{aligned}\tag{4.1.6}$$
where the second equality follows from changing variables $(x_1,x_2)=s(\cos\phi,\sin\phi)$. Using (4.1.4), the identity (4.1.6) implies immediately
$$\|\triangle^r_{1,2,t}f\|_p^p=\frac{1}{\omega_d}\int_0^1 s(1-s^2)^{\frac{d-4}{2}}\int_{\mathbb{S}^{d-3}}\left[\int_0^{2\pi}\left|\overrightarrow{\triangle}^r_t g_{s,y}(\phi)\right|^p d\phi\right]d\sigma(y)\,ds.\tag{4.1.7}$$
In the case of $d=3$, the formula (4.1.6) degenerated to a form in which the integral over $\mathbb{S}^{d-3}$ is replaced by a sum of two terms. Furthermore, by (4.1.1) and (4.1.3), it is easy to see that
$$(-1)^rD^r_{1,2}f(s\cos\phi,s\sin\phi,\sqrt{1-s^2}y)=g^{(r)}_{s,y}(\phi)\tag{4.1.8}$$
and $g_{s,y}(\phi)$ is a 2π-periodic polynomial in ϕ. Consequently, we can deduce the proof of the lemma by applying the corresponding results in Lemma 1.0.3 to $g_{s,y}$ in an obvious manner. □

We also have an analogue of the Marchaud inequality.

LEMMA 4.1.3. *For* $0 < t < \frac{1}{2}$ *and every* $m > r$,

$$\omega_r(f;t)_p \le c_m t^r \int_t^1 \frac{\omega_m(f,u)_p}{u^{r+1}}\,du.$$

And here is an analogue of the Bernstein inequality.

LEMMA 4.1.4. *Let* f *be a polynomial in* Π_n^d. *For* $1 \le i < j \le d$, $r \in \mathbb{N}$,

$$\|D_{i,j}^r f\|_p \le n^r \|f\|_p, \qquad 1 \le p \le \infty. \tag{4.1.9}$$

Both these inequalities can be proved by reduction to one-variable.

Recall the operator $L_n f$ defined in Definition 3.4.2 and its kernel $L_n(x,y)$. For a fixed integer $\ell \in \mathbb{N}$, let

$$G_n(t) = G_{n,\ell}(t) := n^{d-1}(1+nt)^{-\ell}, \qquad t \in [0,\pi].$$

By Theorem 3.4.4, the kernel function L_n is bounded by

$$|L_n(\cos\theta)| \le c_\ell G_{n,\ell}(\theta), \qquad 0 \le \theta \le \pi. \tag{4.1.10}$$

The following two lemmas are important in our proof of the main theorem for approximation on the sphere in the next section. In particular, the first one is the key lemma in the proof.

LEMMA 4.1.5. (*i*) *Suppose that* $f \in L^p(\mathbb{S}^{d-1})$ *for* $1 \le p < \infty$ *and* $\ell > p + d$ *in the estimate* (4.1.10). *Then*

$$\int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} |f(x) - f(y)|^p G_n(d(x,y)) d\sigma(x)\, d\sigma(y) \le c\,\omega(f;n^{-1})_p^p.$$

(*ii*) *For* $x, y \in \mathbb{S}^{d-1}$ *and* $f \in C(\mathbb{S}^{d-1})$,

$$|f(x) - f(y)| \le c\,\omega(f; d(x,y))_\infty,$$

where c *depends only on dimension.*

LEMMA 4.1.6. *For* $1 \le p \le \infty$ *and* $1 \le i \ne j \le d$,

$$\triangle_{i,j,t}^r L_n f = L_n\left(\triangle_{i,j,t}^r f\right).$$

In particular, for $t > 0$,

$$\omega_r(f - L_n f, t)_p \le c\,\omega_r(f,t)_p.$$

4.2. Characterization of best approximation

Recall the quantity $E_n(f)_p$ of best approximation by polynomials defined in Definition 3.4.1. Our main result in this section is a characterization of the best approximation by polynomials in terms of the modulus of smoothness.

THEOREM 4.2.1. *For* $f \in L^p(\mathbb{S}^{d-1})$, $1 \le p < \infty$, *or* $f \in C(\mathbb{S}^{d-1})$ *if* $p = \infty$,

$$E_n(f)_p \le c\,\omega_r(f;n^{-1})_p, \qquad 1 \le p \le \infty. \tag{4.2.1}$$

On the other hand,

$$\omega_r(f;n^{-1})_p \le c\,n^{-r} \sum_{k=1}^n k^{r-1} E_{k-1}(f)_p, \qquad 1 \le p \le \infty, \tag{4.2.2}$$

where $\omega_r(f;t)_p$ *and* $E_n(f)_p$ *are defined in* (4.1.5) *and* (3.4.1), *respectively.*

Proof. When $r = 1$ and $1 \le p < \infty$, we use Lemma 3.4.3, Hölder's inequality and the fact that $\int_{\mathbb{S}^{d-1}} |K_n(\langle x, y\rangle)| d\sigma(y) \le c$ for all $x \in \mathbb{S}^{d-1}$ to obtain

$$\begin{aligned} E_n(f)_p &\le \|f - L_{\lfloor \frac{n}{2} \rfloor} f\|_p \\ &\lesssim \left(\int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} |f(x) - f(y)|^p |K_{\lfloor \frac{n}{2} \rfloor}(\langle x, y\rangle)| \, d\sigma(x) \, d\sigma(y) \right)^{\frac{1}{p}}, \end{aligned}$$

from which (4.2.1) for $r = 1$ follows from Lemma 4.1.5. For $r = 1$ and $p = \infty$, we use (ii) in Lemma 4.1.5 and $L_n f$ to conclude

$$\begin{aligned} E_n(f)_\infty \le \|f - L_{\lfloor \frac{n}{2} \rfloor} f\|_\infty &\le \int_{\mathbb{S}^{d-1}} |f(x) - f(y)| |K_{\lfloor \frac{n}{2} \rfloor}(\langle x, y\rangle)| \, d\sigma(y) \\ &\lesssim \int_{\mathbb{S}^{d-1}} \omega(f, d(x, y))_\infty |K_{\lfloor \frac{n}{2} \rfloor}(\langle x, y\rangle)| \, d\sigma(y) \\ &\lesssim \omega(f; n^{-1})_\infty \int_{\mathbb{S}^{d-1}} (1 + n d(x, y)) |K_{\lfloor \frac{n}{2} \rfloor}(\langle x, y\rangle)| \, d\sigma(y) \\ &\lesssim \omega(f, n^{-1})_\infty, \end{aligned}$$

where the last inequality follows from (4.1.10) and the fact that $\langle x, y\rangle = \cos d(x, y)$.

For $r > 1$, we follow the induction procedure on r using $L_n f$ in Lemmas 3.4.3 and 4.1.6. Assume that we have proven (4.2.1) for some positive integer $r \ge 1$. Let $g = f - L_{\lfloor \frac{n}{2} \rfloor} f$. It suffices to show that $\|g\|_p \le c\,\omega_{r+1}(f, n^{-1})_p$. The definition of $L_n f$ implies that $L_{\lfloor \frac{n}{4} \rfloor} g = 0$, so that

$$\|g\|_p = \|g - L_{\lfloor \frac{n}{4} \rfloor} g\|_p \le c E_{\lfloor \frac{n}{4} \rfloor}(g)_p \le c_1 \omega_r(g; n^{-1})_p.$$

On the other hand, using Lemma 4.1.3, we obtain, for any $m \in \mathbb{N}$,

$$\begin{aligned} \omega_r(g; t)_p &\le c_r t^r \int_t^{2^m t} \frac{\omega_{r+1}(g, u)_p}{u^{r+1}} \, du + c_r 2^{r+1} t^r \|g\|_p \int_{2^m t}^1 u^{-r-1} \, du \\ &\le c_1(m, r) \omega_{r+1}(g, t)_p + c_2(r) 2^{-mr} \|g\|_p, \end{aligned}$$

where $c_2(r)$ is independent of m. Choosing m so that $4^{-1} \le c_1 c_2(r) 2^{-mr} < 2^{-1}$, we deduce from these two equations that

$$\|g\|_p \le c\,\omega_{r+1}(g; n^{-1})_p \le c\,\omega_{r+1}(f; n^{-1})_p,$$

where the last step follows from Lemma 4.1.6. This completes the proof of (4.2.1).

The proof (4.2.2) relies on the Bernstein inequality and it is essentially the same as the proof in the one-variable case. □

Corollary 4.2.2. *Let $f \in L^p(\mathbb{S}^{d-1})$ if $1 \le p < \infty$ or $f \in C(\mathbb{S}^{d-1})$ if $p = \infty$. For $0 < \alpha < r$,*

$$E_n(f)_p \sim n^{-\alpha} \quad \textit{iff} \quad \omega_r(f; t)_p \sim t^\alpha.$$

Besides modulus of smoothness, the smoothness of a function can also be described by a K-functional, which describes how well the function can be approximated by smooth functions in certain sense. We define a K-functional via the differential operators $D_{i,j}$, which turns out to be equivalent to $\omega_r(f; t)_p$, as it is often the case in approximation theory. The K-functional is defined via the Sobolev space and is often easier to apply when the function is known to be differentiable.

For $r \in \mathbb{N}$ and $1 \le p \le \infty$, the Sobolev space $\mathcal{W}_p^r \equiv \mathcal{W}_p^r(\mathbb{S}^{d-1})$ consists of functions $f \in L^p(\mathbb{S}^{d-1})$ with distributional derivatives $D_{i,j}^r f$, $1 \le i < j \le d$, all belong to $L^p(\mathbb{S}^{d-1})$, where $L^p(\mathbb{S}^{d-1})$ is replaced by $C(\mathbb{S}^{d-1})$ when $p = \infty$. The norm of the space is defined by

$$\|f\|_{\mathcal{W}_p^r(\mathbb{S}^{d-1})} := \|f\|_p + \sum_{1 \le i < j \le d} \|D_{i,j}^r f\|_p.$$

Definition 4.2.3. *For $r \in \mathbb{N}_0$ and $t \ge 0$,*

$$K_r(f,t)_p := \inf_{g \in \mathcal{W}_p^r(\mathbb{S}^{d-1})} \left\{ \|f-g\|_p + t^r \max_{1\le i<j\le d} \|D_{i,j}^r g\|_p \right\}. \tag{4.2.3}$$

Theorem 4.2.4. *Let $r \in \mathbb{N}$ and let $f \in L^p(\mathbb{S}^{d-1})$ if $1 \le p < \infty$ and $f \in C(\mathbb{S}^{d-1})$ if $p = \infty$. For $0 < t < 1$,*

$$\omega_r(f;t)_p \sim K_r(f,t)_p, \qquad 1 \le p \le \infty.$$

Proof. By (ii) of Lemma 4.1.2 and the triangle inequality,

$$\|\triangle_{i,j,\theta}^r f\|_p \le \|\triangle_{i,j,\theta}^r (f-g)\|_p + \|\triangle_{i,j,\theta}^r g\|_p \lesssim \|f-g\|_p + \theta^r \|D_{i,j}^r g\|_p$$

from which $\omega_r(f;t)_p \lesssim K_r(f,t)_p$ follows. On the other hand, for $t > 0$ set $n = \lfloor \frac{1}{t} \rfloor$, then by Lemma 3.4.3, (4.2.1) and (iii) of Lemma 4.1.2

$$\begin{aligned} K_r(f,t)_p &\le \|f - L_n f\| + t^r \max_{1\le i<j\le d} \|D_{i,j}^r L_n f\|_p \\ &\lesssim \omega_r(f;n^{-1})_p + t^r n^r \max_{1\le i<j\le d} \|\triangle_{i,j,n^{-1}}^r L_n f\|_p \\ &\lesssim \omega_r(f;n^{-1})_p \lesssim \omega_r(f;t)_p \end{aligned}$$

where the last step follows from (i) of Lemma 4.1.4. □

The proof of the above theorem, together with Lemma 4.1.6 and (iii) of Lemma 4.1.2, yields a realization of the K-functional.

Corollary 4.2.5. *Under the assumption of Theorem 4.2.4,*

$$K_r(f,n^{-1})_p \sim \|f - L_n f\|_p + n^{-r} \max_{1\le i<j\le d} \|D_{i,j}^r L_n f\|_p.$$

In particular, this shows that the best approximation $E_n(f)_p$ can be characterized by the K-functional. Furthermore, we can now consider approximation in the Sobolev space.

Corollary 4.2.6. *If $r \in \mathbb{N}$, $f \in \mathcal{W}_p^r(\mathbb{S}^{d-1})$ and $1 \le p \le \infty$, then*

$$E_n(f)_p \le c n^{-r} \|f\|_{\mathcal{W}_p^r}. \tag{4.2.4}$$

4.3. Computational examples

We give several examples of functions whose modulus of smoothness can be determined. Since $\omega_r(f;t)_p$ is defined in terms of forwarded difference of one variable, it is not difficult to derive its upper bound. The difficulty lies in proving the lower bound.

Example 4.3.1. *For $x \in \mathbb{S}^{d-1}$ and $d \ge 3$, let $f_\alpha(x) = x^\alpha$ with $\alpha = (\alpha_1, \ldots, \alpha_d) \ne 0$. If $0 \le \alpha_i < 1$ for $1 \le i \le d$, then for $r \ge 2$ and $1 \le p \le \infty$,*

$$\omega_r(f,t)_{L^p(\mathbb{S}^{d-1})} \sim t^{\delta+1/p}, \qquad \delta = \min_{\alpha_i \ne 0} \{\alpha_1, \ldots, \alpha_d\}. \tag{4.3.1}$$

Consequently,

$$E_n(f_\alpha)_p \sim n^{-\delta-1/p}, \qquad 1 \le p \le \infty.$$

Example 4.3.2. *For $d \ge 3$ and $\alpha \ne 0$, let $g_\alpha(x) = (1-x_1)^\alpha$, $x = (x_1, \ldots, x_d) \in \mathbb{S}^{d-1}$. Then for $1 \le p \le \infty$,*

$$\omega_2(g_\alpha,t)_{L^p(\mathbb{S}^{d-1})} \sim \begin{cases} t^{2\alpha + \frac{d-1}{p}}, & -\frac{d-1}{2p} < \alpha < 1 - \frac{d-1}{2p}, \\ t^2 |\log t|^{1/p}, & \alpha = 1 - \frac{d-1}{2p}, \quad p \ne \infty, \\ t^2, & \alpha > 1 - \frac{d-1}{2p}. \end{cases} \tag{4.3.2}$$

For $\alpha = 0$, $\omega_2(g_\alpha, t)_p = 0$. *Consequently, for* $-\frac{d-1}{2p} < \alpha < 1 - \frac{d-1}{2p}$ *and* $\alpha \neq 0$,

$$E_n(g_\alpha)_p \sim n^{-2\alpha - \frac{d-1}{p}}, \qquad 1 \le p \le \infty.$$

If neither i nor j equals to 1, then $\triangle^2_{i,j,\theta} g_\alpha(x) = 0$. Thus, we only need to consider $\triangle^2_{1,j,\theta} g_\alpha$ and we can assume $j = 2$. Since $x \in \mathbb{S}^{d-1}$ and $d \ge 3$ imply that $(x_1, x_2) \in \mathbb{B}^2$, by (4.1.4),

$$\begin{aligned}\|\triangle^2_{1,2,\theta} g_\alpha\|_p^p &= c \int_{\mathbb{B}^2} |\triangle^2_{1,2,\theta} g_\alpha(x_1, x_2)|^p (1 - x_1^2 - x_2^2)^{\mu-1} dx \\ &= c \int_0^1 s \int_0^{2\pi} \left| \overrightarrow{\triangle}^2_\theta (1 - s\cos\phi)^\alpha \right|^p d\phi (1 - s^2)^{\mu-1} ds,\end{aligned}$$

where $\mu = \frac{d-2}{2}$ and the forward difference acts on ϕ; for $p = \infty$ the integral is replaced by the maximum taken over $0 \le s \le 1$ and $0 \le \phi \le 2\pi$. This last integral can be shown to give the order in (4.3.2). The proof is elementary but rather involved, see [**12**].

It is interesting to compare the two examples. As functions defined on $\mathbb{R}^d$, the functions x_1^α and $(1 - x_1)^\alpha$ have the same smoothness and a reasonable modulus of smoothness would confirm that. As functions on the sphere $\mathbb{S}^{d-1}$, however, they have different orders of smoothness as seen in Examples 4.3.1 and 4.3.2, and their errors of best approximation are also different as seen in these examples.

CHAPTER 5

Harmonic Analysis and Approximation in Weighted Spaces on the Sphere

There is a recent far reaching extension of spherical harmonics, in which the Lebesgue measure, the only rotation invariant measure, on the sphere is replaced by a family of weighted measures invariant under a finite reflection group, and the Laplace operator is replaced by a sum of square of Dunkl operators, a family of commuting first order differential-difference operators.

5.1. Dunkl operators and h-spherical harmonics

Let v be a nonzero vector in $\mathbb{R}^d$. The reflection σ_v along v is defined by

$$x\sigma_v := x - 2\langle x, v\rangle v/\|v\|^2, \qquad x \in \mathbb{R}^d,$$

where $\langle x, y\rangle$ denotes the usual Euclidean inner product.

DEFINITION 5.1.1. *A root system is a finite set R of nonzero vectors in $\mathbb{R}^d$ such that $u, v \in R$ implies $u\sigma_v \in R$. A reflection group G with root system R is the subgroup of the orthogonal group $O(d)$ generated by the reflections $\{\sigma_u : u \in R\}$.*

If R is not the union of two nonempty orthogonal subsets, the corresponding reflection group G is called irreducible. Fix $u_0 \in \mathbb{R}^d$ such that $\langle u, u_0\rangle \neq 0$. The set of positive roots R_+ with respect to u_0 is defined by $R_+ = \{u \in R : \langle u, u_0\rangle > 0\}$ and $R = R_+ \cup (-R_+)$.

DEFINITION 5.1.2. *A multiplicity function $v \mapsto \kappa_v$ of $R_+ \mapsto \mathbb{R}$ is a function defined on R_+ with the property that $\kappa_u = \kappa_v$ if σ_u is conjugate to σ_v, that is, if there is a w in the reflection group G generated by R_+ such that $uw = v$.*

By its definition, a multiplicity function is invariant under the group G.

DEFINITION 5.1.3. *Let $v \mapsto \kappa_v$ is a multiplicity function associated with a finite reflection group G. The Dunkl operator is defined by, for $1 \le i \le d$,*

$$\mathcal{D}_j f(x) = \partial f(x) + \sum_{v\in R_+} \kappa_v \frac{f(x) - f(x\sigma_v)}{\langle x, v\rangle} v_j.$$

These are the first order differential-difference operators and they commute:

THEOREM 5.1.4. *For $1 \le j \le d$, $\mathcal{D}_j$ maps $\mathcal{P}_n^d$ into $\mathcal{P}_{n-1}^d$. Furthermore, these operators commute, that is,*

$$\mathcal{D}_i\mathcal{D}_j = \mathcal{D}_j\mathcal{D}_i, \qquad 1 \le i, j \le d.$$

The analogue of the Laplace operator Δ_h is defined by

$$\Delta_h := \mathcal{D}_1^2 + \cdots + \mathcal{D}_d^2.$$

The Laplace operator Δ is a central element of the algebra generated by the differentials. There is an analogous element, Δ_κ, in the centre of the commutative algebra generated by the Dunkl operators and it is defined by

$$\Delta_h := \mathcal{D}_1^2 + \cdots + \mathcal{D}_{d+1}^2. \tag{5.1.1}$$

The h-Laplace operator plays the role of the ordinary Laplace operator when the rotation group is replaced by the reflection group.

Definition 5.1.5. *Let $Y \in \mathcal{P}_n^d$ be a homogeneous polynomial. If $\Delta_h Y = 0$, then Y is called an h-harmonic polynomial.*

The h-harmonics of different degrees turn out to be orthogonal with respect to the weighted inner product

$$\langle f, g \rangle_\kappa := \frac{1}{\omega_d^\kappa} \int_{\mathbb{S}^{d-1}} f(x) g(x) h_\kappa^2(x) d\sigma(x), \tag{5.1.2}$$

where h_κ is the weight function,

$$h_\kappa(x) := \prod_{v \in R_+} |\langle x, v \rangle|^{\kappa_v}, \qquad x \in \mathbb{R},$$

which is a homogeneous function of degree $\gamma_\kappa := \sum_{v \in R_+} \kappa_v$ and invariant under G.

Theorem 5.1.6. *Let f and g be h-harmonic polynomials of degree n and m, respectively, with $n \neq m$. Then $\langle f, g \rangle_\kappa = 0$.*

For $n = 0, 1, 2, \ldots$, let $\mathcal{H}_n^d(h_\kappa^2)$ denote the linear space of h-harmonic polynomials of degree n. As a consequence of the above theorem, the following theorem follows exactly as in the case of ordinary spherical harmonics.

Theorem 5.1.7. *For $n = 0, 1, 2, \ldots$, there is a decomposition of $\mathcal{P}_n^d$,*

$$\mathcal{P}_n^d = \bigoplus_{0 \le j \le n/2} \|x\|^{2j} \mathcal{H}_{n-2j}^d(h_\kappa^2). \tag{5.1.3}$$

Furthermore, for $n = 0, 1, 2, \ldots$,

$$\dim \mathcal{H}_n^d(h_\kappa^2) = \dim \mathcal{P}_n^d - \dim \mathcal{P}_{n-2}^d = \binom{n+d-1}{d-1} - \binom{n+d-3}{d-1}. \tag{5.1.4}$$

There is also an analogue of the Laplace-Beltrami operator, denoted by $\Delta_{h,0}$, associated with the reflection invariant weight function. In the rest of this section and the next section, we set for $\kappa = (\kappa_1, \ldots, \kappa_d)$ with $\kappa_i \ge 0$,

$$\lambda_k := |\kappa| + \frac{d-2}{2} = \kappa_1 + \cdots + \kappa_d + \frac{d-2}{2}. \tag{5.1.5}$$

Theorem 5.1.8. *In the spherical-polar coordinates $x = r\xi$, $r > 0$, $\xi \in \mathbb{S}^{d-1}$, the Laplace operator satisfies,*

$$\Delta = \frac{d^2}{dr^2} + \frac{2\lambda_\kappa + 1}{r} \frac{d}{dr} + \frac{1}{r^2} \Delta_{h,0}. \tag{5.1.6}$$

The h-spherical harmonics are eigenfunctions of the $\Delta_{h,0}$,

$$\Delta_{h,0} Y_n^h(\xi) = -n(n+\lambda_k) Y_n^h(\xi), \qquad \forall Y_n^h \in \mathcal{H}_n^d(h_\kappa^2), \quad \xi \in \mathbb{S}^{d-1}. \tag{5.1.7}$$

There is a linear operator V_κ, called the intertwining operator, which acts between ordinary harmonics and h-harmonics and encodes essentially information on the action of the reflection group.

Definition 5.1.9. *A linear operator V_κ is called an intertwining operator if it satisfies*

$$\mathcal{D}_i V_\kappa = V_\kappa \partial_i, \qquad V1 = 1, \qquad V\mathcal{P}_n \subset \mathcal{P}_n. \tag{5.1.8}$$

From (5.1.8) it follows immediately that $\Delta_h V_\kappa = V_\kappa \Delta$ and, consequently, if P is an ordinary harmonic polynomial, then $V_\kappa P$ is an h-harmonic. The operator V_κ is known to be positive in the sense that $f(x) \geq 0$ implies $V_\kappa f(x) \geq 0$.

With the help of the intertwining operator, a large part of the theory for h-harmonics can be developed in parallel to the theory for ordinary spherical harmonics. Let us denote by

$$\operatorname{proj}_n^\kappa : L^2(\mathbb{S}^{d-1}, h_\kappa^2) \mapsto \mathcal{H}_n^d(h_\kappa^2)$$

the orthogonal projection operator from $L^2(\mathbb{S}^{d-1}, h_\kappa^2)$ onto $\mathcal{H}_n^d(h_\kappa^2)$. Just like the projection operator for ordinary spherical harmonics, the operator $\operatorname{proj}_n^\kappa$ can be expressed as an integral

$$\operatorname{proj}_n^\kappa f(x) = \frac{1}{\omega_d^\kappa} \int_{\mathbb{S}^{d-1}} f(y) Z_n^\kappa(x, y) h_\kappa^2(y) d\sigma(y), \tag{5.1.9}$$

where $Z_n^\kappa(\cdot, \cdot)$ is the reproducing kernel of $\mathcal{H}_n^d(h_\kappa^2)$. This kernel can be expressed in terms of an orthonormal basis $\{Y_j^h : 1 \leq j \leq N\}$ of $\mathcal{H}_n^d(h_\kappa^2)$ as

$$Z_n^\kappa(x, y) = \sum_{j=1}^N Y_j^h(x) Y_j^h(y), \quad N = \dim \mathcal{H}_n^d(h_\kappa^2), \quad x, y \in \mathbb{S}^{d-1}, \tag{5.1.10}$$

and it is independent of the particular choice of bases of $\mathcal{H}_n^d(h_\kappa^2)$.

THEOREM 5.1.10. *For $\kappa_i \geq 0$, and $\|y\| \leq \|x\| = 1$,*

$$Z_n^\kappa(x, y) = \|y\|^n \frac{n + \lambda_\kappa}{\lambda_\kappa} V_\kappa \left[C_n^{\lambda_\kappa} \left(\left\langle \cdot, \frac{y}{\|y\|} \right\rangle \right) \right] (x), \tag{5.1.11}$$

where the constant

$$\lambda_\kappa := \sum_{v \in R_+} \kappa_v + \frac{d-2}{2}. \tag{5.1.12}$$

If all $\kappa_i = 0$ then V_κ becomes the identity operator and (5.1.11) becomes the addition formula (2.2.5) for ordinary spherical harmonics. Thus, (5.1.11) is an analogue of the addition formula for ordinary spherical harmonics.

THEOREM 5.1.11. *For $f : \mathbb{R}^d \mapsto \mathbb{R}$ such that both integrals below are defined,*

$$\frac{1}{\omega_d^\kappa} \int_{\mathbb{S}^{d-1}} V_\kappa g(x) h_\kappa^2(x) d\sigma = b_\kappa \int_{\mathbb{B}^d} g(x)(1 - \|x\|^2)^{|\kappa|-1} dx, \tag{5.1.13}$$

where b_κ is a constant such that the right hand side is equal to 1 when $g(x) = 1$.

Example: the group $\mathbb{Z}_2^d$.

The simplest reflection group is $\mathbb{Z}_2^d$. A function $f : \mathbb{R}^d \to \mathbb{C}$ is invariant under the group $\mathbb{Z}_2^d$ if it is invariant under the sign changes in each of its variables. For $1 \leq j \leq d$, let σ_j denote the reflection of x with respect to the coordinate plane $x_j = 0$; that is,

$$x\sigma_j := (x_1, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_d).$$

Then f is invariant under $\mathbb{Z}_2^d$ if $f(x) = f(x\sigma_j)$ for all $1 \leq j \leq d$.

DEFINITION 5.1.12. *Let κ_j, $1 \leq j \leq d$, be nonnegative numbers. For $1 \leq i \leq d$, the Dunkl operators $\mathcal{D}_i$ with respect to the group $\mathbb{Z}_2^d$ are defined by*

$$\mathcal{D}_j f(x) := \frac{\partial f}{\partial x_j} + \kappa_j \frac{f(x) - f(x\sigma_j)}{x_j}, \qquad 1 \leq j \leq d. \tag{5.1.14}$$

The weight function h_κ, invariant under the group $\mathbb{Z}_2^d$, is defined by

$$h_\kappa(x) := \prod_{j=1}^{d} |x_j|^{\kappa_j}, \qquad x \in \mathbb{S}^{d-1}. \tag{5.1.15}$$

In the case of $\mathbb{Z}_2^d$, V_κ is given explicitly as an integral operator.

THEOREM 5.1.13. *Let $\kappa_i \ge 0$. The intertwining operator for $\mathbb{Z}_2^d$ is given by*

$$V_\kappa f(x) = c_\kappa \int_{[-1,1]^d} f(x_1t_1,\ldots,x_dt_d) \prod_{i=1}^{d} (1+t_i)(1-t_i^2)^{\kappa_i-1} dt, \tag{5.1.16}$$

where $c_k = c_{\kappa_1}\cdots c_{\kappa_d}$ with $c_\mu = \Gamma(\mu+1/2)/(\sqrt{\pi}\Gamma(\mu))$, and if any one of $\kappa_i = 0$ then the formula holds under the limit

$$\lim_{\mu\to 0} c_\mu \int_{-1}^{1} f(t)(1-t^2)^{\mu-1} d\mu(t) = \frac{f(1)+f(-1)}{2}.$$

5.2. Convolution and h-harmonics series

Recall that $w_\lambda(x) = (1-x^2)^{\lambda-1/2}$, $x \in (-1,1)$. As suggested by (5.1.9) and (5.1.11), we can define a convolution with respected to the h_κ^2 on the sphere.

DEFINITION 5.2.1. *For $f \in L^1(h_\kappa^2, \mathbb{S}^{d-1})$ and $g \in L^1(w_{\lambda_\kappa}, [-1,1])$,*

$$(f *_\kappa g)(x) := \frac{1}{\omega_d^\kappa} \int_{\mathbb{S}^{d-1}} f(y) V_\kappa[g(\langle \cdot, y\rangle)](x) h_\kappa^2(y) d\sigma(y). \tag{5.2.1}$$

In particular, the projection $\operatorname{proj}_n^\kappa$ is a convolution operator:

$$\operatorname{proj}_n^\kappa f = f *_\kappa Z_n^\kappa, \qquad Z_n^\kappa(t) := \frac{n+\lambda}{\lambda_\kappa} C_n^{\lambda_\kappa}(t). \tag{5.2.2}$$

Recall that the norm of the space $L^p(w_\lambda; [-1,1])$ is denoted by $\|\cdot\|_{\lambda,p}$. We denote by $\lambda \cdot \|_{\kappa,p}$ the norm of the space $L^p(h_\kappa, \mathbb{S}^{d-1})$ for $1 \le p < \infty$ and $\|\cdot\|_{\kappa,\infty} = \|\cdot\|_\infty$ the norm of the space $C(\mathbb{S}^{d-1})$.

THEOREM 5.2.2. *Let $p,q,r \ge 1$ and $p^{-1} = r^{-1}+q^{-1}-1$. For $f \in L^q(h_\kappa^2, \mathbb{S}^{d-1})$ and $g \in L^r(w_{\lambda_\kappa}; [-1,1])$,*

$$\|f *_\kappa g\|_{\kappa,p} \le \|f\|_q \|g\|_{\lambda_\kappa,r}. \tag{5.2.3}$$

PROOF. Following the usual proof of Young's inequality, we only need to show

$$\|G(x,\cdot)\|_{\kappa,r} \le \|g\|_{w_\lambda,r}, \qquad \text{where} \quad G(x,y) = V_\kappa[g(\langle x,\cdot\rangle)](y),$$

which can be proved as follows: the positivity of V_κ implies $|V_\kappa g| \le V_\kappa[|g|]$, so that $\|G(x,\cdot)\|_{\kappa,\infty} \le \|g\|_{w_{\lambda_\kappa},\infty}$ and we deduce by (5.1.13) that

$$\|G(x,\cdot)\|_{\kappa,1} \le \frac{1}{\omega_d^\kappa} \int_{\mathbb{S}^{d-1}} V_\kappa\left[|g(\langle x,\cdot\rangle)|\right](y) h_\kappa^2(y) d\sigma = c_{\lambda_\kappa} \int_{-1}^{1} |g(t)| w_\lambda(t) dt = \|g\|_{\lambda_\kappa,1}.$$

The Riesz-Thorin interpolation theorem shows then $\|G(x,\cdot)\|_{\kappa,r} \le \|g\|_{w_{\lambda_\kappa},r}$. □

The Fourier orthogonal series in h-spherical harmonics are defined exactly as in the case of ordinary spherical harmonics. In analogy to (3.1.6), we have for $f \in L^2(h_\kappa^2, \mathbb{S}^{d-1})$ that

$$f(x) = \sum_{n=0}^{\infty} \operatorname{proj}_n^\kappa f(x), \tag{5.2.4}$$

and an analogue of (3.1.5) holds. Since the convergence of the series (5.2.4) does not go beyond L^2 convergence, we again need to consider summability methods.

For $\delta \in \mathbb{R}$, let $S_n^\delta(h_\kappa^2; f)$ denote the Cesàro (C, δ) means of the series (5.2.4),

$$S_n^\delta\left(h_\kappa^2; f\right) := \frac{1}{A_n^\delta} \sum_{j=0}^{n} A_{n-j}^\delta \operatorname{proj}_j^\kappa f = f * K_n^\delta\left(h_\kappa^2\right), \tag{5.2.5}$$

where $S_n^0(h_\kappa^2; f)$ is the n-th partial sum and, as shown in (3.1.14),

$$K_n^\delta\left(h_\kappa^2; t\right) := \frac{1}{A_n^\delta} \sum_{k=0}^{n} A_{n-k}^\delta \frac{k+\lambda}{\lambda} C_k^\lambda(t) = k_n^\delta\left(w_{\lambda_\kappa}; 1, t\right), \tag{5.2.6}$$

in which $k_n^\delta(w_\lambda; \cdot, \cdot)$ is the kernel of the (C, δ) means of the Fourier orthogonal series in the Gegenbauer polynomials. Recall that the norm of the space $L^p(w_\lambda; [-1, 1])$ is denoted by $\|\cdot\|_{\lambda,p}$. We denote by $\lambda \cdot \|_{\kappa,p}$ the norm of the space $L^p(h_\kappa, \mathbb{S}^{d-1})$ for $1 \le p < \infty$ and $\|\cdot\|_{\kappa,\infty} = \|\cdot\|_\infty$ the norm of the space $C(\mathbb{S}^{d-1})$.

Theorem 5.2.3. *The Cesàro means of the h-spherical harmonics series satisfy*

1. *If $\delta \ge 2\lambda_k + 1$ then $S_n^\delta(h_\kappa^2)$ is a nonnegative operator;*
2. *If $\delta > \lambda_\kappa$ then $S_n^\delta(h_\kappa^2; f)$ converges to f in $L^p(h_\kappa^2; \mathbb{S}^{d-1})$ for $1 \le p \le \infty$.*

A major departure from the theory on ordinary spherical harmonic series is that, in general, the condition $\delta > \lambda_k$ is only a sufficient condition, not the necessary and sufficient condition. In fact, the essential ingredient of the proof is (5.1.13), taking an average of $V_\kappa g$ over the sphere, which however removes V_κ from the scene and, as a result, erases the information of the reflection group inadvertently.

5.3. Cesàro means of h-harmonic series for $\mathbb{Z}_2^d$

In the case of $\mathbb{Z}_2^d$, the intertwining operator is known explicitly, which allows us to establish sharp results on the Cesàro means of h-harmonics series. We define

$$\sigma_\kappa := \tfrac{d-1}{2} + |\kappa| - \kappa_{\min} \quad \text{with} \quad \kappa_{\min} := \min_{1 \le j \le d+1} \kappa_j.$$

Furthermore, we define

$$\mathbb{S}_{int}^{d-1} := \mathbb{S}^{d-1} \setminus \bigcup_{i=1}^{d} \left\{\mathbf{x} \in \mathbb{S}^{d-1} : x_i = 0\right\},$$

the interior region bounded by the intersection of the coordinate plans with $\mathbb{S}^{d-1}$.

Our main results on the Cesàro summation of h-harmonic expansions are the following theorems:

Theorem 5.3.1. (*i*) *The Cesàro means $S_n^\delta(h_\kappa^2; f)$ converges to f in $C(\mathbb{S}^d)$ if and only if $\delta \ge \sigma_\kappa$.*

(*ii*) *Let $f \in C(\mathbb{S}^{d-1})$. If $\delta > (d-2)/2$, then $S_n^\delta(h_\kappa^2; f)$ converge to f for every $\mathbf{x} \in \mathbb{S}_{int}^{d-1}$. Moreover, the convergence is uniform over each compact set contained inside $\mathbb{S}_{int}^{d-1}$.*

Theorem 5.3.2. *Suppose that $f \in L^p(h_\kappa^2, S^d)$, $1 \le p \le \infty$, $|\frac{1}{p} - \frac{1}{2}| \ge \frac{1}{2\sigma_\kappa+2}$ and*

$$\delta > \delta_\kappa(p) := \max\{(2\sigma_\kappa + 1)|\tfrac{1}{p} - \tfrac{1}{2}| - \tfrac{1}{2}, 0\}. \tag{5.3.1}$$

Then $S_n^\delta(h_\kappa^2; f)$ converges to f in $L^p(h_\kappa^2, S^d)$ and

$$\sup_{n \in \mathbb{N}} \|S_n^\delta(h_\kappa^2; f)\|_{\kappa,p} \le c\|f\|_{\kappa,p}.$$

Theorem 5.3.3. *Assume $1 \le p \le \infty$ and $0 < \delta \le \delta_\kappa(p)$. Then there exists a function $f \in L^p(h_\kappa^2, S^d)$ such that $S_n^\delta(h_\kappa^2; f)$ diverges in $L^p(h_\kappa^2; S^d)$.*

For $\kappa = 0$, $h_\kappa(x) \equiv 1$ and the spherical h-harmonic becomes the ordinary spherical harmonics. Hence, Theorem 5.3.2 is the complete analogue of the main result in [**37**], while Theorem 5.3.3 is the analogue of [**4**, Theorem 5.2] for spherical harmonics.

For the projection operator $\mathrm{proj}_n(h_\kappa^2; f)$ we have the following theorem which is a complete analogue of a theorem due to Sogge [**37**] for spherical harmonics.

THEOREM 5.3.4. *Let $d \ge 2$ and $n \in \mathbb{N}$. Then*

(i) for $1 \le p \le \frac{2(\sigma_k+1)}{\sigma_k+2}$,

$$\left\|\mathrm{proj}_n(h_\kappa^2; f)\right\|_{\kappa,2} \le c n^{\delta_\kappa(p)} \|f\|_{\kappa,p},$$

with $\delta_k(p)$ given in (5.3.1);

(ii) for $\frac{2(\sigma_k+1)}{\sigma_k+2} \le p \le 2$,

$$\left\|\mathrm{proj}_n(h_\kappa^2; f)\right\|_{\kappa,2} \le c n^{\sigma_k(\frac{1}{p}-\frac{1}{2})} \|f\|_{\kappa,p}.$$

Furthermore, the estimate (i) is sharp.

The estimate in (ii) is sharp if $\kappa = 0$ as shown in [**37**]. We expect that it is also sharp for $\kappa \ne 0$ but could not prove it at this moment.

For the spherical harmonics, the above theorem is enough for the proof of the boundedness of the Cesàro means. (See [**4**] and [**38**].) For h-harmonics, however, a stronger result is needed since $\delta_\kappa(p) > \delta(p) := \max\{d|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}, 0\}$.

THEOREM 5.3.5. *Suppose that $1 \le p \le \frac{2\sigma_\kappa+2}{\sigma_k+2}$ and f is supported in a spherical cap $\mathsf{c}(\varpi, \theta)$ with $\theta \in (n^{-1}, \pi]$ and $\varpi \in S^d$. Then*

$$\left\|\mathrm{proj}_n(h_\kappa^2; f)\right\|_{\kappa,2} \le c n^{\delta_\kappa(p)} \theta^{\delta_\kappa(p)+\frac{1}{2}} \left[\int_{\mathsf{c}(\varpi,\theta)} h_\kappa^2(x)\, d\sigma(x)\right]^{\frac{1}{2}-\frac{1}{p}} \|f\|_{\kappa,p}.$$

For other reflection groups, no sharp results as listed above are known. The reason lies essentially in the lack of explicit formula of the intertwining operator.

5.4. Weighted approximation on the sphere

For the Lebesgue measure, the classical modulus of smoothness is defined via the spherical means, denoted by $T_\theta f$,

$$T_\theta f(x) = \frac{1}{\omega_{d-1}(\sin\theta)^{d-1}} \int_{\langle x,y\rangle = \cos\theta} f(y) d\sigma(y).$$

For $r > 0$, it is defined by

$$\omega_r(f; t)_p := \sup_{0<\theta\le t} \|(T_\theta - I)^r f\|_p,$$

where I denote the identity operator.

For weighted approximation, we define an analogue of spherical means for the weight function h_κ invariant under a reflection group.

DEFINITION 5.4.1. *For $0 \le \theta \le \pi$, the translation operator T_θ^κ is defined by*

$$\mathrm{proj}_n^\kappa(T_\theta^\kappa f) = \frac{C_n^{\lambda_\kappa}(\cos\theta)}{C_n^{\lambda_\kappa}(1)} \mathrm{proj}_n^\kappa f, \quad n = 0, 1, \ldots. \tag{5.4.1}$$

Although T_θ^κ does not have an explicit integral expression, its properties are similar to those of T_θ, which are summed up in the following:

PROPOSITION 5.4.2. *The translation operator T_θ^κ is well defined for all $f \in L^1(h_\kappa^2, \mathbb{S}^{d-1})$ and it satisfies the following properties,* (*i*) *For* $f \in L^2(h_\kappa^2, \mathbb{S}^{d-1})$ *and* $g \in L^1(w_{\lambda_\kappa}, [-1,1])$,

$$(f *_\kappa g)(x) = c_{\lambda_\kappa} \int_0^\pi T_\theta^\kappa f(x) g(\cos\theta)(\sin\theta)^{2\lambda_\kappa} d\theta; \tag{5.4.2}$$

(*ii*) T_θ^κ *preserves positivity, i.e.,* $T_\theta^\kappa f \geq 0$ *if* $f \geq 0$;

(*iii*) *For* $f \in L^p(h_\kappa^2)$, $1 \leq p < \infty$, *or* $f \in C(S^d)$,

$$\|T_\theta^\kappa f\|_{\kappa,p} \leq \|f\|_{\kappa,p} \qquad \text{and} \qquad \lim_{\theta\to 0} \|T_\theta^\kappa f - f\|_{\kappa,p} = 0; \tag{5.4.3}$$

(*iv*) $T_\theta^\kappa f(-x) = T_{\pi-\theta}^\kappa(x)$.

The properties of $T_\theta^\kappa f$ are parallel to those of $T_\theta f$. They lead to the following definition of an analog of the modulus of smoothness:

DEFINITION 5.4.3. *Let* $r > 0$. *For* $f \in L^p(h_\kappa^2, \mathbb{S}^{d-1})$, $1 \leq p < \infty$, *or* $f \in C(\mathbb{S}^{d-1})$, *define*

$$\omega_r(f,t)_{\kappa,p} := \sup_{0\leq\theta\leq t} \|(I - T_\theta^\kappa)^{r/2}\|_{\kappa,p}.$$

For $r > -0$, the operator $(I - T_\theta^\kappa)^r$ is defined through

$$(I - T_\theta^\kappa)^{r/2} f \sim \sum_{n=0}^\infty (1 - R_n^{\lambda_\kappa}(\cos\theta))^{r/2} Y_n(h_\kappa^2; f), \qquad R_k^\lambda(t) := C_k^\lambda(t)/C_k^\lambda(1).$$

We define the r-th power of the h-Laplace-Beltrami operator, $(-\Delta_{h,0})^r$ analogously, which leads to the following definition of the K-functional:

DEFINITION 5.4.4. *For* $r > 0$, *define the function space* $\mathcal{W}_r^p(h_\kappa^2)$ *by*

$$\mathcal{W}_{r,\kappa}^p := \{f \in L^p(h_\kappa^2, \mathbb{S}^{d-1}) : (-\Delta_{h,0})^r \in L^p(h_\kappa^2)\}.$$

The K-functional between $L^p(h_\kappa^2, \mathbb{S}^{d-1})$ *and* $\mathcal{W}_{r,\kappa}^p$ *is defined by*

$$K_r(f;t)_{\kappa,p} := \inf\left\{\|f - g\|_{\kappa,p} + t^r \|(-\Delta_{h,0})^{r/2} g\|_{\kappa,p},\ g \in \mathcal{W}_r^p(h_\kappa^2)\right\}.$$

The weighted $\omega_r(f,t)_{\kappa,p}$ defined above satisfies the usual properties of the modulus of smoothness. It is also equivalent to the K-functional.

THEOREM 5.4.5. *For* $f \in L^p(h_\kappa^2)$, $1 \leq p \leq \infty$,

$$c_1, \omega_r(f;t)_{\kappa,p} \leq K_r(f;t)_{\kappa,p} \leq c_2\, \omega_r(f;t)_{\kappa,p}.$$

Define the quantity of weighted best approximation by

$$E_n(f)_{\kappa,p} := \inf_{p \in \Pi_n^(\mathbb{S}^{d-1})} \|f - g\|_{\kappa,p}.$$

Then both the direct and inverse theorems for the best approximation hold:

THEOREM 5.4.6. *For* $f \in L^p(h_\kappa^2)$, $1 \leq p \leq \infty$,

$$E_n(f)_{\kappa,p} \leq c\, \omega_r(f; n^{-1})_{\kappa,p}.$$

On the other hand,

$$\omega_r(f, n^{-1})_{\kappa,p} \leq c\, n^{-r} \sum_{k=0}^n (k+1)^{r-1} E_k(f)_{\kappa,p}.$$

One immediate question is how $\omega_r^*(f,t)_p := \omega_r(f,t)_{\kappa,p}|_{\kappa=0}$ compares with $\omega_r(f,t)_p$ defined in Chapter 4. They should be close since both characterize the best approximation. It has been proved in [**12**] that

PROPOSITION 5.4.7. *Let* $f \in L^p(\mathbb{S}^{d-1})$ *and* $1 < p < \infty$. *Then for* $r \in \mathbb{N}$, $\omega_r(f,t)_p \leq c\omega_r^*(f,t)_p$. *Furthermore,* $\omega_r(f,t)_p \sim \omega_r^*(f,t)_p$ *if* $r = 1, 2$ *and* $1 < p < \infty$.

References. The theory of h-harmonics is pioneered by C. Dunkl. The Dunkl operators were introduced in [**17**] and the intertwining operators and the integral kernel appeared in [**18**]. For more results and the proof, we refer to [**19**]. The first study of h-harmonic expansions appeared in [**45, 46**], which includes (5.1.16) and (5.1.11). The convolution and the translation operators were defined in [**50**]. The Theorem 5.3.1 was proved in [**26**]. The rest of the results in the Section 5.3 were established in [**11**]. In the case of $\kappa = 0$, the main results in the Section 5.4 were studied and proved in various cases by several authors (see [**31, 44**] and the references therein) before Rustamov [**35**] finally proved it in the full generality. The extension in the weighted case as presented here were developed in [**50**].

CHAPTER 6

Harmonic Analysis and Approximation on the Unit Ball

The analysis on the unit ball is closely related to the analysis on the unit sphere. We discuss this connection and its consequence below.

6.1. Orthogonal structure on the unit ball

We consider orthogonal polynomials with respect to a weight function on the unit ball $\mathbb{B}^d$. The classical weight function on the unit ball is

$$W_\mu(x) := (1 - \|x\|^2)^{\mu - 1/2}, \qquad \mu > -1/2, \tag{6.1.1}$$

which is a special case of a more general weight function

$$W_\kappa(x) := \prod_{i=1}^{d} |x_i|^{\kappa_i} (1 - \|x\|^2)^{\kappa_{d+1} - 1/2}, \qquad \kappa_i > -1/2. \tag{6.1.2}$$

Definition 6.1.1. *For $n \in \mathbb{N}_0$, let $\mathcal{V}_n^d(W_\kappa)$ denote the space of orthogonal polynomials of degree exactly n with respect to the inner product*

$$\langle f, g \rangle_{W_\kappa} := a_\kappa \int_{\mathbb{B}^d} f(x) g(x) W_\kappa(x) d,$$

where a_κ is the normalization constant of W_κ, $a_\kappa := 1/\int_{\mathbb{B}^d} W_\kappa(x)dx$.

From the Gram-Schmidt process applying to the monomials, it follows

$$\dim \mathcal{V}_n^d(W_\kappa) = \binom{n+d-1}{n}, \quad n = 0, 1, 2, \ldots.$$

The orthogonal structure on the ball is closely related to the corresponding structure on the unit sphere. We start with a simple relation on polynomials over these two domains.

Let $\mathbb{S}_+^{d+1}$ denote the upper sphere $\mathbb{S}_+^d := \{x \in \mathbb{S}^d : x_{d+1} \geq 0\}$. A simple mind relation between the ball and the sphere is

$$x \in \mathbb{B}^d \quad \Longleftrightarrow \quad (x, x_{d+1}) \in \mathbb{S}_+^d, \quad x_{d+1} = \sqrt{1 - \|x\|^2}. \tag{6.1.3}$$

The domain $\mathbb{S}_+^d$ induces a symmetry in the polynomial space. Let $\mathcal{P}_n^+(\mathbb{S}^d)$ denote the subspace of elements in $\mathcal{P}_n(\mathbb{S}^d)$ that are even in its $(d+1)$-th coordinates. The mapping (6.1.3) leads immediately to the following basic result:

Lemma 6.1.2. *For each $n \geq 0$ the equation*

$$\mathcal{P}_n(\mathbb{S}^d) = \Pi_n^d \cup x_{d+1} \Pi_{n-1}^d \tag{6.1.4}$$

holds in the sense that for each $P \in \mathcal{P}_n(\mathbb{S}^d)$ there exist unique elements $p \in \Pi_n^d$ and $q \in \Pi_{n-1}^d$ such that

$$P(x, x_{d+1}) = p(x) + x_{d+1} q(x), \qquad (x, x_{d+1}) \in S^{d+1}.$$

In particular, there is a one-to-one correspondence between Π_n^d and $\mathcal{P}_n^+(\mathbb{S}^d)$.

One immediate consequence of the relation (6.1.3) is the following lemma:

LEMMA 6.1.3. *For any integrable function on* $\mathbb{S}^d$,

$$\int_{\mathbb{S}^d} f(y)d\sigma(y) = \int_{\mathbb{B}^d} \left[f\left(x, \sqrt{1-\|x\|^2}\right) + f\left(x, -\sqrt{1-\|x\|^2}\right) \right] \frac{dx}{\sqrt{1-\|x\|^2}}.$$

The weight function W_κ is closely related to $h_\kappa^2(x) = \prod_{i=1}^{d+1} |x_i|^{2\kappa_i}$ on $\mathbb{S}^d$, which is defined in (5.1.15) but with d replaced by $d+1$. Indeed, by (6.1.3),

$$W_\kappa(x) = h_\kappa^2\left(x, \sqrt{1-\|x\|^2}\right), \quad x \in \mathbb{B}^d. \tag{6.1.5}$$

PROPOSITION 6.1.4. *Let* $\kappa = (\kappa', \kappa_{d+1})$ *and* $\kappa' = (\kappa_1, \ldots, \kappa_d)$. *Then*

$$\mathcal{H}_n^{d+1}\left(h_\kappa^2\right) = \mathcal{V}_n\left(W_\kappa\right) \oplus x_{d+1}\mathcal{V}_{n-1}\left(W_{\kappa', \kappa_{d+1}+1}\right). \tag{6.1.6}$$

Let $P(W_\kappa; \cdot, \cdot)$ denote the reproducing kernel of $\mathcal{V}_n^d(W_\kappa)$. It is uniquely determined by the reproducing property

$$\langle P_n(W_\kappa; x, \cdot), q\rangle_{W_\kappa} = q(x), \quad \forall q \in \mathcal{V}_n^d(W_\kappa).$$

Let $\operatorname{proj}_n(W_\kappa; f)$ denote the projection operator from $L^2(\mathbb{B}^d, W_\kappa)$ to $\mathcal{V}_n^d(W_\kappa)$. Then

$$\operatorname{proj}_{\mathcal{V}_n(W_\kappa)} f(x) = a_\kappa \int_{\mathbb{B}^d} f(y) P_n(W_\kappa; x, y) W_\kappa(y) dy. \tag{6.1.7}$$

Let Z_n^κ be the reproducing kernel of $\mathcal{H}_n^{d+1}(h_\kappa^2)$, defined in (5.1.10) but with d replaced by $d+1$. By (5.1.16) and (5.1.11), the reproducing kernel $Z_n^\kappa(\cdot, \cdot)$ has a closed formula, from which a closed formula of $P_n(W\kappa; \cdot, \cdot)$ follows readily. To state the result, we give the following definition.

DEFINITION 6.1.5. *Let* V_κ *denote the intertwining operator for* h_κ^2 *defined* (5.1.16) *but with* d *replaced by* $d+1$. *Define*

$$V_\kappa^{\mathbb{B}} f(x, x_{d+1}) := \frac{1}{2}\left[V_\kappa f(x, x_{d+1}) + V_\kappa f(x, -x_{d+1})\right], \quad x \in \mathbb{R}^d. \tag{6.1.8}$$

THEOREM 6.1.6. *The reproducing kernel* $P_n(W_\kappa; x, y)$ *satisfies*

$$\begin{aligned} P_n(W_\kappa; x, y) &= \frac{1}{2}\left[Z_n^\kappa\left((x, x_{d+1}), (y, y_{d+1})\right) + Z_n^\kappa\left((x, x_{d+1}), (y, -y_{d+1})\right)\right] \qquad (6.1.9)\\ &= \frac{n+\lambda_k}{\lambda_k} V_\kappa^{\mathbb{B}}\left[C_n^{\lambda_\kappa}\left(\langle (y, y_{d+1}), \cdot\rangle\right)\right](x, x_{d+1}), \end{aligned}$$

where $x_{d+1} = \sqrt{1-\|x\|^2}$ *and* $y_{d+1} = \sqrt{1-\|y\|^2}$.

The formula in the case of classical weight function W_μ is as follows:

COROLLARY 6.1.7. *For* $\mu \ge 0$, *let* $\lambda_\mu = \mu + \frac{d-1}{2}$. *Then*

$$P_n(W_\mu; x, y) = c_\mu \frac{n+\lambda_\mu}{\lambda_\mu} \int_{-1}^1 C_n^{\lambda_\mu}\left(\langle x, y\rangle + \sqrt{1-\|x\|^2}\sqrt{1-\|y\|^2}\, t\right)(1-t^2)^{\mu-1} dt,$$

from which the expression for $\mu = 0$ *becomes a sum of two terms upon taking the limit.*

Using the relation in Proposition 6.1.4, we can also deduce from (5.1.7) that the elements in $\mathcal{V}_n^d(W_\kappa)$ are the eigenfunctions of a differential-diffrence operator, which we state below without proof.

THEOREM 6.1.8. *The orthogonal polynomials in* $\mathcal{V}_n^d(W_\kappa)$ *satisfy*

$$\left[\Delta_h - \langle x, \rangle\nabla^2 - (2|\kappa| + d - 1)\langle x, \nabla\rangle\right] u = -n(n + 2|\kappa| + d - 1)u, \tag{6.1.10}$$

where Δ_h *is defined in* (5.1.1). *In particular, for the classical weigh function* W_μ, (6.1.10) *is a second order differential equation with* Δ_h *replaced by* Δ.

6.2. Convolution and orthogonal expansions

We denote by $\|\cdot\|_{W_\kappa,p}$ the norm of the space $L^p(W_\kappa, \mathbb{B}^d)$,

$$\|f\|_{W_\kappa,p} := \left(a_\kappa \int_{\mathbb{B}^d} |f(x)|^p W_\kappa(x)dx\right)^{1/p}, \qquad 1 \le p < \infty,$$

and, as usual, consider $C(\mathbb{B}^d)$ with $\|f\|_{W_\kappa,\infty} = \|f\|_\infty$ for $p = \infty$.

The operator $V_\kappa^{\mathbb{B}}$ can be used to define a convolution structure $*_{\kappa,\mathbb{B}}$. Recall that $w_\lambda(t) = (1-t^2)^{\lambda-1/2}$ and $\lambda_k = |\kappa| + \frac{d-1}{2}$.

Definition 6.2.1. *For $f \in L^1(W_\kappa, \mathbb{B}^d)$ and $g \in L^1(w_{\lambda_\kappa}, [-1,1])$,*

$$(f *_{\kappa,\mathbb{B}} g)(x) := a_\kappa \int_{\mathbb{B}^d} f(y) \left(V_\kappa^{\mathbb{B}} [g(\langle \cdot, (x, x_{d+1})\rangle)]\right)(y, y_{d+1}) W_\kappa(y) dy, \tag{6.2.1}$$

where $x_{d+1} = \sqrt{1-\|x\|^2}$ and $y_{d+1} = \sqrt{1-\|y\|^2}$.

By (6.1.7) and (6.1.9), the projection operator $\mathrm{proj}_{\mathcal{V}_n(W_\kappa)} f$ is a convolution

$$\mathrm{proj}_{\mathcal{V}_n(W_\kappa)} f = f *_{\kappa,\mathbb{B}} Z_n^\kappa, \qquad Z_n^\kappa(t) := \frac{n+\lambda_\kappa}{\lambda_\kappa} C_n^{\lambda_\kappa}(t), \tag{6.2.2}$$

which is an analogue of (5.2.2). In fact, this convolution structure is related to the convolution structure $*_\kappa$ on the sphere $\mathbb{S}^d$.

Theorem 6.2.2. *Let F be defined by $F(x, x_{d+1}) := f(x)$. Then*

$$(f *_{\kappa,\mathbb{B}} g)(x) = (F *_\kappa g)\left(x, \pm\sqrt{1-\|x\|^2}\right). \tag{6.2.3}$$

In particular, for $f \in L^p(W_\kappa, \mathbb{B}^d)$, $1 \le p < \infty$, or $f \in C(\mathbb{B}^d)$, $p = \infty$,

$$\|f *_{\kappa,\mathbb{B}} g\|_{W_\kappa,p} = \|F *_\kappa g\|_{\kappa,p}, \quad 1 \le p \le \infty. \tag{6.2.4}$$

Proof. From (6.1.8) and Lemma 6.1.3 it follows that ,

$$(f *_{\kappa,\mathbb{B}} g)(x) = \frac{1}{\omega_{d+1}^\kappa} \int_{\mathbb{S}^d} f(y) V_\kappa[g(\langle \cdot, (x, x_{d+1})\rangle)](\widehat{y}) h_\kappa^2(\widehat{y}) d\sigma(\widehat{y}),$$

where $\widehat{y} = (y, y_{d+1})$, which proves (6.2.3) with $x_{d+1} = \sqrt{1-\|x\|^2}$. Since

$$V_\kappa[g(\langle \cdot, (x, x_{d+1})\rangle)](y, -y_{d+1}) = V_\kappa[g(\langle \cdot, (x, -x_{d+1})\rangle)](y, y_{d+1}),$$

a change of variable $y_{d+1} \mapsto -y_{d+1}$ in the last integral proves (6.2.3) with $x_{d+1} = -\sqrt{1-\|x\|^2}$. The equation (6.2.4) follows from (6.2.3) and Lemma 6.1.3. □

As a consequence of the relation (6.2.3), Young's inequality holds for the convolution $*_{\kappa,\mathbb{B}}$: for $p, q, r \ge 1$ and $p^{-1} = r^{-1} + q^{-1} - 1$, $f \in L^q(W_\kappa, \mathbb{B}^d)$ and $g \in L^r(w_{\lambda_\kappa}; [-1,1])$,

$$\|f *_{\kappa,\mathbb{B}} g\|_{W_\kappa,p} \le \|f\|_{W_\kappa,q} \|g\|_{\lambda_\kappa,r}. \tag{6.2.5}$$

The Fourier orthogonal series with respect to W_κ on the ball $\mathbb{B}^d$ are defined in terms of $\mathcal{V}_n^d(W_\kappa)$. For $f \in L^2(W_\kappa, \mathbb{B}^d)$,

$$f(x) = \sum_{n=0}^\infty \mathrm{proj}_{\mathcal{V}_n(W_\kappa)} f(x), \tag{6.2.6}$$

and an analogue of (3.1.5) follows from the usual Hilbert space theory. For convergence of the series (6.2.6) beyond L^2 setting, we again consider summability methods. We denote by $S_n^\delta(W_\kappa; f)$ the Cesàro (C,δ) means of the series (6.2.6),

$$S_n^\delta(W_\kappa; f) := \frac{1}{A_n^\delta} \sum_{j=0}^n A_{n-j}^\delta \, \mathrm{proj}_{\mathcal{V}_j(W_\kappa)} f = f *_{\kappa,\mathbb{B}} K_n^\delta(W_\kappa), \tag{6.2.7}$$

where $K_n^\delta(W;t) = k_n^\delta(w_{\lambda_\kappa}; 1, t)$ just as in (5.2.6). Then we deduce immediately from Theorems 6.2.2 and 6.2.3 the following result:

THEOREM 6.2.3. *The Cesàro means of the orthogonal expansions with respect to W_κ on $\mathbb{B}^d$ satisfy*

1. *If $\delta \geq 2\lambda_k + 1$ then $S_n^\delta(W_\kappa)$ is a nonnegative operator;*
2. *If $\delta > \lambda_\kappa$ then $S_n^\delta(W_\kappa; f)$ converges to f in $L^p(W_\kappa; \mathbb{B}^d)$ for $1 \leq p \leq \infty$.*

Furthermore, analogues of all results in Section 5.3 can be established for the Cesàro means $S_n^\delta(W_\kappa)$.

We can also define a translation operator $T(W_\kappa; f)$.

DEFINITION 6.2.4. *For $0 \leq \theta \leq \pi$, the translation operator T_θ^κ is defined by*

$$\operatorname{proj}_{\mathcal{V}_n(W_\kappa)} T_\theta(W_\kappa; f) = \frac{C_n^{\lambda_\kappa}(\cos\theta)}{C_n^{\lambda_\kappa}(1)} \operatorname{proj}_{\mathcal{V}_n(W_\kappa)} f, \quad n = 0, 1, \ldots. \tag{6.2.8}$$

This operator is closely related to the translation operator $T_\theta^\kappa f$.

PROPOSITION 6.2.5. *The translation operator T_θ^κ is well defined for all $f \in L^1(h_\kappa^2, \mathbb{S}^{d-1})$ and it satisfies the following properties,*

(*i*) *Let $F(x, x_{d+1}) = f(x)$. Then $T_\theta(W_\kappa; f) = T_\theta^\kappa F(x, \sqrt{1 - \|x\|^2})$.*

(*ii*) *For $f \in L^2(W_\kappa, \mathbb{B}^d)$ and $g \in L^1(w_{\lambda_\kappa}, [-1, 1])$,*

$$(f *_{\kappa,\mathbb{B}} g)(x) = c_{\lambda_\kappa} \int_0^\pi T_\theta(W_\kappa; f, x) g(\cos\theta)(\sin\theta)^{2\lambda_\kappa} d\theta; \tag{6.2.9}$$

(*iii*) *For $f \in L^p(W_\kappa, \mathbb{B}^d)$, $1 \leq p < \infty$, or $f \in C(\mathbb{B}^d)$,*

$$\|T_\theta(W_\kappa, f)\|_{W_\kappa,p} \leq \|f\|_{W_\kappa,p} \quad \textit{and} \quad \lim_{\theta \to 0} \|T_\theta(W_\kappa, f) - f\|_{W_\kappa,p} = 0. \tag{6.2.10}$$

As a consequence, we can define an analogue of modulus of smoothness:

DEFINITION 6.2.6. *Let $r > 0$. For $f \in L^p(W_\kappa, \mathbb{B}^d)$, $1 \leq p < \infty$, or $f \in C(\mathbb{B}^d)$, define*

$$\omega_r(f, t)_{W_\kappa,p} := \sup_{0 \leq \theta \leq t} \|(I - T_\theta(W_\kappa))^{r/2}\|_{W_\kappa,p}.$$

With this definition, analogues of all results in Section 5.4 hold. In particular, the modulus of smoothness gives a complete characterization of the best approximation by polynomials on the unit ball.

6.3. Approximation on the unit ball

For the weight function W_μ, we can define a modulus of smoothness in analogue to the modulus of smoothness in Chapter 4, which is computable and gives a characterization of the best approximation.

Given a function f on $\mathbb{B}^d$, we can regard it as a projection onto $\mathbb{B}^d$ of a function F, defined on $\mathbb{S}^{d+m-1}$ by

$$F(x, x') := f(x), \qquad (x, x') \in \mathbb{S}^{d+m-1}, \quad x \in \mathbb{B}^d, \quad x' \in \mathbb{B}^m. \tag{6.3.1}$$

Under such an extension of f, we have the following integral relation:

$$\int_{\mathbb{S}^{d+m-1}} F(y) d\sigma(y) = \sigma_m \int_{\mathbb{B}^d} f(x)(1 - \|x\|^2)^{\frac{m-2}{2}} dx, \tag{6.3.2}$$

where σ_m denotes the surface area of $\mathbb{S}^{m-1}$ for $m \geq 2$ and $\sigma_1 = 2$. Evidently, this relation shows that if $f \in L^p(B^d, W_{\frac{m-1}{2}})$ then $F \in L^p(\mathbb{S}^{d+m-1})$.

We denote by $\widetilde{f}$ the extension of f in (6.3.1) in the case of $m = 1$; that is,

$$\widetilde{f}(x, x_{d+1}) = f(x), \qquad (x, x_{d+1}) \in \mathbb{R}^{d+1}, \quad x \in \mathbb{B}^d. \tag{6.3.3}$$

Recall that $\triangle_{i,j,\theta} = \triangle_{Q_{i,j,\theta}}$ and $Q_{i,j,\theta}$ is the rotation in angle θ in the (x_i, x_j)-plane.

DEFINITION 6.3.1. *Let* $\mu = \frac{m-1}{2}$ *and* $m \in \mathbb{N}$. *Let* $f \in L^p(\mathbb{B}^d, W_\mu)$ *if* $1 \le p < \infty$ *and* $f \in C(\mathbb{B}^d)$ *if* $p = \infty$. *For* $r \in \mathbb{N}$ *and* $t > 0$

$$\omega_r(f,t)_{p,\mu} := \sup_{|\theta| \le t} \Big\{ \max_{1 \le i < j \le d} \|\triangle^r_{i,j,\theta} f\|_{L^p(\mathbb{B}^d, W_\mu)}, \tag{6.3.4}$$

$$\max_{1 \le i \le d} \|\triangle^r_{i,d+1,\theta} \widetilde{f}\|_{L^p(\mathbb{B}^{d+1}, W_{\mu-1/2})} \Big\},$$

where, for $m = 1$, $\|\triangle^r_{i,d+1,\theta} \widetilde{f}\|_{L^p(\mathbb{B}^{d+1}, W_{\mu-1/2})}$ *is replaced by* $\|\triangle^r_{i,d+1,\theta} \widetilde{f}\|_{L^p(\mathbb{S}^d)}$.

To emphasise the dependence on the dimension, we shall write the modulus of smoothness on the sphere as $\omega_r(f,t)_p = \omega_r(f,t)_{L^p(\mathbb{S}^{d-1})}$ in the following.

LEMMA 6.3.2. *Let* $\mu = \frac{m-1}{2}$ *and* $m \in \mathbb{N}$. *Let* $f \in L^p(\mathbb{B}^d, W_\mu)$ *if* $1 \le p < \infty$ *and* $f \in C(\mathbb{B}^d)$ *if* $p = \infty$, *and let* F *be defined as in* (6.3.1). *Then*

$$\omega_r(f,t)_{L^p(\mathbb{B}^d, W_\mu)} \sim \omega_r(F,t)_{L^p(\mathbb{S}^{d+m-1})}.$$

Aa a result, we can deduce the characterization of the best approximation on the unit ball from the corresponding results on the sphere.

THEOREM 6.3.3. *Let* $\mu = \frac{m-1}{2}$ *and* $m \in \mathbb{N}$. *For* $f \in L^p(\mathbb{B}^d, W_\mu)$ *if* $1 \le p < \infty$ *and* $f \in C(\mathbb{B}^d)$ *if* $p = \infty$,

$$E_n(f)_{p,\mu} \le c\, \omega_r(f, n^{-1})_{p,\mu}, \qquad 1 \le p \le \infty; \tag{6.3.5}$$

on the other hand,

$$\omega_r(f, n^{-1})_{p,\mu} \le c\, n^{-r} \sum_{k=1}^{n} k^{r-1} E_k(f)_{p,\mu}. \tag{6.3.6}$$

When $d = 1$, the ball becomes the interval $B^1 = [-1, 1]$. It turns out that our modulus of smoothness appears to be new even in this case. For $\mu = \frac{m-1}{2}$ and $m \in \mathbb{N}$, the definition in (6.3.4) becomes, written out explicitly,

$$\omega_r(f,t)_{p,\mu} := \sup_{|\theta| \le t} \left(c_\mu \int_{B^2} \left| \overrightarrow{\triangle}^r_\theta f(x_1 \cos(\cdot) + x_2 \sin(\cdot)) \right|^p W_{\mu - \frac{1}{2}}(x) dx \right)^{1/p} \tag{6.3.7}$$

for $1 \le p < \infty$ with the usual modification for $p = \infty$, where $c_\mu^{-1} = \int_{B^2} W_{\mu-\frac{1}{2}}(x) dx$. The difference $\overrightarrow{\triangle}^r_\theta$ in this definition can be evaluated at any fixed point $t_0 \in [0, 2\pi]$. More precisely, $\overrightarrow{\triangle}^r_\theta f(x_1 \cos(\cdot) + x_2 \sin(\cdot)) = \overrightarrow{\triangle}^r_\theta g_{x_1,x_2}(t_0)$ for a fixed $t_0 \in [0, 2\pi]$, where $g_{x_1,x_2}(\theta) = f(x_1 \cos\theta + x_2 \sin\theta)$.

EXAMPLE 6.3.4. *For* $\alpha \ne 0$, *let* $f_\alpha(x) = (1 - \|x\|^2)^\alpha$ *for* $x \in \mathbb{B}^d$. *Then*

$$\omega_2(f_\alpha, t)_{L^p(\mathbb{B}^d)} \sim \begin{cases} t^{2\alpha + \frac{2}{p}}, & \text{if } -\frac{1}{p} < \alpha < 1 - \frac{1}{p}; \\ t^2 |\log t|^{\frac{1}{p}}, & \text{if } \alpha = 1 - \frac{1}{p}; \\ t^2, & \text{if } \alpha > 1 - \frac{1}{p}. \end{cases}$$

EXAMPLE 6.3.5. *Let* $f_\alpha(x) = x^\alpha$ *for* $x \in \mathbb{B}^d$ *and* $\alpha = (\alpha_1, \cdots, \alpha_d) \ne 0$. *If* $0 \le \alpha_i < 1$ *for all* $1 \le i \le d$ *then for* $1 \le p \le \infty$,

$$\omega_2(f_\alpha, t)_{L^p(\mathbb{B}^d)} \sim t^{\delta + \frac{1}{p}}, \quad \delta := \min_{\alpha_i \ne 0} \{\alpha_1, \cdots, \alpha_d\}.$$

EXAMPLE 6.3.6. *Let* $\alpha \ne 0$, $d \ge 2$ *and let* $f_\alpha : \mathbb{B}^d \to \mathbb{R}$ *be given by* $f_\alpha(x) = \|x - e_0\|^{2\alpha}$, *where* $e_0 = (1, 0, \ldots, 0) \in \mathbb{B}^d$. *Then*

$$\omega_2(f_\alpha, t)_{L^p(\mathbb{B}^d)} \sim \begin{cases} t^{2\alpha + \frac{d}{p}}, & -\frac{d}{2p} < \alpha < 1 - \frac{d}{2p}, \\ t^2 |\log t|^{\frac{1}{p}}, & \alpha = 1 - \frac{d}{2p}, \quad p \ne \infty, \\ t^2, & \alpha > 1 - \frac{d}{2p}. \end{cases} \tag{6.3.8}$$

References. The connection between the orthogonal structures on the unit ball and on the sphere was developed in [**47, 49**]. The formula in Corollary 6.1.7 appeared first in [**48**]. Further results in the first two sections are developed in [**50, 51**] and [**9, 10, 11**]. The approximation in the last section was developed recently in [**12**].

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