THE CUNTZ SEMIGROUP AND THE CLASSIFICATION OF C^*-ALGEBRAS

ANDREW S. TOMS

ABSTRACT. These notes represent a (very) preliminary version of my contribution to the far more detailed text that will cover the full material of this CRM Advanced Course. Our aim is to introduce the Cuntz semigroup and examine its properties, structure, and relevance to the classification of simple nuclear separable C*-algebras.

1. INTRODUCTION

The study of algebras of bounded linear operators on Hilbert space was initiated by Frank Murray and Jon von Neumann in the 1930s. They concentrated on algebras which were closed under the formation of adjoints and in the strong operator topology (SOT). Recall that

$$T_n \stackrel{SOT}{\longrightarrow} T \ \Leftrightarrow\ T_n x \longrightarrow Tx,\ \forall x \in \mathcal{H},$$

where $\mathcal{H}$ is a Hilbert space. These algebras are now known as *von Neumann algebras*, and are referred to as *factors* if they have trivial center. One of Murray and von Neumann's significant achievements was the so-called type classification of factors. Their classification used the structure of projection operators, i.e., self-adjoint idempotents, and more specifically the notion of rank for such operators. Let's examine this a bit.

Let $\mathcal{M}$ be a factor on $\mathcal{H}$, and let $p, q \in \mathcal{M}$ be projections. One says that p and q are Murray-von Neumann equivalent, written $p \sim q$, if there is $v \in \mathcal{M}$ such that $v^*v = p$ and $vv^* = q$. Note that v must be a partial isometry, and that v^* denotes the adjoint of v. If $\mathcal{M} = B(\mathcal{H})$ with $\mathcal{H}$ separable, where $B(\mathcal{H})$ denotes the factor of all bounded linear operators on $\mathcal{H}$, then $p \sim q$ if and only if the ranges of p and q have the same dimension. For more general $\mathcal{M}$, therefore, Murray-von Neumann equivalence can be viewed as a relativized notion of dimension. In a factor, this relativized dimension can be analyzed very nicely using the notion of *dimension function.* If $P(\mathcal{M})$ denotes the set of projections in $\mathcal{M}$, then a dimension function $D : P(\mathcal{M}) \to \mathbb{R}_{\geq 0}$ is characterized by the following properties:

- $D(q) = D(p)$ whenever $p \sim q$
- $D(p + q) = D(p) + D(q)$ whenever $pq = 0$, i.e., p and q are orthogonal

The type classification of factors is just the answer to the question "What are the possible ranges of a dimension function?" Here is an exhaustive list of the answers, up to normalization:

- If $D(\mathcal{M}) = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$, then we say that $\mathcal{M}$ is of type I_n. In this case $\mathcal{M} \cong \mathrm{M}_n(\mathbb{C})$.
- If $D(\mathcal{M}) = \mathbb{N}$, then $\mathcal{M}$ is of type I_∞. $B(\mathcal{H})$ for $\mathcal{H}$ separable is an example.
- If $D(\mathcal{M}) = [0, 1]$, then we say that $\mathcal{M}$ is of type II_1. The hyperfinite II_1 factor $\mathcal{R}$, also known as the SOT-closure of the tracial GNS representation of the CAR algebra, is a very important example.

- If $D(\mathcal{M}) = [0, \infty]$, then we say that $\mathcal{M}$ is of type II_∞. The von Neumann algebra tensor product of $B(\mathcal{H})$ and $\mathcal{R}$ is an example.
- If $D(\mathcal{M}) = \{0, \infty\}$, then we say that $\mathcal{M}$ is of type III.

What should we take from all this? In a von Neumann algebra, at least, projections considered up to a suitable notion of equivalence can tell you quite a lot about the structure of the algebra.

Here, however, we will be concerned with a more general type of operator algebra, namely, C*-algebras. These are self-adjoint algebras of bounded linear operators which are closed in the norm topology. Recall that the norm of a bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ is

$$\sup_{\|x\| \leq 1} \|Tx\|,$$

and that this norm generates the norm topology. It turns out that one cannot have a type classification for C*-algebras, even when they are simple (no closed two-sided proper ideals). We'll see some details on this later. For now, we'll say simply that projections may behave quite badly in C*-algebras, and may also be very sparse, even absent. There is, however, a type of element in C*-algebras which generalizes projections and which is always found in abundance: positive elements. The Cuntz semigroup is, very roughly, a way of adapting the ideas of Murray and von Neumann to the realm of C*-algebras. The results are beautiful and surprising.

2. The Cuntz semigroup

A caveat: there exists already a fairly in depth treatment of the basic theory of the Cuntz semigroup ([**?**]). Proofs, if not in these notes, may be found there.

Let A be a C*-algebra. We assume that A is separable unless otherwise noted. An element a of A is said to be positive if $a = y^*y$ for some $y \in A$. We use the subscript "+" to denote the set of positive elements in a C*-algebra. If $a, b \in A_+$, then we say that a is *Cuntz subequivalent* to b if there exists a sequence (v_n) in A such that $\|v_n b v_n^* - a\| \to 0$; we denote this relation by $a \precsim b$. If $a \precsim b$ and $b \precsim a$, then we write $a \sim b$ and say that a and b are Cuntz equivalent.

Proposition 2.1. *$\sim$ is an equivalence relation.*

Proof. The existence of an approximate unit guarantees reflexivity, and $\sim$ is antisymmetric by definition. Suppose that $a \precsim b$ and $b \precsim c$. Given $\epsilon > 0$, find w such that $\|wbw^* - a\| < \epsilon/2$. Now find v such that $\|vcv^* - b\| < \epsilon/(2\|w\|^2)$. It follows that

$$\begin{aligned} \|(wv)c(wv)^* - a\| &\leq \|wvcv^*w^* - wbw^*\| + \|wbw^* - a\| \\ &< \|w\|^2 \cdot \|vcv^* - b\| + \epsilon/2 \\ &< \epsilon. \end{aligned}$$

□

Exercise 2.2. *If $A = \mathrm{M}_n(\mathbb{C})$, prove that $a \precsim b$ if and only if $\mathrm{rank}(a) \leq \mathrm{rank}(b)$. It follows that $a \sim b$ if and only if $\mathrm{rank}(a) = \mathrm{rank}(b)$.*

From Exercise 2.2 we see that Cuntz equivalence has something of the flavor of Murray-von Neumann equivalence. Indeed, thinking of Cuntz equivalence as Murray-von Neumann equivalence for positive elements is often a useful heuristic. While not accurate in general, we will see that under some mild conditions and an accounting for projections, this heuristic is precise.

Let $\mathcal{K}$ denote the algebra of compact operators on a separable Hilbert space $\mathcal{H}$. Let $Cu(A)$ denote the set $(A \otimes \mathcal{K})_+/\sim$ of Cuntz equivalence classes. We use

$\langle a \rangle$ to denote the class of a in $Cu(A)$. It is clear that $\langle a \rangle \leq \langle b \rangle \Leftrightarrow a \precsim b$ defines an order on $Cu(A)$. We also endow $Cu(A)$ with an addition operation by setting $\langle a \rangle + \langle b \rangle := \langle a' + b' \rangle$, where a' and b' are orthogonal and Cuntz equivalent to a and b respectively.

Exercise 2.3. *Prove that the choice of a' and b' does not affect the Cuntz class of their sum.*

The semigroup $W(A)$ is defined to be the subsemigroup of $Cu(A)$ of Cuntz classes with a representative in $\bigcup_n \mathrm{M}_n(A)_+$. Here one should think of $\mathrm{M}_n(A)$ as $A \otimes \mathrm{M}_n(\mathbb{C})$, and of $A \otimes \mathcal{K}$ as the limit of

$$A \otimes \mathbb{C} \stackrel{\mathrm{id}\otimes\phi_1}{\longrightarrow} A \otimes \mathrm{M}_2(\mathbb{C}) \stackrel{\mathrm{id}\otimes\phi_2}{\longrightarrow} A \otimes \mathrm{M}_3(\mathbb{C}) \stackrel{\mathrm{id}\otimes\phi_3}{\longrightarrow} \cdots$$

where $\phi_n(a) = \mathrm{diag}(a, 0)$. The union $\cup_n A \otimes \mathrm{M}_n(\mathbb{C})$ is usually denoted by $\mathrm{M}_\infty(A)$.

There are some technical devices which we will need later, and which play a role in virtually any serious application of the Cuntz semigroup. Given $\epsilon > 0$, let $(t-\epsilon)_+ : \mathbb{R} \to \mathbb{R}^+$ be given by the following formula:

$$(t-\epsilon)_+ = \begin{cases} 0 & t \leq \epsilon \\ t-\epsilon & t > \epsilon. \end{cases}$$

Let a be a positive operator. We then define $(a-\epsilon)_+$ via functional calculus: it is $(t-\epsilon)_+$ applied to a. In particular, $(a-\epsilon)_+ \precsim (a-\delta)_+$ whenever $\epsilon \geq \delta$.

Proposition 2.4. *If $a \leq b$ for $a, b \in A_+$, then $a \precsim b$.*

Proof. Since $a \leq b$, we have $a \in \overline{bAb}$. For each n one can find a positive function $f_n(t)$ such that $f_n(t)t$ is zero for $t \leq 0$, 1 for $t \geq 1/n$, and in $[0,1]$ otherwise; in particular, $f_n(b)b$ is an approximate unit for $\overline{bAb}$. Set $w = a^{1/2} f_n(t)^{1/2}$. It follows that

$$wbw^* = a^{1/2} f_n(t) b a^{1/2} \to a,$$

as required. □

Exercise 2.5. *Prove that if f, g are positive continuous functions on a compact Hausdorff space X, then $f \precsim g$ in $C(X)$ if and only if the support of f is contained in the support of g.*

The following lemma is endlessly useful.

Lemma 2.6. *Let A be a C*-algebra with $a, b \in A_+$.*

1. *$a \precsim b$ if and only if $(a-\epsilon)_+ \precsim b$ for every $\epsilon > 0$.*
2. *Suppose that $a \precsim b$. It follows that for every $\epsilon > 0$ there is a $\delta > 0$ such that $(a-\epsilon)_+ \precsim (b-\delta)_+$.*
3. *If $a \precsim b$, then for every $\epsilon > 0$ there is $x \in A$ such that $xbx^* = (a-\epsilon)_+$.*

In Cuntz's original definition of what is now his semigroup, he produced $W(A)$. This semigroup, however, has some flaws. For one thing it is not continuous with respect to inductive limits, i.e.,

$$W(\lim_{i\to\infty} A_i) \neq \lim_{i\to\infty} W(A_i)$$

in general, where the limit on the right hand side is taken in the category of partially ordered Abelian semigroups. This flaw is corrected with $Cu(A)$, because this semigroup belongs to an enriched category of semigroups with "larger" inductive limits. We will discuss this soon, but let us first have some examples of Cuntz semigroups.

Example 2.7. $Cu(\mathbb{C}) = \mathbb{Z}^+ \cup \{\infty\}$ with the obvious ordering. How so? We are considering positive compact operators on a separable Hilbert space $\mathcal{H}$. Let $a, b \in \mathcal{K}_+$ be given. We can diagonalize a and b with respect to orthonormal bases (e_k) and (f_k), respectively. Let U be the unitary operator defined by $Ue_k = f_k$. Then U^*aU is a compact operator Cuntz equivalent a and diagonal with respect to (e_k) (check this!). Thus, we may assume that both a and b are diagonal with respect to (e_k). Suppose that $\mathrm{rank}(a) \leq \mathrm{rank}(b)$. If a is finite rank, then some compression of b by a projection P onto the span of a suitable subset of the $e_k's$ will give finite rank operators a and PbP such that $\mathrm{rank}(PbP) \geq \mathrm{rank} a$. We may view these operators as bounded operators on the span of finitely many $e_k's$, whence the definition of $\precsim$ and Exercise 2.2 give

$$a \precsim PbP \precsim b.$$

In other words, if a has finite rank and $\mathrm{rank}(a) \leq \mathrm{rank}(b)$, then $a \precsim b$.

Now suppose that a is no longer finite rank, and neither is b. We would like to prove that $a \precsim b$ nonetheless. It will suffice to find, given $\epsilon > 0$, a $v \in \mathcal{K}$ such that $\|vbv^* - a\| < \epsilon$. Let P_n be projection onto the span of $\{e_1, \ldots, e_n\}$, so that by compactness, $P_n a P_n \to a$. Find n_0 such that $\|P_{n_0} a P_{n_0} - a\| < \epsilon/2$. By the arguments above, we can find v such that $\|vbv^* - P_{n_0} a P_{n_0}\| < \epsilon/2$. An application of the triangle inequality completes the proof.

Example 2.8. Let $A = C([0,1])$, the continuous complex-valued functions on the unit interval. Then, as a set,

$$Cu(A) = \{f : [0,1] \to \mathbb{Z}^+ \cup \infty \mid f \text{ is lower semicontinuous}\}.$$

The ordering is given by $f \leq g \Leftrightarrow f(x) \leq g(x)$ for each $x \in [0,1]$, and addition is pointwise.

Since $A \otimes \mathcal{K} \cong C([0,1]; \mathcal{K})$, it is clear enough that the calculation above represents some kind of pointwise version of Example 2.7. One should not, however, believe that this sort of simple pointwise generalization holds if $[0,1]$ is replaced with another compact metric space X; the low dimension of $[0,1]$ is crucial. Roughly, it allows one to invoke a result of Elliott which says that a continuous map from $[0,1]$ into $\mathrm{M}_n(\mathbb{C})_+$ can be approximately diagonalized in such a way that the eigenvalues appear along the diagonal in descending order. The passage from this result to the assertion of this example is then similar to that of Example 2.7.

Example 2.9. The Cuntz semigroup of $C(X)$ is a wild beast, even for contractible X. We'll postpone our discussion of this fact until our look at Elliott's program.

Example 2.10. Let A be a simple purely infinite C*-algebra. One of the many equivalent characterizations of such an algebra is that for any positive $a, b \neq 0$, one has $a \sim b$. It follows that $Cu(A) = \{0, \infty\}$, where ∞ is additively absorbing. Thus, the Cuntz semigroup is to a large degree degenerate for purely infinite algebras.

$Cu(A)$ can be defined to consist of equivalence classes of countably generated Hilbert modules over A. The equivalence relation boils down to isomorphism in the case that A has stable rank one, but is rather more complicated in general. We do not require the precise definition of this relation in the sequel, but it should be mentioned that this point of view, introduced by Coward, Elliott, and Ivanescu, is the basis for the category **Cu** considered in the next section.

2.1. **The category Cu.** As was mentioned above the semigroup $W(A)$ has some failings which are corrected by considering $Cu(A)$ instead. This is because the semigroup $Cu(A)$ is an object in a richer category of ordered Abelian monoids denoted by **Cu** which we describe presently. Before stating them, we require the notion of order-theoretic compact containment. Let T be a preordered set with $x, y \in T$. We say that x is compactly contained in y—denoted by $x \ll y$—if for any increasing sequence (y_n) in T with supremum y, we have $x \leq y_{n_0}$ for some $n_0 \in \mathbb{N}$. An object S of **Cu** enjoys the following properties:

- **P1** S contains a zero element;
- **P2** the order on S is compatible with addition: $x_1 + x_2 \leq y_1 + y_2$ whenever $x_i \leq y_i, i \in \{1, 2\}$;
- **P3** every countable upward directed set in S has a supremum;
- **P4** the set $x_{\ll} = \{y \in S \mid y \ll x\}$ is upward directed with respect to both $\leq$ and $\ll$, and contains a sequence (x_n) such that $x_n \ll x_{n+1}$ for every $n \in \mathbb{N}$ and $\sup_n x_n = x$;
- **P5** the operation of passing to the supremum of a countable upward directed set and the relation $\ll$ are compatible with addition: if S_1 and S_2 are countable upward directed sets in S, then $S_1 + S_2$ is upward directed and $\sup(S_1 + S_2) = \sup S_1 + \sup S_2$, and if $x_i \ll y_i$ for $i \in \{1, 2\}$, then $x_1 + x_2 \ll y_1 + y_2$.

We assume further that $0 \leq x$ for any $x \in S$. This is always the case for $Cu(A)$.

Theorem 2.11. *If A is a C*-algebra, then $Cu(A)$ belongs to* **Cu**.

Some of the properties above are simple enough: **P1** and **P2**, for instance. Others, such as **P3**, are quite difficult to establish. The proofs are due originally to Coward, Elliott, and Ivanescu. Let us mention where the sequence (x_n) of **P4** typically comes from: one sets $x_n = \langle (a - 1/n)_+ \rangle$, where $\langle a \rangle = x$. We caution the reader that not every object of **Cu** arises as $Cu(A)$ for some A.

For S and T objects of **Cu**, the map $\phi \colon S \to T$ is a morphism in the category **Cu** if

- **M1** ϕ is order preserving;
- **M2** ϕ is additive and maps 0 to 0;
- **M3** ϕ preserves the suprema of increasing sequences;
- **M4** ϕ preserves the relation $\ll$.

The category **Cu** admits inductive limits, and $Cu(\cdot)$ may be viewed as a functor from C*-algebras into **Cu**; the map on $Cu(\bullet)$ induced by a $*$-homomorphism $\phi : A \to B$ is given by $Cu(\phi)(\langle a \rangle) = \langle \phi(a) \rangle$. A central result of Coward-Elliott-Ivanescu is that if (A_i, ϕ_i) is an inductive sequence of C*-algebras, then

$$Cu\left(\lim_{i \to \infty}(A_i, \phi_i)\right) \cong \lim_{i \to \infty}(Cu(A_i), Cu(\phi_i)).$$

Let $S = \lim_{i \to \infty}(S_i, \phi_i)$ be an inductive limit in the category **Cu**, with $\phi_{i,j} \colon S_i \to S_j$ and $\phi_{i,\infty} \colon S_i \to S$ the canonical maps. We have the following two properties:

- **L1** each $x \in S$ is the supremum of an increasing sequence (x_i) belonging to $\bigcup_{i=1}^{\infty} \phi_{i,\infty}(S_i)$ and such that $x_i \ll x_{i+1}$ for all i;
- **L2** If $x, y \in S_i$ and $\phi_{i,\infty}(x) \leq \phi_{i,\infty}(y)$, then for all $x' \ll x$ there is n such that $\phi_{i,n}(x') \leq \phi_{i,n}(y)$.

For $e \in S$ we denote by $\infty \cdot e$ the supremum $\sup_{n\geq 1} ne$. We say that e is full if $\infty \cdot e$ is the largest element of S. We say that e is compact if $e \ll e$. If a sequence (x_i) in S satisfies $x_i \ll x_{i+1}$ for every i, then we say that the sequence is rapidly increasing.

Exercise 2.12. *Prove that Examples 2.7 and 2.8 belong to* **Cu**.

2.2. **Functionals and Cu.** Let S be a semigroup in the category **Cu**. A functional on S is a map $\lambda\colon S \to [0, \infty]$ that is additive, order preserving, preserves suprema of increasing sequences and satisfies $\lambda(0) = 0$. We use $F(S)$ to denote the functionals on S. We will make use of the following lemma, established in [1].

Lemma 2.13. *If S is in the category* **Cu** *and $\lambda\colon S \to [0, \infty]$ is additive, order preserving, and maps 0 to 0, then $\tilde{\lambda}(x) := \sup_{x' \ll x} \lambda(x')$ defines a functional on S.*

For a C*-algebra A, the functionals on $Cu(A)$ admit a description in terms of 2-quasitraces. Recall that a lower semicontinuous extended 2-quasitrace on A is a lower semicontinuous map $\tau\colon (A \otimes \mathcal{K})_+ \to [0, \infty]$ which vanishes at 0, satisfies the trace identity, and is linear on pairs of positive elements that commute. The set of all such quasitraces is denoted by $\mathrm{QT}_2(A)$. Given $\tau \in \mathrm{QT}_2$ we define a map $d_\tau\colon Cu(A) \to [0, \infty]$ by the following formula:

$$d_\tau(\langle a \rangle) := \lim_{n\to\infty} \tau(a^{1/n}).$$

By Proposition 4.2 of [1] that the association $\tau \mapsto d_\tau$ defines a bijection between $\mathrm{QT}_2(A)$ and $F(Cu(A))$, extending the work of Blackadar and Handelman in [**?**]. In particular, $d_\tau(\langle a \rangle)$ is independent of the representative a of $\langle a \rangle$.

Heuristically, d_τ assigns a normalized notion of rank to a trace τ. Consider the particularly simple case where τ is the normalized trace on $\mathrm{M}_n(\mathbb{C})$ and a is a diagonal positive element. What happens when we form $a^{1/n}$ and let $n \to \infty$? The nonzero diagonal entries tend to 1, while the zero entries remain zero. Taking the normalized trace of this limit matrix yields precisely the normalized rank.

Let's record some basic properties of a functional d_τ:

- If a and b are orthogonal positive elements, the $d_\tau(a + b) = d_\tau(a) + d_\tau(b)$.
- If $a \precsim b$, then $d_\tau(a) \leq d_\tau(b)$. In particular, if $a \sim b$, then $d_\tau(a) = d_\tau(b)$.

Let's stick to unital C*-algebras for a moment, and use the functionals d_τ to extract a numerical invariant from $Cu(A)$. It is well known that if M and N are finitely generated projective modules over $\mathrm{C}(X)$ with X compact metric (read "projections"), and if the rank of M exceeds that of N by at least $\dim(X)/2$, then N is isomorphic to a submodule of M. The converse does not hold—the dimension of X cannot be determined by the comparability of finitely generated projective modules—but this can be both corrected and put into the wider context of a unital C*-algebra A. Define the radius of comparison of A to be

$$rc(A) = \inf\{r \geq 0 \mid d_\tau(a) + r < d_\tau(b) \Rightarrow a \precsim b, \text{where } a, b \in \mathrm{M}_\infty(A)_+\}$$

The connection to covering dimension comes from the following theorem:

Theorem 2.14. *Let X be a finite-dimensional CW complex. It follows that*

$$\left| rc(C(X)) - \frac{\dim(X)}{2} \right| \leq \frac{1}{2}$$

The case of $rc(A) = 0$ is rather special. If A satisfies this condition then it is said to have *strict comparison*. If A is moreover simple, then one can say quite a bit about the structure of the Cuntz semigroup. We'll now examine this in some detail.

3. Structure of the Cuntz semigroup

What does the Cuntz semigroup really look like? We'll see later that the question is out of reach in any reasonable sense without imposing some conditions on your C*-algebra. For this section, we restrict our attention to unital simple separable C*-algebras which have strict comparison of positive elements. In light of Example 2.5, we will also assume that our algebra is stably finite, since the Cuntz semigroup is otherwise degenerate. This means that our algebras admits at least one normalized 2-quasitrace (which will be a trace if the algebra is exact).

The first observation we can make is that there is a fundamental difference between elements which have zero as an accumulation point of their spectrum and those that do not. Consider $a \in A$ positive. Since $\langle a\rangle = \langle \lambda a\rangle$ for any $\lambda > 0$, we may assume that $\|a\| = 1$. From an earlier exercise we know that the Cuntz class of $f(a)$ is the same as that of a provided that $f \in C_0(0,1]$ is positive and has support $(0,1]$. Suppose that there is a gap in the spectrum of a at zero, i.e., that the spectrum $\sigma(a)$ does not meet $(0,\epsilon)$ for some $\epsilon > 0$. Define a continuous map $g_\epsilon : \mathbb{R} \to [0,1]$ as follows:

$$g_\epsilon(t) = \begin{cases} 0 & t \leq 0 \\ 1 & t \geq \epsilon \\ \text{linear else}\,. \end{cases}$$

Then $g_\epsilon(a)$ is a projection, the support projection of a. Could a be Cuntz equivalent to a projection even if zero were an accumulation point of the spectrum? The answer is no, and a nice way to see this is with functionals.

Let's assume for simplicity that our algebra is exact, so that every 2-quasitrace is a trace. (What we are about to say works also for 2-quasitraces, but the examples we'll consider later are all exact.) Let τ be a normalized trace on A. Given a as above, we have that τ induces a regular Borel probability measure μ_τ on $\sigma(a)$. We have

$$\tau(f(a)) = \int f(t)\mu_\tau$$

for any continuous function on $\sigma(a)$. What happens if we try to compute $d_\tau(a)$?

$$d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n}) = \lim_{n\to\infty} \int t^{1/n}\mu_\tau = \mu_\tau(\sigma(a)\backslash\{0\}).$$

More generally, if f is a positive continuous function on $\sigma(a)$ we have

$$d_\tau(f(a)) = \lim_{n\to\infty} \int f(t)^{1/n}\mu_\tau = \mu_\tau(\mathrm{supp}(f)).$$

If a has zero as an accumulation point of the spectrum, then the support of $(a-\epsilon)_+$ is strictly smaller than the support of a. Since A is simple, $\tau(a)$ is faithful. In particular, $\mu_\tau(S) > 0$ for any nonempty subset S of $\sigma(a)$. This means that for any $\epsilon > 0$ we have

$$d_\tau((a-\epsilon)_+) = \mu_\tau(\mathrm{supp}((t-\epsilon)\cap\sigma(a))_+ < \mu_\tau(\mathrm{supp}(t)\cap\sigma(a)) = d_\tau(a).$$

If, on the other hand, p is a projection, then for any $1 > \epsilon > 0$ we have

$$d_\tau((p-\epsilon)_+) = \mu_\tau(\mathrm{supp}((t-\epsilon)\cap\sigma(p))_+ = \mu_\tau(\mathrm{supp}(t)\cap\sigma(p)) = d_\tau(p).$$

Thus, we can divide the Cuntz classes in a simple unital C*-algebra with a trace into two types: those which are Cuntz equivalent to a projection, characterized by a gap in the spectrum at zero; and those which are not, characterized by having zero as an accumulation point of the spectrum. We call the latter type of element *purely positive*, and use A_{++} to denote the set of all such. Of course, to analyze the Cuntz semigroup we need to look at positive elements in $A \otimes \mathcal{K}$ rather than just in A, and there we no longer have a bounded trace. We do, however, have an unbounded traces defined on positive elements, and the analysis above goes through more or less intact with these unbounded traces. Let's flesh this out.

Let A be a unital C*-algebra and $a \in A \otimes \mathcal{K}$ be positive. Assume A is exact and let $T(A)$ denote the space of tracial states on A. Let $e_n \in \mathcal{K}$ be an increasing sequence of projections with $\mathrm{rank}(e_n) = n$, and put $P_n = 1 \otimes e_n \in A \otimes \mathcal{K}$. Then,

$$P_1 a P_1 \precsim P_2 a P_2 \precsim P_3 a P_3 \cdots$$

in $Cu(A)$ and $P_n a P_n \to a$ in norm. Let $b = \sup_n \langle P_n a P_n \rangle \in Cu(A)$ (suprema of increasing sequences in the Cuntz semigroup always exist). Then, given $\epsilon > 0$, there is some $n \in \mathbb{N}$ such that

$$(a - \epsilon)_+ \precsim P_n a P_n \precsim b.$$

It follows that $a \precsim b$ and $P_n a P_n \precsim a$ for each n. Since the supremum is unique, $a \sim b$. For each trace $\tau \in T(A)$ and positive element $a \in A \otimes \mathcal{K}$ we define a function $\iota\langle a \rangle : T(A) \to \mathbb{R}^+ \cup \{\infty\}$ as follows:

$$\iota\langle a \rangle(\tau) = \sup_n d_\tau(P_n a P_n).$$

It turns out (see [**?**]) that ι is independent of the choice of P_n.

Let use $Cu(A)_{++}$ to denote the Cuntz classes of elements of $(A \otimes \mathcal{K})_{++}$. Let $b \in (A \otimes \mathcal{K})_{++}$ and $a \in (A \otimes \mathcal{K})_+$. Since

$$\langle a \rangle + \langle b \rangle = \left\langle \left[\begin{array}{cc} a & 0 \\ 0 & b \end{array} \right] \right\rangle,$$

and since the spectrum of the matrix on the right hand side is $\sigma(a) \cup \sigma(b)$, we see that $Cu(A)_{++}$ is absorbing in $Cu(A)$, i.e., if $x \in Cu(A)$ and $y \in Cu(A)_{++}$, then $x + y \in Cu(A)_{++}$. In particular, $Cu(A)_{++}$ is a subsemigroup of $Cu(A)$.

What about $Cu(A) \backslash Cu(A)_{++}$? We know that these are the Cuntz classes of all projections, and since the orthogonal sum of projections is again a projection, this set is also a subsemigroup of $Cu(A)$. This subsemigroup is actually a familiar object.

Exercise 3.1. *Let A be a stably finite C*-algebra. Prove that projections p and q are Murray-von Neumann equivalent if and only if they are Cuntz equivalent, and that $p \precsim q$ if and only if p is equivalent to a subprojection of q.*

(Note that this fails without stable finiteness. Consider purely infinite simple C*-algebras: they may have nontrivial K-theory but always have degenerate Cuntz semigroups.) By considering all projections in $A \otimes \mathcal{K}$ up to Murray-von Neumann equivalence and forming an ordered semigroup using exactly the same procedure we used for $Cu(A)$, one obtains the Murray-von Neumann semigroup $V(A)$. This semigroup is the precursor to the K_0-group, which we'll discuss further soon. Let's summarize our findings:

Proposition 3.2. *Let A be a simple unital C*-algebra with a trace. It follows that*

$$Cu(A) = V(A) \sqcup Cu(A)_{++}.$$

Both components are subsemigroups, and the second component is absorbing.

We haven't discussed how the order looks under the decomposition above. This requires that we invoke strict comparison.

Lemma 3.3. *Let A be a simple separable unital C*-algebra with a trace and strict comparison of positive elements. If $a \in A_{++}$ and $b \in A_+$, and if $d_\tau(a) \leq d_\tau(b)$ for every trace on A, then $a \precsim b$.*

Proof. By Lemma 2.6, we need only prove that $(a-\epsilon)_+ \precsim b$ for every $\epsilon > 0$. From our discussion of purely positive elements above we know that

$$d_\tau((a-\epsilon)_+) < d_\tau(a) \leq d_\tau(b)$$

for every trace on A. By strict comparison, then, we have $(a-\epsilon)_+ \precsim b$, as required. □

The following corollary is immediate.

Corollary 3.4. *Let A be as in Lemma 3.3. If $a, b \in A_{++}$, then $a \precsim b$ if and only if $d_\tau(a) \leq d_\tau(b)$ for every trace τ. In particular, $a \sim b$ if and only if $d_\tau(a) = d_\tau(b)$ for every trace.*

The upshot of this corollary is that the Cuntz class of $a \in A_{++}$ is determined by the map $\hat{a} : QT_2(A) \to \mathbb{R}$ given by $\hat{a}(\tau) = d_\tau(a)$. This function is easily seen to be affine. It is also lower semicontinuous because is it the supremum of the increasing sequence of continuous affine functions $f_{a,n}(\tau) = \tau(a^{1/n})$. In fact, $\hat{a}$ coincides with $\iota\langle a\rangle$ defined above. Once again we have to pass from A to $A \otimes \mathcal{K}$, and this time things get a little sticky. What happens is this: you can have the analogue of Corollary 3.4 provided that you have the following condition. Given any continuous strictly positive affine function g on $T(A)$ and $\epsilon > 0$, there is a positive $b \in \mathrm{M}_n(A)$ such that $|\iota\langle a\rangle(\tau) - g(\tau)| < \epsilon$ for every τ. Let's call this Condition (D) for "dense".

Let $SAff(T(A))$ denote the set of functions on $T(A)$ which are pointwise suprema of increasing sequences of continuous, affine, and strictly positive functions on $T(A)$. Define an addition operation on the disjoint union $V(A) \sqcup SAff(T(A))$ as follows:

(1) if $x, y \in V(A)$, then their sum is the usual sum in $V(A)$;
(2) if $x, y \in SAff(T(A))$, then their sum is the usual (pointwise) sum in $SAff(T(A))$;
(3) if $x \in V(A)$ and $y \in SAff(T(A))$, then their sum is the usual (pointwise) sum of $\hat{x}$ and y in $SAff(T(A))$, where $\hat{x}(\tau) = \tau(x)$, $\forall \tau \in T(A)$.

Equip $V(A) \sqcup SAff(T(A))$ with the partial order $\leq$ which restricts to the usual partial order on each of $V(A)$ and $SAff(T(A))$, and which satisfies the following conditions for $x \in V(A)$ and $y \in SAff(T(A))$:

(1) $x \leq y$ if and only if $\hat{x}(\tau) < y(\tau)$, $\forall \tau \in T(A)$;
(2) $y \leq x$ if and only if $y(\tau) \leq \hat{x}(\tau)$, $\forall \tau \in T(A)$.

Theorem 3.5. *Let A be a unital simple exact and tracial C*-algebra with strict comparison. Assume that A satisfies Condition (D). It follows that*

$$Cu(A) \cong V(A) \sqcup SAff(T(A))$$

as ordered semigroups, with the ordered semigroup structure on the right hand side as described above.

This is all well and good, but worthless if we can't find any examples which satisfy the hypotheses of the theorem. Fortunately, examples abound.

Let $\mathcal{Z}$ denote the Jiang-Su algebra, a simple, unital, separable and nuclear C*-algebra without nontrivial projections. It also has trivial K_1-group (the unitary group has one connected component) and a unique tracial state. In some sense, it's like a noncommutative analogue of the complex numbers. If C*-algebras were animals, then $\mathcal{Z}$ would be a unicorn.

A C*-algebra A can be (and frequently is) $\mathcal{Z}$-absorbing, i.e., has the property that $A \otimes \mathcal{Z} \cong A$. This property has lots of nice consequences. For one thing, a $\mathcal{Z}$-absorbing (or "$\mathcal{Z}$-stable") C*-algebra which is simple, unital, and exact has both strict comparison and Property (D). Let's see why.

Without saying too much about the construction of the Jiang-Su algebra, we'll at least note this: there is a $*$-homomorphism $\phi : C([0,1]) \to \mathcal{Z}$ with the property that the unique tracial state on $\mathcal{Z}$ induces Lebesgue measure on the spectrum of the source algebra. For any $0 < t < 1$, let $z_t = \phi(f_t)$, where f_t is a positive function whose support has measure t. Let γ be any tracial state on A, where A is $\mathcal{Z}$-stable, and let τ be the unique tracial state on $\mathcal{Z}$. Then if $a \in A$ is positive, we have

$$d_{\gamma\otimes\tau}(a \otimes z_t) = \lim_{n\to\infty} (\gamma \otimes \tau)(a^{1/n} \otimes z_t^{1/n}) = \lim_{n\to\infty} \gamma(a^{1/n})\tau(z_t^{1/n}) = td_\gamma(a).$$

In other words, functions of the form $\gamma \to d_\gamma(a)$ for some positive a in a $\mathcal{Z}$-stable C*-algebra are scalable.

Lemma 3.6. *Let X be a compact metric space and $f \in Lsc(T(C(X) \otimes \mathcal{Z}))$ be a nonnegative lower semicontinous function. Then for any $\epsilon > 0$ there exists an element $\langle a\rangle \in Cu(C(X) \otimes \mathcal{Z})$ such that $\|\iota\langle a\rangle - f\|$.*

Proof. Since the tracial simplex of $C(X) \otimes \mathcal{Z}$ is affinely homeomorphic to that of $C(X)$, we are again in the situation of a Bauer simplex, i.e., a simplex whose extreme boundary is compact. This means that the affine lower semicontinuous functions on this simplex are in bijective correspondence with the lower semicontinuous functions on the extreme boundary via restriction.

We first handle the case that $f = \chi_O$, where $O \subseteq X$ is an open set. Define $a \in C(X)$ to be any function which is positive precisely on O and one has $\iota\langle a\rangle = \chi_O$. We can even hit multiples of such characteristic functions this characteristic function is scalable–see above. This, however, completes the proof since linear combinations of such characteristic functions are uniformly dense in the lower semicontinuous functions. □

4. Elliott's program

4.1. **The conjecture and some confirmations.** See survey.

4.2. **Counterexamples.** Here we discuss the construction of two nonisomorphic C*-algebras which are simple, unital, separable, and nuclear (indeed, AH), and have the same K-theory and traces. We employ the notion of a rapid dimension growth AH algebra, discovered first by Villadsen. We will distinguish our algebras using the Cuntz semigroup; specifically, we will arrange for the presence or absence of *perforation* in the Cuntz semigroup. A simple ordered Abelian group (G, G^+) is said to be *weakly unperforated* if whenever $nx > 0$ then $x > 0$. An ordered Abelian semigroup is said to be *almost unperforated* if $(n+1)x \leq ny$ implies $x \leq y$. We will first see how to arrange for perforation in the K_0-group of a simple C*-algebra.

Let X be a connected topological space, and let ω and γ be (complex) vector bundles over X of fibre dimensions k and m, respectively. Recall that the Euler class $e(\omega)$ is an element of $H^{2k}(X;\mathbb{Z})$ with the following properties:

(1) $e(\omega \oplus \gamma) = e(\omega) \cdot e(\gamma)$, where "" denotes the cup product in $H^*(X;\mathbb{Z})$;
(2) $e(\theta_l) = 0$, where θ_l denotes the trivial vector bundle over X of (complex) fibre dimension 1.

The Chern class $c(\omega) \in H^*(X;\mathbb{Z})$ is a sum

$$c(\omega) = 1 + c_1(\omega) + c_2(\omega) + \ + c_k(\omega),$$

where $c_i(\omega) \in H^{2i}(X;\mathbb{Z})$. Its properties are similar to those of the Euler class:

(1) $c(\omega \oplus \gamma) = c(\omega) \cdot c(\gamma)$;
(2) $c(\theta_l) = 1$.

The key connection between these characteristic classes is that $e(\eta) = c_1(\eta)$ for every line bundle. The next lemma is due essentially to Villadsen.

Lemma 4.1. *Let X be a finite CW-complex, and let $\eta_1, \eta_2, ..., \eta_k$ be complex line bundles over X. If $l < k$ and $\prod_{i=1}^k e(\eta_i) \neq 0$, then*

$$[\eta_1 \oplus \eta_2 \oplus \cdots \oplus \eta_k] - [\theta_l] \notin K^0(X)^+.$$

Proof. If $[\eta_1 \oplus \eta_2 \oplus \cdots \oplus \eta_k] - [\theta_l] \in K^0(X)^+$, then there is a vector bundle γ of dimension $k - l$ and $d \in \mathbb{N}$ such that

$$\eta_1 \oplus \eta_2 \oplus \cdots \oplus \eta_k \oplus \theta_d \cong \gamma \oplus \theta_{d+k-l}.$$

Applying the Chern class to both sides of this equation we obtain

$$\prod_{i=1}^{k} c(\eta_i) = c(\gamma).$$

Expanding the left hand side yields

$$\prod_{i=1}^{k} (1 + c_1(\eta_i)) = \prod_{i=1}^{k} (1 + e(\eta_i)).$$

The last product has only one term in $H^{2k}(X;\mathbb{Z})$, namely, $\prod_{i=1}^k e(\eta_i)$, and this is non-zero. On the other hand, $c(\gamma)$ has no non-zero term in $H^{2i}(X;\mathbb{Z})$ for $i > k - l$. Thus, we have a contradiction, and must conclude that

$$[\eta_1 \oplus \eta_2 \oplus \cdots \oplus \eta_k] - [\theta_l] \notin K^0(X)^+. \qquad \square$$

Villadsen's Lemma gives us a way to establish perforation in the K_0-group of (matrices over) certain commutative C*-algebras. One can take $X = (S^2)^n$, let ξ be any nontrivial line bundle over S^2, and set $\eta_i = \pi_i^*(\xi)$, where $\pi_i : (S^2)^n \to S^2$ is the i^{th} co-ordinate projection. With the following construction, due also to Villadsen, one even gets examples of this perforation in the setting of simple C*-algebras.

Given X, let $\pi_i : X^k \to X$ be the i^{th} co-ordinate projection. Suppose that $l, m \in \mathbb{N}$ with l dividing m. Let $x_1, x_2, \ldots, x_t$ be points in X with $t + k = m/l$. Define $\phi : \mathrm{M}_l(C(X)) \to \mathrm{M}_m(C(X^k))$ by

$$\phi(f) = \mathrm{diag}(f \circ \pi_1, \ldots, f \circ \pi_k, f(x_1), \ldots, f(x_t)).$$

It is straightforward to check that this map sends projections corresponding to trivial bundles to the same type of projection. What is special about the map is that it preserves (when $t \ll n$) perforation. Indeed, if we take $X = (S^2)^n$ and η_i as above we have

$$K^0(\phi)([\eta_1 \oplus \eta_2 \oplus \cdots \oplus \eta_n] - [\theta_1]) = [\eta_1 \oplus \eta_2 \oplus \cdots \oplus \eta_k n] - [\theta_t]$$

so that a non-positive element is sent to another such provided that t is small compared with kn. We can iterate this type of map and through a judicious choice of the points $x_1, \ldots, x_t$ at each stage arrange for the limit algebra to be simple. At the same time we need not use too many such points—it suffices to throw them in once every geological eon—and so we can preserve perforation at each stage, and hence in the limit.

Villadsen's example is surely not among the K-theoretically classifiable C*-algebras, but this is hard to prove. The reason is that the K_0-group of his example has an extremely complicated order structure, one that cannot be duplicated easily, much less in an algebra that can somehow be distinguished from the original example. The way around this is to adapt the argument to positive elements while gaining control of K-theory. In the construction above, replace S^2 with $[0,1]^3$, and fix an immersed copy of S^2 inside this cube. The projections η_i and θ_1, viewed as living over this copy of S^2, can be extended to an open collar of S^2. Multiply these projections by a scalar function $f : [0,1]^3 \to [0,1]$ which is 1 on S^2 and vanishes off the said open collar. Let's call these new positive elements γ_i and ν, respectively. We've now produced perforation in the Cuntz semigroup of $C([0,1]^{3n})$, because Cuntz comparability, restricted to our copy of S^2, entails Murray-von Neumann comparability. This is obstructed by Villadsen's Lemma. One needs to be a bit careful here since preserving perforation in the Cuntz semigroup from one stage to the next does **not** guarantee perforation in the Cuntz semigroup of the limit algebra. Here, however, we're OK.

4.3. **Winter's Theorems and new confirmations.** Here we cover the state-of-the-art in classification theorems, and indicate the role of the Cuntz semigroup.

Theorem 4.2 (Winter, Lin-Niu)**.** *Let $\mathcal{C}$ be a class of simple separable nuclear unital C*-algebras satisfying the UCT. Suppose that each element of $\mathcal{C}$ is $\mathcal{Z}$-stable, and that the collection $\mathcal{D}$ obtained by tensoring the elements of $\mathcal{C}$ with a UHF algebra of type p^∞ is Elliott-classifiable (in a strong sense). It follows that $\mathcal{C}$ is Elliott classifiable. This results applies, in particular, when the elements of $\mathcal{D}$ having tracial rank zero.*

Definition 4.3. *Let A be a nuclear C*-algebra. We say that A has nuclear dimension at most n if there is a net $(\phi_\lambda, F_\lambda, \psi_\lambda)$ of completely positive approximations of the identity map (converging pointwise to the identity) having the following property: each F_λ can be decomposed into a direct sum of at most $n+1$ ideals $F_\lambda^{(i)}$ with the property that the restriction of ψ_λ to $F_\lambda^{(i)}$ is a completely positive contraction which preserves orthogonality, i.e., a c.p.c. order zero map. The nuclear dimension of A is the smallest n for which this holds.*

The nuclear dimension generalizes covering dimension for locally compact Hausdorff spaces. It is insensitive to taking tensor products with matrices, and has the rather nice property that nuclear dimension zero corresponds to the algebra being AF.

Definition 4.4. *Let A be a C*-algebra. A has locally finite nuclear dimension if for every finite subset F of A and every $\epsilon > 0$, there is a subalgebra B of A with finite nuclear dimension and the property that each $f \in F$ is within ϵ of some $g \in B$.*

Theorem 4.5. *(Winter) Let A be a simple separable nuclear unital C*-algebra with locally finite nuclear dimension. Suppose further that A has strict comparison and Property (D). It follows that A is $\mathcal{Z}$-stable.*

In fact the hypotheses of the theorem above can be weakened somewhat, and this turns out to be quite useful. One can replace the hypothesis of strict comparison with n-comparison, defined as follows: A unital simple separable C*-algebra has n-comparison if

$$a \precsim d_1 \oplus d_2 \oplus \cdots \oplus d_n$$

whenever $a, d_1, d_2, \ldots, d_n$ are nonzero positive contractions satisfying

$$d_\tau(a) < d_\tau(d_i), \ \forall i, \ \forall \tau \in T(A).$$

Theorem 4.6 (Robert). *Let A be a unital simple separable C*-algebra. If A has nuclear dimension at most n, then A has the n-comparison property.*

One can also weaken the Property (D) hypothesis in Winter's theorem to a property that can be shown to hold in the case of a simple unital separable C*-algebra of finite nuclear dimension. Combining all of this yields:

Corollary 4.7. *Let A be a simple unital separable C*-algebra of finite nuclear dimension. It follows that A is $\mathcal{Z}$-stable.*

We now turn to two preprints to see these general theorems in action.

References

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Department of Mathematics
Purdue University
150 N. University St.
West Lafayette IN
USA
47907
E-mail address: atoms@purdue.edu