C^*-algebras and topological dynamics: finite approximation and paradoxicality

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Chapter 1

Introduction

The aim of these notes is to survey several recent developments at the crossed product interface of the subjects of C^*-algebras and group actions on compact spaces, especially in connection with the classification program for separable nuclear C^*-algebras. Groups and group actions have from the beginning provided a rich source of examples in the theory of operator algebras, and the struggle to obtain an algebraic understanding of dynamical phenomena has to a great extent driven, and continues to drive, the development of structure and classification theory for both von Neumann algebras and C^*-algebras. While our focus is on the topological realm of C^*-algebras, we haved nevertheless endeavored to take a broad perspective that incorporates both the measurable and the topological in a unifying framework. This enables us not only to illuminate the conceptual similarities and technical differences between the two sides, but also to emphasize that topological-dynamical and C^*-algebraic concepts themselves can range from the more measure-theoretic (involving weak-type approximation of multiplicative structure, or norm approximation of linear structure via bounded or completely bounded maps, like entropy and nuclearity) to the more topological (involving norm approximation of multiplicative structure, like periodicity and approximate finite-dimensionality). This variation is important for predicting and understanding the role of various phenomena in C^*-classification theory.

One of our major themes is the distinction between internal and external approximation. For a C^*-algebra A, internal approximation means modelling the structure of A locally via C^*-subalgebras or C^*-algebras which map into A, while in external approximation this modelling is done via C^*-algebras into which A maps. One can similarly speak of internal and external approximation for a discrete group G, as we can model the structure either using subsets of G or groups into which G maps. The same distinction also applies to group actions on spaces, and in which case we apply the internal/external terminology by thinking of the action C^*-algebraically. Thus, for an action of a discrete group G on a compact Hausdorff space X, internal finite approximation in the strongest sense would be a clopen

partition of X whose elements are permuted by the action, which corresponds to a G-invariant finite-dimensional *-subalgebra of $C(X)$, while external finite approximation in the strongest sense would be a finite orbit, which corresponds to a G equivariant homomorphism from $C(X)$ into $C(E)$ for some finite set E on which G acts.

The notion of external finite or finite-dimensional approximation is very flexible and broadly applicable (for example in defining invariants like entropy), but by itself it is of limited value if one is seeking the kind of refined internal structural information that C^*-classification theory demands. In fact it is by applying the external and internal viewpoints together in a back and forth way that one arrives at the key idea in classification theory, namely the intertwining argument, which was developed by Elliott in his seminal K-theoretic classification of AF algebras [25]. The goal of an intertwining is to lift an isomorphism between K-theoretic invariants of two C^*-algebras by proceeding locally along the given inductive limit decompositions, passing alternately from building blocks of one algebra to those of the other. Success depends on the existence and uniqueness (modulo approximate conjugacy) of maps between the kind of building blocks along which one is doing the intertwining. One can view existence as an external problem, and uniqueness as an internal one, as the basic case of AF algebras makes clear. The hope is that, under the assumption of nuclearity or some topological strengthening of it, external approximations exist and are unique up to approximate conjugacy modulo the obvious K-theoretic obstructions.

The original K-theoretic formulation of the classification program for simple separable nuclear C^*-algebras has enjoyed and continues to enjoy spectacular successes [29]. However, it has had to come to terms with examples of Villadsen [96, 97], Rørdam [84], and Toms [94] that have indicated the need either to enlarge the invariant beyond K-theory and traces or to identify regularity properties that suitably restrict the class of C^*-algebras. The latter has been the subject of remarkable progress over the last several years and has led to the various notions of tracial rank zero, decomposition rank, nuclear dimension, radius of comparison, and $\mathcal{Z}$-stability, all of which we will discuss in connection with crossed products in Chapter 4.

A basic principle that has emerged is that one should look for regularity properties that are noncommutative topological expressions of zero-dimensionality. While zero-dimensionality is a rather restrictive condition for compact metrizable spaces, and indeed uniquely identifies the Cantor set under the additional assumption of no isolated points, the introduction of noncommutativity at the algebraic level produces a dimension-lowering effect to the point where in the extreme case of simple C^*-algebras one might expect higher dimensional phenomena to be the exception rather than the rule, at least if our algebras are derived by combining matrices and spaces in some asymptotic way and not by using more absolute noncommutative means like freeness. We can at least say, as a consequence of classification theory, that the class of "zero-dimensional" simple separable nuclear C^*-algebras is extremely rich. It includes crossed products of minimal homeo-

morphisms of compact metrizable spaces whose covering dimension is finite (see Section 4.4). This is a noncommutative manifestation of the fact that minimality, being the dynamical analogue of C^*-algebraic simplicity, produces the same kind of dimension lowering. The degree to which dimension is lowered under the dynamics is captured by a dimensional version of entropy called mean dimension (see Section 3.4). Entropy measures the degree to which cardinality is lowered at fixed observational scales and can thus be viewed in comparison as the "mean cardinality" of a system (see Section 3.2). Mean dimension can be nonzero for minimal homeomorphisms, and crossed products of such systems can fail to exhibit certain key regularity properties, but the precise relationship between these properties and mean dimension has yet to be worked out. This is discussed in Chapter 4.

Table 1.1: Measure-theoretic and topological finite approximation properties for discrete groups

	property of G	structure of $\mathcal{L}G$
internal:	*amenable* ($\Leftrightarrow$ not paradoxical)	*hyperfinite*
external:	*sofic*	R^ω-*embeddable*

	property of G	structure of $C_\lambda^*(G)$
internal:	*locally finite*	*AF algebra*
external:	*locally embeddable into finite groups*	*MF algebra*

The framework which lays out the broad conceptual relationships between all of the phenomena that we will encounter is presented in Tables 1.1 to 1.4. Appearing in the headings on the right sides are the group von Neumann algebra $\mathcal{L}G$, the von Neumann algebra crossed product $L^\infty(X,\mu) \rtimes G$, the reduced group C^*-algebra $C_\lambda^*(G)$, and the reduced C^*-crossed product $C(X) \rtimes_\lambda G$, the latter two of which are reviewed below. The horizontal pairing between group-theoretic or dynamical properties and properties of the corresponding von Neumann algebra or C^*-algebra are structural analogies that are known in some cases to translate at the technical level to an equivalence and in many other cases to a forward implication. In the remaining couple of cases some further massaging must be done in order to come up with a precise general statement. The categories with question marks indicate the target of current research in the C^*-classification theory of topological dynamics on higher dimensional spaces, where a definitive picture is far from being attained. While the purely infinite case has barely been

Table 1.2: Incompressible measurable and topological dynamics: finite-dimensional approximation

	property of $G \curvearrowright (X, \mu)$	structure of $L^\infty(X, \mu) \rtimes G$
internal:	*amenable* ($\Leftrightarrow G$ is amenable)	*hyperfinite*
external:	*sofic*	R^ω-*embeddable*

	property of $\mathbb{Z} \curvearrowright X$	structure of $C(X) \rtimes_\lambda \mathbb{Z}$
internal:	*clopen Rokhlin property*	*A$\mathbb{T}$ algebra*
	??? *(zero mean dimension, higher-dimensional Rokhlin property, . . .)*	*???* *($\mathcal{Z}$-stable, finite nuclear dimension, strict comparison, AH algebra, . . .)*
external:	*chain recurrence*	*MF algebra*

addressed, some remarkable progress has recently been achieved for minimal $\mathbb{Z}$-actions, although there are still many pieces of the puzzle that have yet to be put together. Notice that in Table 1.2 we have restricted ourselves to $\mathbb{Z}$-actions, as this has been the primary focus of C^*-classification and the picture becomes much hazier already for $\mathbb{Z}^2$. For minimal $\mathbb{Z}$-actions the construction of Rokhlin towers based on first return time maps has played a fundamental role in classifying crossed products, and so one might say that the successes of classification in this case are predicated on the fact that $\mathbb{Z}$ is the only nontrivial group that is both free and amenable.

Table 1.1 presents the array of local finite approximation properties for discrete groups that one obtains by taking all four cross-pairings of the categories internal/external and measure-theoretic/topological. Here we see in primal combinatorial form many of the approximation phenomena that arise in operator algebras. Note that, since we are only considering discrete groups, "topological" means the same thing here as "purely group-theoretic", while measure-theoretic properties involve basic combinatorial approximation. On the topological side one has the finite approximation properties of local finiteness (internal) and local embeddability into finite groups (external), while on the measure-theoretic side one has the respectively weaker finite approximation properties of amenability (internal) and soficity (external). This schema summarizes the organization of the main body of these notes into four chapters, each of which begins with a discussion of the corresponding finite approximation property for groups and its C^*-algebraic analogue as can be found on the right side of Table 1.1.

Table 1.2 applies the same logic to incompressible measurable and topological dynamics, with a restriction to $\mathbb{Z}$-actions in the latter case. "Incompressible" should be interpreted for measurable systems as meaning probability-measure-preserving, and more loosely for topological systems as referring to some kind of generalized recurrence. We will say little about measure-preserving systems per se, although invariant probability measures will appear in our analysis of topological systems. The bulk of Chapters 4 and 5 will be devoted to the study of internal and external incompressible structure in topological dynamics and C^*-algebras.

Table 1.3: Compressible measurable and topological dynamics: internal finite-dimensional approximation

property of $G \curvearrowright (X,\mu)$	structure of $L^\infty(X,\mu) \rtimes G$
amenable	*hyperfinite*

property of $G \curvearrowright X$	structure of $C(X) \rtimes_\lambda G$
(topologically) amenable	*nuclear*

Finally, Tables 1.3 and 1.4 treat compressible dynamics, by which we essentially mean the absence of finite G-invariant measures. Here we see that the amenable/paradoxical dichotomy for discrete groups (Theorem 2.1.2) breaks down so as to allow for the dynamical coexistence of both phenomena, whose C^*-algebraic manifestations are nuclearity and proper infiniteness. This is the main subject of Chapter 2. As evidenced by its direct connection to nuclearity, dynamical amenability is a measure-theoretic concept. On the other hand, where exactly paradoxical decomposability for topological dynamics should be situated is less obvious. In its more generous multi-level/matricial sense, paradoxicality it is directly tied to the lack of invariant measures via the Goodearl-Handelman theorem. However, it is the conventional two-into-one sense of paradoxicality that in global form translates C^*-algebraically into pure infiniteness, at least for actions on the Cantor set, and this is what is of immediate interest from the perspective of C^*-classification theory. The relation between these two degrees of paradoxicality is connected to a topological question involving perforation in the associated type semigroup (see Section 2.3). For group actions on sets without extra structure the difference between the two collapses, and this is part of the proof of the theorem of Tarski relating paradoxical decomposability and the absence of invariant measures, which is discussed prior to Lemma 2.4.3 and yields the amenable/paradoxical dichotomy for discrete groups. It is on account of Theorems 2.4.7, 2.5.3, and 2.5.5 that we have decided to treat paradoxical decomposability for topological dy-

Table 1.4: Compressible measurable and topological dynamics: paradoxicality

property of $G \curvearrowright (X, \mu)$	structure of $L^\infty(X, \mu) \rtimes G$
countable paradoxical decomposability of measurable sets ($\Leftrightarrow$ no equivalent finite invariant measure)	*purely infinite (type III)*

property of $G \curvearrowright X$	structure of $C(X) \rtimes_\lambda G$
paradoxical decomposability of clopen sets	*purely infinite*
??? *(some more general kind of paradoxical decomposability?)*	*purely infinite*

namics in Chapter 2, although Theorem 2.3.8 would perhaps more properly belong to Chapter 4 given that the problem about whether perforation can occur (Question 2.3.9) remains unresolved.

The remaining Chapter 3 is devoid of specifically C^*-algebraic concepts, as there does not seem to be anything special to say here without simply passing to a von Neumann algebra closure, where one can talk about the external finite-dimensional approximation property of hyperlinearity. We will however spend some time in this chapter exploring a couple of topological-dynamical notions that, in their most general conventional formulations, hinge on measure-theoretic finite approximation. These are entropy and mean dimension, the latter of which, as hinted above, is the subject of some tantalizing questions at the frontier of the classification program for crossed products of $\mathbb{Z}$-actions that are examined in Chapter 4.

In these notes we have restricted our attention to actions of groups, as opposed to semigroups. In the latter case the theory turns out somewhat differently [81, 23, 2]. Already by passing from $\mathbb{Z}$-actions to $\mathbb{N}$-actions, i.e., from homeomorphisms to possibly noninvertible continuous maps, one can readily encounter the kind of purely infinite behaviour that for group actions can only be achieved in the nonamenable case (see Sections 2.3 to 2.5). See [20] for some recent progress in this direction. Also, as our focus is on crossed products, we do not discuss the groupoid C^*-algebras associated to hyperbolic-type dynamics that appear in [87, 80, 91].

We round out the introduction with some basic terminology and notation. Throughout these notes G will always be a discrete group with identity element e. In applications to C^*-algebras we are mainly interested in countable G, and

we will make this assumption starting in Section 2.3. In Sections 2.1 and 2.2 we will not impose any cardinality hypothesis since the basic theory of amenability works equally well for uncountable G. The *full group C^*-algebra $C^*(G)$* of G is the completion of the group ring $\mathbb{C}G$ in the norm $\|a\| = \sup_\pi \|\pi(a)\|$ where π ranges over all *-representations on Hilbert spaces. The *reduced group C^*-algebra $C^*_\lambda(G)$* is the norm closure of $\mathbb{C}G$ in the left regular representation on $\ell^2(G)$. The full group C^*-algebra has the universal property that for every unitary representation $u\colon G \to \mathcal{B}(\mathcal{H})$ there is a unique *-homomorphism $\pi\colon C^*(G) \to \mathcal{B}(\mathcal{H})$ such that $\pi(s) = u_s$ for all $s \in G$. When G is amenable, the reduced and full group C^*-algebras coincide canonically.

Actions of G on a compact Hausdorff space are always assumed to be continuous. Let $G \curvearrowright X$ be such an action. Write α for the induced action on $C(X)$, i.e., $\alpha_s(f)(x) = f(s^{-1}x)$. Denote by $C_c(G, C(X))$ the space of finitely supported functions on G with values in $C(X)$. This is a *-algebra with the convolution product

$$\left(\sum_{s \in G} a_s s \right) * \left(\sum_{s \in G} b_s s \right) = \sum_{s,t \in G} a_s \alpha_s(b_t) st$$

and involution

$$\left(\sum_{s \in G} a_s s \right)^* = \sum_{s \in G} \alpha_{s^{-1}}(a_s^*) s^{-1}.$$

The *-representations of $C_c(G, C(X))$ on Hilbert spaces correspond to *covariant representations* $(u, \pi, \mathcal{H})$, which consist of a unitary representation $u\colon G \to \mathcal{B}(\mathcal{H})$ and a representation $\pi\colon C(X) \to \mathcal{B}(\mathcal{H})$ such that $u_s \pi(f) u_s^* = \pi(\alpha_s(f))$ for all $f \in C(X)$ and $s \in G$. The *full crossed product $C^*(X) \rtimes G$* is the completion of $C_c(G, C(X))$ in the norm $\|a\| = \sup_\pi \|\pi(a)\|$ where π ranges over all *-representations of $C_c(G, C(X))$ on Hilbert spaces. To define the reduced crossed product, start with a faithful representation $C(X) \subseteq \mathcal{B}(\mathcal{H})$ and amplify this to the representation $\pi\colon C(X) \to \mathcal{B}(\mathcal{H} \otimes \ell^2(G))$ given by

$$\pi(f)\xi \otimes \delta_s = (\alpha_{s^{-1}}(f)\xi) \otimes \delta_s$$

where $\{\delta_s\}_{s \in G}$ is the canonical orthonormal basis of $\ell^2(G)$. Along with the amplification $\mathrm{id} \otimes \lambda$ on $\mathcal{H} \otimes \ell^2(G)$ of the left regular representation of G, this yields a covariant representation, and the *reduced crossed product $C(X) \rtimes_\lambda G$* is the norm closure of $C_c(G, C(X))$ under the resulting *-representation. This can be seen not to depend on the initial faithful representation of $C(X)$. There is a canonical conditional expectation $C(X) \rtimes_\lambda G \to C(X)$ which on elements of $C_c(G, C(X))$ is evaluation at e. As for group C^*-algebras, when G is amenable the reduced and full crossed products canonically coincide.

The action $G \curvearrowright X$ is *topologically free* if the set of all $x \in X$ such that $sx \neq x$ for every $s \in G \setminus \{e\}$ is dense in X. The action is *minimal* if there are no nonempty proper closed G-invariant subsets of X, which is equivalent to every G-orbit being dense. Minimal topologically free actions will be our primary focus,

as they connect to the classification theory of simple nuclear C^*-algebras via the following specialization to commutative dynamics of a result from [5], which will be used frequently.

Theorem 1.0.1. *Suppose that the action $G \curvearrowright X$ is minimal and topologically free. Then $C(X) \rtimes_\lambda G$ is simple.*

The rough strategy of the proof in [5] is to take a nonzero closed ideal I in $C(X) \rtimes_\lambda G$ and argue using the canonical expectation $C(X) \rtimes_\lambda G \to C(X)$ and topological freeness that $I \cap C(X)$ is a nonzero G-invariant ideal of $C(X)$, which by minimality must be all of $C(X)$, so that $I = C(X) \rtimes_\lambda G$.

The study of more general actions and closed ideals in reduced crossed products raises a number of interesting and subtle questions which we will not address (see for example [88]).

Finally, we direct the reader to the book [19] by Brown and Ozawa for a reference on groups, group actions, and crossed products in the framework of finite-dimensional approximation, and to the book [8] by Blackadar for a general reference on operator algebras. For background on the classification program for separable nuclear C^*-algebras see [83], and for a recent survey see [29]. We also mention the article [7] of Blackadar on the "algebraization of dynamics", which has some overlap with the present notes but focuses to a greater extent on operator algebras.

Chapter 2

Internal measure-theoretic phenomena

2.1 Amenable groups and nuclear C^*-algebras

The notion of amenability in its most basic combinatorial sense captures the idea of internal finite approximation from a measure-theoretic perspective. It plays a pivotal role not only in combinatorial and geometric group theory but also in the theory of operator algebras through its various linear manifestations like hyperfiniteness, semidiscreteness, injectivity, and nuclearity. In this section we will review the theory of amenability for discrete groups (see [36, 75] for general references), and then move in Section 2.2 to amenable actions and their reduced crossed products.

A discrete group G is said to be *amenable* if on $\ell^\infty(G)$ there exists a left invariant mean, i.e., a state σ satisfying $s \cdot \sigma = \sigma$ for all $s \in G$ where $(s \cdot \sigma)(t) = \sigma(s^{-1}t)$ for all $t \in G$. By Gelfand theory, left invariant means on $\ell^\infty(G)$ correspond to invariant Borel probability measures for the associated action of G on the spectrum of $\ell^\infty(G)$, which is the Stone-Čech compactification βG of G. It follows that G is amenable if and only if every action of G on a compact Hausdorff space X admits a G-invariant Borel probability measure, since every invariant Borel probability measure on βG can be pushed forward to one on any compact Hausdorff space X on which G acts by selecting a point $x \in X$ and applying the continuous equivariant map $\beta G \to X$ sending s to sx, which exists by the universal property of βG. Amenability for groups is thus an inherently dynamical concept.

From the combinatorial viewpoint, amenability can be expressed by the Følner property, which is the existence of a net $\{F_i\}$ of nonempty finite sets such that $\lim_i |sF_i \cap F_i|/|F_i| = 1$ for all $s \in G$. Given such a Følner net $\{F_i\}$ one can produce a left invariant mean on $\ell^\infty(G)$ by taking any weak* cluster point of

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the sequence of normalized characteristic functions $|F_i|^{-1}1_{F_i}$ viewed as states on $\ell^\infty(G)$ via the embedding $\ell^1(G) \hookrightarrow \ell^1(G)^{**} = \ell^\infty(G)^*$. The converse operation of teasing out approximate finiteness from simple invariance is trickier. Starting with a left invariant mean σ on $\ell^\infty(G)$, an application of the Hahn-Banach separation theorem shows that the set $P(G)$ of positive norm-one functions in $\ell^1(G)$ is weak* dense when viewed as a subset of the state space of $\ell^\infty(G)$ via duality. It follows that there is a net $\{g_i\}$ in $P(G)$ converging weak* to σ, which means that, for a given nonempty finite set $F \subseteq G$, the net $\{(s \cdot g_i - g_i)_{s \in F}\}$ converges weakly to zero in $\ell^1(G)^F$ (Day's trick). Since the norm and weak closures of a convex subset of a Banach space coincide by Mazur's theorem, there is a net $\{h_i\}$ of convex combinations of the functions g_n such that $\{(s \cdot h_i - h_i)_{s \in F}\}$ converges to zero in norm. Using the directed set of finite subsets of G we can thereby construct a net $\{k_i\}$ in $P(G)$ such that $\|s \cdot k_i - k_i\|_1 \to 0$ for every $s \in G$ (Reiter's property). To finish one can perturb the functions k_i so as to have finite support and then show by a summation argument that for each h_i there is some $r_i > 0$ so that the sets $\{s \in G : h_i(s) > r_i\}$ are approximately invariant to an asymptotically vanishing degree for each $s \in G$ in accordance with the definition of the Følner property. One can now see for example that all Abelian groups are amenable since the Følner property clearly holds in view of the structure theorem for finitely generated Abelian groups.

The above ideas also play a key role in the much more difficult proof of Connes' analogous result for von Neumann algebras with separable predual that says that injectivity (the existence of a norm-one projection from $\mathcal{B}(\mathcal{H})$ onto the algebra as acting on the Hilbert space $\mathcal{H}$) is equivalent to hyperfiniteness (the existence of an increasing sequence of finte-dimensional *-subalgebras with ultraweakly dense union). In this setting the term amenability itself refers to the equivalent cohomological property that every bounded derivation from the von Neumann algebra to a normal dual Banach bimodule over the algebra is inner. Likewise, a C^*-algebra A is said to be amenable if every bounded derivation from A to a dual Banach module A-bimodule. By work of Connes, Choi-Effros, Kirchberg, and Haagerup, this is equivalent to each of the following conditions (see Section IV.3 of [8]):

(i) there is a unique C^*-tensor norm on $A \otimes B$ for every C^*-algebra B,

(ii) the enveloping von Neumann algebra A^{**} is amenable,

(iii) for every representation $\pi\colon A \to \mathcal{B}(\mathcal{H})$ the von Neumann algebra $\pi(A)''$ is amenable,

(iv) there is a net of contractive complete positive maps $\varphi_n\colon A \to M_{k_n}$ and $\psi_n\colon M_{k_n} \to A$ through matrix algebras such that $\|\psi_n \circ \varphi_n(a) - a\| \to 0$ for all $a \in A$.

If A is unital then the maps in (iv) may be taken to be unital, which together with complete positivity automatically implies contractivity. Property (i) is called

nuclearity and property (iv) the *completely positive approximation property*. Nuclearity tends to be the preferred term for this class of C^*-algebras, and we will adhere to this convention. While C^*-algebras as such are to be thought of as topological objects (as opposed to von Neumann algebras, which, in addition to being C^*-algebras, have measure-theoretic structure), nuclearity is a measure-theoretic property. This evident in the fact that nuclearity can be expressed in terms of the von Neumann algebra A^{**} according to condition (ii). It is also reflected in the fact that in (iv) the maps are not required to interact in any way with the multiplication in the C^*-algebra, which is where the topological structure resides. So the following result should not be too surprising (see Section 2.6 of [19] and Section IV.3.5 of [8]).

Theorem 2.1.1. *G is amenable if and only if $C^*_\lambda(G)$ is nuclear.*

Proof. To show that if G is amenable then $C^*_\lambda(G)$ is nuclear, one can verify the completely positive approximation property by starting with a Følner net $\{F_i\}$ for G and for each n defining $\varphi_i \colon C^*_\lambda(G) \to M_{F_i}$ to be the cut-down to $\mathcal{B}(\ell^2(F_i)) \subseteq \mathcal{B}(\ell^2(G))$ as identified with M_{F_i}, and $\psi_i \colon M_{F_i} \to C^*_\lambda(G)$ to be the map defined on matrix units by $e_{s,t} \mapsto |F_i|^{-1}\lambda_{st^{-1}}$. In the converse direction, suppose that we have a net of unital completely positive maps $\varphi_i \colon C^*_\lambda(G) \to M_{F_i}$ and $\psi_i \colon M_{F_i} \to C^*_\lambda(G)$ such that $\|\psi_i \circ \varphi_i(a) - a\| \to 0$ for all $a \in C^*_\lambda(G)$. By Arveson's extension theorem we can extend φ_i to a unital completely positive map $\tilde{\varphi}_i \colon \mathcal{B}(\ell^2(G)) \to M_{k_i}$. Take a point-ultraweak cluster point γ of the net $\{\psi \circ \tilde{\varphi}_i\}$. Then γ is a unital completely positive map to the von Neumann algebra $C^*_\lambda(G)''$ which restricts to the identity on $C^*_\lambda(G)$. Denoting by τ the tracial state $a \mapsto \langle a\delta_e, \delta_e \rangle$ on $C^*_\lambda(G)''$, one then checks that the restriction of $\tau \circ \gamma$ to $\ell^\infty(G) \subseteq \mathcal{B}(\ell^2(G))$ is a left invariant mean using the fact that $C^*_\lambda(G)$ lies in the multiplicative domain of γ. $\qquad\square$

Note that in the forward direction of the above proof we don't need the full combinatorial power of the Følner property. It would be enough to use the fact that amenability is equivalent to Reiter's property, which, as mentioned above, asserts the existence of a net of functions $\{h_i\}$ in $P(G)$ such that $\|s \cdot h_i - h_i\|_1 \to 0$. By a simple perturbation argument we may assume that the support F_i of k_i is finite for each i, and then define the map $\psi_i \colon M_{F_i} \to C^*_\lambda(G)$ above instead by $e_{s,t} \mapsto \sqrt{h_n}\lambda_{st^{-1}}$. We mention in connection with this that Reiter's property is equivalent to its ℓ^2 version, i.e., the existence of a net of functions $\{h_i\}$ in the unit ball of $\ell^2(G)$ such that $\|s \cdot h_i - h_i\|_2 \to 0$.

It is also the case that G is amenable if and only if the group von Neumann algebra $\mathcal{L}G = C^*_\lambda(G)'' \subseteq \mathcal{B}(\ell^2(G))$ is amenable. We remark that Theorem 2.1.1 fails in the nondiscrete setting, as $C^*_\lambda(G)$ is nuclear for every separable connected locally compact group G. However, $C^*(G) = C^*_\lambda(G)$ is equivalent to amenability for all locally compact groups G.

We infer from this discussion that one cannot expect to say anything very general about the topological structure of nuclear C^*-algebras. A large part of the classification program for nuclear C^*-algebras attempts to do precisely that, at

least in its execution, and what is surprising is the success it has achieved in a great many cases. On the other hand, all but the topologically simplest classifiable C^*-algebras require the incorporation of traces, and hence measure theory, into the classifying invariant.

Returning our focus to groups, let us now discuss the prevalence of amenability and the conditions under which it fails to hold. As mentioned above, Abelian group are amenable, and amenability is closed under taking subgroups, extensions, and increasing unions. The groups that can be generated by these operations starting with the cyclic groups are called the *elementary amenable groups*. They do not exhaust all amenable groups, as the Grigorchuk group demonstrates. What distinguishes the Grigorchuk group from the elementary amenable groups is that the growth as $n \to \infty$ of the number of distinct words of length n with respect to a finite generating set is neither polynomial nor exponential. See Chapter VIII of [45].

The prototype of a nonamenable group is the free group F_2 on two generators. The lack of an left invariant mean on $\ell^\infty(F_2)$ is an immediate consequence of the fact that F_2 admits a *paradoxical decomposition*, which for a group G means a partition $\{C_1, \ldots, C_n, D_1, \ldots, D_m\}$ of G and elements $s_1, \ldots, s_n, t_1, \ldots, t_m \in G$ such that both $\{s_1 C_1, \ldots, s_n C_n\}$ and $\{t_1 D_1, \ldots, t_m D_m\}$ are partitions of G. For F_2 with generators a and b one can take $C_1 = W(a)$, $C_2 = W(a^{-1})$, $D_1 = W(b) \cup \{e, b^{-1}, b^{-2}, \ldots\}$, and $D_2 = W(b^{-1}) \setminus \{b^{-1}, b^{-2}, \ldots\}$, where $W(\cdot)$ denotes the set of all words beginning with the indicated element, and verify that $\{C_1, aC_2\}$ and $\{D_1, bD_2\}$ are both partitions of F_2. Whether a group G is nonamenable if and only if it contains a copy of F_2 was an open problem for many years following the introduction of amenability by von Neumann. It is true for groups of matrices by a result of Tits, but was shown to be false in general by Olshanskii, who constructed nonamenable torsion groups. Nevertheless, we have the following remarkable theorem of Tarski, which establishes a dichotomy between amenability and paradoxical decomposability.

Theorem 2.1.2. *G is amenable if and only if it does not admit a paradoxical decomposition.*

The *Tarski number* of a group G is defined as the smallest possible number of pieces in a paradoxical decomposition of G, or infinity if no paradoxical decomposition exists. It then turns out that G contains F_2 if and only if its Tarski number is is the smallest possible value, namely 4.

For an extensive account of the subject of paradoxicality see the book [102].

Paradoxical decomposability has both local and global C^*-algebra analogues, namely proper and pure infiniteness, which we will explore in Sections 2.3 through 2.5. The novelty in the dynamical setting is that amenability and paradoxical decomposability can coexist, so that the Tarski dichotomy no longer holds.

2.2   Amenable actions and nuclear and exact C^*-algebras

Motivated by Zimmer's notion of amenability for measurable dynamical systems, Anantharaman-Delaroche introduced the following topological analogue [1, 3, 4]. As in the previous section, we write $P(G)$ for the set of all probability measures on G, which we identify with the positive functions in $\ell^1(G)$ of norm one.

Definition 2.2.1. An action of G on a compact Hausdorff space X is said to be *amenable* if there is a net of continuous maps m_i from X to $P(G)$ with the weak* topology such that $\|m_i^{sx} - s \cdot m_i^x\|_1 \to 0$ for all $s \in G$.

    Note that the amenability of a group G is equivalent to the amenability of its action on a singleton, which simply reduces to Reiter's property. Also, every action of an amenable group is amenable, as one can take constant maps m_i whose images are elements of $P(G)$ which are asymptotically invariant in the sense of Reiter's property.

    It is possible for nonamenable groups to admit an amenable action. The prototype for this is the action of the free group F_2 on its Gromov boundary ∂G, which is the Cantor set consisting of all infinite reduced words $x_1 x_2 \cdots$ in the generators a and b and their inverses, equipped with the relative product topology as a subset of $\{a, b, a^{-1}, b^{-1}\}^{\mathbb{N}}$. The action is defined by left concatenation followed by cancellation as necessary. For each $n \in \mathbb{N}$ one defines the map $m_n \colon \partial G \to P(G)$ by sending a reduced infinite word $x_1 x_2 \cdots$ to $n^{-1} \sum_{k=1}^{n} \delta_{x_1 \cdots x_k}$. It is then easy to see that $\|m_n^{sx} - s \cdot m_n^x\|_2 \to 0$ for all $s \in G$. More generally, every Gromov hyperbolic group acts amenably on its Gromov boundary (see Section 5.3 of [19]). The existence of groups which admit no amenable action leads to the subject of exactness, which we will treat below.

    We will next reformulate amenability for actions from the C^*-algebra perspective as a generalization of the 2-norm version of Reiter's property for groups and use this to establish the theorem below from [1] (see also Sections 4.3 and 4.4. of [19]). For an action $G \curvearrowright X$ we can view the convolution algebra $C_c(G, C(X))$ as a pre-Hilbert $C(X)$-module with the $C(X)$-valued inner product $\langle S, T \rangle = \sum_{s \in G} S(s)^* T(s)$ and norm $\|S\|_A = \|\langle S, S \rangle\|^{1/2}$.

Proposition 2.2.2. *An action $G \curvearrowright X$ on a compact Hausdorff space is amenable if and only if there is a net of functions $T_i \in C_c(G, C(X))$ such that (i) for each i one has $T_i(s) \geq 0$ for all $s \in G$ and $\langle T_i, T_i \rangle = 1$, and (ii) $\|\delta_s * T_i - T_i\|_A \to 0$ for all $s \in G$, where δ_s is the function in $C_c(G, C(X))$ taking value 1 at s and zero otherwise.*

Theorem 2.2.3. *An action $G \curvearrowright X$ on a compact Hausdorff space is amenable if and only if $C(X) \rtimes_\lambda G$ is nuclear.*

Proof. We denote by α the induced action on $C(X)$, i.e., $\alpha_s(f)(x) = f(s^{-1}x)$. Suppose first that the action is amenable. Take a net of functions $T_i \in C_c(G, C(X))$

as given by Proposition 2.2.2. Write D_i for the support of T_i. For each i define the unital completely positive compression map $\varphi_i \colon C(X) \rtimes_\lambda G \to C(X) \otimes M_{D_i}$ by

$$\varphi_i(f u_t) = \sum_{s \in D_i \cap t D_i} \alpha_s^{-1}(f) \otimes e_{s, t^{-1} s}.$$

Write R_i for the self-adjoint element $\sum_{s \in D} \alpha_s^{-1}(T_i(s)) \otimes e_{s,s}$. Define $\psi_i \colon C(X) \otimes M_{D_i} \to C(X) \rtimes_\lambda G$ by composing the unital completely positive cut-down $a \mapsto R_i a R_i$ from $C(X) \otimes M_{D_i}$ to itself with the unital map from $C(X) \otimes M_{D_i} \to C(X) \rtimes_\lambda G$ given by $\psi_i(f \otimes e_{s,t}) = \alpha_s(f) u_{st^{-1}}$, which is readily checked to be completely positive. A short computation then shows that $\psi_i \circ \varphi_i(f u_s) = (T_i * T_i^*(s)) f u_s$, from which we deduce that $\|\psi_i \circ \varphi_i(f u_s) - f u_s\| \to 0$ for all $f \in C(X)$ and $s \in F$. Since $C(X) \otimes M_D$ is nuclear, we can then find $n_i \in \mathbb{N}$ and unital completely positive maps $\theta_i \colon C(X) \otimes M_{D_i} \to M_{n_i}$ and $\gamma_i \colon M_{n_i} \to C(X) \otimes M_{D_i}$ such that $\|\psi_i \circ \gamma_i \circ \theta_i \circ \varphi_i(f u_s) - f u_s\| \to 0$ for all $f \in C(X)$ and $s \in G$. This verifies the complete positive approximation approximation property and hence the nuclearity of $C(X) \rtimes_\lambda G$.

For the converse direction, suppose that $C(X) \rtimes_\lambda G$ is nuclear. Let F be a finite subset of G and $\varepsilon > 0$. Then there are unital completely positive maps $\varphi \colon C(X) \rtimes_\lambda G \to M_n$ and $\psi \colon M_n \to C(X) \rtimes_\lambda G$ such that $\|\psi \circ \varphi(u_s) - u_s\| < \varepsilon$ for all $s \in F$. We would like the map φ to send u_s to zero for all s outside of a finite set, and this can be arranged as follows. Let π be the representation of $C(X) \rtimes_\lambda G$ on $\mathcal{H} \otimes \ell^2(G) \otimes \ell^2(\mathbb{N})$ which is the countably infinite amplication of the canonical representation on $\mathcal{H} \otimes \ell^2(G)$. Then π is essential, and thus, by the matricial version of Glimm's lemma, given any finite set $\Omega \subseteq C(X) \rtimes_\lambda G$ and $\varepsilon > 0$ there is an isometry $V \colon \ell_n^2 \to \mathcal{H} \otimes \ell^2(G) \otimes \ell^2(\mathbb{N})$ such that the unital completely positive map $\varphi' \colon C(X) \rtimes_\lambda G \to M_n$ given by $a \mapsto V^* \pi(a) V$ satisfies $\|\varphi'(a) - \varphi(a)\| < \varepsilon$ for all $a \in \Omega$. By a perturbation we may assume that the image of V lies in $\mathcal{H} \otimes \ell^2(\tilde{F}) \otimes \ell^2(\mathbb{N})$ for some finite set $\tilde{F} \subseteq G$, which has the effect that $\varphi(u_s) = 0$ for all $s \notin \tilde{F} \tilde{F}^{-1}$, as desired.

Now define $h \in C_c(G, C(X))$ by $h(s) = E((\psi \circ \varphi(u_s)) u_s^{-1})$ where $E \colon C(X) \rtimes_\lambda G \to C(X)$ is the canonical conditional expectation. Then

$$\|1 - h(s)\| = \|E((u_s - \psi \circ \varphi(u_s)) u_s^{-1})\| \leq \|u_s - \psi \circ \varphi(u_s)\| < \varepsilon.$$

One can also verify using the G-equivariance of E that h is a positive-type function in the sense that for any $s_1, \ldots, s_n \in G$ the element $[\alpha_{s_i}^{-1}(h(s_i s_j^{-1}))]_{i,j}$ of $M_n(C(X))$ is positive. Viewing h as an element of $C(X) \rtimes_\lambda G$, it follows that for every finite set $D \subseteq g$ the cut-down of h by the orthogonal projection onto $\mathcal{H} \otimes \ell^2(D)$ in $\mathcal{H} \otimes \ell^2(G)$ is positive, which implies that h itself is positive as an element of $C(X) \rtimes_\lambda G$. It follows that there is a $g \in C_c(G, C(X))$ such that $\|g^* * g - h\| < \varepsilon$. We may assume that $E(g^* * g) = 1$ by replacing g with $g E(g^* * g)^{1/2}$ and recalibrating ε. Now define $T \in C_c(G, C(X))$ by $T(s)(x) = |g(s^{-1} x, s^{-1})|$. Then, as is readily checked, for each $s \in G$ we have $\langle T, \delta_s * T \rangle(x) = |(g^* * g)(s)(x)|$ for all $x \in X$, and

in particular $\langle T, T \rangle = 1$. Furthermore, for $s \in F$ we have

$$\|\delta_s * T - T\|_2^2 = \|2 - \langle T, \delta_s * T \rangle - \langle \delta_s * T, T \rangle\| \leq 2\|h(s) - 1\| < 2\varepsilon$$

We conclude by Proposition 2.2.2 that the action is amenable. $\qquad\square$

The fact that a nonamenable group can admit an amenable action crossed product brings us to the notions of nuclearly embeddability and exactness. A C^*-algebra A is said to be *nuclearly embeddable* if there is a C^*-algebra D and an injective *-homomorphism $\iota \colon A \to D$ which is nuclear in the sense that there exist a net of contractive completely positive contractive linear maps $\varphi_n \colon A \to M_{k_n}$ and $\psi_n \colon M_{k_n} \to D$ through matrix algebras such that $\|\psi_n \circ \varphi_n(a) - a\| \to 0$ for all $a \in A$. In particular, C^*-subalgebras of nuclear C^*-algebras are nuclearly embeddable. By a deep theorem of Kirchberg, nuclear embeddability for a C^*-algebra A is equivalent to *exactness*, which means that for every C^*-algebra and closed two-sided ideal J in B the sequence

$$0 \to J \otimes A \to B \otimes A \to (B/J) \otimes A \to 0$$

is exact. Another deep theorem of Kirchberg asserts that every separable exact C^*-algebra arises as a C^*-subalgebra of a nuclear C^*-algebra, which moreover can always be taken to be the Cuntz algebra O_2.

Given an amenable action $G \curvearrowright X$, the crossed product $C(X) \rtimes_\lambda G$ is nuclear by the above theorem, and so $C_\lambda^*(G)$, as a C^*-subalgebra of $C(X) \rtimes_\lambda G$, is nuclearly embeddable. In fact we have the following equivalences from [3].

Theorem 2.2.4. *The following are equivalent.*

1. *G is exact,*

2. *the action of G on βG is amenable,*

3. *G admits an amenable action on a compact metrizable space,*

4. *for every action $G \curvearrowright X$ the crossed product $C(X) \rtimes_\lambda G$ is exact.*

We thus see that, from the viewpoint of both dynamics and general C^*-algebras, exactness, like amenability, should be thought of as a measure-theoretic property. In particular, exactness sees nothing in the dynamics that is separate from the exactness of the group itself.

2.3 The type semigroup, invariant measures, and pure infiniteness

The following three sections are primarily based on the work of Rørdam and Sierakowski [86]. We will however take a slightly different approach in our treatment of simple purely infinite crossed products which will allow us to obtain some

extra information concerning minimal actions of groups which are not necessarily exact, including a dichotomy for the reduced crossed products of universal minimal actions in Section 2.4. We will also concentrate exclusively on minimal topologically free actions, for which the associated reduced crossed product is simple. Thus we do not have to worry about ideals, which are handled by Rørdam and Sierakowski in the nonsimple case under an exactness assumption, which permits one to relate closed ideals in the reduced crossed product to closed invariant sets. Consistent with this more concentrated scope, we give a proof of Theorem 2.5.5 that uses exactness only to ensure nuclearity of the reduced crossed product, which enables us at the same time to establish Theorem 2.5.3. The key to doing this is to work with the universal minimal system instead of the action $G \curvearrowright \beta G$, so that we do not have to negotiate ideals.

In the translation from groups to C^*-algebras, paradoxical decomposability becomes more of a dynamical concept compared to the existence of a left invariant mean. While $C^*_\lambda(G)$ detects group-theoretic amenability by way of invariant means and the completely positive approximation property, for nonamenable G one cannot see paradoxicality in a direct way by looking at $C^*_\lambda(G)$, which is always stably finite due to the presence of the faithful tracial state given by $a \mapsto \langle a\delta_e, \delta_e \rangle$ in the canonical representation on $\ell^2(G)$. The existence of a left invariant mean can be detected C^*-algebraically either according to its definition as a state $\ell^\infty(G)$ which is invariant under the left translation, or dually via the nuclearity of $C^*_\lambda(G)$. To find the C^*-algebraic manifestation of paradoxical behaviour one must look instead inside the fusion of these two dual objects, namely the crossed product $\ell^\infty(G) \rtimes_\lambda G$. Note that $\ell^\infty(G) \rtimes_\lambda G$ contains all reduced crossed products by actions of G on compact Hausdorff spaces possessing a dense orbit. This follows from the fact that every such system $G \curvearrowright X$ can be realized as a factor of $G \curvearrowright \beta G$ by picking an $x \in X$ with dense orbit and using the universal property of the Stone Čech compactification to extend the map $G \to X$ given by $s \mapsto sx$ to βG.

Thus to study paradoxical decomposability as it is reflected in the C^*-algebraic notions of proper and pure infiniteness we must work in the context of the dynamics of G acting on compact Hausdorff spaces, and not in $C^*_\lambda(G)$ as in the case of the original invariant mean formulation of amenability. This has the interesting effect that, whereas amenability and paradoxical decomposability for groups are mutually exclusive, pure infiniteness can coexist with nuclearity if one considers crossed products of actions instead of simply $C^*_\lambda(G)$.

The analogue of paradoxical decomposability for nonzero positive elements of a C^*-algebra is proper infiniteness. For positive elements a, b in a C^*-algebra A we write $a \precsim b$ if there is a sequence $\{t_k\}_k$ in A such that $\lim_{k \to \infty} t_k^* b t_k = a$. A nonzero positive element a in A is said to be *infinite* if $a \oplus b \precsim a$ for some nonzero positive $b \in A$, and *properly infinite* if $a \oplus a \precsim a$. A unital C^*-algebra A is said to be *infinite* if the projection 1_A is infinite, and *properly infinite* if 1_A is properly infinite. Given an infinite projection p in a C^*-algebra A there exists a sequence of nonzero mutually orthogonal subprojections of p, as we can take a partial isometry $v \in A$ satisfying $v^*v = p$ and $vv^* \lneq p$ and set $p_1 = p - vv^*$ and $p_n = v p_{n-1} v^*$ for

$n \geq 2$. The converse is true for the projection 1_A in a simple unital C^*-algebra A. Moreover when A is simple and unital the following are equivalent:

1. A is infinite,

2. A is properly infinite,

3. there is a sequence of nonzero mutually orthogonal and mutually equivalent projections in A,

4. there is a sequence of mutually orthogonal projections in A which are all equivalent to 1_A.

A C^*-algebra A is *purely infinite* if it has no one-dimensional quotients and $a \precsim b$ for all positive elements $a, b \in A$ such that $a \in \overline{AbA}$. If every nonzero hereditary C^*-subalgebra of a C^*-algebra A contains an infinite positive element, then A is purely infinite (it is unknown if the converse is true). For a simple unital C^*-algebra A not isomorphic to $\mathbb{C}$ the following are equvalent:

1. A is purely infinite,

2. for every nonzero $a \in A$ there exist $x, y \in A$ such that $xay = 1$,

3. every nonzero hereditary C^*-algebra of A contains an infinite projection.

In particular, simple unital purely infinite C^*-algebras contain many projections. The C^*-algebra A is purely infinite if and only if each of its nonzero positive elements is properly infinite [57]. In the case that A is simple, this is equivalent to every projection p in A being properly infinite in the sense that $p \oplus p$ is Murray-von Neumann subequivalent to p.

As paradoxical decomposability for groups is a measure-combinatorial concept, one might regard pure infiniteness for C^*-algebras in a similar way. Indeed simple purely infinite C^*-algebras tend to behave more like measure-theoretic or even combinatorial objects: they all have real rank zero, and in the case that the algebras are separable, nuclear, and satisfy the universal coefficient theorem there is the classification of Kirchberg and Phillips [56, 77] that is in terms of K-theory alone (in contrast to the stably finite setting, where traces are needed in the invariant to handle the topological phenomena which appear there). On the other hand, there are topological obstructions which prevent one from making general statements equating tracelessness with pure infiniteness in the spirit of Tarski's dichotomy for groups. See Section 9 of [58] and [84]. Rørdam constructed in [84] a simple separable nuclear C^*-algebra in the UCT class containing both an infinite and a nonzero finite projection. The ordered K_0 group of such a C^*-algebra is perforated, and we will see in Theorem 2.3.8 how perforation plays a role in our dynamical context.

Such pathologies are inherently topological and do not occur in von Neumann algebras. A von Neumann algebra decomposes as a direct sum of a finite part, a

nonfinite semifinite part, and a purely infinite (or type III) part. The finite part splits into type I and II_1 parts, and the nonfinite semifinite part splits into type I and II_∞ parts. Recall that a factor is a von Neumann algebra with trivial centre, and that every von Neumann algebra with separable predual can be written as a direct integral of factors. Since a factor is indecomposable it is either (i) finite, in which case it is either type I_n (i.e., isomorphic to M_n) for some $n \in \mathbb{N}$ or type II_1, (ii) nonfinite and semifinite, in which case it is either type I_∞ (i.e., isomorphic to $\mathcal{B}(\mathcal{H})$ for some infinite-dimensional Hilbert space $\mathcal{H}$) or type II_∞, or (iii) purely infinite/type III. Finite factors admit a unique normal tracial state, while nonfinite semifinite factors admit a unique semifinite normal tracial state. Thus the invariant mean/paradoxical decomposability dichotomy for groups translates in a direct way to von Neumann algebras. One underlying reason for this is that one can perform countable cutting and pasting operations with projections, which is not possible in a general C^*-algebra. This can be seen in a prototypical way in Murray and von Neumann's construction of the trace on a II_1 factor.

Murray and von Neumann's method was used by Nadkarni in [72] to show that a Borel automorphism of a standard Borel space X admits an invariant Borel probability measure if and only if X is not compressible in the following sense. For a Borel action of a countable group G on Borel space X we say that two Borel sets $A, B \subseteq X$ are *countably equidecomposable* and write $A \sim_\infty B$ if there is a countable Borel partition $\{A_i\}_{i=1}^\infty$ of A, a countable Borel partition $\{B_i\}_{i=1}^\infty$ of B, and a sequence $\{s_i\}_{i=1}^\infty$ in G such that $s_i A_i = B_i$ for every $i \in \mathbb{N}$. We then say that X is *compressible* if there exist disjoint Borel subsets A and B of X such that $A \sim_\infty X$ and every G-orbit meets B. Moreover, we say that X is *countably G-paradoxical* if there exist disjoint Borel subsets A and B of X such that $A \sim_\infty B \sim_\infty X$. Compressibility and countable G-paradoxicality are the Borel action analogues of infinite and properly infinite projections.

Becker and Kechris observed in [6] that Nadkarni's argument can be applied to obtain the same conclusion in the more general setting of Borel actions of countable groups on Borel spaces, and in fact even more generally for countable equivalence relations on a standard Borel space, in which the notion of invariant measure and compressibility still make sense. Building on this result Becker and Kechris then proved the following, which applies in particular to actions of countable groups on compact Hausdorff spaces.

Theorem 2.3.1. *Let $G \curvearrowright X$ be a Borel action of a countable group on a standard Borel space. Then there is a G-invariant Borel probability measure on X if and only if X is not countably G-paradoxical.*

Thus the invariant mean/paradoxical decomposability dichotomy persists here as in the von Neumann algebra setting. In the topological context of group actions on compact spaces and their crossed products, however, one cannot perform the same kind of countable cutting and pasting operations without being forced to pass to the Borel structure. In the case that most resembles measure theory, namely that of zero-dimensional spaces, we expect clopen sets to play the role of

measurable sets in the analysis of paradoxical decomposability and so we will be forced to work with finite partitions. This will mean that we must consider the kind of multilevel/matricial version of paradoxicality that appears in the proof of Tarski's theorem via the type semigroup but collapses there into the basic form of paradoxicality by virtue of an axiom of choice argument. In the topological setting we cannot appeal to the axiom of choice to form partitions and we thus run into the phenomenon of perforation in ordered semigroups. This is what we turn to next.

Definition 2.3.2. Suppose that G acts on a set X. Let $\mathscr{S}$ be a collection of subsets of X. Let k and l be integers with $k > l > 0$. We say that a set $A \subseteq X$ is $(G, \mathscr{S}, k, l)$-*paradoxical* (or simply $(G, \mathscr{S})$-*paradoxical* when $k = 2$ and $l = 1$) if there exist $A_1, \ldots, A_n \in \mathscr{S}$ and $s_1, \ldots, s_n \in G$ such that $\sum_{i=1}^{n} 1_{A_i} \geq k \cdot 1_A$ and $\sum_{i=1}^{n} 1_{s_i A_i} \leq l \cdot 1_A$. The set A is said to be *completely* $(G, \mathscr{S})$-*nonparadoxical* if it fails to be $(G, \mathscr{S}, k, l)$-paradoxical for all integers $k > l > 0$.

Remark 2.3.3. Suppose that $\mathscr{S}$ is actually a subalgebra of the power set $\mathscr{P}_X$, which will always be the case in our applications. Then we can express the $(G, \mathscr{S}, k, l)$-paradoxicality of a set A in $\mathscr{S}$ by partitioning copies of A instead of merely counting multiplicities. More precisely, A is $(G, \mathscr{S}, k, l)$-paradoxical if and only if for each $i = 1, \ldots, k$ there exist an $n_i \in \mathbb{N}$ and $A_{i,1}, \ldots, A_{i,n_i} \in \mathscr{S}$, $s_{i,1}, \ldots, s_{i,n_i} \in G$, and $m_{1,1}, \ldots, m_{i,n_i} \in \{1, \ldots, l\}$ so that $\bigcup_{j=1}^{n_i} A_{i,j} = A$ for each $i = 1, \ldots, k$ and the sets $s_{i,j} A_{i,j} \times \{m_{i,j}\} \subseteq A \times \{1, \ldots, l\}$ for $j = 1, \ldots, n_i$ and $i = 1, \ldots, k$ are pairwise disjoint. For the nontrivial direction, observe that if $A_1, \ldots, A_n$ and $s_1, \ldots, s_n$ are as in the definition of $(G, \mathscr{S}, k, l)$-paradoxicality then the sets of the form

$$A \cap \left(\left(\bigcap_{i \in P} A_i \right) \setminus \bigcup_{i \in \{1, \ldots, n\} \setminus P} A_i \right) \cap s_j^{-1} \left(\left(\bigcap_{i \in Q} s_i A_i \right) \setminus \bigcup_{i \in \{1, \ldots, n\} \setminus Q} s_i A_i \right),$$

where P and Q are subsets of $\{1, \ldots, n\}$ with $|P| \leq k$ and $|Q| \leq l$ and $j \in Q$, can be relabeled so as to produce the desired $A_{i,j}$.

For a compact Hausdorff space X we write $\mathscr{C}_X$ for the collection of clopen subsets of X and $\mathscr{B}_X$ for the collection of Borel subsets of X.

Suppose that G acts on a set X. Let $\mathscr{S}$ be a G-invariant subalgebra of the power set $\mathscr{P}_X$ of X. The *type semigroup* $S(X, G, \mathscr{S})$ of the action with respect to $\mathscr{S}$ is the preordered semigroup

$$\left\{ \bigcup_{i \in I} A_i \times \{i\} : I \text{ is a finite subset of } \mathbb{N} \text{ and } A_i \in \mathscr{S} \text{ for each } i \in I \right\} \Big/ \sim$$

where $\sim$ is the equivalence relation under which $P = \bigcup_{i \in I} A_i \times \{i\}$ is equivalent to $Q = \bigcup_{i \in J} B_i \times \{i\}$ if there exist $n_i, m_i \in \mathbb{N}$, $C_i \in \mathscr{S}$, and $s_i \in G$ for $i = 1, \ldots, k$

such that $P = \bigsqcup_{i=1}^{k} C_i \times \{n_i\}$ and $Q = \bigsqcup_{i=1}^{k} s_i C_i \times \{m_i\}$. Addition is defined by

$$\left[\bigcup_{i \in I} A_i \times \{i\}\right] + \left[\bigcup_{i \in J} B_i \times \{i\}\right] = \left[\left(\bigcup_{i \in I} A_i \times \{i\}\right) \cup \left(\bigcup_{i \in J + \max I} B_i \times \{i\}\right)\right],$$

and for the preorder we declare that $a \leq b$ if $b = a + c$ for some c.

Paradoxical decomposability can now be reexpressed as $a + a \leq a$, in formal analogy with proper infiniteness for nonzero positive elements in a C^*-algebra.

The following is a standard observation.

Lemma 2.3.4. *Let X be a compact metrizable space and $G \curvearrowright X$ an action. Let B be a nonempty Borel subset of X. Suppose that there is a G-invariant Borel probability measure μ on X with $\mu(B) > 0$. Then U is completely $(G, \mathscr{B}_X)$-nonparadoxical.*

Proof. Let μ be a G-invariant Borel probability measure on X with $\mu(B) > 0$. Suppose that B fails to be completely $(G, \mathscr{B}_X)$-nonparadoxical. Then there are $k, l \in \mathbb{N}$ with $k > l$ for each $i = 1, \ldots, k$ there are an $n_i \in \mathbb{N}$ and $B_{i,1}, \ldots, B_{i,n_i} \in \mathscr{B}_X$, $s_{i,1}, \ldots, s_{i,n_i} \in G$, and $m_{1,1}, \ldots, m_{i,n_i} \in \{1, \ldots, l\}$ so that $\bigcup_{j=}^{n_i} B_{i,j} = B$ for each $i = 1, \ldots, k$ and the sets $s_{i,j} B_{i,j} \times \{m_{i,j}\} \subseteq B \times \{1, \ldots, l\}$ for $j = 1, \ldots, n_i$ and $i = 1, \ldots, k$ are pairwise disjoint. Since μ is G-invariant we have

$$k\mu(B) \leq \sum_{j=1}^{k} \sum_{i=1}^{n} \mu(B_i) = \sum_{j=1}^{k} \sum_{i=1}^{n} \mu(s_{i,j} B_i) = \mu\left(\bigcup_{j=1}^{k} \bigcup_{i=1}^{n} \mu(s_{i,j} B_i)\right) \leq l\mu(B),$$

and dividing by $\mu(B)$ yields $k \leq l$, a contradiction. We conclude that B is completely $(G, \mathscr{B}_X)$-nonparadoxical. $\square$

Lemma 2.3.5. *Let $G \curvearrowright X$ be an action on a compact metrizable space X. Let V be a completely $(G, \mathscr{C}_X)$-nonparadoxical nonempty clopen subset of X such that $G \cdot V = X$. Then there is G-invariant Borel probability measure μ on X such that $\mu(V) > 0$.*

Proof. There exists a state σ on $S(X, G, \mathscr{B}_X)$ such that $\sigma([V]) > 0$, for otherwise the Goodearl-Handelman theorem [34, Lemma 4.1] would yield $k, l \in \mathbb{N}$ such that $k > l$ and $k[V] \leq l[V]$, contradicting the complete $(G, \mathscr{C}_X)$-nonparadoxicality of V. Since the clopen subsets of X generate the Borel σ-algebra of X, σ induces a G-invariant Borel measure μ on X by first setting $\mu(U) = \sigma([U])$ for all $U \in \mathscr{C}_X$, showing that this is a premeasure on $\mathscr{C}_X$ using compactness to reduce countable additivity to finite additivity, and then extending (Lemma 5.1 of [86]). By compactness there is a finite set $F \subseteq G$ such that $F \cdot V = X$, and so

$$\mu(X) = \mu(F \cdot V) \leq \bigcup_{s \in F} \mu(sV) = |F|\mu(V).$$

Setting $\nu(\cdot) = \mu(\cdot)/\mu(X)$ then yields a G-invariant Borel probability measure on X. $\square$

Proposition 2.3.6. *Let $G \curvearrowright X$ be a minimal action on a zero-dimensional compact metrizable space. Then the following are equivalent:*

1. *there exists a G-invariant Borel probability measure on X,*

2. *X is completely $(G, \mathscr{C}_X)$-nonparadoxical,*

3. *there exists a nonempty clopen subset of X which is completely $(G, \mathscr{C}_X)$-nonparadoxical.*

Proof. Lemma 2.3.4 yields (1)$\Rightarrow$(2), while (2)$\Rightarrow$(3) is trivial. Since for any clopen set $V \subseteq X$ we have $G \cdot V = X$ by minimality, we obtain (3)$\Rightarrow$(1) from Lemma 2.3.5. $\qquad\square$

Lemma 2.3.7. *Let $G \curvearrowright X$ be a topologically free minimal action on a zero-dimensional compact Hausdorff space. Then $C(X) \rtimes_\lambda G$ is purely infinite if and only if every nonzero projection in $C(X)$ is infinite in $C(X) \rtimes_\lambda G$.*

Proof. By Theorem 1.0.1, the crossed product $C(X) \rtimes_\lambda G$ is simple. Since every projection in a simple purely infinite C^*-algebra is properly infinite, we obtain the forward direction. Suppose then that every nonzero projection in $C(X)$ is infinite in $C(X) \rtimes_\lambda G$ and let us show that $C(X) \rtimes_\lambda G$ is purely infinite. By simplicity, it suffices to show that every nonzero hereditary C^*-subalgebra $C(X) \rtimes_\lambda G$ contains an infinite projection. Let A be such a C^*-subalgebra. Take a nonzero positive element a in A such that $\|E(a)\| = 1$. Using topological freeness it is straightforward to construct an $f \in C(X)^+$ such that $\|f\| = 1$, $\|fE(a)f - faf\| \leq 1/4$, and $\|fE(a)f\| \geq \|E(a)\| - 1/4 = 3/4$. Setting $g = (fE(a)f - 1/2)_+$ we then have $g \neq 0$ since $\|fE(a)f\| > 1/2$, and $g \precsim faf$ since $\|fE(a)f - faf\| < 1/2$, so that $g \precsim a$. Since X is zero-dimensional there exists a nonzero projection $p \in \overline{gC(X)g}$, which is infinite in $C(X) \rtimes_\lambda G$ by our hypothesis. Since $p \precsim g$ [57, Prop. 2.7] we have $p \precsim a$ and so there exists a $w \in C(X) \rtimes_\lambda G$ such that $p = waw^*$ [57, Prop. 2.6]. Then $a^{1/2}ww^*a^{1/2}$ is a projection in A which is equivalent to p and hence is infinite. $\qquad\square$

The next theorem, which is a specialization of a result of Rørdam and Sierakowski to the minimal case, shows that if the invariant mean/paradoxical decomposability dichotomy fails for topologically free minimal actions on zero-dimensional compact Hausdorff spaces then any examples witnessing this failure must exhibit perforation, or more precisely the lack of almost perforation, in the type semigroup. The type semigroup $S(G, X, \mathscr{S})$ is *almost unperforated* if $a \leq b$ whenever a and b are elements of $S(G, X, \mathscr{S})$ for which there are $k, l \in \mathbb{N}$ satisfying $k > l$ and $ka \leq lb$.

Theorem 2.3.8. *Let $G \curvearrowright X$ be a topologically free minimal action on a zero-dimensional compact Hausdorff space. Consider the following conditions:*

1. *$S(X, G, \mathscr{C}_X)$ is purely infinite,*

2. *every clopen subset of X is $(G, \mathscr{C}_X)$-paradoxical,*

3. *$C(X) \rtimes_\lambda G$ is purely infinite,*

4. *$C(X) \rtimes_\lambda G$ does not admit a tracial state,*

5. *$S(X, G, \mathscr{C}_X)$ does not admit a nontrivial state.*

Then $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$. Moreover, if $S(X, G, \mathscr{C}_X)$ is almost unperforated then all five conditions are equivalent.

Proof. $(1) \Rightarrow (2)$. This is a straightforward consequence of the definitions.

$(2) \Rightarrow (3)$. Let U be a $(G, \mathscr{C}_X)$-paradoxical clopen subset of X. Then there is a clopen partition $\{C_1, \ldots, C_n\}$ of U and $s_1, \ldots, s_n \in G$ such that $V = \bigcup_{i=1}^n s_i C_i$ is a proper subset of U. Set $z = \sum_{i=1}^n u_{s_i} 1_{C_i}$. Then $z^* z = 1_U$ and $zz^* = 1_V$, so that 1_U is an infinite projection in $C(X) \rtimes_\lambda G$. It follows by Lemma 2.3.7 that $C(X) \rtimes_\lambda G$ is purely infinite.

$(3) \Rightarrow (4)$. A unital purely infinite C^*-algebra does not admit a tracial state since the unit is properly infinite.

$(4) \Rightarrow (5)$. As the proof of Lemma 2.3.5 demonstrates, from any nonzero state on $S(X, G, \mathscr{C}_X)$ we can construct a G-invariant Borel probability measure on X, from which we obtain a tracial state on $C(X) \rtimes_\lambda G$ by composing with the canonical conditional expectation onto $C(X)$.

Finally, if $S(X, G, \mathscr{C}_X)$ is almost unperforated then to obtain $(5) \Rightarrow (1)$ we observe that the absence of nontrivial states on $S(X, G, \mathscr{C}_X)$ implies by the Goodearl-Handelman theorem [34, Lemma 4.1] that for any given $x \in S(X, G, \mathscr{C}_X)$ there are $k, l \in \mathbb{N}$ such that $k > l$ and $kx \leq lx$. Taking a large enough $n \in \mathbb{N}$ such that $2(l^n + 1) \leq k^n$ we obtain $2(l^n + 1)x \leq k^n x \leq l^n x$, in which case $2x \leq x$ by almost unperforation. $\qquad\square$

Question 2.3.9. Does the type semigroup $S(X, G, \mathscr{C}_X)$ associated to an action as in Theorem 2.3.8 (or any action on a zero-dimensional compact Hausdorff space) ever fail to be almost unperforated?

In the next two sections we will examine actions as in Theorem 2.3.8 for which the type semigroup $S(X, G, \mathscr{C}_X)$ is almost unperforated.

Example 2.3.10. Consider the free group $F_r = \langle a_1, \ldots, a_r \rangle$ of rank r acting on its Gromov boundary ∂F_r. Since $(G, \mathscr{C}_X)$-paradoxicality for clopen sets is preserved under finite disjoint unions and every clopen set is a finite disjoint union of members of a given clopen basis, we need only verify that ∂F_r has a basis of clopen $(G, \mathscr{C}_X)$-paradoxical sets. Let $w_1 \cdots w_n$ be a reduced word in the generators and their inverses. Let U be the clopen subset of ∂F_r consisting of all infinite reduced words beginning with $w_1 \cdots w_n$. Such clopen sets form a basis for X. Take distinct elements $x, y \in \{a, b, a^{-1}, b^{-1}\} \setminus \{w_1^{-1}, w_n^{-1}\}$. Then $w_1 \cdots w_n x U$ and $w_1 \cdots w_n y U$ are disjoint clopen subsets of U, so that U is $(G, \mathscr{C}_X)$-paradoxical.

The action of $F_r \curvearrowright \partial F_r$ is an example of a strong boundary action. A action of G on an infinite compact Hausdorff space X is a *strong boundary action* if for for every pair U and V of nonempty open subset of X there is an $s \in G$ such that $s(X \setminus U) \subseteq V$. The action is *n-filling* if for every collection of n nonempty open subsets $U_1, \ldots, U_n$ of X there are $s_1, \ldots, s_n \in G$ such that $s_1 U_1 \cup \cdots \cup s_n U_n = X$. For $n = 2$ this is the same as being a strong boundary action. Note that an n-filling action is minimal, for if U is a nonempty G-invariant open subset of X then we can take $U_1, \ldots, U_n$ to be all equal to U to deduce that $U = X$.

In the case that X is zero-dimensional the n-filling property implies that every nonempty clopen subset of X is $(G, \mathscr{C}_X)$-paradoxical, which can be seen as follows. Since the action is minimal and X is assumed to be infinite, X contains no isolated points in X. Thus given a nonempty clopen set $U \subseteq X$ we can take a partition of U into $2n$ nonempty clopen sets $U_1, \ldots U_n, V_1, \ldots V_n$. Then by the n-filling property there are $s_1, \ldots, s_n, t_1, \ldots, t_n \in G$ such that $s_1 U_1 \cup \cdots \cup s_n U_n = t_1 V_1 \cup \cdots \cup t_n V_n = X$. For $i = 1, \ldots, n$ set $A_i = U \cap (s_i U_i \setminus \bigcup_{j=1}^{i-1} s_j U_j)$ and $B_i = U \cap (t_i V_i \setminus \bigcup_{j=1}^{i-1} t_j V_j)$. Then the nonempty sets among the A_i form a clopen partition of U, as do the nonempty sets among the B_i, and $s_1^{-1} A_1, \ldots, s_n^{-1} A_n, t_1^{-1} B_1, \ldots, t_n^{-1} B_n$ are pairwise disjoint subsets of U, showing that U is $(G, \mathscr{C}_X)$-paradoxical.

In [60] Laca and Spielberg showed that the reduced crossed product of a strong boundary action is purely infinite, and in [47] Jolissaint and Robertson obtained the same conclusion more generally for n-filling actions. Strong boundary actions include word hyperbolic groups acting on their Gromov boundary, of which $F_r \curvearrowright \partial F_n$ is the prototype, where hyperbolicity is exhibited in its most extreme tree form. Note that the Gromov boundary need not be zero-dimensional, as happens for example for a Fuchsian group of the first kind having a compact fundamental domain in the closed unit disk $\mathbb{D}$, in which case the Gromov boundary action is the same as the action on the boundary of $\mathbb{D}$.

2.4 The universal minimal system

Like any action of G on a compact Hausdorff space, the action $G \curvearrowright \beta G$ admits a minimal subsystem by Zorn's lemma. By the universal property of βG, each minimal subsystem of $G \curvearrowright \beta G$ factors onto every minimal action of G on a compact Haudorff space. It turns out that there is, up to conjugacy, a unique minimal action of G satisfying this universal property (see [39] for a short proof of the uniqueness). We will write this universal minimal action as $G \curvearrowright M$ and view M as a minimal closed G-invariant subset of βG.

Lemma 2.4.1. *Let $G \curvearrowright X$ be an action on a compact Hausdorff space X. Let U be a $(G, \mathscr{C}_X)$-paradoxical subset of X. Then 1_U is properly infinite in $C(X) \rtimes_\lambda G$.*

Proof. By hypothesis there exist pairwise disjoint clopen sets $C_1, \ldots, C_n$, $D_1, \ldots, D_m \subseteq U$ and $s_1, \ldots, s_n, t_1, \ldots, t_m \in G$ such that both $\{s_1 C_1, \ldots, s_n C_n\}$

and $\{t_1 D_1, \ldots, t_m D_m\}$ are partitions of U. Set $a = \sum_{i=1}^n u_{s_i} 1_{C_i}$ and $b = \sum_{i=1}^m u_{t_i} 1_{D_i}$. Then $a^* a = b^* b = 1_U$ and $aa^* + bb^* = 1_{C_1 \cup \cdots \cup C_n \cup D_1 \cup \cdots \cup D_m} \leq 1_U$, so that 1_U is properly infinite in $C(X) \rtimes_\lambda G$. $\qquad\square$

Lemma 2.4.2. *Let $G \curvearrowright X$ be an action on a compact Hausdorff space and let $\sigma_0 \colon C(X)_+ \to [0, \infty]$ be a G-invariant lower semicontinuous tracial functional. Then $\sigma = \sigma_0 \circ E \colon (C(X) \rtimes_\lambda G)_+ \to [0, \infty]$ is a lower semicontinuous tracial functional, where E is the canonical conditional expectation $C(X) \rtimes_\lambda G \to C(X)$.*

Proof. That σ is $\mathbb{R}_+$-linear and lower semicontinuous is clear. Since lower semicontinuity implies that σ commutes with norm convergent sums of positive elements, one can readily verify that σ is tracial by expressing $E(aa^*)$ and $E(a^*a)$ for a given $a \in C(X) \rtimes_\lambda G$ by the respective sums $\sum_{s \in G} E(xu_s^*) E(xu_s^*)^*$ and $\sum_{s \in G} \alpha_t(E(xu_{s^{-1}}^*)^* E(xu_{s^{-1}}^*))$, which are norm convergent since they converge on states and hence converge uniformly as functions on the state space by Dini's theorem. $\qquad\square$

Suppose that G acts on a set X and E is a nonempty subset of X. We will simply say that E is *G-paradoxical* if it is $(G, \mathscr{P}_X)$-paradoxical according to Definition 2.3.2, where $\mathscr{P}_X$ is the power set of X. This means that there are pairwise disjoint subsets $A_1, \ldots, A_m, B_1, \ldots, B_n$ of E and $s_1, \ldots, s_m, t_1, \ldots, t_n \in G$ such that $E = \bigcup_{i=1}^m s_i A_i = \bigcup_{i=1}^n t_i B_i$. A theorem of Tarski asserts that there is a finitely additive G-invariant measure on $\mathscr{P}_X$ with $\mu(E) = 1$ if and only if E is not G-paradoxical (see [102]). This result is the basis for Theorem 2.1.2 and will be used in the proof below.

Lemma 2.4.3. *Let $E \subseteq G$. Then the following are equivalent.*

1. *E is G-paradoxical,*

2. *1_E is properly infinite in $C(X) \rtimes_\lambda G$,*

3. *$(1_E)^{\oplus n}$ is properly infinite in $C(X) \rtimes_\lambda G$ for some $n \in \mathbb{N}$,*

4. *$\varphi(1_E) \in \{0, \infty\}$ for every tracial linear functional $\varphi \colon (C(X) \rtimes_\lambda G)^+ \to [0, 1]$.*

Proof. (1)$\Rightarrow$(2). Since subsets of G correspond spectrally to clopen subsets of βG via the identification $\ell^\infty(G) \cong C(\beta G)$, we can apply Lemma 2.4.1.

(2)$\Rightarrow$(3). Trivial.

(3)$\Rightarrow$(4). Suppose that $(1_E)^{\oplus n}$ is properly infinite in $C(X) \rtimes_\lambda G$. Then it can be written as $p_1 + p_2 + q$ for some projections $p_1, p_2, q \in M_n(C(X) \rtimes_\lambda G)$ with $p_1 \sim p_2 \sim (1_E)^{\oplus n}$. Now given a tracial functional $\varphi \colon (C(X) \rtimes_\lambda G)_+ \to [0, \infty]$, we can extend it to a tracial functional on $(M_n(C(X) \rtimes_\lambda G))^+$ by setting $\varphi((a_{i,j})) = \sum_i \varphi(a_{i,i})$, and $n\varphi(1_E) = \varphi(p_1) + \varphi(p_2) + \varphi(q) = 2n\varphi(1_E) + \varphi(q)$, which implies that $\varphi(1_E)$ is equal to either 0 or ∞.

(4)$\Rightarrow$(1). Suppose that E is not G-paradoxical. By the theroem of Tarski recalled above, there is a finitely additive G-invariant measure μ on $\mathscr{P}(G)$ such

that $\mu(E) = 1$. Write $\mathscr{P}_\mu(G)$ for the upwards directed set of all sets $D \subseteq G$ such that $\mu(D) < \infty$. For $D \in \mathscr{P}_\mu(G)$ define a finitely additive bounded measure μ_D on $\mathscr{P}(G)$ by $\mu_D(C) = \mu(C \cap D)$. Then we obtain a bounded positive linear functional $\psi_D : \ell^\infty(G)$ such that $\psi(1_C) = \mu_D(C)$ for all $C \in \mathscr{P}(G)$ by first defining it on simple functions and then extending by continuity. Now set $\varphi_0(f) = \sup_{D \in \mathscr{P}_\mu(G)} \psi_D(f)$ for $f \in \ell^\infty(G)$ to obtain a functional which is $\mathbb{R}_+$-linear since $\psi_D \leq \psi_{D'}$ whenever $D \subseteq D'$, is lower semicontinuous since each ψ_D is continuous, and takes the value 1 at 1_E since $\psi_D(E) = \mu(E)$ for every $D \in \mathscr{P}_\mu(G)$. Moreover one can readily verify the G-invariance of φ_0 by checking it on characteristic functions. We can thus appeal to Lemma 2.4.2 to extend φ_0 to a tracial linear functional $\varphi \colon (C(X) \rtimes_\lambda G)^+ \to [0, \infty]$, and since $\varphi(1_E) = 1$ we obtain the negation of (4). $\qquad\square$

Proposition 2.4.4. *For a minimal action $G \curvearrowright X$ on a compact Hausdorff space, the following are equivalent:*

1. *there exists a G-invariant Borel probability measure on X,*

2. *$C(X) \rtimes_\lambda G$ admits a faithful tracial state,*

3. *$C(X) \rtimes_\lambda G$ is stably finite.*

Proof. (1)$\Rightarrow$(2). Every G-invariant Borel probability measure on X produces a tracial state on $C(X) \rtimes_\lambda G$ via composition with the canonical conditional expectation $C(X) \rtimes_\lambda G \to C(X)$, and this state is faithful since the measure has full support by minimality.

(2)$\Rightarrow$(3). It is well known and easy to verify that the existence of a faithful tracial state on a unital C^*-algebra implies stable finiteness.

(3)$\Rightarrow$(1). If $C(X) \rtimes_\lambda G$ is stably finite then it admits a quasitrace $\tau \colon (C(X) \rtimes_\lambda G)_+ \to [0, \infty)$ which we may assume to be normalized so that $\tau(1) = 1$. By the definition of quasitrace, τ defines via restriction a tracial state on every unital commutative C^*-subalgebra and satisfies $\tau(a^*a) = \tau(aa^*)$ for all $a \in C(X) \rtimes_\lambda G$. Thus τ defines via restriction a tracial state on $C(X)$ and for all $f \in C(X)_+$ and $s \in G$ we have $\tau(u_s f u_s^*) = \tau((u_s f^{1/2})(u_s f^{1/2})^*) = \tau(f)$, so that the Borel probability measure on X induced by τ is G-invariant. $\qquad\square$

In the proof of Theorem 2.4.7 we will use the fact that the universal minimal action of G is free [30]. This is contained in Lemma 2.4.6, which provides some additional information for the purposes of Section 2.5.

Lemma 2.4.5. *Let $t \in G \setminus \{e\}$. Then there is a partition of G into three sets E_1, E_2, and E_3 such that $E_i \cap tE_i = \emptyset$ for each $i = 1, 2, 3$.*

Proof. Take a maximal set $H \subseteq G$ with the property $H \cap tH = \emptyset$ and define $E_1 = H$, $E_2 = tH$, and $E_3 = G \setminus (E_1 \cup E_2)$. Then $E_2 \cap tE_2 = t(H \cap tH) = \emptyset$ and, since $t^{-1}H \cup H \cup tH = G$ by the maximality of H,

$$G \setminus (E_3 \cap tE_3) = (G \setminus E_3) \cup t(G \setminus E_3) = t(t^{-1}H \cup H \cup tH) = G. \qquad\square$$

Lemma 2.4.6. *There is a countable set $\Omega \subseteq C(M)$ such that for every G-invariant C^*-subalgebra A of $C(M)$ containing Ω the action of G on $\widehat{A}$ is free.*

Proof. By Lemma 2.4.5, for every $t \in G \setminus \{e\}$ there is a partition of G into three sets $E_{t,1}$, $E_{t,2}$, and $E_{t,3}$ such that $E_{t,i} \cap tE_{t,i} = \emptyset$ for each $i = 1, 2, 3$. Write $p_{t,i}$ for the image of the projection $1_{E_{t,i}}$ under the restriction map $\ell^\infty(G) \cong C(\beta G) \to C(M)$ and set $\Omega = \{p_{t,i} : t \in G \setminus \{e\}, i = 1, 2, 3\}$. Since $p_{t,i} \perp tp_{t,i}$ for every $t \in G \setminus \{e\}$ and $i = 1, 2, 3$, we see that Ω has the desired property. $\square$

We can now establish the analogue for the universal minimal system of the invariant mean/paradoxical decomposability dichotomy for groups.

Theorem 2.4.7. $C(M) \rtimes_\lambda G$ *either has a tracial state or is purely infinite depending on whether or not G is amenable.*

Proof. If G is amenable then every continuous action $G \curvearrowright X$ on a compact Hausdroff space admits a G-invariant Borel probability measure, and if the action is minimal then every such measure has full support and hence produces a faithful tracial state on $C(X) \rtimes_\lambda G$ via composition with the canonical conditional expectation onto $C(X)$.

Suppose now that G is nonamenable and let us show that $C(M) \rtimes_\lambda G$ is purely infinite. By Lemma 2.4.6 the action $G \curvearrowright M$ is free, and so by Lemma 2.3.7 it suffices to show that every nonzero projection in $C(M)$ is infinite in $C(M) \rtimes_\lambda G$. So let p be a nonzero projection in $C(M)$. Then it has the form 1_U for some clopen subset U of M. We claim that there is a clopen subset V of βG and a finite set $F \subseteq G$ such that $V \cap M = U$ and $\bigcup_{s \in F} sV = \beta G$. To see this, take a clopen subset W of βG such that $W \cap M = U$. Since $\beta G \setminus M$ is open, we can write it as union of the collection $\{U_i\}_{i \in I}$ of clopen subsets of βG which do not intersect M. Since the action on M is minimal there is a finite set $F \subseteq G$ such that $\bigcup_{s \in F} sU = M$. Then the clopen sets $\bigcup_{s \in F} s(W \cap U_i)$ for $i \in I$ cover βG and hence by compactness there is a finite set $J \subseteq I$ such that $\bigcup_{i \in J} \bigcup_{s \in F} s(W \cap U_i) = \beta G$. We can then take $V = \bigcup_{i \in J}(W \cap V_i)$ to verify the claim.

Let E be the subset of G which spectrally corresponds to V under the identification $\ell^\infty(G) \cong \beta G$. Then $1 \leq \sum_{s \in F} \alpha_s(1_E)$ where 1 is the unit in $\ell^\infty(G)$. Setting $n = |F|$, this implies that $1 \precsim (1_E)^{\oplus n}$ relative to $\ell^\infty(G) \cong \beta G$. Now since G is nonamenable it has a paradoxical decomposition by Tarski's theorem, and so 1 is properly infinite in $\ell^\infty(G) \rtimes_\lambda G$ by Lemma 2.4.3. Consequently $(1_E)^{\oplus 2n} \precsim 1^{\oplus 2n} \precsim 1 \precsim (1_E)^{\oplus n}$, which shows that $(1_E)^{\oplus n}$ is properly infinite in $\ell^\infty(G) \rtimes_\lambda G$. By Lemma 2.4.3, 1_E is properly infinite in $\ell^\infty(G) \rtimes_\lambda G$. Therefore p, viewed as the image of 1_E under the canonical quotient map $\ell^\infty(G) \rtimes_\lambda G \to C(M) \rtimes_\lambda G$, is (properly) infinite, completing the proof. $\square$

Note that every G admits a minimal action $G \curvearrowright X$ with an invariant Borel probability measure [46], in which case the reduced crossed product is stably finite. In the case that G is amenable all minimal actions have this property. So we ask

if the invariant mean/paradoxical decomposability dichotomy for groups persists in the natural dynamical context that produces simple reduced crossed products:

Question 2.4.8. Is it true that for every minimal topologically free action $G \curvearrowright X$ of a nonamenable group the reduced crossed product $C(X) \rtimes_\lambda G$ either has a tracial state or is purely infinite?

2.5 Minimal actions, pure infiniteness, and nuclearity

Lemma 2.5.1. *Suppose that G is nonamenable. Let p be a projection in $C(M)$. Then there is a countable set $\Lambda \subseteq C(M)$ such that p is properly infinite in $A \rtimes_\lambda G$ for every G-invariant C^*-subalgebra $A \subseteq C(M)$ which contains $\{p\} \cup \Lambda$.*

Proof. By Lemma 2.3.7 the projection p is properly infinite in $C(M) \rtimes_\lambda G$, and so there are partial isometries $x, y \in C(M) \rtimes_\lambda G$ such that $x^*x + y^*y = p$ and $xx^* + yy^* \leq p$. Take sequences $\{x_n\}$ and $\{y_n\}$ in $C_c(G, C(M))$ which converge in norm to x and y, respectively. Then for each n the set K_n of all elements in $C(M) \rtimes_\lambda G$ of the form $E(x_n u_t^*)$ or $E(y_n u_t^*)$ for $t \in G$ is finite, and for every G-invariant C^*-subalgebra A of $C(M)$ containing K_n the crossed product $A \rtimes_\lambda G$ contains x_n and y_n. Thus for every G-invariant C^*-subalgebra A of $C(M)$ that contains the countable set $\bigcup_{n=1}^\infty K_n$ the crossed product $A \rtimes_\lambda G$ contains x and y and hence also p as a properly infinite projection. $\square$

Lemma 2.5.2. *Let Ω be a countable subset of $C(M)$. Then there is a separable G-invariant C^*-subalgebra A of $C(M)$ which is generated by projections, contains Ω, and has the property that each of its projections is properly infinite in $A \rtimes_\lambda G$.*

Proof. We will recursively construct countable subsets $\Omega_0 = \Omega, \Omega_1, \Omega_2, \ldots$ of $C(M)$ and countable G-invariant subsets $P_0, P_1, P_2, \ldots$ consisting of projections in $C(M)$ such that for each n the set Q_n of projections in $C^*(P_n)$ is contained in Ω_{n+1} and each member of Q_n is properly infinite in $C^*(P_{n+1}) \rtimes_\lambda G$.

Since M is zero-dimensional, every element of $C(M)$ can be approximated in norm by linear combinations of projections. It follows that every countable subset of $C(M)$ lies in the C^*-algebra generated by a countable set of projections, and by applying G to this set we may take it to be G-invariant. So take a countable G-invariant set P_0 of projections in $C(M)$ such that $\Omega \subseteq C^*(P_0)$. Since $C^*(P_0)$ is separable the set Q_0 of projections in $C^*(P_0)$ is countable, and so by Lemma 2.5.1 we can find for each $p \in Q_0$ a countable set $\Lambda_p \subseteq C(M)$ such that p is properly infinite in $A \rtimes_\lambda F$ for every G-invariant C^*-subalgebra $A \subseteq C(M)$ which contains $\{p\} \cup \Lambda_p$. Set $\Omega_1 = Q_0 \cup \bigcup_{p \in Q_0} \Lambda_p$. Since Ω_1 is countable we can find as before a countable G-invariant set $P_1 \subseteq C(M)$ of projections such that $\Omega_1 \subseteq C^*(P_1)$. Now continue in the same fashion to generate sets $\Omega_0, \Omega_1, \Omega_2, \ldots$ and $P_0, P_1, P_2, \ldots$ with the desired properties.

Let A be the C^*-subalgebra of $C(M)$ generated by the G-invariant countable set of projections $\bigcup_{n=0}^\infty P_n$. Then A is G-invariant and $\Omega \subseteq A$. Moreover, if p is

a projection in A then it is equivalent to a projection in $C^*(P_n)$ for some n and therefore p is properly infinite in $C^*(P_{n+1}) \rtimes_\lambda G$ and hence also in $A \rtimes_\lambda G$. $\square$

Theorem 2.5.3. *G is nonamenable if and only if there exists a free minimal action $G \curvearrowright X$ on the Cantor set such that $C(X) \rtimes_\lambda G$ is purely infinite.*

Proof. If G is amenable then every action on a compact Hausdorff space admits a G-invariant Borel probability measure, and such an invariant measure yields a tracial state on the reduced crossed product via composition with the canonical conditional expectation onto $C(X)$, which implies that the reduced crossed product is not purely infinite.

Suppose then that G is nonamenable. Combining Lemmas 2.4.6, 2.5.1, and 2.5.2 we obtain a separable G-invariant C^*-subalgebra A of $C(M)$ such that $\widehat{A}$ is zero-dimensional, the action of G on $\widehat{A}$ is free and minimal, and every projection in A is properly infinite in $A \rtimes_\lambda G$. By Lemma 2.4.3, $A \rtimes_\lambda G$ is purely infinite. Now since G is nonamenable it is infinite, and so $\widehat{A}$ cannot contain any isolated points in view of the minimality of the action. Since $\widehat{A}$ is metrizable by virtue of the separability of A, it follows that $\widehat{A}$ is the Cantor set. $\square$

By Theorem 2.2.4, G is exact if and only if there is an action $G \curvearrowright X$ which is amenable. We record next some more precise information in the forward direction.

Lemma 2.5.4. *Suppose that G is exact. Then there is a countable set $\Upsilon \subseteq C(M)$ such that for every G-invariant unital C^*-subalgebra A of $C(M)$ containing Υ the action of G on $\widehat{A}$ is amenable.*

Proof. By Theorem 2.2.4 the action $G \curvearrowright \beta G$ is amenable, and hence so is its restriction to M. Let $\{T_i\}_{i \in I}$ be a net in $C_c(G, C(X))$ which witnesses the amenability of the action as in Proposition 2.2.2. Since G is countable we may assume I to be countable. Then we can take $\Upsilon = \{T_i(t) : i \in I,\ t \in G\}$. $\square$

The main point of the following result of Rørdam and Sierakowski is that the crossed products in question fall under the purview of the Kirchberg-Phillips classification theorem for simple separable purely infinite nuclear C^*-algebras (i.e., Kirchberg algebras) which satisfy the universal coefficient theorem (UCT) [56, 77]. The classifying invariant in the unital case is K-theory paired with the K_0 class of the unit, and it is complete.

Theorem 2.5.5. *G is exact and nonamenable if and only if there exists a free minimal action $G \curvearrowright X$ on the Cantor set such that $C(X) \rtimes_\lambda G$ is a Kirchberg algebra in the UCT class.*

Proof. By Theorems 2.2.3, 2.2.4, and 2.5.3, we need only show the forward direction, and this follows by incorporating the use of Lemma 2.5.4 into the proof of (1)$\Rightarrow$(3) in Theorem 2.5.3, applying Theorem 2.2.3, and noting that the UCT property a consequence of a result of Tu [95]. $\square$

Chapter 3

External measure-theoretic phenomena

3.1 Sofic groups, sofic actions, and hyperlinearity

As discussed in Section 2.1, for discrete groups the basic idea of internal measure-theoretic finite approximation is captured by the Følner set characterization of amenability. At the same time we can view Følner sets as furnishing external finite approximations in the following way. Let F be a nonempty finite subset of a discrete group G. For every $s \in G$ choose a bijection $\tilde{\sigma}_s \colon F \setminus s^{-1}F \to F \setminus sF$ and define an element σ_s the permutation group $\mathrm{Sym}(F)$ of F by $\sigma_s(t) = st$ if $st \in F$ and $\sigma_s(t) = \tilde{\sigma}_s(t)$ otherwise. This defines a map $\sigma \colon G \to \mathrm{Sym}(F)$, and if F is approximately invariant under translation by a given finite set $E \subseteq G$ in the sense that $|sF \cap F|/|F|$ is small for all $s \in E$ then σ is approximately multiplicative and free on E in the sense that $|\{t \in F : \sigma_{rs}(t) = \sigma_r\sigma_s(t)\}|/|F|$ is small for all $r, s \in E$ and $|\{t \in F : \sigma_r(t) \neq \sigma_s(t)\}|/|F|$ is small for all distinct $r, s \in E$. The existence of such approximately multiplicative and free maps into the permutation group of a finite set leads us to the following notion of a sofic group, which was conceived by Gromov in [37] (see also [104]).

We say that a countable discrete group G is *sofic* if for $i \in \mathbb{N}$ there are a sequence $\{d_i\}_{i=1}^{\infty}$ of positive integers and a sequence $\{\sigma_i\}_{i=1}^{\infty}$ of maps $s \mapsto \sigma_{i,s}$ from G to $\mathrm{Sym}(d_i)$ which is asymptotically multiplicative and free in the sense that

$$\lim_{i \to \infty} \frac{1}{d_i} \big| \{k \in \{1, \dots, d_i\} : \sigma_{i,st}(k) = \sigma_{i,s}\sigma_{i,t}(k)\} \big| = 1$$

for all $s, t \in G$ and

$$\lim_{i \to \infty} \frac{1}{d_i} \big| \{k \in \{1, \dots, d_i\} : \sigma_{i,s}(k) \neq \sigma_{i,t}(k)\} \big| = 1$$

for all distinct $s, t \in G$. Such a sequence $\{\sigma_i\}_{i=1}^{\infty}$ is called a *sofic approximation sequence* for G. In the theory of sofic entropy discussed in the next section we assume, in order to avoid pathologies, that $\lim_{i \to \infty} d_i = \infty$, which is automatic if G is infinite. To treat uncountable G one simply replaces sequences with nets.

In addition to amenable groups, all residually finite groups are sofic, since we can produce genuinely multiplicative maps $\sigma_i : G \to \mathrm{Sym}(G/G_i)$ where G_i is a finite-index normal subgroup and the action on G/G_i is by left translation. The image of a group element s under such a map will be a genuinely free permutation if s is not contained in the normal subgroup. More generally, all groups that are locally embeddable into finite groups (see Section 5.1) are sofic, and soficity can be viewed as the measure-theoretic analogue of this topological property (note that since we are talking about discrete groups, "topological" is synonymous here with "purely group-theoretic"). Free groups are sofic because they are residually finite. It is not known whether there is a countable discrete group that is not sofic.

One can also regard the σ_i above as maps into permutation matrices in M_{d_i}, and in this way one can formally weaken soficity by merely requiring σ_i to map into unitaries, with the approximate multiplicativity and freeness expressed using the 2-norm arising from the unique tracial state on M_{d_i}. This gives us the notion of *hyperlinearity*, which is equivalent to the embeddability of the group von Neumann algebra $\mathcal{L}G$ into an ultrapower R^ω of the hyperfinite II$_1$ factor. In fact, using a 2×2 matrix trick one can deduce that $\mathcal{L}G$ embeds into R^ω merely knowing that G embeds into the unitary group of R^ω, i.e., without the zero-trace condition on nontrivial group elements [55, 82] (see Prop. 7.1 in [74]). In contrast to the internal finite approximation picture where it is known that the countable discrete group G is amenable if and only if the group von Neumann algebra $\mathcal{L}G$ is hyperfinite, it is not known whether soficity and hyperlinearity are the same.

Whether there exist countable discrete groups which are not hyperlinear is a specialization of Connes' embedding problem, which asks whether every separable II$_1$ von Neumann algebra embeds into R^ω. Thus R^ω-embeddability is the algebraic analogue of the combinatorial property of soficity, and one could also apply this notion to C^*-algebras with a faithful tracial state. However, insofar as they take us beyond the realm of amenability, soficity and R^ω-embeddability do not appear to be directly relevant to the classification theory for nuclear C^*-algebras. We note however that by a result of Kirchberg [54, 74] Connes' embedding problem is equivalent to asking whether every separable C^*-algebra is the quotient of a C^*-algebra with the weak expectation property (QWEP), and it would be very interesting to test these ideas by examining actions nonamenable groups and their reduced crossed products.

As is implicit in the definition of sofic entropy reviewed in the next section, the concept of soficity also applies to measure-preserving and topological dynamics, and more generally to groupoids with appropriate structure. One asks that the associated inverse semigroup of partial transformations can be locally modelled using partial permutation matrices in much the same way as for groups using permutation matrices. Because of the rigid matricial nature of the modelling, for

topological systems this implies the existence of an invariant Borel probability measure. It follows that for actions of nonamenable groups on compact Hausdorff spaces the properties of amenability and soficity are mutually exclusive. On the other hand, for amenable acting groups they always both hold.

The topological analogues of hyperlinearity and soficity play an important role in C^*-structure theory, and we will examine this subject in Chapter 5. Also, soficity has recently come to be recognized as the missing ingredient in our understanding of the role of internal and external finite approximation and the relation between them in the theories of entropy and mean dimension, to which we turn next.

3.2 Entropy

The dynamical concept of topological entropy is based, in its most general sofic form, on the idea of counting finite models which are distinguished up to some observational error. As such it is directly connected to external finite-dimensional approximation in C^*-algebras via the crossed product construction, and specifically to the question of what the size or volume of the number of finite-dimensional local approximations for a C^*-algebra can tell us about its structure. Not surprisingly, given our analysis of the relationship between internal and external structure for groups, topological entropy can also be interpreted as reflecting internal structure in certain specialized contexts, like that of integer actions on the Cantor set, which will be discussed in connection with combinatorial independence in Section 3.3.

If we try to count all external finite-dimensional local models of a C^*-algebra then we are in the realm of free entropy dimension, a subject which we will not pursue here (compare however the internal notions of dimension in Section 4.2) although it is linked to certain phenomena like approximate unitary equivalence that one encounters in C^*-classification theory [101, 44]. In contrast, topological entropy counts the number dynamical models relative to the group or some fixed approximation thereof, and the growth is in the much smaller exponential regime. One might expect that this coordinatized information typically gets washed away when passing to the crossed product. For minimal homeomorphisms of the Cantor set this is exactly what happens: within every class of such homeomorphisms yielding the same crossed product, every possible value of entropy occurs [16, 89, 90]. On the other hand, if one measures the exponential growth of the asymptotic number of finite models for the dynamics (instead of just taking a supremum or infimum as for entropy) as the precision with which one distinguishes these models gets finer and finer, then one can identify dimensional phenomena that lie at the heart of certain key issues in the classification of simple stably finite nuclear C^*-algebras. This variation on entropy is called mean dimension and will be treated in Section 3.4, and it will be our main interest in connection with C^*-structure. For context and motivation we will begin with a review of entropy theory.

There are two mathematical approaches to the notion of entropy, and they are connected by Stirling's approximation. One is captured by Shannon's information theory, while the other has its origin in the work of Boltzmann in statistical mechanics and is based on the idea of counting finite models. Both of these viewpoints can be used to generate invariants for dynamical systems. Conventionally one does this in a measure-theoretic way using partial orbits even if one is treating actions on compact spaces, as one has a variational principle that allows one to pass between measurable and topological dynamics. One the other hand, the computation of topological entropy in many cases involves the counting of finite orbits, which has a topological flavour (see Example 3.2.2). One could thus define a more genuinely topological notion of entropy by counting partial orbits which are close to being finite orbits, as opposed to arbitrary partial orbits according to the conventional definitions. This "topological" topological entropy would be in line with the kinds of finite-dimensional approximation that are specific to C^*-algebras, as opposed to von Neumann algebras. We will return to this point, although we will spend the bulk of our time discussing the conventional definitions of dynamical entropy.

The information-theoretic approach to entropy is internal in nature and, as expected from our prior discussions, works in the dynamical setting most generally for amenable acting groups. This is the domain of the classical theory of dynamical entropy [103, 32, 73]. The statistical mechanical approach is external and applies in dynamics most generally for sofic acting groups. The theory of sofic entropy was only recently pioneered by Bowen in [14] and further developed by Kerr and Li in [51]. Just as soficity for groups includes amenability as a special case, the sofic version of entropy subsumes the classical amenable one [15, 52]. Dynamical entropy in its most general sofic form measures the exponential growth of the number of finite models for the dynamics at fixed but arbitrarily fine levels of precision. For single transformations these finite models can be taken to be partial orbits which we only distinguish to within a given error, with the exponential growth measured relative to the number of iterations. More generally, for actions of a countable amenable group one similarly computes the exponential growth of partial orbits along a Følner sequence. Most generally, for actions of a countable sofic group one replaces the internal counting of orbits with an external picture that counts the number of models for the dynamics that are compatible with a given sofic approximation of the group by permutations of a finite set, and then measures exponential growth along a fixed sequence of such approximations which asymptotically witness the soficity of the group. The Ornstein-Weiss quasitiling machinery can then be used to show that this sofic notion of entropy reduces to the partial orbit definition in the amenable case.

The notion of entropy was first introduced into dynamics in the work of Kolmogorov and Sinai using Shannon's idea of information. In its original form, Kolmogorov-Sinai measure entropy is expressed in terms of partial orbits of partitions (internal picture) rather than points (external picture), although one can equivalently take the latter viewpoint. Because of its internal nature it works most

generally for actions of amenable groups. We now recall the definition. Let (X, μ) be a probability space. Let $\mathcal{P}$ be a measurable partition of X. The *information function* $I \colon X \to [0, \infty]$ assigns to a point x the value $-\log \mu(A)$ where A is the member of $\mathcal{P}$ that contains x. This is meant to quantify the amount of information that one gains in learning the partition member A to which a prescribed but unrevealed point x belongs. The idea is that the amount of information we gain about x should be inversely proportional to the measure of A, with the additional application of a logarithm designed to produce additive behaviour. The entropy $H(\mathcal{P})$ of the partition $\mathcal{P}$ is defined as

$$H(\mathcal{P}) = \int_X I(x)\, d\mu(x) = \sum_{A \in \mathcal{P}} -\mu(A) \log \mu(A),$$

that is, the average amount of information gained in learning that x lies in its particular partition member as x ranges over all of X.

Now given a measure-preserving transformation $T \colon X \to X$ one defines

$$h_\mu(T, \mathcal{P}) = \lim_{n \to \infty} \frac{1}{n} H(\mathcal{P} \vee T^{-1}\mathcal{P} \vee \cdots \vee T^{-n+1}\mathcal{P})$$

with the limit existing by subadditivity. This quantity represents the amount of information gained, on average in both space and time, in learning that the trajectory of a point visits a certain sequence of members of $\mathcal{P}$. Thus the more chaotic or mixing the dynamics are, the larger we expect $h_\mu(T, \mathcal{P})$ to be, since we should be able to hone in more quickly on the location of a random point by knowing what members of $\mathcal{P}$ its partial trajectories visit. The Kolmogorov-Sinai entropy of T is defined by

$$h_\mu(T) = \sup_{\mathcal{P}} h_\mu(T, \mathcal{P})$$

where $\mathcal{P}$ ranges over the finite measurable partitions of X. For a Bernoulli shift $T \colon (Y, \nu)^{\mathbb{Z}} \to (Y, \nu)^{\mathbb{Z}}$, where (Y, ν) is a probability space and the action is by translation, the entropy is the logarithm of the entropy of Y, which is defined as $H(\mathcal{P})$ if Y is atomic and $\mathcal{P}$ is the partition of Y into its atoms, and $+\infty$ otherwise. Ornstein showed that entropy is a complete invariant for Bernoulli shifts. Prior to the introduction of entropy, all of the known measure-dynamical invariants (e.g., ergodicity, mixing, weak mixing) were of a spectral nature in the sense that they depend on the associated unitary representation of $\mathbb{Z}$ on $L^2(X, \mu)$ given by composition with the powers of T. For a nontrivial Bernoulli shifts one always obtains the regular representation with infinite multiplicity on the orthogonal complement of the constant functions.

Although one could define an invariant for measure-preserving actions $G \curvearrowright (X, \mu)$ of any countable discrete group by averaging as above but over arbitrary but fixed finite subsets of G instead of the intervals $\{0, 1, \ldots, n-1\}$ in $\mathbb{Z}$, it is only by taking these finite sets to be Følner sets that one can compare the values of entropy on different partitions, which is crucial for the purpose of computation. Thus this approach works most generally for actions of amenable groups.

For homeomorphisms of compact Hausdorff spaces, Adler, Konheim, and McAndrew introduced a notion of topological entropy modelled on the Kolomogorov-Sinai definition. One replaces the computation of the Shannon entropy of a partition with the counting of the minimal cardinality of a subcover of an open cover. Thus for a homeomorphism $T \colon X \to X$ of a compact Hausdorff space we define

$$h_{\text{top}}(T) = \sup_{\mathcal{U}} \lim_{n \to \infty} \frac{1}{n} \log N(\mathcal{U} \vee T^{-1}\mathcal{U} \vee \cdots \vee T^{-n+1}\mathcal{U})$$

where $\mathcal{U}$ ranges over the finite open covers of X and $N(\cdot)$ denotes the minimum cardinality of a subcover. The limit exists by the subadditivity of $N(\cdot)$ with respect to joins. As for Kolmogorov-Sinai entropy, the internal averaging over subsets of $\mathbb{Z}$ can be done more generally over Følner sets for an amenable acting group G in order to obtain a meaningful invariant $h_{\text{top}}(X, G)$. In this case the entropy of the shift $G \curvearrowright \{1, \ldots, k\}^G$ is easily computed to be $\log k$.

R. Bowen subsequently gave an equivalent formulation in terms of ε-separated partial orbits with respect to a compatible metric. This facilitates computation in many cases and runs as follows. Let X be a compact space with compatible metric d and let $T \colon X \to X$ be a homeomorphism. For $\varepsilon > 0$ and $n \in \mathbb{N}$ we say that a set $E \subseteq X$ is (n, ε)-*separated* if $\max_{k=0,\ldots,n-1} d(T^k x, T^k y) > \varepsilon$ for any two distinct $x, y \in E$. Then

$$h_{\text{top}}(T) = \sup_{\varepsilon > 0} \limsup_{n \to \infty} \frac{1}{n} \log \operatorname{sep}_n(\varepsilon)$$

where $\operatorname{sep}_n(\varepsilon)$ is the maximum cardinality of an (n, ε)-separated subset of X.

The classical variational principle asserts that, for an action $G \curvearrowright X$ of a countable amenable group on a compact Hausdorff space, the topological entropy is equal to the supremum of the measure entropies over all invariant Borel probability measures on X. This is one expression of the fact that topological entropy is really a measure-theoretic construct. Compare the equivalence, for a C^*-algebra A, of nuclearity with the hyperfiniteness of $\pi(A)''$ for every representation π of A.

R. Bowen's (n, ε)-separated set formulation gives us a hint of how entropy can be extended to actions of sofic groups by externalizing the averaging to some finite set on which the group approximately acts. In the case of an amenable group we can take these finite sets to be Følner sets in order to recover the classical definition. This idea of sofic entropy was introduced by L. Bowen, who showed that one can generate a computable invariant for a measure-preserving action by modelling orbits of partitions inside sofic approximations for the group [14]. Kerr and Li subsequently demonstrated that sofic entropy can be expressed, in the spirit of R. Bowen, by using points instead of partitions [51, 52]. This approach also produces a topological dynamical invariant, and the variational principle extends to this context. For an action $G \curvearrowright X$ one fixes a sequence of sofic approximations for the acting group G and then measures the exponential growth of the number of approximately equivariant maps from the sofic approximations into X, which

one only distinguishes up to an observational ε-error. The precise definition is as follows.

Let X be a compact metrizable space and $G \curvearrowright X$ an action of a countable sofic group. Let ρ be a continuous pseudometric on X. For $d \in \mathbb{N}$ we define on the set of all maps from $\{1, \ldots, d\}$ to X the pseudometric

$$\rho_2(\varphi, \psi) = \left(\frac{1}{d} \sum_{a=1}^{d} (\rho(\varphi(a), \psi(a)))^2 \right)^{1/2}.$$

Write $\mathrm{Sym}(d)$ for the group of permutations of $\{1, \ldots, d\}$. Given a nonempty finite set $F \subseteq G$, a $\delta > 0$, and a map $\sigma \colon G \to \mathrm{Sym}(d)$, we define $\mathrm{Map}(\rho, F, \delta, \sigma)$ to be the set of all maps $\varphi \colon \{1, \ldots, d\} \to X$ such that $\rho_2(\varphi \circ \sigma_s, \alpha_s \circ \varphi) < \delta$ for all $s \in F$.

Definition 3.2.1. Let $\Sigma = \{\sigma_i \colon G \to \mathrm{Sym}(d_i)\}$ be a sofic approximation sequence for G. Let F be a nonempty finite subset of G and $\delta > 0$. For $\varepsilon > 0$ we define, writing $N_\varepsilon(\cdot, \rho_2)$ for the maximum cardinality of an ε-separated set with respect to ρ_2,

$$
\begin{aligned}
h_\Sigma^\varepsilon(\rho, F, \delta) &= \limsup_{i \to \infty} \frac{1}{d_i} \log N_\varepsilon(\mathrm{Map}(\rho, F, \delta, \sigma_i), \rho_2), \\
h_\Sigma^\varepsilon(\rho, F) &= \inf_{\delta > 0} h_\Sigma^\varepsilon(\rho, F, \delta), \\
h_\Sigma^\varepsilon(\rho) &= \inf_F h_\Sigma^\varepsilon(\rho, F), \\
h_\Sigma(\rho) &= \sup_{\varepsilon > 0} h_\Sigma^\varepsilon(\rho),
\end{aligned}
$$

where F in the third line ranges over the nonempty finite subsets of G. If $\mathrm{Map}(\rho, F, \delta, \sigma_i)$ is empty for all sufficiently large i, we set $h_\Sigma^\varepsilon(\rho, F, \delta) = -\infty$.

We note that one could substitute the pseudometric

$$\rho_\infty(\varphi, \psi) = \max_{a=1, \ldots, d} \rho(\varphi(a), \psi(a))$$

for ρ_2 without changing the value of $h_\Sigma(\rho)$.

We say that the continuous metric ρ is *dynamically generating* if for any distinct points $x, y \in X$ one has $\rho(sx, sy) > 0$ for some $s \in G$. It is easily checked that $h_\Sigma(\rho)$ has a common value over all dynamically generating continuous metrics ρ, and we define the topological entropy $h_\Sigma(X, G)$ of the system to be this common value. Note that this could depend on Σ. The prototypical example of the shift action $G \curvearrowright \{1, \ldots, k\}^G$ has entropy $\log k$, independently of Σ.

In the case that G is amenable, every sofic approximation for G approximately decomposes into copies of Følner sets (and is therefore essentially internal to G), so that the elements of $\mathrm{Map}(\rho, F, \delta, \sigma_i)$ are essentially unions of partial orbits. As a result $h_\Sigma(X, G)$ reduces to the classical topological entropy, and in particular does not depend on Σ [52]. For general G an element of $\mathrm{Map}(\rho, F, \delta, \sigma_i)$ can be viewed as a system of interlocking approximate partial orbits.

Example 3.2.2. Let G be a countable discrete group, and let f be an element in the integral group ring $\mathbb{Z}G$. Then G acts on $\mathbb{Z}G/\mathbb{Z}Gf$ by left translation, and this yields by Pontrjagin duality an action α_f of G by automorphisms on the compact Abelian dual group $X_f := \widehat{\mathbb{Z}G/\mathbb{Z}Gf}$. When f is equal to k times the unit this gives the shift action $G \curvearrowright \{1, \dots, k\}^G$. Now if G is residually finite and $\{G_i\}_{i=1}^\infty$ is a sequence of finite-index normal subgroups of G with $\bigcap_{j=1}^\infty \bigcup_{i=j}^\infty G_i = \{e\}$, and f is invertible as an element in the full group C^*-algebra $C^*(G)$, then the topological entropy of α_f with respect to the sofic approximation sequence Σ arising from $\{G_i\}_{i=1}^\infty$ via left translations on the quotients G/G_i is equal to the exponential growth rate of the number of G_i-fixed points and to the logarithm of the Fuglede-Kadison determinant of f in the group von Neumann algebra of G [51]. The topologcal entropy is also equal to logarithm of the Fuglede-Kadison determinant of f when G is amenable and f is invertible in $C^*(G)$ [62].

The above examples are natural from the viewpoint of entropy structure but are far from being minimal, and one would like to be able to say something about the prevalence or even possibility of nonzero entropy for minimal actions on various spaces. In Section 4.5 we will construct minimal homeomorphisms that have nonzero mean dimension, which implies infinite entropy.

If in addition to an action $G \curvearrowright X$ we have a G-invariant Borel probability measure μ on X, then one can define the sofic measure entropy $h_{\Sigma,\mu}(X, G)$ in the same way as $h_\Sigma(X, G)$, except that now one must ask that the push forward to X of the uniform measure on the sofic approximation space be weak* close to μ. One can moreover show that $h_{\Sigma,\mu}(X, G)$ is in fact a measure-dynamical invariant, i.e., for an abstract probability-measure-preserving action $G \curvearrowright (X, \mu)$ we obtain the same value over all topological models for the action. Extending the classical variational principle, for an action $G \curvearrowright X$ on a compact metrizable space one has

$$h_\Sigma(X, G) = \sup_\mu h_{\Sigma,\mu}(X, G)$$

where μ ranges over all G-invariant Borel probability measures on X. In particular, for $h_\Sigma(X, G)$ not to be $-\infty$ there must exist a G-invariant Borel probability measure on X. Thus if G is nonamenable then $h_\Sigma(X, G) = -\infty$ for all amenable actions of G.

In the case that X is zero-dimensional (e.g., a Cantor set) and G is amenable, one has

$$h_{\text{top}}(X, G) = \sup_{\mathcal{U}} \limsup_{n \to \infty} \frac{1}{n} \log |\mathcal{U} \vee T^{-1}\mathcal{U} \vee \cdots \vee T^{-n+1}\mathcal{U}|$$

where $\mathcal{U}$ ranges over all clopen partitions of X. Thus the computation reduces to merely counting sets, as opposed to the subtler problem of trying determine minimum cardinalities of subcovers, and we have an internal expression of topological entropy that is structurally the nicest possible from a C^*-algebra viewpoint, in the sense of the algebraic finite-dimensional approximation that defines an AF algebra

(see Section 4.1). In [100] Voiculescu introduced another approach to formulating topological entropy that uses the completely positive approximation property and works most generally for amenable G, and this provides a clear manifestation of the internal viewpoint for general X.

3.3   Combinatorial independence

Does knowing the value of entropy give us any structural information about the dynamical system, especially as it might impact the structure of the crossed product? As a single number, entropy by itself tells us little about global structure, although one can localize its study ("local entropy theory" [33]) in order to identify phenomena that collectively say something about the system as a whole. It turns out that, for amenable acting groups, nonzero entropy occurs precisely when the dynamics exhibits a product structure along a positive density subset of the group. Something similar happens for sofic groups, with the positive density condition being externalized to the sofic approximation space. The shift $G \curvearrowright \{1, \ldots, k\}^G$ exhibits this product structure in a prototypical way along the whole group. The formal notion is what we call *combinatorial independence*, which since the 1970s has played an important role in Banach space theory via the study of ℓ_1 and its relation to other spaces. For simplicity we will now restrict our attention to single homeomorphisms, although one can develop the theory for amenable acting groups, and also to a certain extent for sofic acting groups.

A collection $\{(A_{i,0}, A_{i,1})\}_{i \in I}$ of pairs of disjoint subsets of a given set is said to be *independent* if for every finite set $F \subseteq I$ and $\sigma \in \{0,1\}^F$ we have $\bigcap_{i \in F} A_{i,\sigma(i)} \neq \emptyset$. Given a compact Hausdorff space X, if one takes an independent collection $\{(A_{i,0}, A_{i,1})\}_{i \in I}$ of pairs of disjoint closed subsets of X and functions $f_i \in C(X, \mathbb{R})$ satisfying $\|f_i\| \leq 1$, $f_i|_{A_{i,0}} = 1$, and $f_i|_{A_{i,1}} = -1$, then the map that sends the ith coordinate basis vector in $\ell_{\mathbb{R}}^1$ to f_i is easily seen to extend to an isometric linear embedding. In fact it is only by uniformly separating the values of functions in this way over an independent collection that one can embed $\ell_{\mathbb{R}}^1$ (which has the largest norm among sequence spaces) isomorphically in $C(X, \mathbb{R})$ (which is a continuous analogue of $\ell_{\mathbb{R}}^\infty$, which has the smallest norm among sequence spaces).

Let X be a compact metric space and $T \colon X \to X$ a homeomorphism. We say that a set $J \subseteq \mathbb{Z}$ is an *independence set* for a pair (A_0, A_1) of subsets of X if the collection $\{(T^{-i}A_0, T^{-i}A_1) : i \in I\}$ is independent. The *upper density* of a set $I \subseteq \mathbb{Z}_{\geq 0}$ is the quantity $\limsup_{n \to \infty} |I \cap \{0, \ldots, n-1\}|/n$. Now suppose that there is a pair (A_0, A_1) of disjoint nonempty closed sets of X with an independence set $I \subseteq \mathbb{Z}_{\geq 0}$ of positive upper density. Take an $\varepsilon > 0$ which is smaller than the Hausdorff distance between A_0 and A_1. Given an $n \in \mathbb{N}$, for every $\sigma \in \{0,1\}^{I \cap \{0, \ldots, n-1\}}$ we can find an $x_\sigma \in X$ such that $T^i x \in A_{\sigma(i)}$ for all $i \in I \cap \{0, \ldots, n-1\}$, and the collection of all such x_σ is (n, ε)-separated and has cardinality $2^{|I \cap \{0, \ldots, n-1\}|}$.

It follows that

$$h_{\mathrm{top}}(T) \geq \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{sep}_n(\varepsilon) \geq \limsup_{n\to\infty} \frac{1}{n} |I \cap \{0, \ldots, n-1\}| \log 2 > 0.$$

This gives one direction of the following theorem.

Theorem 3.3.1. $h_{\mathrm{top}}(T) > 0$ *if and only if there is a pair of disjoint nonempty closed sets of X which has an independence set of positive upper density.*

The forward direction can be established by means of a hard combinatorial argument that will not be reproduced here [50]. When X is zero-dimensional the combinatorics are simpler and one can appeal to the classical Sauer-Shelah lemma, which in crude form asserts the following.

Lemma 3.3.2. *For every $\beta > 0$ there is a $d > 0$ such that, for all $n \in \mathbb{N}$, if $S \subseteq \{0,1\}^{\{1,\ldots,n\}}$ has cardinality at least $e^{\beta n}$ then there is an $I \subseteq \{1, \ldots, n\}$ such that $|I| \geq dn$ and $S|_I = \{0,1\}^I$.*

To complete the proof of the forward direction in Theorem 3.3.1 under the assumption that X is zero-dimensional, take a clopen partition $\mathcal{U}$ with $h_{\mathrm{top}}(T, \mathcal{U}) > 0$, where $h_{\mathrm{top}}(T, \mathcal{U}) = \limsup_{n\to\infty} n^{-1} \log |\mathcal{U} \vee T^{-1}\mathcal{U} \vee \cdots \vee T^{-n+1}\mathcal{U}|$. By refining $\mathcal{U}$ we may assume that it has the form $\mathcal{U}_1 \vee \cdots \vee \mathcal{U}_k$ for some two-element clopen partitions $\mathcal{U}_1, \ldots, \mathcal{U}_k$. Then one can verify that $h_{\mathrm{top}}(T, \mathcal{U}) \leq \sum_{i=1}^{k} h_{\mathrm{top}}(T, \mathcal{U}_i)$ so that $h_{\mathrm{top}}(T, \mathcal{U}_i) > 0$ for some i, and so we may assume that $\mathcal{U}$ itself is a two-element clopen partition $\{A_0, A_1\}$. Now apply the Sauer-Shelah lemma to a suitable sequence of intervals in $\mathbb{N}$ to construct an independence set of positive upper density for the pair (A_0, A_1).

In fact, using some ergodic theory one can show that the existence of an independence set of positive upper density for disjoint nonempty closed sets of an arbitrary X implies the existence of an independence set of positive lower density.

As mentioned in Section 3.2, for minimal homeomorphisms of the Cantor set the entropy gives absolutely no information about the crossed product. In particular, the product behaviour that is manifest through combinatorial independence along positive density sets of iterates cannot be detected. On the other hand, if this kind of independence occurs at a dimensional level as exhibited by the shift $G \curvearrowright [0,1]^G$, then this touches directly on some of the key issues in the classification program for nuclear C^*-algebras. This brings us to the concept of mean dimension.

3.4   Mean dimension

As for entropy, mean dimension has one version that uses open covers ("topological mean dimension", or just "mean dimension") and another that counts partial orbits or embedded sofic approximations that are ε-separated with respect to a

metric ("metric mean dimension"). Unlike for entropy, however, the relationship between these two versions is not completely understood in general, although it is known in some important cases.

Topological mean dimension is a dynamicization of covering dimension and was introduced by Gromov in [38] and further developed by Lindenstrauss and Weiss in [71, 70]. One can also formulate it in terms of Urysohn width, and both flavours appear in Gromov's paper. Let X be a compact Hausdroff space, and let $\mathcal{U}$ be a open cover of X. We set

$$\text{ord}(\mathcal{U}) = \max_{x \in X} \sum_{U \in \mathcal{U}} 1_U(x) - 1$$

and define $D(\mathcal{U})$ to be the minimum of $\text{ord}(\mathcal{V})$ over all open covers $\mathcal{V}$ refining $\mathcal{U}$. The covering dimension $\dim(X)$ of X is defined as the minimum of $D(\mathcal{U})$ over all open covers $\mathcal{U}$ of X.

Now let $T \colon X \to X$ be a homeomorphism. We define its *mean dimension* by

$$\text{mdim}(T) = \sup_{\mathcal{U}} \lim_{n \to \infty} \frac{1}{n} D(\mathcal{U} \vee T^{-1}\mathcal{U} \vee \cdots \vee T^{-n+1}\mathcal{U}).$$

where $\mathcal{U}$ ranges over all finite open covers of X. The limit exists because of the subadditivity of $D(\cdot)$ with respect to joins of finite open covers. We can similarly define $\text{mdim}(X, G)$ for an action $G \curvearrowright X$ of an amenable group by averaging over Følner sets. We can furthermore define $\text{mdim}(X, G)$ for an action $G \curvearrowright X$ of a sofic group by externalizing the averaging to finite sets on which a sofic group approximately acts according to the definition of soficity. In the latter case one pulls back open covers to the finite sofic approximation space and computes $\text{ord}(\cdot)$ there [63]. For a finite-dmensional compact metrizable space K the left shift action $G \curvearrowright K^G$ satisfies $\text{mdim}(K^G, G) \leq \dim(K)$, and if $K = [0, 1]^d$ then $\text{mdim}(K^G, G) = d$. At the other extreme, if X itself has finite covering dimension then the mean dimension of every action on X is zero. Since our applications in the next chapter concern integer actions, we will concentrate on that case from now on.

We next define the metric version of mean dimension. Let $T \colon X \to X$ be a homeomorphism, and let d be a compatible metric on X. Recall from the section on entropy that $\text{sep}_n(\varepsilon)$ denotes the maximal cardinality of a subset A of X which is (n, ε)-separated in the sense that $\max_{i=0,\ldots,n-1} d(T^i x, T^i y) > \varepsilon$ for all distinct $x, y \in A$. The *metric mean dimension* of T is defined by

$$\text{mdim}_{\text{M}}(T, d) = \liminf_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \left(\limsup_{n \to \infty} \frac{1}{n} \log \text{sep}_n(\varepsilon) \right).$$

This is the exponential rate at which the entropy at the scale ε grows as $\varepsilon \to 0$. One has $\text{mdim}(T) \leq \text{mdim}_{\text{M}}(T, d)$ for all compatible metrics on X [71], and if (X, T) is an extension of a minimal system, and in particular if T itself is minimal, then

$$\text{mdim}(T) = \sup_d \text{mdim}_{\text{M}}(T, d)$$

where d ranges over all compatible metrics on d [70].

For entropy one has a stark internal interpretation in the zero-dimensional case using clopen partitions. Because of its dimensional scale, it is not so clear what circumstances might permit an analogous internal description for mean dimension. This is especially relevant to the study of crossed product structure within the terms of classification theory. On the other hand, one should think of mean dimension itself as being the abstract object of study, without viewing the space as something separate from the action on which we might impose conditions to facilitate a more definitive but less general analysis. This is consistent with the fact that mean dimension is a dynamical version of covering dimension. As such, for C^*-classification purposes we are most interested in the problem of when the mean dimension is zero, given that zero-dimensionality for C^*-algebras as expressed in various different ways (tracial rank zero, finite decomposition rank or nuclear dimension, $\mathcal{Z}$-stability, zero radius of comparison) is closely tied to K-theoretic classifiability.

In [68] it is shown that if X is a compact metric space with finite covering dimension and $T \colon X \to X$ is a minimal homeomorphism such that the image of $K_0(C(X) \rtimes \mathbb{Z})$ in the space of affine functions space over the tracial state space of $C(X) \rtimes \mathbb{Z}$ is dense, then $C(X) \rtimes \mathbb{Z}$ has tracial rank zero, and hence by classification theory is an AH algebra with real rank zero. To verify tracial rank zero, crucial use is made of the fact that X, having finite covering dimension, contains an abundant supply of closed sets whose boundaries all have measure zero for every T-invariant Borel probability measure on X. In fact such a small boundary property is, among minimal homeomorphisms, characteristic of mean dimension zero. Given a compact metric space X and a homeomorphism $T \colon X \to X$, the *orbit capacity* of a set $E \subseteq X$ is defined as $\lim_{n \to \infty} \sup_{x \in X} n^{-1} \sum_{i=0}^{n-1} 1_E(T^i x)$, with the limit existing by subadditivity. We say that T has the *small boundary property* if for every $x \in U$ and open neighbourhood U of x there is a neighbourhood of x contained in U with zero orbit capacity. The first part of the following was established in [71], and the second part in [70].

Theorem 3.4.1. *If T has the small boundary property then it has mean dimension zero, and if (X, T) is assumed to be an extension of a minimal system then the converse also holds.*

The converse in the above theorem does not hold in general, as the presence of periodic points can create an obstruction. Consider for example the homeomophism $T \colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ given by $T(x, y) = (x, y + x \bmod 1)$, which has zero mean dimension since $\mathbb{R}/\mathbb{Z}$ has finite covering dimension. The boundary of every small enough neighbourhood of the point $(0, 0)$ contains at least two points of the form $(0, y)$, which are fixed by T, and so T fails to have the small boundary property.

In Section 4.5 we will give a construction of a minimal $\mathbb{Z}$-action with nonzero mean dimension as in [71] and show that the structure that is responsible for the lower bound in this example produces nonzero radius of comparison in the crossed product.

Chapter 4

Internal topological phenomena

4.1 Locally finite groups and AF algebras

The topological analogue of amenability for discrete group is local finiteness. In contrast to the setting of C^*-algebras, which we will turn to below, for discrete groups the topological notion of perturbation is trivial and thus, unlike the combinatorial measure-theoretic viewpoint, does not give us anything new beyond the merely group-theoretic. The group G is said to be *locally finite* if every finite subset of G generates a finite subgroup. Equivalently, G is the increasing union of finite subgroups. Obviously every finite group is locally finite. An example of a countably infinite locally finite group is the group of all permutations of $\mathbb{N}$ which fix all but finitely many elements. There are uncountably many pairwise nonisomorphic countable locally finite groups, and there is a countable locally finite group U which has the universal property that it contains a copy of every finite group and any two monomorphisms of a finite group into U are conjugate by an inner automorphism [49]. Note that every locally finite group is torsion. The converse is the general Burnside problem and is false, as was first shown by Golod and Shafarevich. In fact a torsion group need not even be amenable, which is the measure-theoretic analogue of local finiteness to be discussed below. It is true however that torsion implies local finiteness for subgroups of $\mathrm{GL}(n, \mathbb{K})$ for an $n \in \mathbb{N}$ and a field $\mathbb{K}$. Also, solvable torsion groups are locally finite.

AF algebras are the C^*-algebraic analogue of residually finite groups and are the prototype of internal finite-dimensional approximation in C^*-algebra theory. A C^*-algebra is *approximately finite-dimensional* or is an *AF algebra* if it can be written as the closure of an increasing union of (or, equivalently, as an inductive limit of) finite-dimensional C^*-algebras, i.e., of finite direct sums of matrix algebras. These were classified first by Bratteli in terms of diagrams [17], and then by Elliott using K-theory in a step that opened the door to the classification program [25]. Like local finiteness for groups, approximate finite-dimensionality

for C^*-algebras also has a purely local description: A separable C^*-algebra is AF if and only if for every finite set $F \subseteq A$ and $\varepsilon > 0$ there is a finite-dimensional *-subalgebra B of A which ε-contains F in the sense that for every $a \in F$ there is a $b \in B$ with $\|a-b\| < \varepsilon$. To produce an inductive limit representation from this local approximability one uses the fact that sufficiently good approximate containment of one finite-dimensional C^*-algebra in another implies that the smaller algebra can be conjugated into the larger one by a unitary close to 1. One of the major obstacles in C^*-algebra classification theory is that while one can often show local approximation by more general but still manageable building blocks this does not necessarily imply an inductive limit representation of the algebra in terms of the same type of building blocks. The implication does still hold in the case of direct sums of matrix algebras tensored with the algebra of continuous functions over the circle, but is not clear if one substitutes compact spaces of higher dimension.

If G is locally finite then $C^*_\lambda(G)$ is an AF algebra. This follows from the observation that if H is subgroup of a group G then setting $\pi(u_s)\delta_t = \delta_s$ for all $s \in H$ and $t \in G$, where δ_t is the canonical basis element in $\ell^2(G)$ associated to t, defines an injective *-homomorphism $\pi \colon C^*_\lambda(H) \to C^*_\lambda(G)$ since the unitary representation $s \mapsto \pi(u_s)$ of H decomposes as a direct sum of copies of the left regular representation via the restrictions to the right coset subspaces $\ell^2(Hs) \subseteq \ell^2(G)$ for $s \in G$. The following problem however seems to be unresolved.

Question 4.1.1. If $C^*_\lambda(G)$ is an AF algebra, must G be locally finite?

Note that if $C^*_\lambda(G)$ is both AF and commutative then G is locally finite by Pontrjagin duality theory.

In [61] it is shown that a large class of inductive limits of finite alternating groups, including the simple ones, can be classified in terms of the Bratteli diagram that is associated to both the group and its group C^*-algebra, so that the relation to AF algebras is rigidly structural in this case.

4.2   Dimension and K-theoretic classification

As far as classifying crossed products is concerned, the most immediate interest is in minimal $\mathbb{Z}$-actions, where we now have a fairly broad, yet still far from complete, understanding. In this section we will lay out the classification background that will set the stage for our study of crossed products of such actions, which will be the focus of the remainder of the chapter. Of particular importance will be the idea of noncommutative topological dimension in the various forms of tracial rank, decomposition rank, nuclear dimension, and radius of comparison.

By the Pimsner-Voiculescu exact sequence for $\mathbb{Z}$-crossed products, for a homeomorphism $T \colon X \to X$ of a compact metrizable space the canonical unitary in the crossed product always has nontrivial K_1 class and this prevents $C(X) \rtimes \mathbb{Z}$ from being an AF algebra. When T is minimal and X is zero-dimensional, the K_1

class of the canonical unitary is in some sense the only obstruction to approximate finite-dimensionality and, as shown by Putnam [79], the crossed product turns out to be an A$\mathbb{T}$ algebra with real rank zero, as will be explained in the next section. Recall that an A$\mathbb{T}$ algebra an inductive limit of algebras of the form $M_{j_1}(C(\mathbb{T})) \oplus \cdots \oplus M_{j_m}(C(\mathbb{T}))$, while real rank zero means that self-ajoint elements can be approximately by self-adjoint elements with finite spectrum. For irrational rotations of the circle, Elliott and Evans showed that the crossed product is also an A$\mathbb{T}$ algebra with real rank zero [27]. These examples are all captured by Elliott's classification theorem from [26]. Q. Lin and Phillips [69] proved that the crossed product of a minimal diffeomorphism of a compact smooth manifold is an inductive limit of recursive subhomogeneous algebras, as defined in Section 4.4. As a consequence, many of these crossed products fall under the purview of the classification of real rank zero AH algebras with slow dimension growth.

As more powerful classification methods have developed, it has become no longer necessary to directly exhibit inductive limit decompositions in order to be able to situate various crossed products within certain classes of K-theoretically classifiable C^*-algebras. The key to verifying K-theoretic classifiability along more abstract lines is to show that the C^*-algebra satisfies a suitable noncommutative version of topological zero-dimensionality, which in conjunction with other hypotheses will permit the appeal to an appropriately high-powered classification theorem. In particular this will actually imply that the crossed product has certain inductive limit structure in view of the cumulative way in which classification theory has developed. Thus for higher-dimensional X we would like to know when the crossed product still exhibits the kind of noncommutative topological zero-dimensionality that one sees in the strictest sense in AF algebras and to a slightly weaker degree in A$\mathbb{T}$ algebras. What topological dimension might mean in this noncommutative context, especially as it impacts classification, has been the subject of much investigation and has led to development of the various notions of real rank, stable rank, tracial rank, radius of comparison, decomposition rank, and nuclear dimension. The long established notions of real rank zero and stable rank zero are finite spectrum approximability conditions on single elements and thus do not provide enough structural leverage in themselves to lead to classification results. On the other hand, tracial rank, decomposition rank, and nuclear dimension, like the venerable concept of approximate finite-dimensionality, all involve the modelling of arbitrary finite subsets of the C^*-algebra by finite-dimensional structure in a robust enough way to be directly consequential for classification.

As we will see in the following sections, the basic strategy for estimating the dimension of a crossed product is to obtain a Rokhlin tower decomposition via a first return time map. The additive dynamical structure that is thereby revealed takes on a matricial form in the crossed product. What is also of interest is the tension between this additive structure and the multiplicative structure that underlies entropy and mean dimension, and indeed we will see how multiplicative structure at the dimensional scale affects classifiability via the radius of comparison in Section 4.5.

A simple unital C^*-algebra has *tracial rank zero* if for every finite set $\Omega \subseteq A$, $\varepsilon > 0$, and nonzero positive element $c \in A$ there is a projection $p \in A$ and a unital finite-dimensional subalgebra B of pAp such that

1. $\|[a, p]\| < \varepsilon$ for all $a \in \Omega$,

2. $pap \subseteq_\varepsilon B$ for all $a \in \Omega$, and

3. $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{cAc}$.

Building on ideas from the inductive limit classification work of Elliott and Gong [28], H. Lin proved the following classification theorem in [64]. Recall that the Elliott invariant for a unital C^*-algebra consists of (i) the ordered K_0 group with the class of the unit, (ii) the K_1 group, (iii) the tracial state space, and (iv) the pairing between the K_0 group and the tracial state space given by evaluation.

Theorem 4.2.1. *Simple unital nuclear C^*-algebras which have tracial rank zero and satisfy the UCT are classified by their Elliott invariant.*

The C^*-algebras in the above theorem turn out to be the simple unital AH algebras (AH meaning an inductive limit of homogeneous algebras) with real rank zero and slow dimension growth. Slow dimension growth refers to the asymptotic vanishing of the ratios of the dimensions of the space to the rank of the cut-down projections in some inductive limit presentation with homogeneous building blocks, and among the simple unital AH algebras with real rank zero it is equivalent to the existence of an inductive limit presentation with homogeneous building blocks whose spaces have dimension at most three [21]. In [68] H. Lin and Phillips proved that, for a minimal homeomophism of an infinite compact metrizable space X with finite covering dimension, if the canonical map from $K_0(C(X) \rtimes \mathbb{Z})$ to the space of affine functions on the tracial state space of $C(X) \rtimes \mathbb{Z}$ has dense image then $C(X) \rtimes \mathbb{Z}$ has tracial rank zero. By appealing to H. Lin's classification theorem they deduced that such crossed products are AH algebras with real rank zero. This extends the classification of crossed products of minimal homeomorphisms of the Cantor set and of irrational rotations of the circle.

Meanwhile Winter introduced the idea of refining the completely positive approximation property that is characteristic of nuclearity so that it additionally picks up some topological (i.e., multiplicative) information in a way that can be used to extend the notion of covering dimension to noncommutative C^*-algebras. There are some variations on how this can be done, the most important of which for classification purposes are decomposition rank [59] and the more flexible nuclear dimension [109], both of which we now review.

Let A be a C^*-algebra.

1. A completely positive map φ from a C^*-algebra B to A said to have *order zero* if $\varphi(a)\varphi(b) = 0$ for all self-adjoint $a, b \in B$ satisfying $ab = 0$. It turns out these are precisely the maps $\varphi \colon B \to A$ that arise as $a \mapsto \pi(a)h$ for some

*-homomorphism $\pi\colon B \to A$ and positive element h in the commutant of $C^*(\varphi(B))$ inside its multiplier algebra.

2. We say that a completely positive map φ from a finite-dimensional C^*-algebra B to A is *n-decomposable* if we can write $B = B_0 \oplus \cdots \oplus B_n$ so that the restriction of φ to each B_i has order zero.

3. The *decomposition rank* of A is the least integer n such that for every finite set $\Omega \subseteq A$ and $\varepsilon > 0$ there are a finite-dimensional C^*-algebra B and completely positive contractions $\varphi\colon A \to B$ and $\psi\colon B \to A$ such that $\|\psi \circ \varphi(a) - a\| < \varepsilon$ for all $a \in \Omega$ and *psi* is *n*-decomposable.

4. The *nuclear dimension* of A of A is the least integer n such that for every finite set $\Omega \subseteq A$ and $\varepsilon > 0$ there are a finite-dimensional C^*-algebra B, completely positive contraction $\varphi\colon A \to B$, and a completely positive map $\psi\colon B \to A$ such that $\|\psi \circ \varphi(a) - a\| < \varepsilon$ for all $a \in \Omega$ and *psi* is *n*-decomposable with contractive order zero components.

5. A has *locally finite nuclear dimension* if for every finite set $\Omega \subseteq A$ and $\varepsilon > 0$ there is a C^*subalgebra $D \subseteq A$ with finite nuclear dimension such that $\Omega \subseteq_\varepsilon D$.

Incorporating the notions of decomposition rank and nuclear dimension, Winter initiated an innovative approach to classification in the stably finite realm whose broad scheme parallels what was done in the settings of injective II_1 factors and purely infinite nuclear C^*-algebras, with a new twist involving the passage from UHF-stable classification to $\mathcal{Z}$-stable classification. The outcome is the following result [106, 108].

Theorem 4.2.2. *The class of unital simple separable nonelementary C^*-algebras with finite decomposition rank which satisfy the UCT and have the property that projections separate traces is classified by the Elliott invariant.*

The strategy of the proof can be broken into two parts:

1. Show that if A is unital, simple, and separable, has finite decomposition rank, and its projections separate traces, then $A \otimes \mathcal{U}$ has tracial rank zero for every UHF algebra $\mathcal{U}$ of infinite type, and then apply a tracial rank zero classification theorem [65, 67] to deduce that such C^*-algebras are classified by their Elliott invariant up to stabilization by $\mathcal{Z}$.

2. Show that every unital simple separable nonelementary C^*-algebra with finite nuclear dimension is $\mathcal{Z}$-stable.

Step (2) was originally established for finite decomposition rank in [108] but later the more flexible nuclear dimension was shown to be sufficient [107]. However, decomposition rank is still at present needed to clinch the first part of step (1), and one would like to be able to replace it here as well with nuclear dimension.

Toms and Winter showed that, for a minimal homeomophism of an infinite compact metrizable space X with finite covering dimension, the crossed product has nuclear dimension at most $2\dim(X) + 1$. Although this is not sufficient to be able to appeal directly to Theorem 4.2.2, which requires finite decomposition rank, Toms and Winter were nevertheless able, by applying an argument of H. Lin and Phillips from [68], to verify step (1) above under the extra assumption that projections separate the traces, which is automatic if the action is uniquely ergodic. Toms and Winter also proved $\mathcal{Z}$-stability under the same hypotheses, but this now follows from the general result of Winter on nuclear dimension and $\mathcal{Z}$-stability. We will present the argument of Toms and Winter for nuclear dimension in Section 4.4, after first analyzing minimal homeomorphisms of the Cantor set and their crossed products in the next section.

Despite the remarkable successes discussed above, examples of Villadsen, Rørdam, and Toms have shown that the classification program for nuclear C^*-algebras in its K-theoretic formulation necessarily requires some regularity assumptions like $\mathcal{Z}$-stability, or the addition of finer invariants. In these examples, topological behaviour of a dimensional nature also plays an important role. Villadsen showed how perforation in the ordered K_0 of certain manifolds like $\mathbb{T}^4$ can be propagated across building blocks in an AH algebra so as to persist in the limit. Toms did the same but in the Cuntz semigroup instead of K_0, and was thereby able to produce a simple AH algebra A whose tensor product with the universal UHF algebra is a simple AI algebra with the same Elliott invariant as A but is not isomorphic to A. The perforation exhibited by the Cuntz semigroup of the C^*-algebra A means that it lies outside of the class of $\mathcal{Z}$-stable C^*-algebras. The C^*-algebra A also has the related property that the ordering on the Cuntz semigroup is not determined by the ordering on lower semicontinuous dimension functions. Toms introduced the radius of comparison as a numerical measure of the failure of the ordering to be determined in this way.

To explain these notions, we first we recall the definition of the Cuntz semigroup $W(A)$ for a C^*-algebra A. For elements a, b in $M_\infty(A)^+ = \bigcup_{n=1}^\infty M_n(A)^+$ (viewing $M_n(A)^+$ as an upper left-hand corner in $M_m(A)^+$ for $m > n$), we write $a \precsim b$ if there is a sequence $\{t_k\}_k$ in $M_{m,n}(A)$ such that $\lim_{k\to\infty} t_k^* b t_k = a$, and $a \sim b$ if $a \precsim b$ and $b \precsim a$. Set $W(A) = M_\infty(A)^+ / \sim$ and write $\langle a \rangle$ for the equivalence class of a. For $a \in M_n(A)^+$ and $b \in M_m(A)^+$ set $\langle a \rangle + \langle b \rangle = \langle a \oplus b \rangle$ where $a \oplus b = \operatorname{diag}(a, b) \in M_{n+m}(A)^+$. We declare that $\langle a \rangle \leq \langle b \rangle$ when $a \precsim b$. This gives $W(A)$ with the structure of a positively ordered Abelian semigroup.

Associated to a quasitrace τ on A is the lower semicontinuous map $s_\tau \colon M_\infty(A)^+ \to \mathbb{R}^+$ given by $s_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n})$. The value $s_\tau(a)$ depends only on the Cuntz equivalence class of a, and we thereby regard s_τ as a state on $W(A)$. These states are called *lower semicontinuous dimension functions*. When A is exact the states on $W(A)$ can be identified with the quasitraces on A [9] and hence with the tracial states on A [40, 41].

The Cuntz semigroup $W(A)$ is *almost unperforated* if for all $a, b \in W(A)$ we have $a \precsim b$ whenever $(n+1)a \leq nb$ for some $n \in \mathbb{N}$. We say that A has

strict comparison if for all $a, b \in M_\infty(A)^+$ we have $a \precsim b$ whenever $s(a) < s(b)$ for all lower semicontinuous dimension functions s on $W(A)$. Rørdam proved in [85] that $\mathcal{Z}$-stability implies strict comparison, and that, for a simple unital exact C^*-algebra A, if $W(A)$ is almost unperforated then A has strict comparison.

A theorem of Kirchberg demonstrates that, for traceless simple separable nuclear C^*-algebras, $\mathcal{Z}$-stability and strict comparison are equivalent (see [85]). It is conjectured that for simple unital separable nuclear C^*-algebras the following are equivalent: (i) finite nuclear dimension, (ii) $\mathcal{Z}$-stability, and (iii) strict comparison. The implication (i)$\Rightarrow$(ii) is a result of Winter that goes into the proof of his Theorem 4.2.2, while (ii)$\Rightarrow$(iii) is contained in the results of Rørdam mentioned above [85]. For AH algebras the equivalence of all three conditions was proved in [93, 92].

We say that A has *r-comparison* if for all $a, b \in M_\infty(A)^+$ we have $\langle a \rangle \leq \langle b \rangle$ whenever $s(\langle a \rangle) + r < s(\langle b \rangle)$ for all lower semicontinuous dimension functions s on $W(A)$. The *radius of comparison* of A is the infimum of the set of all $r \in \mathbb{R}^+$ for which A has r-comparison, unless this set is empty, in which case it is defined to be ∞. Note that if A is simple, unital, and exact then $\mathcal{Z}$-stability implies that the radius of comparison is zero. The radius of comparison scales under tensoring with a matrix algebra, and indeed it is the asymptotic ratio between the matrix size and the topological dimension of the base spaces that controls the value in an inductive limit of homogeneous algebras. In contrast, decomposition rank and nuclear dimension are stable under tensoring with matrix algebras, and, especially in view of Theorem 4.2.2, one might argue that finite values of these invariants in the simple case should be thought of as a manifestation of zero-dimensionality in the way that we conceive of it for ordinary spaces.

In the next three sections we will see how all of these ideas play out in the study of crossed products of minimal $\mathbb{Z}$-actions.

4.3 Minimal homeomorphisms of zero-dimensional spaces

The topological setting in which one has the most direct analogue of the Rokhlin tower lemma for measure-preserving transformations is that of minimal homeomorphisms of the Cantor set (or more generally zero-dimensional compact metric spaces, but by minimality the only extra examples we would obtain are the cyclic permutations of a finite set). This Rokhlin lemma, which unlike the measure-preserving case must admit several towers in general, establishes the link to matricial structure in the crossed product. Since the canonical unitary in the crossed product yields, via the Pimsner-Voiculescu exact sequence, a K_1 obstruction to approximate finite-dimensionality, the crossed product will not be an AF algebra, but rather the next closest thing, namely an A$\mathbb{T}$ algebra whose K_1 group is equal to $\mathbb{Z}$. By erasing that part of the crossed product that reflects the dynamics connecting

the tops of the Rokhlin towers to the base, one effectively cuts the K_1 obstruction and obtains a finite-dimensional subalgebra of the crossed product. (As will be discussed in the next section, these ideas can be applied more generally, and to great effect, to homeomorphisms of finite-dimensional compact metric spaces with finite-dimensional subalgebras being replaced by recursive subhomogeneous subalgebras.) By taking a sequence of finer and finer Rokhlin tower decompositions one can construct an AF algebra out of these finite-dimensional subalgebras which contains all of the ordered K_0 information of the crossed product in a canonical way. However, if we are only concerned with showing that $C(X) \rtimes \mathbb{Z}$ is an $A\mathbb{T}$ algebra, one can apply Berg's technique in conjunction with two nested iterations of the Rokhlin lemma to get local approximability of a prescribed finite set of crossed product elements by some subalgebra of the form $(M_{j_0} \otimes C(\mathbb{T})) \oplus M_{j_1} \oplus \cdots \oplus M_{j_r}$. Such "circle" subalgebras are semiprojective and thus can be assembled, with the appropriate unitary twisting, into an inductive limit that expresses $C(X) \rtimes \mathbb{Z}$ as an $A\mathbb{T}$ algebra. We will now show how to produce the local approximation by circle algebras, as demonstrated by Putnam in [79]

Theorem 4.3.1. *Let $T \colon X \to X$ be a minimal homeomorphism of the Cantor set. Then for every finite set $\Omega \subseteq C(X) \rtimes \mathbb{Z}$ and $\varepsilon > 0$ there is a unital C^*-algebra A of $C(X) \rtimes \mathbb{Z}$ which is *-isomorphic*

$$(M_{n_1} \otimes C(\mathbb{T})) \oplus M_{n_2} \oplus \cdots \oplus M_{n_l}$$

for some $n_1, \ldots, n_l \in \mathbb{N}$ and approximately contains Ω to within ε.

Proof. The result will follow upon showing that, given an $\varepsilon > 0$ and a clopen partition $\mathcal{P}$ of X, there is an B as in the theorem statement such that $C(\mathcal{P}) \subseteq A$ and $\|u - \tilde{u}\| < \varepsilon$ for some unitary $\tilde{u} \in B$. The idea is to first generate one Rokhlin tower decomposition to produce a unital finite-dimensional C^*-sublagebra A_0 of $C(X) \rtimes \mathbb{Z}$ that contains $C(\mathcal{P})$ diagonally, and then generate a second tower decomposition nested in the first from which we can produce a unitary u_1 which commutes with A and contains all of the first return information in the second tower. This unitary will be responsible for the $C(\mathbb{T})$ part in the circle algebra, and by multiplying it by the permutation unitary v_1 that mirrors the second tower decomposition without the first return information we recover the canonical crossed product unitary u. The only problem is that v_1 will not be approximately contained in the C^*-subalgebra generated by A_0 and u_1. To fix this one applies Berg's technique to construct a unitary z that gradually exchanges the actions of v_1 and its counterpart v_0 for the first tower decomposition along sufficiently long segments of the towers in the second decomposition. This done by applying the functional calculus to $v_1^* v_0$, which lies in the finite-dimensional C^*-subalgebra generated by the second tower decomposition with the first return information erased. The unitary z will commute with u_1 and $z A_0 z^*$ will approximately contain v_1 and hence also $u = u_1 v_1$, and so the C^*-subalgebra B we are looking for is generated by $z A z^*$ and u_1. We now describe how all of this works in more detail.

Starting with an $\varepsilon > 0$ and a clopen partition $\mathcal{P}$, choose an $m \in \mathbb{N}$ such that $\pi/m < \varepsilon$. Take a nonempty clopen subset Y of X such that the sets $Y, TY, \ldots, T^m Y$ are pairwise disjoint and for each $n = 0, \ldots, m$ the set $T^n Y$ is contained is some member of $\mathcal{P}$. We build Rokhlin towers over Y by considering the first return time map $\gamma \colon Y \to \mathbb{N}$ given by

$$\gamma(x) = \inf\{n \in \mathbb{N} : T^n x \in Y\}.$$

Since Y is clopen this map is continuous and hence takes on only finitely many values $n_1, \ldots, n_l$. For $k = 1, \ldots, l$ define Y_k to be the clopen set $\gamma^{-1}(n_k)$. Thus we have for each $k = 1, \ldots, l$ a tower whose levels $TY_k, T^2 Y_k, \ldots, T^{n_k} Y$ we can picture as being shifted upward under T except for the top one. The union $\bigcup_{k=1}^{l} T^{n_k} Y$ of the tops of all of the towers, which is equal to Y, gets mapped under T to the union $\bigcup_{k=1}^{l} TY_k$ of all of the bases, and by minimality this operation will involve some mixing between different towers. Note that since the union of all of the towers is closed and T-invariant, by minimality the collection $\mathcal{P}_0$ of all towers must be a partition of X. We may moreover assume, by dividing up towers if necessary, that $\mathcal{P}_0$ refines $\mathcal{P}$.

Now we construct the unital finite-dimensional C^*-subalgebra $A_0 \cong M_{n_1} \oplus \cdots \oplus M_{n_l}$ of $C(X) \rtimes \mathbb{Z}$ by defining for each $k = 1, \ldots, K$ the matrix units $e_{ij}^{(k)} = 1_{T^i Y_k} u^{i-j}$ where $1 \leq i, j \leq n_k$. Then $C(\mathcal{P}_0)$ is the span of the diagonals of these K matricial summands, which therefore contains $C(\mathcal{P})$. Since M_n is generated as a C^*-algebra by the diagonal matrices and the matrix with ones on the subdiagonal and zeros elsewhere, we see that A_0 is the C^*-algebra generated by $C(\mathcal{P}_0)$ and $u1_{X \setminus Y}$.

Define the unitary

$$v_0 = \sum_{k=1}^{l} \sum_{i=2}^{n_k} \left(e_{i,i-1}^{(k)} + e_{1,n_k}^{(k)} \right) \in A_0,$$

which acts via conjugation on $C(\mathcal{P}_0)$ by shifting all but the top levels of each tower upward, just like u, but with the top level of the each tower shifted to the bottom level of the same tower so as to produce the identity on the base after cycling through all of the levels. Then the unitary $u_0 = v_0^{-1} u$ contains all of the first return information of u and is the identity in the cut-down of the crossed product by the characteristic function of the complement of the unions of the tower tops.

Now set $Z = T^{n_1} Y_1$ and construct a second Rokhlin tower decomposition $\mathcal{P}_1 = \{T^i Z_k : 1 \leq k \leq l' \text{ and } 1 \leq i \leq n_k'\}$. We may assume that $\mathcal{P}_1$ is finer than $\mathcal{P}_0$ and that $1_Y \in C(\mathcal{P}_1)$. As before we define the finite-dimensional C^*-subalgebra A_1 generated by $C(\mathcal{P}_1)$ and $u1_{X \setminus Z}$, and unitaries v_1 and u_1 such that $u = v_0 u_0$. The unitary u_1 conjugates each of the characteristic functions of the levels of the first tower decomposition, and thus commutes with A_0. To complete the argument we will show that there is a unitary $z \in A_1$ which commutes with u_1 and $C(\mathcal{P})$ and conjugates v_0 to an element close to v_1. The C^*-algebra generated by $z A_0 z^*$ and u_1 will then have the properties that we are seeking.

Since $v_1 v_0^*$ is contained in the finite-dimensional algebra A_1 and hence has finite spectrum, we can apply the functional calculus to produce a unitary $w \in A_1$ such that $w^m = v_1 v_0^*$ and $\|w - 1\| < \pi/m \le \varepsilon$. We then define the unitary $z \in A_1$ so that (i) its cut-down by $1_{X \setminus TY \cup \cdots \cup T^m Y}$ is equal to $1_{X \setminus TY \cup \cdots \cup T^m Y}$, and (ii) its cut-down by $1_{TY \cup \cdots \cup T^m Y}$, viewed as an $m \times m$ block diagonal matrix with respect to the levels $TY, \ldots, T^m Y$, is diagonal with the $(n-1)$st entry down the diagonal equal to $u^n w^{m-n} u^{-n}$. One can then verify that for each $n = 0, \ldots, m-1$ we have

$$(z v_0 - v_1 z) 1_{T^n Y} = u^n w^{m-n} (1 - w) u^{1-n}$$

and hence $\|(z v_0 - v_1 z) 1_{T^n Y}\| < \pi/m \le \varepsilon$. Since $(z v_0 - v_1 z) 1_{X \setminus TY \cup \cdots \cup T^m Y} = 0$ it follows that $\|z v_0 - v_1 z\| < \varepsilon$.

Now define A to be the C^*-algebra generated by $z A_0 z^*$ and u_1. Then $C(\mathcal{P}) \subseteq A$ since $C(\mathcal{P}) \subseteq A_0$ and z commutes with $C(\mathcal{P})$. Furthermore, the unitary $\tilde{u} = (z v_0 z^*) u_1 \in A$ satisfies $\|u - \tilde{u}\| = \|z v_0 - v_1 z\| < \varepsilon$. It remains to show that $A \cong (M_{n_1} \otimes C(\mathbb{T})) \oplus M_{n_2} \oplus \cdots \oplus M_{n_K}$.

Via conjugation A is *-isomorphic to the C^*-algebra B generated by A_0 and u_1. In this algebra we define the unitary

$$u' = \sum_{i=1}^{n_1} e_{i,n_1}^{(1)} u_1 e_{i_1,n}^{(1)} + \sum_{k=2}^{l} p_k$$

where p_k is the characteristic function of the union of the levels of the kth tower in the first decomposition. Then A_0 and u' generate B, u' commutes with A_0, and $u' p_k = p_k u' = p_k$ for all $k = 2, \ldots, K$. Finally we check that the spectrum of u' is the entire unit circle. Note that in K_1 we have $[v_1] = 0$ since v_1 has finite spectrum and $[u] \ne 0$ by the Pimsner-Voiculescu sequence, so that $[u_1] \ne 0$. Therefore u_1, and hence also u', has spectrum equal to the unit circle. $\qquad \square$

The problem of establishing local approximation by subhomogeneous C^*-algebras as in the above proof becomes substantially more difficult for homeomorphisms of higher-dimensional spaces. Additionally, we do not expect the crossed products to be $A\mathbb{T}$ algebras, whose circle algebra building blocks are semiprojective, and hence local approximation will not by itself be enough to obtain an inductive limit decomposition. On the other hand, C^*-classification technology has been developed to such a powerful degree that to establish classifiability of the crossed product it is sufficient to verify some abstract properties like tracial rank zero, finite nuclear dimension, or $\mathcal{Z}$-stability, and this does not need anywhere near the full strength of an inductive limit decomposition, or even local approximation in the strictest sense. One can then conclude the existence of inductive limit decomposition for the crossed product by applying classification results. To show that many crossed products possess one or more of the above abstract properties, one uses the analogue of the C^*-algebra A_0 in the proof of Theorem 4.3.1 in a fundamental way that avoids having to locally approximate

the canonical unitary itself, specifically as it reflects the first return information in a Rokhlin tower decomposition. This we discuss next.

4.4 Minimal homeomorphisms of finite-dimensional spaces

Let X be a compact metric space and let $T\colon X \to X$ be a minimal homeomorphism. Let Y be a closed subset of X with nonempty interior. By generating Rokhlin towers according to the first return to Y as in the Cantor set case in the last section, we would like to build a C^*-subalgebra of $C(X) \rtimes \mathbb{Z}$ with a simple structure that reflects the approximate periodicity of the Rokhlin decomposition with the information about the first return dynamics erased. The problem is that if Y is not clopen then we will have overlapping boundaries amongst the levels of different towers. However, this overlapping can be controlled as one moves from shorter towers to taller ones in such a way that we can realize the desired subalgebra by a recursive procedure involving pullbacks of subhomogeneous C^*-algebras. C^*-algebras that arise in this way are called recursive subhomogeneous algebras and they form a particularly tractable and rather broad class of subhomogeneous algebras.

If A, B, and C are C^*-algebras and $\varphi\colon A \to C$ and $\psi\colon B \to C$ are homomorphisms then the *pullback* $A \oplus_C B = A \oplus_{C,\varphi,\psi} B$ is defined as $\{(a,b) \in A \oplus B : \varphi(a) = \psi(b)\}$. This is the noncommutative generalization of the operation of gluing together two locally compact Hausdorff spaces along a common closed subspace. If we are given a C^*-algebra A, a compact Hausdorff space X, a closed subset $X^{(0)} \subseteq X$, and a unital homomorphism $\varphi\colon A \to C(X^{(0)}, M_n)$, then taking the restriction homomorphism $\psi\colon C(X, M_n) \to C(X^{(0)}, M_n)$ we can form the associated pullback $A \oplus_{C(X^{(0)}, M_n)} C(X, M_n)$. A *recursive subhomogeneous algebra* is a C^*-algebra that can be obtain by the recursive application of such pullbacks starting with a homogeneous algebra $C(X, M_n)$ and taking the algebra A in the construction of the $(k+1)$st pullback to be the kth pullback. As a special case we can produce a direct sum $C(X_0, M_{n_0}) \oplus \cdots \oplus C(X_r, M_{n_r})$ by taking $X^{(0)}$ to be empty at every stage.

Recursive subhomogeneous algebras form a subclass of the subhomogeneous C^*-algebras, which are defined by the existence of a finite bound on the dimension of the irreducible representations. It is shown in [105] that a subhomogeneous C^*-algebra has decomposition rank n if and only if it has a recursive subhomogeneous decomposition with topological dimension n, which means that the maximum covering dimension of the spaces appearing in the decomposition is n. We will need this fact in the proof of Theorem 4.4.6, which, although the conclusion concerns nuclear dimension, requires an appeal to the global contractivity of n-decomposable maps in the definition of decomposition rank as applied to the recursive subhomogeneous algebras that we now describe.

Let X be a compact metrizable space and $T\colon X \to X$ is minimal homeomorphism. For a closed set $Y \subseteq X$ we define A_Y to be the C^*-subalgebra of $C(X) \rtimes \mathbb{Z}$ generated by $C(X)$ and $uC_0(X \setminus Y)$.

Theorem 4.4.1. *Let X be an infinite compact metrizable space and $T\colon X \to X$ a minimal homeomorphism. Let Y be a closed subset of X with nonempty interior. Then the C^*-subalgebra $A_Y \subseteq C(X) \rtimes \mathbb{Z}$ has a recursive subhomogeneous decomposition with topological dimension equal to $\dim(X)$.*

The proof of this theorem of Q. Lin, which we will briefly outline, relies on a Rokhlin tower decomposition as in the Cantor set situation. Define the first return time map $\gamma\colon Y \to \mathbb{N}$ by

$$\gamma(x) = \inf\{n \in \mathbb{N} : T^n x \in Y\}.$$

Since Y is closed this map is upper semi-continuous and hence takes on only finitely many values $n_1, \ldots, n_K$. This determines a partition $Y_1, \ldots, Y_K$ of Y. These sets are not necessarily closed, although $Y_1 \cup \cdots \cup Y_k$ is closed for every $k = 1, \ldots, K$. The sets $T^j Y_k$ for $k = 1, \ldots, K$ and $j = 1, \ldots, j_k$ partition X into K towers, and the union $\bigcup_{k=1}^{K} Y_k$ of the tops of all of the towers, which is equal to Y, gets mapped under T to the union $\bigcup_{k=1}^{K} TY_k$ of all of the bases.

Now set $B_Y = \bigoplus_{k=1}^{l} C(\overline{Y}_k, M_{n_k})$. Then one has a canonical unital embedding $\rho\colon A_Y \to B_Y$ where in $C(\overline{Y}_k, M_{n_k}) \cong M_{n_k}(C(\overline{Y}_k))$ the component of $\rho(f)$ for $f \in C(X)$ is the diagonal matrix

$$\begin{pmatrix} f|_{TY_k} \circ T & 0 & \cdots & 0 \\ 0 & f|_{T^2 Y_k} \circ T^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f|_{T^{n_k} Y_k} \circ T^{n_k} \end{pmatrix}$$

and the component of $\rho(u)$ in $C(\overline{Y}_k, M_{n_k})$ is the subdiagonal matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Since B_Y is homogeneous this shows that A_Y is subhomogeneous, but we would like to know that A_Y is in fact recursive subhomogeneous. To this end, for each $m = 1, \ldots, l$ we define the projection map $\pi_m\colon B_Y \to \bigoplus_{k=1}^{m} C(\overline{Y}_k, M_{n_k})$ onto the first k summands and set $A_m = \pi_m(A_Y)$. One can then demonstrate, as an effect of how the points in ∂Y_m return to Y at times before n_m, that

$$\varphi_m(b_1, \ldots, b_{m-1}) = b_m|_{\partial Y_m \cap (Y_1 \cup \cdots \cup Y_m)}$$

gives a well-defined homomorphism $\varphi_m \colon A_{m-1} \to C(\partial Y_m \cap (Y_1 \cup \cdots \cup Y_m), M_{n_m})$, and that the pullback $A_{m-1} \oplus_{C(\partial Y_m \cap (Y_1 \cup \cdots \cup Y_m), M_{n_m})} C(\overline{Y}_m, M_{n_m})$ obtained from φ_m and the restriction map $C(\overline{Y}_m, M_{n_m}) \to C(\partial Y_m \cap (Y_1 \cup \cdots \cup Y_m), M_{n_m})$ is isomorphic to A_m. As $A_Y = A_l$, this yields Theorem 4.4.1.

Using the Theorem 4.4.1, H. Lin and Phillips proved the following [68].

Theorem 4.4.2. *Let X be an infinite compact metrizable space with finite covering dimension and $T \colon X \to X$ a minimal homeomorphism. Suppose that the canonical map from $K_0(C(X) \rtimes \mathbb{Z})$ to the space of affine functions on the tracial state space of $C(X) \rtimes \mathbb{Z}$ has dense image. Then $C(X) \rtimes \mathbb{Z}$ has tracial rank zero.*

Theorem 4.4.3. *The class of crossed products in Theorem 4.4.2 is classified by the Elliott invariant, and they are all AH algebras with real rank zero.*

We will now give a proof of a more recent result of Toms and Winter that says that if X an infinite compact metrizable space with finite covering dimension and $T \colon X \to X$ a minimal homeomorphism then $C(X) \rtimes \mathbb{Z}$ has nuclear dimension at most $2 \dim(X) + 1$. This relies on the recursive subhomogeneous structure described above, and it leads to the crossed product classification result recorded below as Theorem 4.4.8. Note that the hypotheses that X is infinite and the homeomorphism T is minimal imply that the action is free, so that $C(X) \rtimes \mathbb{Z}$ simple by Theorem 1.0.1.

Lemma 4.4.4. *Let A be a separable C^*-algebra. Let $d \in \mathbb{N}$. Suppose that for every finite set $\Omega \subseteq A$ and $\varepsilon > 0$ there are C^*-subalgebras $A_0, A_1 \subseteq A$ with decomposition rank at most d and a $h \in A$ such that $\mathrm{dist}(ah, A_0) < \varepsilon$, $\mathrm{dist}(a(1-h), A_1) < \varepsilon$, and $\|[a, h]\| < \varepsilon$ for all $a \in \Omega$. Then A has nuclear dimension at most $2d + 1$.*

Proof. Let Ω be a finite subset of A and $\varepsilon > 0$. By hypothesis there are C^*-subalgebras $A_0, A_1 \subseteq A$ with decomposition rank at most d, an $h \in A$, and $b_{a,0} \in A_0$ and $b_{a,1} \in A_1$ for $a \in \Omega$ such that for every $a \in \Omega$ the quantities $\|ah - b_{a,0}\|$ and $\|a(1-h) - b_{a,1}\|$ are less than $\varepsilon/8$ and $\|[a, h]\|$ is sufficiently small to guarantee that $\|h^{1/2}ah^{1/2} - ah\| < \varepsilon/8$ and $\|(1-h)^{1/2}a(1-h)^{1/2} - a(1-h)\| < \varepsilon/8$. Since A_0 and A_1 have decomposition rank at most d we can find for each $i = 0, 1$ a finte-dimensional C^*-algebra B_i and completely positive contractions $\varphi_i \colon A_i \to B_i$ and $\psi_i \colon B_i \to A_i$ such that ψ_i is d-decomposable and $\|\psi_i \circ \varphi_i(b_{a,i}) - b_{a,i}\| < \varepsilon/8$ for all $a \in \Omega$. For each $i = 0, 1$ apply Arveson's extension theorem to extend φ_i to a completely positive contraction $\tilde{\varphi}_i \colon A \to F_i$. Define a completely positive contraction $\varphi \colon A \to B_0 \oplus B_1$ by

$$\varphi(a) = \tilde{\varphi}_0\big(h^{1/2}ah^{1/2}\big) + \tilde{\varphi}_1\big((1-h)^{1/2}a(1-h)^{1/2}\big)$$

and a completely positive map $\psi \colon B_0 \oplus B_1 \to A$ by

$$\psi((b_0, b_1)) = \psi_0(b_0) + \psi_1(b_1).$$

Then for $a \in \Omega$ we have

$$\|\psi_0 \circ \tilde\varphi_0(h^{1/2}ah^{1/2}) - ah\| \leq \|\psi_0 \circ \tilde\varphi_0(h^{1/2}ah^{1/2} - ah)\| + \|\psi_0 \circ \tilde\varphi_0(ah - b_{a,0})\|$$
$$+ \|\psi_0 \circ \tilde\varphi_0(b_{a,0}) - b_{a,0}\| + \|b_{a,0} - ah\|$$
$$< \frac{\varepsilon}{2}$$

and similarly $\|\psi_1 \circ \tilde\varphi_1((1-h)^{1/2}a(1-h)^{1/2}) - a(1-h)\| < \varepsilon/2$, so that

$$\|\psi \circ \varphi(a) - a\| \leq \|\psi_0 \circ \tilde\varphi_0(h^{1/2}ah^{1/2}) - ah\|$$
$$+ \|\psi_1 \circ \tilde\varphi_1((1-h)^{1/2}a(1-h)^{1/2}) - a(1-h)\|$$
$$< \varepsilon.$$

Since ψ_0 and ψ_1 are d-decomposable we see that ψ is $(2d+1)$-decomposable, and thus, since φ is contractive, we conclude that A has nuclear dimension at most $2d+1$. $\square$

Lemma 4.4.5. *Let X be an infinite compact metrizable space and $T\colon X \to X$ a minimal homeomorphism. Let Ω be a finite subset of $C(X) \rtimes \mathbb{Z}$ and $\delta > 0$. Then there is an $h \in C(X)$ with $0 \leq h \leq 1$ such that $\|[h,a]\| < \delta$ for all $a \in \Omega$ and $h^{-1}(\{0\})$ and $h^{-1}(\{1\})$ both have nonempty interior.*

Proof. By a straightforward approximation argument using the fact that elements of $C(X) \rtimes \mathbb{Z}$ can be approximated by polynomials in u with coefficients in $C(X)$, it is enough to find a positive $h \in C(X)$ such that $\|[h,u]\| < \varepsilon$ for a sufficiently small $\varepsilon > 0$ depending on δ and Ω. Set $n = \lceil 1/\varepsilon \rceil + 1$. Since X is infinite T has no periodic points and so we can find a nonempty open set $U \subseteq X$ which is small enough so that the sets $U, TU, \ldots, T^{2n}U$ are pairwise disjoint. Take a nonempty open set V such that $\overline{V} \subseteq U$ and then take an $f \in C_0(U)$ with $0 \leq f \leq 1$ such that $f|_V = 1$. Define $h \in C_0(U \cup TU \cup \cdots \cup T^{2n}U)$ so that on T^iU it is equal to $(k/n)(f \circ T^{-k})$ for $k = 0, \ldots, n$ and $(2 - k/n)(f \circ T^{-k})$ for $k = n+1, \ldots, 2n$. Then $0 \leq h \leq 1$, $h|_V = 0$, and $h|_{T^nV} = 1$. Finally, since $uhu^* = h \circ T^{-1}$ we see that $\|[u,h]\| = \|uhu^* - h\| = \|f\|/n = 1/n < \varepsilon$. $\square$

Theorem 4.4.6. *Let X be an infinite compact metrizable space with finite covering dimension and let $T\colon X \to X$ a minimal homeomorphism. Then $C(X) \rtimes \mathbb{Z}$ has nuclear dimension at most $2\dim(X) + 1$.*

Proof. To establish the result we verify the hypotheses of Lemma 4.4.4 for $A = C(X) \rtimes \mathbb{Z}$ and $d = 2\dim(X) + 1$. Let Ω be a finite subset of $C(X) \rtimes \mathbb{Z}$ and $\varepsilon > 0$. In order to derive the conclusion of Lemma 4.4.4 we may assume by perturbing Ω that there is a $m \in \mathbb{N}$ such that each element of Ω is a sum of at most m elements of the form $f_1u^{j_1}f_2u^{j_2}\cdots f_mu^{j_m}$ where $f_1, \ldots, f_m \in C(X)$ and $j_1, \ldots, j_m \in \{-1,0,1\}$. By Lemma 4.4.5 there are closed sets $Y_0, Y_1 \subseteq X$ with nonempty interior and an $h \in C(X)$ with $0 \leq h \leq 1$, $h|_{Y_0} = 0$, and $h|_{Y_1} = 1$ such that $\|[u,h]\|$ is

small enough to ensure by a functional calculus argument that $\|[u, h^{1/m}]\|$ are in turn small enough so that $(f_1 u^{j_1} f_2 u^{j_2} \cdots f_m u^{j_m})h$ is within distance ε/m to the element $f_1 v_1 f_2 v_2 \cdots f_m v_m$ of A_{Y_0}, where v_k is equal to $h^{1/m} u^{j_k}$ if $j_k = -1$ and $u^{j_k} h^{1/m}$ otherwise. Summing up we then obtain for every $a \in \Omega$ a $b_{a,0} \in A_{Y_0}$ such that $\|ah - b_{a,0}\| < \varepsilon$. By a similar argument we may assume that $\|[u, (1-h)^{1/m}]\|$ is small enough so that we can find a $b_{a,1} \in A_{Y_1}$ such that $\|a(1-h) - b_{a,1}\| < \varepsilon$. Since A_{Y_0} and A_{Y_1} have decomposition rank equal to $\dim(X)$, we conclude by Lemma 4.4.4 that $C(X) \rtimes \mathbb{Z}$ has nuclear dimension at most $2\dim(X) + 1$. $\square$

In view of the above theorem we ask the following.

Question 4.4.7. Does the crossed product of a minimal homeomorphism of a compact metrizable space X with zero mean dimension have finite nuclear dimension?

As mentioned in Section 4.2, using classification theory one deduces from Theorem 4.4.6 the following result of Toms and Winter. This also requires adapting the argument that H. Lin and Phillips used in the proof of Theorem 4.4.2 in order to deduce that the tensor products of $C(X) \rtimes \mathbb{Z}$ with certain UHF algebras have tracial rank zero.

Theorem 4.4.8. *Let $\mathcal{C}$ be the class of C^*-algebras in which every member (i) arises as a crossed product of a minimal homeomorphism of infinite compact metrizable space with finite covering dimension, and (ii) has the property that projections separate traces. Then $\mathcal{C}$ is classified by the Elliott invariant, and each member is an AH algebra with real rank zero.*

4.5 Mean dimension and comparison in the Cuntz semigroup

Here we construct a minimal homeomorphism whose crossed product has nonzero radius of comparison [31], in the spirit of the AH algebra examples of Toms from [94]. This crossed product will have the same Elliott invariant as an A$\mathbb{T}$ algebra to which it is not isomorphic. The strategy, which was pioneered by Villadsen [96] and refined by Toms, is to propagate an Euler class obstruction across building blocks by ensuring, roughly speaking, that the topological dimension growth outpaces the matricial growth. We do this by means of a standard recursive blocking procedure that enables one to construct minimal subsystems of a shift system $\mathbb{Z} \curvearrowright K^{\mathbb{Z}}$. Using this construction, Lindenstrauss and Weiss exhibited examples of minimal $\mathbb{Z}$-actions with nonzero mean dimension [71]. Mean dimension, as we saw in Section 3.4, is an entropy-like invariant that measures dimension growth in dynamical systems, in close analogy with the radius of comparison, and for these minimal subshifts it is the same underlying structure that is responsible for nonzero values of both invariants. However, it is an open problem to determine the direct relationship between mean dimension and the radius of comparison of

the crossed product. For minimal $\mathbb{Z}$-actions at least, one might expect them to be roughly the same.

Question 4.5.1. For minimal homeomorphisms of compact metrizable spaces, what is the relation between the mean dimension of the homeomorphism and the radius of comparison of the crossed product?

To construct our minimal subshift, start with a compact space Y with a compactible metric ρ gives Y diameter at most one. Define a compatible metric d on $Y^{\mathbb{Z}}$ by $d(x, w) = \sum_{k \in \mathbb{Z}} 2^{-|k|} \rho(x_k, w_k)$ where $x = (x_k)_k$ and $w = (w_k)_k$. Let $T \colon Y^{\mathbb{Z}} \to Y^{\mathbb{Z}}$ be the shift $(x_k)_k \mapsto (x_{k+1})_k$. By a *block* we mean a subset of Y^l of the form $D_1 \times \cdots \times D_l$ for closed sets $D_1, \ldots, D_l \subseteq Y_l$. For a block $B \subseteq Y^l$ and an $i \in \{1, \ldots, l\}$ we write $X_{B,i}$ for the set of all $(x_k)_k \in Y^{\mathbb{Z}}$ such that $(x_{i+sl}, x_{i+sl+1}, \ldots, x_{i+sl+l-1}) \in B$ for every $s \in \mathbb{Z}$. Thus $X_{B,i}$ is the set of sequences that can be partitioned, with a fixed phase described by i, into segments of length l belonging to B. Note that the sets $X_{B,i}$ might not be disjoint. Write X_B for the closed T-invariant subset $\bigcup_{i=1}^l X_{B_i}$ of $Y^{\mathbb{Z}}$.

Let $0 < d < 1$. Our minimal subshift will be defined as the intersection of a decreasing sequence $X_{B_1} \supseteq X_{B_2} \supseteq \ldots$ where the B_n are blocks of the form $Y_{n,1} \times \cdots \times Y_{n,l_n}$ where l_n divides l_{n+1} and

1. for all $x, w \in X_{B_n}$ there is a $k \in \mathbb{Z}$ such that $d(T^k x, w) \leq 2^{-n+3}$, and

2. $Y_{n,i}$ is equal to Y for all i in a subset of $\{1, \ldots, l_n\}$ of cardinality greater than dl_n and is a singleton otherwise.

The block B_{n+1} will be constructed as a subset of $B_n^{l_{n+1}/l_n}$ formed by taking the product of a large number of copies of B_n along with some singletons in Y. These singletons are needed to ensure condition (1), which guarantees that the restriction of T to the intersection of the X_{B_n} is minimal.

To begin with, set $l_1 = 1$ and $B_1 = Y$. Suppose next that we have constructed l_n and $B_n = Y_{n,1} \times \cdots \times Y_{n,l_n}$ satisfying (1) and (2) above. Take a $(z_k)_k \in X_{B_n}$ which contains as a substring the concatenation of a finite collection of l_n-tuples in $B_n \times B_n \times B_n$ which is sufficiently dense to guarantee the existence of an integer $b \geq 2$ such that for all $w = (w_k)_k \in X_{B_n}$ there are an $s \in \{1, \ldots, b-2\}$ and a $j \in \{1, \ldots, l_n\}$ for which $\rho(z_{sl_n+j+k}, w_k) \leq 2^{-n-3}$ for all k in the interval $E_n = \{-l_n, -l_n+1, \ldots, l_n\}$. Let a be a positive integer to be specified in a moment. Set $Y_{n+1,sl_n+i} = Y_{n,i}$ for $s = 0, \ldots, a-1$ and $i = 1, \ldots, l_n$, and $Y_{n+1,(a+s)l_n+i} = \{z_{sl_n+i}\}$ for $s = 0, \ldots, b-1$ and $i = 1, \ldots, l_n$. Put $l_{n+1} = (a+b)l_n$ and $B_{n+1} = Y_{n+1,1} \times \cdots \times Y_{n+1,l_{n+1}}$. Then by taking a sufficiently large realtive to b we can arrange for condition (2) to hold for $Y_{n+1,1}, \ldots, Y_{n+1,l_{n+1}}$ given that it holds for $Y_{n,1}, \ldots, Y_{n,l_n}$.

Finally, to verify (1) let $w = (w_k)_k, (x_k)_k \in X_{B_{n+1}}$. Since $w \in X_n$ we can find an $s \in \{1, \ldots, b-2\}$ and a $j \in \{1, \ldots, l_n\}$ such that $\rho(z_{sl_n+j+k}, w_k) \leq 2^{-n-3}$ for all $k \in E_n$. Since x is contained in one of the sets $X_{B_{n+1},1}, \ldots, X_{B_{n+1},l_{n+1}}$ we

can find an integer m such that $x_{m+sl_n+j+k} \in Y_{n+1,(a+s)l_n+j+k}$ for all $k \in E_n$. Thus, since Y has ρ-diameter at most one and assuming that $l_n \geq n$ as we may,

$$d(T^{m+sl_n+j}x, w)$$
$$\leq \sum_{k \in E_n} 2^{-|k|} \left[\rho(x_{m+sl_n+j+k}, z_{sl_n+j+k}) + \rho(z_{sl_n+j+k}, w_k) \right] + \sum_{k \in \mathbb{Z} \setminus E_n} 2^{-|k|}$$
$$\leq 3 \cdot 2^{-n-2} + 2 \cdot 2^{-l_n} \leq 2^{-n+2},$$

completing the recursive construction.

Now set $X = \bigcap_{n=1}^{\infty} X_n$, which a closed T-invariant subset of $Y^{\mathbb{Z}}$. The restriction of T to X will again be written T. By (1) the system (X, T) is minimal. Since there are no periodic points by construction, the system (X, T) is also free, and so $C(X) \rtimes \mathbb{Z}$ is simple by Theorem 1.0.1.

In the case that Y is the cube $[0, 1]^{3q}$, we will show that, for d above satisfying $1 - 1/q < d < 1$, the radius of comparison is bounded below by $q - 1$. To do this we proceed in two steps:

1. Represent $C(X) \rtimes \mathbb{Z}$ as an inductive limit of the the crossed products $C(X_{B_n} \rtimes \mathbb{Z})$ via the quotients $C(X) \to C(X_{B_n})$ given by restriction.

2. By using an Euler class obstruction for vector bundles over spheres embedded into Y, construct two elements a and b in $M_{2q}(C(Y))$ such that a has smaller rank than b but fails to be Cuntz subequivalent to b in a uniform way under the canonical embedding of $C(Y)$ in each $C(X_{B_n} \rtimes \mathbb{Z})$. The failure of Cuntz subequivalence then passes to the limit $C(X \rtimes \mathbb{Z})$. In each $C(X_{B_n} \rtimes \mathbb{Z})$ one witnesses the Euler class obstruction by mapping into $M_{l_n} \otimes C(B_n)$.

Step 1 is contained in the following lemma.

Lemma 4.5.2. *Let X_1 be a compact Haudorff space and $T \colon X_1 \to X_1$ a homeomorphism. Let $X_2 \subseteq X_3 \subseteq \ldots$ be closed T-invariant subsets of X_1 and set $X = \bigcap_{n=1}^{\infty}$. Let*

$$C(X_1) \rtimes \mathbb{Z} \xrightarrow{\varphi_1} C(X_2) \rtimes \mathbb{Z} \xrightarrow{\varphi_2} C(X_3) \rtimes \mathbb{Z} \xrightarrow{\varphi_3} \ldots$$

be the inductive system with connecting maps induced from the quotients $C(X_n) \to C(X_{n+1})$ via the universal property of the full crossed product. Let $\gamma \colon \varinjlim C(X_n) \rtimes \mathbb{Z} \to C(X) \rtimes \mathbb{Z}$ be the map arising from the maps $\gamma_n \colon C(X_n) \rtimes \mathbb{Z} \to C(X) \rtimes \mathbb{Z}$ induced by the universal property of the full crossed product from the quotients $C(X_n) \to C(X)$. Then γ is a $*$-isomorphism.

Proof. We write u for the canonical unitary in $C(X) \rtimes \mathbb{Z}$ and u_n for the canonical unitary in $C(X_n) \rtimes \mathbb{Z}$. Given a finite sum $\sum_{k \in I} f_k u^k$ where $f_k \in C(X)$, by Tietze's extension theorem we can find, for every $k \in I$, a $g_k \in C(X_1)$ which restricts to f_k on X . For each n the element $\sum_{k \in I} (g_k|_{X_n}) u_n^k \in C(X_n) \rtimes \mathbb{Z}$ is sent to $\sum_{k \in I} (g_k|_{X_{n+1}}) u_{n+1}^k$ under φ_n. Thus $\sum_{k \in I} f_k u^k$ lies in the image of γ, and since

such finite sums are dense in $C(X) \rtimes \mathbb{Z}$ and the image of γ is closed we deduce that γ is surjective.

To establish injectivity, first observe that $C(X)$ can be expressed as the inductive limit $\varinjlim C(X_n)$. Consider on each of our crossed products the dual action of the circle [76, Prop. 7.8.3] which on the canonical unitary is given by $(\lambda, u) \mapsto \lambda u$ and has the canonical commutative C^*-subalgebra as its fixed point subalgebra. We also have a circle action on $\varinjlim C(X_n) \rtimes \mathbb{Z}$ as induced by he dual actions on the crossed products $C(X_n) \rtimes \mathbb{Z}$.

Let a be a positive element in the kernel of γ. The map γ intertwines the circle actions, and so if we integrate the orbit of a over the circle we obtain a positive element b which is contained in both the fixed point subalgebra of $\varinjlim C(X_n) \rtimes \mathbb{Z}$ and the kernel of γ. To conclude that γ is faithful it is then enough to show that $b = 0$, since integration with respect to the dual action is faithful. Now given an $\varepsilon > 0$ there are an $m \in \mathbb{N}$ and a $c \in C(X_m) \rtimes \mathbb{Z}$ such that $\|\gamma_m(c) - b\| < \varepsilon$. We may assume, by integrating with respect to the dual action, that $c \in C(X_m)$. It follows that $\gamma_m(c)$ lies in $\varinjlim C(X_n)$ viewed as a C^*-subalgebra of $\varinjlim C(X_n) \rtimes \mathbb{Z}$, whence

$$\|b\| \leq \|\gamma_m(c)\| + \varepsilon = \|\gamma(\gamma_m(c) - b)\| + \varepsilon < 2\varepsilon.$$

Since ε was an arbitrary positive number we conclude that $b = 0$, as desired. $\square$

Now return to our minimal homeomorphism T of $X = \bigcap_{n=1}^{\infty} X_{B_n}$ and complete step 2. We write θ_r to refer to the trivial vector bundle of rank r over the space in question and $\xi^{\times r}$ for the r-fold Cartesian product of the vector bundle ξ. When regarding vector bundles as projections in matrix algebras over a space, the Cartesian product $\xi^{\times r}$ translates into the sum $(p \otimes 1 \otimes \cdots \otimes 1) + (1 \otimes p \otimes 1 \otimes \cdots \otimes 1) + \cdots + (1 \otimes \cdots \otimes 1 \otimes p)$ of r-fold elementary tensors where p is the projection representing ξ.

Theorem 4.5.3. *Let $q \geq 2$ and suppose that $1 - 1/q < d < 1$ where d is as in the construction. Then for $Y = I^{3q}$ the radius of comparison of $C(X) \rtimes \mathbb{Z}$ is bounded below by $q - 1$.*

Proof. We first construct positive elements $a, b \in M_{2q}(C(Y))$ in the same manner as in [94]. Take a rank one vector bundle ξ on S^2 with nonzero Euler class and a continuous embedding ε of $(S^2 \times [0, 1])^q$ into Y. Let f be a continuous function on Y with $0 \leq f \leq 1$ such that f is one on $\varepsilon((S^2 \times \{\frac{1}{2}\})^q)$ and zero outside of $\varepsilon((S^2 \times (0, 1))^q)$. Writing $\pi \colon S^2 \times [0, 1] \to S^2$ for the projection onto the first coordinate and viewing vector bundles as projections, define a and b to be the positive elements θ_1 and $(1 - f)\theta_q + f\pi^*(\xi)^{\times q}$, respectively, of $M_{2q}(C(Y))$.

Let $\lambda \colon C(Y) \to C(X) \rtimes \mathbb{Z}$ be the composition of the embedding $C(Y) \hookrightarrow C(X)$ arising from the projection $X \to Y$ onto the zeroeth coordinate and the canonical emebdding $C(X) \hookrightarrow C(X) \rtimes \mathbb{Z}$. Consider the positive elements $a_\infty = (\mathrm{id}_{M_{2q}} \otimes \lambda)(a)$ and $b_\infty = (\mathrm{id}_{M_{2q}} \otimes \lambda)(b)$ in $M_{2q} \otimes (C(X) \rtimes \mathbb{Z})$. Since the system (X, T) is minimal, the states on $W(C(X) \rtimes \mathbb{Z})$ are precisely the compositions of the

canonical conditional expectation $C(X) \rtimes \mathbb{Z} \to C(X)$ with T-invariant states on $C(X)$, which correspond via integration to T-invariant Borel probability measures on X (see Section VIII.3 [22]). Hence $s(\langle a_\infty \rangle) = 1$ and $s(\langle b_\infty \rangle) \geq q$ for every state s on $W(C(X) \rtimes \mathbb{Z})$. It thus remains to show that $\langle a_\infty \rangle \not\precsim \langle b_\infty \rangle$.

Given an $n \in \mathbb{N}$, write ψ_n for the composition $C(Y) \to C(X_n) \hookrightarrow C(X_n) \rtimes \mathbb{Z}$ where the second *-homomorphism is the canonical embedding and the first *-homomorphism is the restriction map. Set $a_n = (\mathrm{id}_{M_{2q}} \otimes \psi_n)(a)$ and $b_n = (\mathrm{id}_{M_{2q}} \otimes \psi_n)(b)$. Then a_n and b_n map to a_∞ and b_∞, respectively, under the canonical quotient $M_{2q} \otimes (C(X_n) \rtimes \mathbb{Z}) \to M_{2q} \otimes (C(X) \rtimes \mathbb{Z})$. It is now enough to prove that for each n we have $\|t^* b_n t - a_n\| \geq 1/2$ for all $t \in M_{2q} \otimes (C(X_n) \rtimes \mathbb{Z})$, since this permits us to deduce, in view of Lemma 4.5.2, that $\|t^* b_\infty t - a_\infty\| \geq 1/2$ for all $t \in M_{2q} \otimes (C(X) \rtimes \mathbb{Z})$ and hence that $\langle a_\infty \rangle \not\precsim \langle b_\infty \rangle$.

Let K_n be the set of all $k \in \{1, \ldots, l_n\}$ such that $Y_{n,k} = Y$. Let $\gamma_n \colon C(B_n) \to C(((S^2)^q)^{K_n})$ be the *-homomorphism induced from the continuous embedding $((S^2)^q)^E \to B_n$ under which the kth coordinate of the image of $(x_j)_{j \in K_n} \in ((S^2)^q)^{K_n}$ is $\varepsilon(x_k, \frac{1}{2})$ if $k \in K_n$ and y_k otherwise, where y_k is the unique point in $Y_{n,k}$.

Let $\omega \colon B_n \to X_{B_n}$ be the continuous injection which sends $(x_1, \ldots, x_{l_n})$ to the sequence whose $sl_n + i$ coordinate is x_i for $s \in \mathbb{Z}$ and $i = 1, \ldots, l_n$. The image of ω is the set of l_n-periodic points in $X_{B_n,1}$. By the universal property of the full crossed product there is a *-homomorphism $\varphi_n \colon C(X_B) \rtimes \mathbb{Z} \to M_1 \otimes C(B_n) \cong M_l(C(B_n))$ which sends a function $f \in C(X_{B_n})$ to the diagonal matrix

$$\begin{pmatrix} f \circ \omega & 0 & \cdots & 0 \\ 0 & f \circ T \circ \omega & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f \circ T^{1-l_n} \circ \omega \end{pmatrix}$$

and the canonical unitary u to the shift matrix

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

as we clearly have $\varphi_n(u)\varphi_n(f)\varphi_n(u)^* = \varphi_n(f \circ T^{-1})$. Write ζ_n for the composition

$$C(Y) \xrightarrow{\psi_n} C(X_{B_n}) \rtimes \mathbb{Z} \xrightarrow{\varphi_n} M_{l_n} \otimes C(B_n) \xrightarrow{\mathrm{id} \otimes \gamma_n} M_{l_n} \otimes C(((S^2)^q)^{K_n}).$$

Viewing bundles as projections in matrix algebras, we have $(\mathrm{id}_{M_{2q}} \otimes \zeta_n)(a) = \theta_{l_n}$ and $(\mathrm{id}_{M_{2q}} \otimes \zeta_n)(b) = \xi^{\times q|E|} \oplus \theta_{q(l_n - |E|)}$. Since ξ has nonzero Euler class and $q(l_n - |E|) \leq q l_n(1 - d) < l_n$ by our hypothesis on d, by Lemma 1 of [96] the trivial bundle θ_{l_n} on $((S^2)^q)^E$ is not subequivalent to $\xi^{\times q|E|} \oplus \theta_{q(l_n - |E|)}$. It

follows by Lemma 2.1 of [94] that $\|t^*(\xi^{\times q|E|} \oplus \theta_{q(l_n-|E|)})t - \theta_{l_n}\| \geq 1/2$ for all $t \in M_{2q} \otimes M_{l_n} \otimes C(((S^2)^q)^E)$. Since *-homomorphisms are contractive, we thus obtain $\|t^*b_n t - a_n\| \geq 1/2$ for all $t \in M_{2q} \otimes (C(X_n) \rtimes \mathbb{Z})$, as desired. $\qquad\square$

If Y is contractible then $K^1(X) = \varinjlim K^1(X_n) = 0$ and one can deduce from the Pimsner-Voiculescu six-term exact sequence for $\mathbb{Z}$-crossed products that $K_1(C(X) \rtimes \mathbb{Z}) \cong \mathbb{Z}$. In general the K_0 group of $C(X) \rtimes \mathbb{Z}$ will be complicated. Suppose however that for each n the sets $X_{B_n,1}, \ldots, X_{B_n,l_n}$ are pairwise disjoint, which can be arranged by replacing Y with $Y \times [0,1]$ and the factors equal to Y in the blocks B_n by sets of the form $Y \times \{x\}$ for different $x \in [0,1]$. One can then check in this case that that the system (X,T) is an extension of the odometer system defined as addition by $(1,0,0,\ldots)$ with carry over on the sequence space $\prod_{n=1}^{\infty}\{1,\ldots,l_{n+1}/l_n\}$, and that the ordered K_0 group of $C(X) \rtimes \mathbb{Z}$ is identical to that of the crossed product of this odometer. If for example that every positive integer divides some l_n, then this ordered K_0 group, along with the class of the unit, will be naturally isomorphic $(\mathbb{Q}, \mathbb{Q}^+, 1)$. It follows by classification theory that there is a simple $A\mathbb{T}$ algebra which has the same Elliott invariant as $C(X) \rtimes \mathbb{Z}$ but, in view of Theorem 4.5.3, is not isomorphic to it. One can also see that $C(X) \rtimes \mathbb{Z}$ does not have real rank zero, since this would mean that the linear span of the projections is dense and hence that K_0 separates tracial states, contradicting the fact that $K_0(C(X) \rtimes \mathbb{Z})$ has a unique state.

Chapter 5

External topological phenomena

5.1 Groups which are locally embeddable into finite groups

Complementing the internal finite modelling property of local finiteness is the external property of local embeddability into finite groups, which is the topological (i.e., purely group-theoretic, since our groups are discrete) analogue of soficity. The group G is said to be *LEF (locally embeddable into finite groups)* if for every finite set $F \subseteq G$ there is a finite group H and a map $\sigma \colon G \to H$ such that $\sigma(st) = \sigma(s)\sigma(t)$ for all $s, t \in F$ and $\sigma|_F$ is injective. This notion was introduced by Gordon and Vershik in [35].

Local embeddability into finite groups is a purely local version of residual finiteness, which requires global multiplicativity of the maps to finite groups. The group G is *residually finite* if it has a separating family of finite quotients, i.e., for every finite set $F \subseteq G$ there is a finite group H and a homomorphism $\pi \colon G \to H$ such that $\pi(s) \neq \pi(t)$ for all distinct $s, t \in F$. Equivalently, there exists a sequence $\{G_n\}$ of finite-index normal subgroups of G such that $\bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} G_n = \{e\}$.

The C^*-algebraic analogue of an LEF group is an *MF algebra*, which is defined in [10, Defn. 3.2.1] as an inductive limit of a generalized inductive system of finite-dimensional C^*-algebras. Assuming that A is separable, this is equivalent to each of the following conditions:

1. A embeds into $\prod_{n=1}^{\infty} M_{k_n} / \bigoplus_{n=1}^{\infty} M_{k_n}$ for some sequence $\{k_n\}_{n=1}^{\infty}$ in $\mathbb{N}$,

2. for every $\varepsilon > 0$ and finite set $\Omega \subseteq A$ there are a $k \in \mathbb{N}$ and a $*$-linear map $\varphi \colon A \to M_k$ such that $\|\varphi(ab) - \varphi(a)\varphi(b)\| < \varepsilon$ and $\big|\|\varphi(a)\| - \|a\|\big| < \varepsilon$ for all $a, b \in \Omega$.

Condition (2) makes clear the analogy with LEF groups, and that being an MF algebra is a local property in the strictest possible sense. Condition (1) is the

topological analogue of R^ω-embeddability. In comparison with the relationship between soficity/hyperlinearity and R^ω-embeddability, the technical passage from LEF groups to MF algebras does not factor in such a direct way through the left regular representation, which is understandable given that the ultrapower R^ω is defined with respect to the trace norm. Thus we cannot derive a general result about the MF structure of $C_\lambda^*(G)$ itself, unless G is amenable.

Theorem 5.1.1. *Let G be a countable discrete LEF group. Then there is a unitary representation π of G such that $C^*(\pi(G))$ is an MF algebra and factors canonically onto $C_\lambda^*(G)$.*

Proof. Let $F_1 \subseteq F_2 \subseteq \ldots$ be an increasing union of finite subsets of G whose union is equal to G. Since G is LEF, for every $n \in \mathbb{N}$ there is a finite group H_n and a map $\sigma_n \colon G \to H_n$ such that $\sigma_n(st) = \sigma_n(s)\sigma_n(t)$ for all $s, t \in F_n$. Define for each n a map ρ_n from G to the unitary group of $M_{H_n} = \mathcal{B}(\ell^2(H_n))$ by setting $\rho(s)\delta_t = \delta_{\rho_n(s)}$ for all $s \in G$ and $t \in H_n$, where $\{\delta_t\}_{t \in H_n}$ is the standard basis of $\ell^2(H_n)$. Let $\pi \colon G \to \prod_{n=1}^\infty M_{H_n} / \bigoplus_{n=1}^\infty M_{H_n}$ be the composition of the map $s \mapsto (\rho_1(s), \rho_2(s), \ldots)$ from G to $\prod_{n=1}^\infty M_{H_n}$ with the quotient map $\prod_{n=1}^\infty M_{H_n} \to \prod_{n=1}^\infty M_{H_n} / \bigoplus_{n=1}^\infty M_{H_n}$. Then π is a group homomorphism into unitaries and thus extends to a *-homomorphism $\Phi \colon C^*(G) \to \prod_{n=1}^\infty M_{H_n} / \bigoplus_{n=1}^\infty M_{H_n}$, whose image is an MF algebra by definition. Finally, to show that $C^*(\pi(G))$ factors canonically onto $C_\lambda^*(G)$ we need only verify that if $a = \sum_{s \in F} c_s u_s$ is a finite linear combination of canonical unitaries in $C^*(G)$ then $\|\Phi(a)\| \geq \|\sum_{s \in F} c_s \lambda_s\|$. This can be done spatially by approximating the norm of a with $\|a\xi\|$ for some norm-one vector $\xi = \sum_{s \in E} d_s \delta_s \in \ell^2(G)$ having finite support with respect to the standard basis, and then showing that for all sufficiently large n the vector $\xi_n = \sum_{s \in E} d_s \delta_s \in \ell^2(H_n)$ has norm one and $\|(\sum_{s \in F} c_s \sigma_n(s))\xi_n\| \geq \|a\xi\| - \varepsilon$ for a prescribed $\varepsilon > 0$. $\qquad\square$

Corollary 5.1.2. *Let G be a countable discrete amenable LEF group. Then $C_\lambda^*(G)$ is an MF algebra.*

Proof. By the theorem there is a unitary representation π of G such that $C^*(\pi(G))$ is an MF algebra and there is a surjective *-homomorphism $C^*(\pi(G)) \to C_\lambda^*(G)$ sending $\pi(s)$ to u_s for each $S \in G$. By the universal property of the full group C^*-algebra there is a canonical surjective *-homomorphism $C^*(G) \to C^*(\pi(G))$. Thus, since $C^*(G) \cong C^*(\pi(G))$ canonically by the amenability of G, the map $C^*(\pi(G)) \to C_\lambda^*(G)$ must be an isomorphism, yielding the result. $\qquad\square$

If G is residually finite in Theorem 5.1.1 then the maps σ_n can be taken to be group homomorphisms, in which case one already gets a *-homomorphism $C^*(G) \to \prod_{n=1}^\infty M_{H_n}$ without having to pass to the quotient $\prod_{n=1}^\infty M_{H_n} / \bigoplus_{n=1}^\infty M_{H_n}$. One deduces from this that $C_\lambda^*(G)$ is residually finite-dimensional, i.e., has a separating family of finite-dimensional quotients. Note that residually finite-dimensionality, like residual finiteness, is not a local property in the strictest sense.

The free groups on two of more generators are residually finite but not amenable. By a result of Choi, $C^*(G)$ is residually finite-dimensional. For the reduced group C^*-algebra we have the following deep theorem of Haagerup and Thorbjørnsen [42].

Theorem 5.1.3. *$C^*_\lambda(F_r)$ is an MF algebra for $r \in \{2, 3, \ldots, \infty\}$.*

Question 5.1.4. Is there a discrete G for which $C^*_\lambda(G)$ fails to be an MF algebra?

The introduction of MF algebras in [10] was motivated by the study of quasidiagonality, a much older concept which has its origins in operator theory. A set Ω of bounded operators on a separable Hilbert space $\mathcal{H}$ is *quasidiagonal* if there is an increasing sequence $P_1 \leq P_2 \leq \ldots$ of finite-rank orthogonal projections on $\mathcal{H}$ converging strongly to 1 such that $\lim_{n\to\infty} \|P_n a - a P_n\| = 0$ for all $a \in \Omega$. A representation $\pi \colon A \to \mathcal{B}(\mathcal{H})$ of a C^*-algebra on a separable Hilbert space is *quasidiagonal* if $\pi(A)$ is a quasidiagonal set of operators. A separable C^*-algebra A is *quasidiagonal* if it admits a faithful quasidiagonal representation on a separable Hilbert space. As shown by Voiculescu, this is equivalent to the condition that for every $\varepsilon > 0$ and finite set $\Omega \subseteq A$ there are a $k \in \mathbb{N}$ and a contractive completely positive map $\varphi \colon A \to M_k$ such that $\|\varphi(ab) - \varphi(a)\varphi(b)\| < \varepsilon$ and $\big| \|\varphi(a)\| - \|a\| \big| < \varepsilon$ for all $a, b \in \Omega$. This is a local property, since by Arveson's extension theorem every contractive completely positive map from an operator system $V \subseteq A$ to matrix algebra M_k admits a contractive completely positive extension to A.

Notice that the only difference between Voiculescu' abstract characterization of quasidiagonality and characterization (2) above for MF algebras is the contractive completely positive condition on the maps, which situates quasidiagonality as a kind of hybrid property that combines topology (approximate multiplicativity) and measure theory (complete positive maps into matrix algebras, which are the matrix-valued version of positive functionals). This measure-theoretic aspect is manifest in the following link to amenability observed by Rosenberg.

Theorem 5.1.5. *Let G be a countable discrete group. Suppose that $C^*_\lambda(G)$ is quasidiagonal. Then G is amenable.*

Under the measure-theoretic assumption of nuclearity, quasidiagonality is equivalent to being an MF algebra. Blackadar and Kirchberg defined a separable C^*-algebra A to be an *NF algebra* if it can be expressed as the inductive limit of a generalized inductive system with contractive completely positive connecting maps, and showed that the following are equivalent:

1. A is an NF algebra,

2. A is a nuclear MF algebra,

3. A is nuclear and quasidiagonal.

Given an embedding $A \hookrightarrow \prod_{n=1}^{\infty} M_{k_n} / \bigoplus_{n=1}^{\infty} M_{k_n}$ as in characterization (2) of MF algebras, if A is nuclear then one can apply the Choi-Effros lifting theorem to produce a contractive completely positive lift $A \to \prod_{n=1}^{\infty} M_{k_n}$ which can then be cut-down to suitable finite sets of coordinates farther and farther out in order to verify Voiculescu's characterization of quasidiagonality.

Question 5.1.6. Is there a countable discrete amenable group for which $C_\lambda^*(G)$ fails to be quasidiagonal?

5.2   Chain recurrence, residually finite actions, and MF algebras

Let X be a compact metrizable space and $T\colon X \to X$ a homeomorphism. Let d be a compatible metric on X. For $x, y \in X$ and $\varepsilon > 0$, an *ε-chain* from x to y is a finite sequence $\{x_1 = x, x_2, \ldots, x_n = y\}$ in X such that $n > 1$ and $d(Tx_i, x_{i+1}) < \varepsilon$ for every $i = 1, \ldots, n-1$. The point x is said to be *chain recurrent* if for every $\varepsilon > 0$ there is an ε-chain from x to itself. This is equivalent to x being pseudo-nonwandering in the sense of [78]. Note that the set of chain recurrent points is a closed T-invariant subset of X. We say that T is *chain recurrent* if every point in X is chain recurrent. This occurs for example if there is a dense set of recurrent points, and in particular if T is minimal.

Theorem 5.2.1. *Let X be a compact metrizable space and $T\colon X \to X$ a homeomorphism. Then the following are equivalent:*

1. *T is chain recurrent,*

2. *$C(X) \rtimes_\lambda \mathbb{Z}$ can be embedded into an AF algebra,*

3. *$C(X) \rtimes_\lambda \mathbb{Z}$ is quasidiagonal,*

4. *$C(X) \rtimes_\lambda \mathbb{Z}$ is stably finite.*

Note that quasidiagonality here is equivalent to being an MF algebra, since all of the above crossed products are nuclear. The implications (2)$\Rightarrow$(3)$\Rightarrow$(4) hold for any C^*-algebra. The implications (3)$\Rightarrow$(2) and (4)$\Rightarrow$(3) are false for general C^*-algebras, as witnessed by $C^*(F_2)$ (which is not exact, and hence is not a C^*-subalgebra of a nuclear C^*-algebra by a theorem of Kirchberg) and $C_\lambda^*(F_2)$, respectively. To prove (4)$\Rightarrow$(1) one uses the following characterization of chain recurrence by the incompressibility of open sets.

Proposition 5.2.2. *A homeomorphism $T\colon X \to X$ of a compact metric space is chain recurrent if and only if there is a nonempty open set $U \subseteq X$ such that $T\overline{U}$ is a proper subset of U.*

Proof. Suppose first that there is a nonempty open set $U \subseteq X$ such that $T\overline{U}$ is a proper subset of U. Take an $x \in U \setminus T\overline{U}$. Then there is no ε-chain from x to itself for ε equal to the distance between x and $T\overline{U}$.

Conversely, suppose that there exists an $x \in X$ and an $\varepsilon > 0$ such that there is no ε-chain from x to itself. Let U be the open set consisting of all points $y \in X$ such that there is an ε'-chain from x to y for every $0 < \varepsilon' < \varepsilon$. Then one can easily check that $T\overline{U} \subseteq U$ and $x \in U \setminus T\overline{U}$. $\square$

Supposing that T is not chain recurrent, Pimsner uses the above proposition along with some index theory to construct a nonunitary isometry in $C(X) \rtimes_\lambda \mathbb{Z}$, the existence of which obstructs stable finiteness. This can be done more directly when X is the Cantor set, as the proposition then easily implies the existence a nonempty clopen set $U \subseteq X$ such that TU is a proper subset of U, in which case $v = 1_{X \setminus U} + 1_{TU} u$ is an isometry satisfying $vv^* = 1_{X \setminus U} + 1_{TU} \neq 1$. We will not delve into the much more involved argument for (1)$\Rightarrow$(2), although we will abstract some of its ideas to obtain a generalization of (1)$\Rightarrow$(3) in Theorem 5.2.8 below (see the discussion following the proof of Theorem 5.2.8 for the relation to quasidiagonality).

Quasidiagonality can be usefully strengthened in certain ways, especially through its conjunction with internal approximation so as to capture more rigid finite-dimensional structure. As applied to $\mathbb{Z}$-crossed products, these properties can also be characterized dynamically in terms of chain recurrence. First we record the following theorem of Hadwin [43, Thm. 25], whose proof makes use of Berg's technique and the theory of induced representations. Recall that a C^*-algebra is *strongly quasidiagonal* if each of its representations is quasidiagonal.

Theorem 5.2.3. *Let T be a homeomorphism of a compact metrizable space X. Then $C(X) \rtimes_\lambda \mathbb{Z}$ is strongly quasidiagonal if and only every restriction of T to a closed invariant subset of X is chain recurrent.*

Recall from the previous section that a separable C^*-algebra A is an NF algebra if it can be expressed as the inductive limit of a generalized inductive system with contractive completely positive connecting maps. We say that A is a *strong NF algebra* if the connecting maps in the definition of NF algebra can be chosen to be complete order embeddings [10]. By [10, 12, 11] the following are equivalent:

1. A is a strong NF algebra,

2. for every $\Omega \in \mathscr{F}A$ and $\varepsilon > 0$ there are a finite-dimensional C^*-algebra B and a complete order embedding $\varphi \colon B \to A$ such that for each $x \in \Omega$ there is a $b \in B$ with $\|x - \varphi(b)\| < \varepsilon$,

3. A is nuclear and inner quasidiagonal,

4. A is nuclear and has a separating family of irreducible quasidiagonal representations.

For a $\lambda > 1$, a C^*-algebra A is said to be an $\mathcal{OL}_{\infty,\lambda}$ *space* if for every finite set $\Omega \subseteq A$ and $\varepsilon > 0$ there is a finite-dimensional C^*-algebra B and an injective linear map $\varphi \colon B \to A$ with $\Omega \subseteq B$ such that $\|\varphi\|_{\mathrm{cb}}\|\varphi^{-1} \colon \varphi(B) \to B\|_{\mathrm{cb}} < \lambda$. Write $\mathcal{OL}_{\infty}(A)$ for the infimum over all $\lambda > 1$ for which A is an $\mathcal{OL}_{\infty,\lambda}$ space. These notions were introduced in [48] so as to furnish a quantitative means for analyzing the relationships between properties like nuclearity, quasidiagonality, inner quasidiagonality, and stable finiteness using local operator space structure. A straightforward perturbation argument shows that if A is a strong NF algebra then $\mathcal{OL}_{\infty}(A) = 1$. It is not known whether the converse is true, although in [24] it was shown to hold under the assumption that A has a finite separating family of primitive ideals. By localizing the arguments used by Hadwin in proving Theorem 5.2.3, one can show the following [53].

Theorem 5.2.4. *Let* $T \colon X \to X$ *be a homeomorphism of a compact metrizable space. Then the following are equivalent:*

1. *there is a collection* $\{X_i\}_{i \in I}$ *of* T-*invariant closed subsets of* X *such that* $\bigcup_{i \in I} X_i$ *is dense in* X *and the restriction of* T *to each* X_i *has a dense orbit and is chain recurrent,*

2. $C(X) \rtimes_{\lambda} \mathbb{Z}$ *is strong NF,*

3. $\mathcal{OL}_{\infty}(C(X) \rtimes_{\lambda} \mathbb{Z}) = 1$.

Using Theorem 5.2.4 one can give dynamical constructions of C^*-algebras which are NF but not strong NF (compare Examples 5.6 and 5.19 of [12]). Perhaps the simplest such example consists in taking two copies of translation on $\mathbb{Z}$ each compactified with two fixed points $\pm\infty$ and identifying $+\infty$ from each copy with $-\infty$ of the other copy. The resulting homeomorphism is chain recurrent, but its restriction to the closure of each copy of $\mathbb{Z}$ fails to be chain recurrent, so that $C(X) \rtimes_{\lambda} \mathbb{Z}$ is NF but not strong NF. This is the dynamical analogue of Example 3.2 in [24].

For actions of groups other than $\mathbb{Z}$ it quickly becomes difficult to say anything very general relating external finite approximation and C^*-algebra structure. Some of the main difficulties occur in the modelling of relations within the group itself. For instance, given a pair of unitaries in a matrix algebra that almost commute to within a prescribed tolerance, no matter how small, it might not be possible to perturb them to commuting unitaries [98]. Thus already for the group $\mathbb{Z}^2$ we will have difficulty extracting finite models for the dynamics given finite-dimensional topological models for the crossed product. In fact the following question of Voiculescu from [99] is still open.

Question 5.2.5. Let $\mathbb{Z}^2$ act on a compact metrizable space X. What dynamical condition is equivalent to $C(X) \rtimes \mathbb{Z}^2$ being embeddable into an AF algebra?

However, H. Lin has obtained a definitive result about embeddability into simple AF algebras by analyzing the structure of iterated crossed products [66]:

Theorem 5.2.6. *Let $r \in \mathbb{N}$ and let $\mathbb{Z}^r$ act on a compact metrizable space X. Then $C(X) \rtimes \mathbb{Z}^r$ embeds into a simple AF algebra if and only if there is a $\mathbb{Z}^r$-invariant Borel probability measure on X of full support.*

When the acting group is free, we do not have to worry about handling approximate relations, and so we might expect to be able to extract dynamical information from matrical approximation of the crossed product. For actions on the Cantor set this can be done (Theorem 5.2.9 below), and the relevant dynamical notion is the following generalization of chain recurrence, which is the topological analogue of soficity for measure-preserving actions.

Definition 5.2.7. An action of a countable discrete group G on a compact metrizable space with compatible metric d is said to be *residually finite* if for every finite set $F \subseteq G$ and $\varepsilon > 0$ there are a finite set E, an action of G on E, and a map $\zeta \colon E \to X$ such that $\zeta(E)$ is ε-dense in X and $d(\zeta(sz), s\zeta(z)) < \varepsilon$ for all $z \in E$ and $s \in F$.

Residual finiteness is easily seen not to depend on the metric d, and thus is a topological-dynamical invariant. It implies that $C(X) \rtimes_\lambda G$ is an MF algebra whenever $C^*_\lambda(G)$ is an MF algebra [53]:

Theorem 5.2.8. *Let $G \curvearrowright X$ be a residually finite action on a compact metrizable space. Suppose that $C^*_\lambda(G)$ is an MF algebra. Then $C(X) \rtimes_\lambda G$ is an MF algebra.*

Proof. Fix a compatible metric d on X. We write α for the induced action of G on $C(X)$, i.e., $\alpha_s(f)(x) = f(s^{-1}x)$. Since X is separable we can construct a sequence $\{x_n\}_{n=1}^\infty$ in X such that $\{n \in \mathbb{N} : x_n \in U\}$ is infinite for every nonempty open set $U \subseteq X$. Then we can view $C(X) \rtimes \mathbb{Z}$ as acting on $\ell^2(\mathbb{N}) \otimes \ell^2(G)$ by $fu_s(\delta_n \otimes \delta_t) = f((st)^{-1}x_n)\delta_n \otimes \delta_{st}$ where $\{\delta_n\}_{n\in\mathbb{N}}$ and $\{\delta_s\}_{s\in G}$ are the standard orthonormal bases for $\ell^2(\mathbb{N})$ and $\ell^2(G)$, respectively. Since the action is residually finite there are for each $n \in \mathbb{N}$ a finite set E_n, an action γ_n of G on $C(E_n)$, and a unital *-homomorphism $\varphi_n \colon C(X) \to C(Y_n)$ such that

1. $\lim_{n\to\infty} \|\varphi_n(f)\| = \|f\|$ for all $f \in C(X)$, and

2. $\lim_{n\to\infty} \|\varphi_n(\alpha_s(f)) - \gamma_{n,s}(\varphi_n(f))\| = 0$ for all $f \in C(X)$ and $s \in G$.

The action γ_n gives rise to a unitary representation $w_n \colon G \to M_{E_n}$ by permutation matrices, and so by viewing $C(E_n)$ as the diagonal in the matrix algebra M_{E_n} we may write $\gamma_{n,s}(f)$ as $w_{n,s} f w^*_{n,s}$ for all $f \in C(E_n)$ and $s \in G$.

Let Ω be a finite subset of the algebraic crossed product $C(X) \rtimes_{\mathrm{alg}} G$ and let $\varepsilon > 0$. To conclude that $C(X) \rtimes_\lambda G$ is MF it suffices to show the existence of a $d \in \mathbb{N}$ and a *-linear map $\beta \colon C(X) \rtimes_{\mathrm{alg}} G \to M_d$ such that $\|\beta(ab) - \beta(a)\beta(b)\| < \varepsilon$ and $\| \|\beta(f)\| - \|f\| \| < \varepsilon$ for all $a, b \in \Omega$.

We regard M_{E_n} as acting on $\ell_2(E_n)$. For $N \in \mathbb{N}$ define the *-homomorphism $\Phi_N \colon C(X) \to \bigoplus_{n=N}^\infty M_{E_n}$ by $\Phi_N(f) = (\varphi_N(f), \varphi_{N+1}(f), \dots)$. Let D_n be the C^*-subalgebra of $\mathcal{B}(\ell_2(E_n) \otimes \ell_2(G))$ generated by $M_{E_n} \otimes 1$ and the operators $w_{n,s} \otimes \lambda_s$

for $s \in G$. Let $\Theta_N \colon A \rtimes_{\mathrm{alg}} G \to \bigoplus_{n=N}^{\infty} D_n$ be the *-linear map defined by setting

$$\Theta_N(f u_s) = (\varphi_N(f) w_{N,s} \otimes \lambda_s, \varphi_{N+1}(f) w_{N+1,s} \otimes \lambda_s, \dots)$$

for $a \in A$ and $s \in G$, which can be done since the subspaces $C(X) u_s$ for $s \in G$ are orthogonal with respect to the canonical conditional expectation from $C(X) \rtimes_\lambda G$ onto $C(X)$. We will now argue that if N is large enough then we will have $\big|\|\Theta_N(a)\| - \|a\|\big| < \varepsilon/2$ for all $a, b \in \Omega$.

Let $\varepsilon' > 0$ be such that, for all $n \in \mathbb{N}$ and $a \in \Omega$, if $\big|\|\theta_n(a)\|^2 - \|c\|^2\big| < \varepsilon'$ then $\big|\|\theta_n(a)\| - \|a\|\big| < \varepsilon/2$. Take a finite set $F \subseteq G$ such that for every $a \in \Omega$ we can write $a = \sum_{s \in f} f_{a,s} u_s$ where $f_{a,s} \in C(X)$ for every $s \in F$. Put $M = \max_{a \in \Omega, s \in F} \|f_{a,s}\|$. For each $a \in \Omega$ we can find a unit vector $\eta_a \in \ell^2(\mathbb{N}) \otimes \ell^2(G)$ such that $\|a\| \leq \|a\eta_a\| + \varepsilon/2$ and $\eta_a = \sum_{t \in K} \xi_{a,s} \otimes \delta_t$ for some finite set $K \subseteq G$, which we may assume to be the same for all $a \in \Omega$, and vectors $\xi_{a,s} \in \ell^2(\mathbb{N})$.

Choose a $\delta > 0$ such that $(1 + 2|F||K|)|F|^2 \delta^2 \leq \varepsilon'/2$. Set $E = \coprod_{n=1}^{\infty} E_n$. The conditions on the sequence $\{x_n\}_{n=1}^{\infty}$ and the *-homomorphisms φ_n permit us to find a bijection $\sigma \colon \mathbb{N} \to E$ such that $d(S \circ \sigma(n), \sigma(n)) < \delta$ for all $n \in \mathbb{N}$ where $S|_{E_n} \colon E_n \to X$ is the map corresponding spectrally to φ_n. By taking δ small enough we may ensure that the unitary operator $U \colon \ell^2(\mathbb{N}) \to \ell^2(E)$ defined on standard basis vectors by $U\delta_n = \delta_{\theta(n)}$ satisfies

$$\big\| U\Phi_N(\alpha_{st}^{-1}(f_{a,s}))U^{-1} - \alpha_{st}^{-1}(f_{a,s}) \big\| < \frac{\delta}{2}$$

for all $a \in \Omega$, $s \in F$, and $t \in K$ (this is a simple embryonic case of Voiculescu's theorem). For $s \in G$ set $w_s = (w_{N,s}, w_{N+1,s}, \dots) \in \bigoplus_{n=N}^{\infty} M_{E_n}$. By the asymptotic equivariance of the maps φ_k, we may assume that N is large enough so that

$$\big\| \Phi_N(\alpha_{st}^{-1}(f_{a,s})) - w_{st}^* \Phi_N(f_{a,s}) w_{st} \big\| < \frac{\delta}{2}$$

and hence

$$\big\| w_{st} U^{-1} \alpha_{st}^{-1}(f_{a,s}) - \Phi_N(f_{a,s}) w_{st} U^{-1} \big\| < \delta$$

for all $a \in \Omega$, $s \in F$, and $t \in K$. As is clear from the definition Θ, we may additionally assume that N is large enough so that $\|\Theta_N(ab) - \Theta_N(a)\Theta_N(b)\| < \varepsilon/2$ for all $a, b \in \Omega$.

Let $a \in \Omega$. Write $\tilde{U}$ for the unitary operator from $(\bigoplus_{n=N}^{\infty} \ell_2(E_n)) \otimes \ell_2(G)$ to $\mathcal{H} \otimes \ell_2(G)$ given by $\tilde{U}(\zeta \otimes \delta_t) = U w_t^{-1} \zeta \otimes \delta_t$. Using the crude bound

$$\|\Theta_N(c)\tilde{U}^{-1}\eta_c\| = \left\| \sum_{s \in F} \sum_{t \in K} \Phi_N(f_{a,s}) w_{st} U^{-1} \xi_{a,t} \otimes \delta_{st} \right\|$$

$$\leq \sum_{s \in F} \sum_{t \in K} \|\Phi_N(f_{a,s})\| \leq |F||K|.$$

and applying the general inequality

$$\left\|\sum_{i=1}^{n} x_i\right\|^2 \le \left\|\sum_{i=1}^{n} y_i\right\|^2 + \left(1 + 2\left\|\sum_{i=1}^{n} y_i\right\|\right)\left\|\sum_{i=1}^{n}(x_i - y_i)\right\|^2$$

we then have

$$\|a\eta_a\|^2 = \left\|\tilde{U}^{-1}\sum_{s\in F}\sum_{t\in K}\alpha_{st}^{-1}(f_{a,s})\xi_{c,t}\otimes\delta_{st}\right\|^2$$

$$= \left\|\sum_{s\in F}\sum_{t\in K}w_{st}U^{-1}\alpha_{st}^{-1}(f_{a,s})\xi_{c,t}\otimes\delta_{st}\right\|^2$$

$$\le \|\Theta_N(c)\tilde{U}^{-1}\eta_a\|^2 + \left(1 + 2\|\Theta_N(a)\tilde{U}^{-1}\eta_a\|\right)$$

$$\times \left(\sum_{s\in F}\left\|\sum_{t\in K}\left(w_{st}U^{-1}\alpha_{st}^{-1}(f_{a,s}) - \Phi_N(f_{a,s})w_{st}U^{-1}\right)\xi_{a,t}\otimes\delta_{st}\right\|\right)^2$$

$$\le \|\Theta_N(c)\|^2 + (1 + 2|F||K|)|F|^2\delta^2$$

$$= \|\Theta_N(c)\|^2 + \frac{\varepsilon'}{2}.$$

so that $\|\Theta_N(a)\|^2 \ge \|a\eta_a\|^2 - \varepsilon'/2 \ge \|a\|^2 - \varepsilon'$.

Next let us show that $\|a\|^2 \ge \|\Theta_N(a)\|^2 - \varepsilon'$. Take a unit vector η in $\left(\bigoplus_{n=N}^{\infty}\ell_2(E_n)\right)\otimes\ell^2(G)$ such that $\|\Theta_N(a)\| \le \|\Theta_N(a)\tilde{U}^{-1}\eta\| + \varepsilon'/2$. We argue as in the above paragraph, only reversing the roles of a and $\Theta_N(a)$ and replacing η_a with η. Unlike for η_a we have no control of the support of η with respect to the standard basis, but the bound on $\|\Theta_N(a)\tilde{U}^{-1}\eta_a\|$ which required this control for η_a can be replaced here simply by $\|\tilde{U}^{-1}a\eta\| \le \|a\|$, in which case

$$\|\Theta_N(a)\tilde{U}^{-1}\eta\|^2$$

$$= \left\|\sum_{s\in F}\sum_{t\in K}\Phi_N(f_{a,s})w_{st}U^{-1}\xi_{a,t}\otimes\delta_{st}\right\|^2$$

$$\le \|\tilde{U}^{-1}a\eta\|^2 + \left(1 + 2\|\tilde{U}^{-1}c\eta\|\right)$$

$$\times \left(\sum_{s\in F}\left\|\sum_{t\in K}\left(\Phi_N(f_{a,s})w_{st}U^{-1} - w_{st}U^{-1}\alpha_{st}^{-1}(f_{a,s})\right)\xi_{c,t}\otimes\delta_{st}\right\|\right)^2$$

$$\le \|a\|^2 + 3|F|^2\delta^2$$

$$= \|a\|^2 + \frac{\varepsilon'}{2}$$

and hence $\|a\|^2 \ge \|\Theta_N(a)\tilde{U}^{-1}\eta\|^2 - \varepsilon'/2 \ge \|\Theta_N(a)\|^2 - \varepsilon'$. By our choice of ε' we conclude that $\left|\|\Theta_N(a)\| - \|a\|\right| < \varepsilon/2$ for all $a \in \Omega$, so that the map Θ_N satisfies the desired properties.

Now let M be an integer greater than or equal to N to be determined shortly. Set $D = \bigoplus_{n=N}^{M} D_n$. Define the *-linear map $\theta\colon\colon A \rtimes_{\mathrm{alg}} G \to D$ by declaring that

$$\theta(au_s) = (\varphi_N(a)w_{n,s} \otimes \lambda_s, \ldots, \varphi_M(a)w_{n,s} \otimes \lambda_s)$$

for $a \in A$ and $s \in G$ and extending linearly. In view of the properties of Θ_N we have $\|\theta(ab) - \theta(a)\theta(b)\| < \varepsilon/3$ for all $a, b \in \Omega$ and, by taking M large enough, $\big|\|\theta(a)\| - \|a\|\big| < \varepsilon/2$ for all $a \in \Omega$.

Set $k = \sum_{n=N}^{M} |E_n|$. Note that D is a C^*-subalgebra of $\mathcal{B}(\bigoplus_{n=N}^{M} \ell_2(E_n)) \otimes C_\lambda^*(G)$ as canonically represented on $(\bigoplus_{n=N}^{M} \ell_2(E_n)) \otimes \ell_2(G)$ where the latter is identified in the standard way with $\bigoplus_{n=N}^{M}(\ell_2(E_n) \otimes \ell_2(G))$. Via some fixed identification of $\mathcal{B}(\bigoplus_{n=N}^{M} \ell_2(E_n))$ with M_k we view D as a C^*-subalgebra of $M_k \otimes C_\lambda^*(G)$. By hypothesis $C_\lambda^*(G)$ is MF, and hence so is $M_k \otimes C_\lambda^*(G)$, for if $C_\lambda^*(G) \hookrightarrow \prod_{n=1}^{\infty} M_{E_n} / \bigoplus_{n=1}^{\infty} M_{E_n}$ is an embedding witnessing the fact that $C_\lambda^*(G)$ is MF then we obtain an embedding of $M_k \otimes C_\lambda^*(G)$ into

$$M_k \otimes \left(\prod_{n=1}^{\infty} M_{E_n} \Big/ \bigoplus_{n=1}^{\infty} M_{E_n} \right) \cong \prod_{n=1}^{\infty} \left(M_k \otimes M_{E_n} \right) \Big/ \bigoplus_{n=1}^{\infty} \left(M_k \otimes M_{E_n} \right).$$

Thus we can find an $l \in \mathbb{N}$ and a *-linear map $\varphi\colon M_k \otimes C_\lambda^*(G) \to M_l$ such that, for all $a, b \in \Omega$,

1. $\|\varphi(\theta(a)\theta(b)) - \varphi(\theta(a))\varphi(\theta(b))\| < \varepsilon/3$,

2. $\|\varphi(\theta(ab) - \theta(a)\theta(b))\| < \|\theta(ab) - \theta(a)\theta(b)\| + \varepsilon/3$, and

3. $\big|\|\theta(a)\| - \|a\|\big| < \varepsilon/2$.

Set $\beta = \varphi \circ \theta$. Then β is *-linear, and for all $a, b \in \Omega$ we have

$$\|\beta(ab) - \beta(a)\beta(b)\| \leq \|\varphi(\theta(ab) - \theta(a)\theta(b))\| + \|\varphi(\theta(a)\theta(b)) - \varphi(\theta(a))\varphi(\theta(b))\|$$

$$\leq \|\theta(ab) - \theta(a)\theta(b)\| + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon$$

and

$$\big|\|\beta(a)\| - \|a\|\big| \leq \big|\|\varphi(\theta(a))\| - \|\theta(a)\|\big| + \big|\|\theta(a)\| - \|a\|\big| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

completing the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

If $C_\lambda^*(G)$ is assumed to be quasidiagonal in the above proposition, then one can conclude that $A \rtimes_\lambda G$ is quasidiagonal. Indeed by a result of Rosenberg (see the appendix of [43]) the quasidiagonality of $C_\lambda^*(G)$ implies that G is amenable, which implies that $C(X) \rtimes_\lambda G$ is nuclear (see Section IV.3.5 of [8]). Since quasidiagonality imples MF, by the proposition $C(X) \rtimes_\lambda G$ is an MF algebra, and thus, since

separable nuclear MF algebras are quasidiagonal [10, Thm. 5.2.2], we conclude that $C(X) \rtimes_\lambda G$ is quasidiagonal.

Theorem 5.2.8 yields one direction of the next result from [53]. For the other direction one extracts the finite dynamical approximations directly from the matricial structure by a series of perturbation arguments.

Theorem 5.2.9. *Let r be an integer greater than one and let $F_r \curvearrowright X$ be an action on a zero-dimensional compact metrizable space. Then the action is residually finite if and only if $C(X) \rtimes_\lambda F_r$ is an MF algebra.*

Question 5.2.10. Can the zero-dimensional hypothesis be removed in the above theorem?

Lemma 5.2.11. *Let X be a compact metrizable space and $F_r \curvearrowright X$ an action. Suppose there exists an F_r-invariant Borel probability measure μ on X with full support. Then the action is residually finite.*

Proof. We may assume that r is finite in view of the definition of residual finiteness. Write S for the standard generating set for F_r. Let $\varepsilon > 0$. Take a finite measurable partition $\mathcal{P}$ of X whose elements have nonzero measure and diameter less than ε. Write $\mathcal{Q}$ for the collection of sets in the join $\bigvee_{s \in S} s\mathcal{P}$ which have nonzero measure. Consider, for each $P \in \mathcal{P}$ and $s \in S$, the homogeneous linear equation $\sum_{Q \in \mathcal{Q}, Q \subseteq P} x_Q = \sum_{Q \in \mathcal{Q}, Q \subseteq sP} x_Q$ in the variables x_Q for $Q \in \mathcal{Q}$. This system of equations has the solution $x_Q = \mu(Q)$ for $Q \in \mathcal{Q}$, and, since the rational solutions are dense in the set of real solutions by virtue of the rationality of the coefficients, we can find a solution consisting of rational x_Q which are close enough to the corresponding quantities $\mu(Q)$ to be all nonzero. Pick a positive integer M such that Mx_Q is an integer for every $Q \in \mathcal{Q}$. For each $Q \in \mathcal{Q}$ take a set E_Q of cardinality Mx_Q and define E to be the disjoint union of these sets. Take a map $\zeta \colon E \to X$ which sends E_Q into Q for each $Q \in \mathcal{Q}$. Now for every $P \in \mathcal{P}$ and $s \in S$ the sets $\bigcup_{Q \in \mathcal{Q}, Q \subseteq P} E_Q$ and $\bigcup_{Q \in \mathcal{Q}, Q \subseteq sP} E_Q$ have the same cardinality and so we can define an action of F_r on E by having a generator s send $\bigcup_{Q \in \mathcal{Q}, Q \subseteq P} E_Q$ to $\bigcup_{Q \in \mathcal{Q}, Q \subseteq sP} E_Q$ in some arbitrarily chosen way for each $P \in \mathcal{P}$. Then ζ and this action on E witness the definition of residual finiteness with respect to ε and the generating set S. $\square$

Theorem 5.2.12. *Let X be a compact metrizable space and $F_r \curvearrowright X$ a minimal action. Then the following are equivalent:*

1. *the action is residually finite,*

2. *there is an F_r-invariant Borel probability measure on X,*

3. *$C(X) \rtimes_\lambda F_r$ is an MF algebra,*

4. *$C(X) \rtimes_\lambda F_r$ is stably finite.*

If moreover X is zero-dimensional then we can add the following conditions to the list:

5. *every nonempty clopen subset of X is completely $(F_r, \mathscr{C}_X)$-nonparadoxical.*

6. *there exists a nonempty clopen subset of X which is completely $(F_r, \mathscr{C}_X)$-nonparadoxical.*

Proof. $(1)\Rightarrow(2)$. Every residually finite action admits an invariant Borel probability measure, as can be obtained by pushing forward the uniform measure under the maps witnessing residual finiteness and taking a weak* cluster point.

$(2)\Rightarrow(1)$. By minimality every G-invariant Borel probability measure on X has full support, and so Lemma 5.2.11 applies.

$(1)\Rightarrow(3)$. By Theorem 5.2.8.

$(3)\Rightarrow(4)$. By Proposition 3.3.8 of [10].

$(4)\Rightarrow(2)$. Stable finiteness implies the existence of a quasitrace (see Section V.2 of [8]) and restricting a quasitrace on $C(X) \rtimes_\lambda F_r$ to $C(X)$ yields a G-invariant Borel probability measure on X.

Finally, in the case that X is zero-dimensional $(2)\Leftrightarrow(5)\Leftrightarrow(6)$ is a special case of Proposition 2.3.6. $\square$

Question 5.2.13. For minimal actions $F_r \curvearrowright X$, is the crossed product always either purely infinite or an MF algebra? When X is the Cantor set, is the type semigroup always unperforated? This would imply the purely infinite/MF algebra dichotomy by Theorem 2.3.8.

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