DYNAMICS OF THE THIRD ORDER LYNESS' DIFFERENCE EQUATION

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Abstract. This paper studies the iterates of the third order Lyness' recurrence $x_{k+3} = (a + x_{k+1} + x_{k+2})/x_k$, with positive initial conditions, being a also a positive parameter. It is known that for $a = 1$ all the sequences generated by this recurrence are 8-periodic. We prove that for each $a \neq 1$ there are infinitely many initial conditions giving rise to periodic sequences which have almost all the even periods and that for a full measure set of initial conditions the sequences generated by the recurrence are dense in either one or two disjoint bounded intervals of $\mathbb{R}$. Finally we show that the set of initial conditions giving rise to periodic sequences of odd period is contained in a codimension one algebraic variety (so it has zero measure) and that for an open set of values of a it also contains all the odd numbers, except finitely many of them.

Contents

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1. Introduction and main results

1.1. **The third order Lyness' difference equation.** The excellent unpublished paper of Zeeman [12] about the celebrated Lyness' second order difference equation

$$x_{n+2} = \frac{a + x_{n+1}}{x_n}, \quad \text{with} \quad a > 0\,,\, x_1 > 0\,,\, x_2 > 0, \tag{1}$$

gives the key points for understanding the behaviour of the sequences generated by (1). In this reference it is proved that the map induced by (1),

$$f(x, y) = \left(y, \frac{a + y}{x}\right), \tag{2}$$

defined on $\{(x, y) \in \mathbb{R}^2 \,:\, x > 0\,,\, y > 0\}$ leaves invariant the level curves of the first integral $V(x, y) = (x + 1)\,(y + 1)\,(a + x + y)\,/(xy)$ and, which is more important, that on each set $\{V(x, y) = h\}$, the map f is conjugated to a rotation of the circle with rotation number $\rho_a(h)$. By using this result, Zeeman explains the behavior of all the sequences generated by (1), modulus a conjecture, the monotonous dependence of $\rho_a(h)$ with respect to h once $a \neq 1$ is fixed. Recall that when $a = 1$, except for the fixed point, all the sequences generated by (1) are 5−periodic. This conjecture has been proved to be true in [2]. The study of the periods that can appear in the Lyness equation, as well as the study of the rotation number has also been done in [1].

This paper studies a similar problem to the one considered by Zeeman but in dimension three, and proves that in this case the dynamics are also described by rotations. The fact that we are in a higher dimension makes the problem more difficult.

Concretely, we consider the third order Lyness' recurrence,

$$x_{n+3} = \frac{a + x_{n+2} + x_{n+1}}{x_n}, \tag{3}$$

for $a > 0$ and positive initial conditions x_1, x_2 and x_3, *i.e.* such that $(x_1, x_2, x_3) \in O^+ := \{(x, y, z) \in \mathbb{R}^3 \,:\, x > 0\,,\, y > 0\,,\, z > 0\}$. This recurrence is also known as Todd's recurrence, see [7, 10]. Recall that if some initial condition is such that $(x_1, x_2, x_3) = (x_{1+p}, x_{2+p}, x_{3+p})$, and p is the minimal positive number satisfying this property it is said that this initial condition gives rise to a p−periodic sequence. It is well known that when $a = 1$ for any positive initial condition it holds that all the initial conditions in O^+ of (3) are $8, 2$ or 1-periodic. We are interested to understand which is the situation when $a \neq 1$. Our main result is:

Theorem 1. *Consider the third order Lyness' recurrence (3) for $a > 0$ and positive initial conditions x_1, x_2 and x_3.*

(i) *If $a \neq 1$ there is a computable value $q_0(a) \in \mathbb{N}$ such that for any $q > q_0(a)$ there exist continua of initial conditions giving rise to $2q$–periodic sequences.*
(ii) *The set of even periods arising when $a \in (0, \infty)$ contains all the even numbers except possibly $4, 6, 10, 12, 16, 18, 24, 28$ and 40.*
(iii) *The set of initial conditions giving rise to odd periods is contained in an algebraic codimension one subset of O^+. Moreover, there is an open set $\mathcal{U} \subset (0, 1) \cup (1, \infty)$ of values of a for which the set of the odd periods contains all the odd numbers except possibly finitely many of them.*
(iv) *If $a \neq 1$ then there exist a dense set of initial conditions in O^+ such that the sequence generated by (3) is dense in either one or two disjoint bounded intervals of $\mathbb{R}$.*

The above theorem makes one to wonder the following natural questions: Are there some $a > 0$ and some initial condition in O^+ such that the recurrence at this initial condition is periodic of period $4, 6, 10, 12, 16, 18, 24, 28$ or 40? Fixed $a > 0$, which are exactly all the even periods of the recurrence? And the odd periods? Is the set $\mathcal{U}$ introduced in Theorem 1 (iii) $U = (0, 1) \cup (1, \infty)$?

We want to remark that when the recurrence (3) is considered with initial conditions in the whole $\mathbb{R}^3$, the periods that can appear can be different. For instance in [3] it is proved that for some values of a there are initial (non positive) conditions giving rise to periodic sequences with periods 2,3,4,5,6,7 and $4p$ for any $p \geq 3$.

The paper is organized as follows. In Section 1.2 we state our results on the discrete dynamical system generated by F thus obtaining the proof of Theorem 1. All the results stated in Section 1.2 are proved in the following sections, moving some large proofs of the technical results to specific subsections and the appendices in order to improve the readability of the paper.

1.2. **Study from a dynamical systems viewpoint.** As usual we reduce the study of the recurrence (3) to the study of the discrete dynamical system generated by the map $F(x, y, z) = (y, z, (a + y + z)/x)$ defined in O^+. Note that this map is a diffeomorphism from O^+ to O^+. A complete description of the discrete system generated by F gives a complete answer to the questions posed in Section 1.1, and in particular a proof of Theorem 1 (see the end of this section). Our analysis of this dynamical system is done in two steps:

1. We will see that the phase space of F is foliated by invariant curves (sometimes degenerated to isolated points) which are given by the level curves of two functionally independent first integrals. The first step is to characterize the topology of this level sets, which turn to be diffeomorphic to circles (when they are not isolated points).

2. The second step is to study the dynamics of F restricted to these invariant sets. As we will see, one of our main tools, at this stage, will be the study of an ordinary differential equation associated to the discrete dynamical system generated by F. This is an approach different to the ones in [1], [12] (and even to the one in [2] although our starting point is the same of this last reference). Our approach turns out to be also effective for studying other difference equations, see [5].

Nevertheless there are some problems, named there as Questions 1 and 2, that have resisted our analysis. We remark that an answer to them would also allow to clarify the answers to the questions about (3) stated in Section 1.1.

Fixed $a > 0$, consider the diffeomorphism

$$F(x, y, z) = \left(y, z, \frac{a + y + z}{x} \right) \tag{4}$$

defined in $O^+ := \{(x, y, z) \in \mathbb{R}^3 \,:\, x > 0\,,\, y > 0\,,\, z > 0\}$.

We begin by introducing some sets in O^+ which are invariant under the action of F, in terms of the level surfaces of the well–known ([4, 6, 8, 11]) couple of functionally independent first integrals of F, given by:

$$V_1(x, y, z) = \frac{(x+1)\,(y+1)\,(z+1)\,(a+x+y+z)}{xyz},$$
$$V_2(x, y, z) = \frac{(1+y+z)(1+x+y)(a+x+y+z+xz)}{xyz}.$$

Let $L_k = \{(x, y, z) \in O^+ \,:\, V_1(x, y, z) = k\}$ and $M_h = \{(x, y, z) \in O^+ \,:\, V_2(x, y, z) = h\}$ be the level surfaces of V_1 and V_2 respectively.

The orbits of F lie in $I_{k,h} = L_k \cap M_h$ for $k \geq k_c$ and $h \geq h_c$, where k_c and h_c denote the values attached at the global minima in O^+ of V_1 and V_2 respectively. For a given fixed $h > h_c$, there exists $k_1 = k_1(h), k_2 = k_2(h)$ satisfying $k_c < k_1 < k_2$ and such that $I_{k,h} \neq \emptyset$ only when $k \in [k_1, k_2]$. See Theorem 2 below, or Proposition 11 in Section 2 for more details about the topology of $I_{k,h}$. We will use the notation $A \cong B$ to denote that the two manifolds A and B are diffeomorphic.

Now, we introduce two interesting invariant sets that will play a very important role. The first one is

$$\mathcal{L} := \{(x, (x+a)/(x-1), x) \in \mathbb{R}^3 \text{ such that } x > 1\} \subset O^+.$$

It is easy to see that the set $\mathcal{L}$ is a curve filled by two-periodic points of F and that it contains the unique fixed point in O^+: (x_c, x_c, x_c), where $x_c = 1 + \sqrt{1+a}$. The second one is

$$\mathcal{G} := \{(x, y, z) \in O^+ \text{ such that } G(x, y, z) = 0\},$$

where

$$G(x, y, z) = -y^3 - (x+z+a+1)y^2 - (x+z+a)y + xz(x+1)(z+1). \quad (5)$$

The set $\mathcal{L} \cup \mathcal{G}$ is formed by the points in O^+ where the gradients of $V_1(x, y, z)$ and $V_2(x, y, z)$ are parallel.

In particular, it is not difficult to check that

$$G(F(x, y, z)) = -\frac{a+y+z}{x^2}\, G(x, y, z). \quad (6)$$

Note that this relation implies that $\mathcal{G}$ is invariant by F and that F maps the zone $\{G > 0\}$ into the zone $\{G < 0\}$ and viceversa. Furthermore it implies that the dynamics of F^2 on the zone $\{G > 0\}$ and on $\{G < 0\}$ are conjugated, being the map F itself the conjugation. Figure 1 gives an example of the more generic position of L_k, M_h, $\mathcal{G}$ and $\mathcal{L}$.

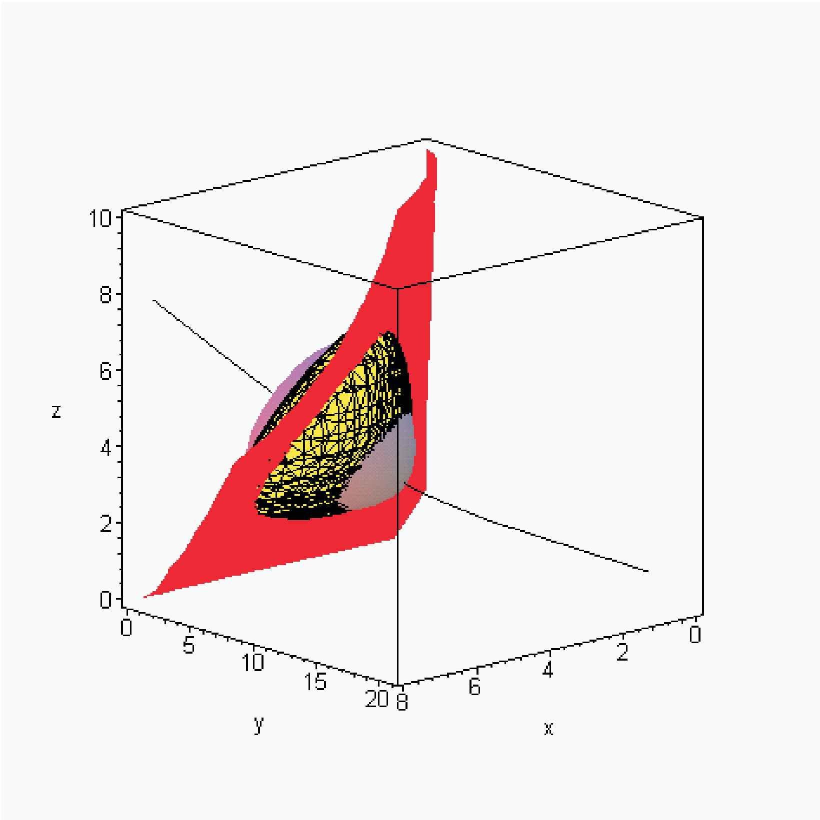

Figure 1: For $a = 3$, the level surfaces L_{31} and $M_{42.5}$ which are diffeomorphic to spheres, the invariant surface $\mathcal{G}$ and the line of 2-period points $\mathcal{L}$.

Our main result about the dynamics of F, which proved in Section 3, is:

Theorem 2. *For each fixed $h > h_c$. The following statements hold*

(i) $M_h = \cup_{k\in[k_1,k_2]} I_{k,h}$, where the values k_1, k_2 are given in Proposition 11. Moreover, for each $k \in (k_1, k_2)$, $I_{k,h}$ splits into two disjoint connected components, that is: $I_{k,h} = I^+_{k,h} \cup I^-_{k,h}$ where $I^+_{k,h} := I_{k,h} \cap \{G > 0\} \cong \mathbb{S}^1$ and $I^-_{k,h} := I_{k,h} \cap \{G < 0\} \cong \mathbb{S}^1$, $F(I^\pm_{k,h}) = I^\mp_{k,h}$, $F^2(I^\pm_{k,h}) = I^\pm_{k,h}$ and the restriction of F^2 on each of these sets is conjugated to a rotation of the circle with rotation number $\rho_{F^2}(k, h)$.

(ii) The set $\mathcal{G}$ is invariant by F and the restriction of F to each set $\mathcal{G} \cap \{V_1(x, y, z) = k\}$, $k > k_c$ is conjugated to a rotation of the circle with

rotation number $\rho_F(k) = \frac{\rho_{F^2}(k,h(k))}{2}$, *where* $h(k)$ *is a suitable known function.*

The proof of the above result is given in two steps. The first one is the study of the topology of the invariant sets $I^{\pm}_{k,h}$ (this is done in Section 2). The second step is to prove that over these invariant sets F is conjugated to a rotation (this is done in Section 3). The proof relies on some results that relate the rotation numbers associated to the invariant sets $I^{\pm}_{k,h}$ of F with the properties of a flow constructed from F which has the same invariant sets.

Next results give some rotation numbers and periods appearing in the dynamical system generated by F. Their proofs use the regularity of the rotation numbers varying h, k and a. This regularity is studied in Section 4.

Theorem 3. *For* $a > 0$ *define*

$$\rho_a := \frac{1}{2\pi} \arccos\left(\frac{(1-a)\sqrt{1+a}}{2(1+\sqrt{1+a})(a+1+\sqrt{1+a})}\right).$$

Then for each $a \neq 1$ *there are circles of initial conditions in* $\{G > 0\} \setminus \mathcal{L}$ *and in* $\{G < 0\} \setminus \mathcal{L}$ *such that such that* F^2 *restricted to them is conjugated to a rotation with rotation number taking any value in* $\left(\frac{1}{4}, \rho_a\right)$, *if* $a > 1$, *and any value in* $\left(\rho_a, \frac{1}{4}\right)$ *if* $0 < a < 1$.

In Section 5 we give a constructive algorithmic approach to the problem of determining which are all the denominators of irreducible fractions which belong to a given interval, see Theorem 25 and Corollary 26. In particular, by using these results and the above theorem, we prove:

Corollary 4. *(i) For any* $a \neq 1$ *there exists a computable value* $q_0(a) \in \mathbb{N}$ *such that for any* $q > q_0(a)$ *there exist a continua of initial conditions giving rise to* $2q$*–periodic orbits for* F.

(ii) Set

$$I_{\text{rot}} = \left(\frac{\pi - 2\,\arcsin(1/8)}{4\pi}, \frac{1}{3}\right).$$

Then, for each number ρ *in* I_{rot} *there exists some* $a > 0$ *and a circle of initial conditions such that such that* F^2 *restricted to it is conjugated to a rotation with rotation number* ρ. *In particular, for all the irreducible rational numbers* $p/q \in I_{\text{rot}}$, *there exist periodic orbits of* F^2 *of period* q.

(iii) The set of even periods arising from the family $\{F(x,y,z) = (y, z, a + y + z/x) : a > 0\}$ *contains all the even periods except possibly* $4, 6, 10, 12, 16, 18, 24, 28,$ *and* 40.

The knowledge that we have of the odd periodic orbits of F is not so detailed as our knowledge of the even periods. We collect all our results in the following proposition:

Proposition 5. *(i) All the initial conditions giving rise to odd periods of F in O^+ are contained in $\mathcal{G}$.*

(ii) There is an open set $\mathcal{U} \subset (0,1) \cup (1,\infty)$ such that for each $a \in \mathcal{U}$ the map F over $\mathcal{G}$ has all the periods except possibly a finite number of them.

(iii) Set

$$J_{\rm rot} = \left(\frac{\arcsin(3/4)}{2\pi}, \frac{1}{6} \right).$$

For each $\rho \in J_{\rm rot}$ there exists some $a > 0$ and a circle of initial conditions contained in $\mathcal{G}$ such that the map F restricted to them is conjugated to a rotation with rotation number ρ. Therefore, for all the irreducible rational numbers $p/q \in J_{\rm rot}$, there exists periodic orbits of F of period q.

All the above results and our numeric simulations of the functions $\rho_{F^2}(k,k)$ and $\rho_F(k)$, detailed in Section 6, make as to propose the following questions. Note that the first one is similar to Zeeman's Conjecture.

Question 1. *Is it true that the function $\rho_{F^2}(k,h)$ varies monotonically when either k or h vary?*

If the answer is affirmative then all the rotation numbers, as well as the set of even periods given in Theorem 1 and Corollary 4, are the only possible ones on $O^+ \setminus \mathcal{G}$.

Question 2. *Is it true that for each $a \neq 1$, the rotation number ρ_F is not identically constant on $\mathcal{G}$? Which is limit of ρ_F when the initial conditions go to infinity over $\mathcal{G}$?*

For instance, the proof of Proposition 5 (ii) follows from the fact that, for a neighbourhood of values of $a = \frac{3-4\cos(2\pi/7)}{(2\cos(2\pi/7)-1)^2}$, the rotation number on $\mathcal{G}$ is not constant. Unfortunately, we have not been able to obtain a general proof of this fact. Our numerical simulations for $a = 3$ and $a = 7/9$ (see Tables 1 and 3 in Section 6) also show the same situation. If the answers to the above questions were affirmative we would obtain that in Proposition 5, $\mathcal{U} = (0,1) \cup (1,\infty)$.

The computation of the limit of ρ_F over $\mathcal{G}$ at infinity would give us useful quantitative information about which would be these odd periods of F.

Remark 6. *Note that when $a \neq 1$, each map F has infinitely many different periods and has sensible dependence with respect to the initial conditions. This last fact is because two close initial condition belong to two close sets, both diffeomorphic to circles, but over each one of them the rotation number is slightly different.*

Proof of Theorem 1. Parts (i) and (ii) are a direct consequence of Corollary 4. Part (iii) follows from Proposition 5.

To prove (iv) observe that all initial conditions in $O^+ \setminus \mathcal{L}$ give rise to rotations for F^2 (respectively F). Moreover for most of these conditions, in the sense of Lebesgue measure, the associated rotation numbers are irrational. Therefore the orbits through these initial conditions are dense in a subset of $\mathcal{G}$ (resp. $O^+ \setminus \mathcal{G}$) which is diffeomorphic to $\mathbb{S}^1$ (resp. the disjoint union of two $\mathbb{S}^1$). The projection into the x-axis of the orbit of F coincides with the sequence generated by (3). This projection is formed by one or two disjoint closed intervals. Both situations are possible depending if the initial conditions are near the two periodic orbit or near $\mathcal{G}$. Hence the theorem follows. □

2. Topology of the invariant sets of F.

2.1. **The results.** This section is devoted to prove the following weaker version of Theorem 2. Note that the difference between both results is that in this second one the dynamics of F or F^2 on each of the invariant circles is not yet described. The description of these dynamics is the goal of next sections. We use the following notations: $A \pitchfork B$ means that A has a transversal intersection with B, and $A \sqcup B$ means the union of A and B and that both sets are disjoint. Recall also that we say $A \cong B$ when A and B are two diffeomorphic varieties.

Theorem 7. (Topology of the invariant sets) *Fix $h > h_c$. Then*

(i) $M_h = \cup_{k\in[k_1,k_2]} I_{k,h}$, where the values k_1, k_2 are given in Proposition 11. For each $k \in (k_1, k_2)$, $I_{k,h} = I^+_{k,h} \sqcup I^-_{k,h}$ and each one of these sets is diffeomorphic to a circle. Moreover $F(I^{\pm}_{k,h}) = I^{\mp}_{k,h}$ and $F^2(I^{\pm}_{k,h}) = I^{\pm}_{k,h}$.

(ii)The set $\mathcal{G}$ is foliated by the fix point of F and the sets $\mathcal{G} \cap \{V_1 = k\}$, $k > k_c$, which are invariant by F and diffeomorphic to circles.

The proof of the above theorem is done at the end of this section. To prove it, we first study the level sets of V_1 and V_2 in O^+ and afterwards their relative position.

Let $L_k = \{(x, y, z) \in O^+ : V_1(x, y, z) = k\}$ and $M_h = \{(x, y, z) \in O^+ : V_2(x, y, z) = h\}$ be the level surfaces of V_1 and V_2 respectively. It is well known that V_1 has a global minimum at (x_c, x_c, x_c), where $x_c = 1+\sqrt{1 + a}$. We set

$$k_c = V_1(x_c, x_c, x_c) = \frac{\left(2 + \sqrt{1 + a}\right)^3 \left(a + 3 + 3\sqrt{1 + a}\right)}{\left(1 + \sqrt{1 + a}\right)^3}.$$

Thus L_k is not empty for $k \geq k_c$, and $L_{k_c} = (x_c, x_c, x_c)$.

Similarly, V_2 also has a minimum at (x_c, x_c, x_c). We set

$$h_c = V_2(x_c, x_c, x_c) = \frac{\left(3 + 2\sqrt{1+a}\right)^2 \left(2\,a + 5 + 5\sqrt{1+a}\right)}{\left(1 + \sqrt{1+a}\right)^3}.$$

Then M_h is not empty for $h \geq h_c$, and $M_{h_c} = (x_c, x_c, x_c)$.

Proposition 8 (proved in Section 2.2), states that except at the fix point all the level surfaces in O^+ of V_1 and V_2 are diffeomorphic to spheres. Note that this result proves in particular that all the orbits of F starting at O^+ lay in compact sets.

Proposition 8. (General properties of L_k and M_h)

(a) *For $k > k_c$, L_k is diffeomorphic to $\mathbb{S}^2$.*
(b) *For $h > h_c$, M_h is diffeomorphic to $\mathbb{S}^2$.*

Theorem 7 follows from the knowledge of the relative positions of the level surfaces L_k and M_h. First we describe the set $\mathcal{F}$ where L_k and M_h are not transversal and give the relative position of $\mathcal{F}$ and L_k.

Lemma 9. (Locus of non-transversality of L_k and M_h) *Let $\mathcal{F}$ be the subset of O^+ where ∇V_1 and ∇V_2 are linearly dependent, i.e $\mathcal{F} := \{\nabla V_1 \| \nabla V_2\} \cap O^+$. Then $\mathcal{F} = \mathcal{L} \cup \mathcal{G}$.*

Proof. Some computations show that

$$\left|\begin{array}{cc} (V_1)_x & (V_2)_x \\ (V_1)_y & (V_2)_y \end{array}\right| = -\,(z+1)\,(1+x+y)\,(a+z+y-xy)\left(ay^2+ay- xz^2 - x^2z + y^2 + yz + y^3 + xy - xz + y^2z - x^2z^2 + xy^2\right)/(x^3y^3z^2),$$

$$\left|\begin{array}{cc} (V_1)_x & (V_2)_x \\ (V_1)_z & (V_2)_z \end{array}\right| = -\,(y+1)\,(x-z)\,(a+x+y+z+xz)\left(ay^2+ay- xz^2 - x^2z + y^2 + yz + y^3 + xy - xz + y^2z - x^2z^2 + xy^2\right)/(x^3y^2z^3), \qquad \text{and}$$

$$\left|\begin{array}{cc} (V_1)_y & (V_2)_y \\ (V_1)_z & (V_2)_z \end{array}\right| = (x+1)\,(1+y+z)\,(a+x+y-yz)\left(ay^2+ay- xz^2 - x^2z + y^2 + yz + y^3 + xy - xz + y^2z - x^2z^2 + xy^2\right)/(x^3y^3z^2).$$

The solutions in O^+ of the above three functions equated to zero satisfy either

$$ay^2 + ay - xz^2 - x^2z + y^2 + yz + y^3 + xy - xz + y^2z - x^2z^2 + xy^2 = 0,$$

which is precisely $\mathcal{G}$ or $\{(x,y,z)\,:\,a+y-xy+z=0,\ x-z=0,\ a+x+y-yz=0\}$ which coincides with $\mathcal{L}=\{(x,y,z)\,:\,y=(x+a)/(x-1),\ z=x\}$, as we wanted to prove. □

The topology of $\mathcal{F}$ is given by the next result, which is proved in Appendix C.

Proposition 10. (Topology of the nontransversality locus) *Fix $k>k_c$. Then*

(i) *$\mathcal{L}\cap L_k$ consists of two points which are a 2–periodic orbit of F.*
(ii) *$\mathcal{G}\pitchfork L_k$.*
(iii) *$\mathcal{G}\cap L_k\cong\mathbb{S}^1$.*

To describe the relative positions of the level surfaces L_k and M_h we keep M_h with $h>h_c$ fixed and consider L_k for all $k>k_c$, obtaining:

Proposition 11. (Relative positions of L_k and M_h) *For a given fixed $h>h_c$, there exists $k_1:=k_1(h),k_2:=k_2(h)$ satisfying $k_c<k_1<k_2$ and such that the following statements hold:*

(a) *If $k\in[k_c,k_1)$ then $I_{k,h}=\emptyset$.*
(b) *If $k=k_1$ then either $I_{k_1,h}=\mathcal{L}\cap M_h$ (which are two points describing a 2–periodic orbit), or $I_{k_1,h}=\mathcal{G}\cap M_h\cong\mathbb{S}^1$.*
(c) *If $k\in(k_1,k_2)$ then $I_{k,h}\cong\mathbb{S}^1\sqcup\mathbb{S}^1$. More precisely, $I_{k,h}\cap\{G>0\}\cong\mathbb{S}^1$ and $I_{k,h}\cap\{G<0\}\cong\mathbb{S}^1$.*
(d) *For $k=k_2$ then either $I_{k_2,h}=\mathcal{L}\cap M_h$ if $I_{k_1,h}=\mathcal{G}\cap M_h$, or $I_{k_2,h}=\mathcal{G}\cap M_h\cong\mathbb{S}^1$, if $I_{k_1,h}=\mathcal{L}\cap M_h$.*
(e) *If $k>k_2$ then $I_{h,k}=\emptyset$.*

The proof of Proposition 11 is given in Subsection 2.3. Now we can prove Theorem 7:

Proof of Theorem 7. The result follows directly from Proposition 11 and the above explained consequences of expression (6). □

2.2. **Proof of proposition 8.** To study the surfaces L_k, solving $V_1(x,y,z)=k$, we get that they can be written as the union of the graphs of the two functions z_- and z_+, given by:

$$z_\pm(x,y;a,k)=\frac{\alpha(x,y;a,k)\pm\sqrt{\Delta(x,y;a,k)}}{\beta(x,y;a,k)},\tag{7}$$

defined in $\{(x,y)\in\mathbb{R}^2:\Delta(x,y;a,k)\geq 0\}$, where

$\alpha(x,y;a,k)=-a-1-(a+2)\,x-(a+2)\,y-x^2-(a-k+3)\,xy-y^2-x^2y-xy^2,$

$$\beta(x,y;a,k) = 2(1+x+y+xy),$$

and
$\Delta(x,y;a,k) = (a-1)^2 + 2a\,(a-1)\,x + 2a\,(a-1)\,y + \left(2a-2+a^2\right)x^2 + \left(-2k-2ka+4a^2-2\right)xy + \left(2a-2+a^2\right)y^2 + 2ax^3 + (-4k-2ka-2+ 6a+2a^2\big)\,x^2y + \left(-4k-2ka-2+6a+2a^2\right)xy^2 + 2ay^3 + x^4 + (4a-2k+2)\cdot \cdot\, x^3y + \left(-2ka-6k+a^2+3+6a+k^2\right)x^2y^2 + (4a-2k+2)\,xy^3 + y^4 + 2x^4y + (-2k+4+2a)\,x^3y^2 + (-2k+4+2a)\,x^2y^3 + 2xy^4 + x^4y^2 + 2x^3y^3 + x^2y^4.$

Observe that

$$(\alpha^2-\Delta)(x,y;a,k) = 4\,(y+1)^2\,(x+1)^2\,(x+y+a) > 0, \tag{8}$$

for $(x,y) \in Q^+ := \{(x,y) : x>0, y>0\}$. Hence $z_\pm(x,y;a,k) \neq 0$ on Q^+. This means that either $z_\pm(x,y;a,k) \in O^+$ for all $(x,y) \in Q^+$, or $z_\pm(x,y;a,k) \in O^- := \{(x,y,z) : x>0, y>0, z<0\}$ for all $(x,y) \in Q^+$. In particular, each connected component of L_k with $x>0$ and $y>0$ is completely contained either in O^+ or in O^- for all $k>k_c$.

On the other hand notice that each level surface has an equator described by

$$z_\pm(x,y;a,k)|_{\Delta(x,y;a,k)=0}.$$

A description of the planar algebraic curves $\Gamma_k := \{(x,y) \in \mathbb{R}^2 : x>0, y>0, \Delta(x,y;a,k)=0\}$ is given in the next lemma, which will be proved in Appendix A. See Figure 2 for more details. It is the key result to prove Proposition 8 (a).

Lemma 12. *For $k \geq k_c$ the planar algebraic curve $\Gamma_k := \{(x,y) \in \mathbb{R}^2 : x>0, y>0, \Delta(x,y;a,k)=0\}$ consists of*

(a) *If $k>k_c$: two concentric ovals γ_k and ζ_k surrounding the point (x_c,x_c). Furthermore γ_k shrinks to (x_c,x_c) when $k \to k_c$, and if $a<1$ then the oval ζ_k has a contact with the axis $\{x=0\}$ and $\{y=0\}$ at $(0,1-a)$ and $(1-a,0)$ respectively.*
(b) *If $k=k_c$: one oval ζ_{k_c} and the point $(x_c,x_c) \in \mathrm{Int}(\zeta_k)$.*

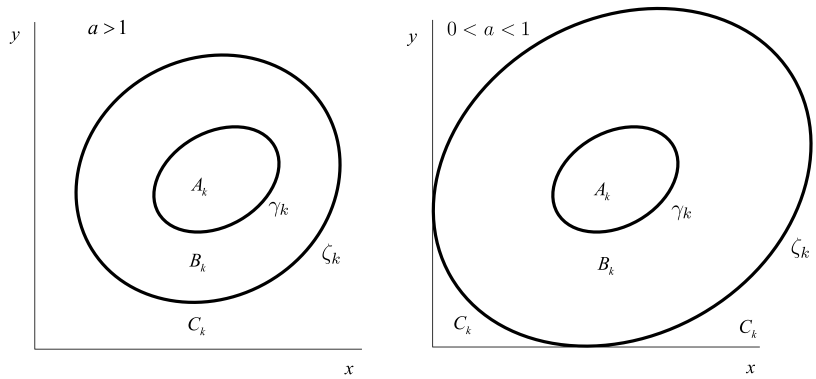

Figure 2: The curve Γ_k of Lemma 12.

Proof of Proposition 8 (a). By Lemma 12, for any $k > k_c$, Q^+ is split in the regions A_k, B_k and C_k, as is shown in Figure 2, defined in the following way:

$$
\begin{aligned}
A_k &= \mathrm{Int}(\gamma_{\mathrm{k}}),\\
B_k &= \mathrm{Int}(\zeta_{\mathrm{k}}) \setminus \{\gamma_{\mathrm{k}} \cup \mathrm{Int}(\gamma_{\mathrm{k}})\},\\
C_k &= Q^+ \setminus \{\zeta_k \cup \mathrm{Int}(\zeta_{\mathrm{k}})\}.
\end{aligned}
$$

Now we will see that for any $k > k_c$ we have $\Delta(x, y; a, k) > 0$ for all $(x, y) \in A_k \cup C_k$; and $\Delta(x, y; a, k) < 0$ for all $(x, y) \in B_k$. This means that the surface $\{V_1 = k\}$, defined by (7) only exists for $(x, y) \in A_k \cup C_k$.

Indeed, we can write $\Delta(x_c, x_c; a, k) = x^2y^2\, k^2 + p_1(a)k + p_0(a)$, and on the other hand $\Delta(x_c, x_c; a, k) = 0$ for $k = k_1$ and $k = k_c$, such that $k_1 < k_c$. Hence, for $k > k_c$ we have $\Delta(x_c, x_c; a, k) > 0$, and this proves that $\Delta(x, y; a, k) > 0$ for all $(x, y) \in A_k$. On the other hand $\Delta(0, 0; a, k) = (1 - a)^2 > 0$, hence $\Delta(x, y; a, k) > 0$ for all $(x, y) \in C_k$.

Finally, it can be seen that the zeros of $\Delta(x, y; a, k) = 0$ on Q^+ are simple so that $\Delta(x, y; a, k) < 0$ for all $(x, y) \in B_k$.

Now we observe that $\lim_{y\to 0^+} \alpha(x, y; a, k) = \alpha(x, 0; a, k) = -x^2 - (a + 2)\, x - a - 1 < 0$, and that $\lim\limits_{y\to+\infty} \alpha(x, y; a, k) = -\infty$, for all $x > 0$ and k. This means together with the above observation concerning equation (8), that $\{V_1 = k\} \subset O^-$, for all $(x, y) \in C_k$.

Observe that $z_\pm(x_c, x_c; a, k_c) = x_c > 0$, hence by continuity $z_\pm(x_c, x_c; a, k) > 0$ for $k \gtrsim k_c$. But, as seen before, by equation (8), each connected component of $\{V_1 = k\}$ with $x > 0$ and $y > 0$ is completely contained either in O^+ or in O^- for all $k > k_c$. So $z_\pm(x, y; a, k) > 0$ for $(x, y) \in A_k$.

Therefore L_k is given by (7) for $(x, y) \in A_k$, hence L_k is a topological sphere. To see that indeed it is diffeomorphic to a sphere, by using the implicit function Theorem, it suffices to prove that the function V_1 has no critical points on L_k. Computing the partial derivatives of V_1 we get:

$$\frac{\partial V_1}{\partial x} = -\frac{(y+1)(z+1)(-x^2+a+y+z)}{x^2yz},$$
$$\frac{\partial V_1}{\partial y} = -\frac{(x+1)(z+1)(-y^2+a+x+z)}{xy^2z},$$
$$\frac{\partial V_1}{\partial z} = -\frac{(x+1)(y+1)(-z^2+a+x+y)}{xyz^2}.$$

Hence the critical points of V_1 which lie on O^+ have to satisfy $x^2 = a+y+z$, $y^2 = a+x+z$, $z^2 = a+x+y$ which easily implies $x = y = z = x_c$. So the only critical point of V_1 on O^+ is the fixed point. Hence part (a) of the proposition follows. □

To prove Proposition 8 (b) we proceed in a similar way that in case (a). Solving $V_2(x, y, z) = h$, we get that the surface $\{V_2 = h\}$ can we written as the union of the graph of the two functions:

$$z_{\pm}(x, y; a, h) = \frac{\alpha(x, y; a, h) \pm \sqrt{\Delta(x, y; a, h)}}{\beta(x, y; a, h)}, \tag{9}$$

where

$$\alpha(x, y; a, h) = -ya-2x^2-3x-a-y^2x-yx^2-5yx-3y+kxy-xa-2y^2-1,$$

$$\beta(x, y; a, h) = 2 + 2x^2 + 4x + 2y + 2yx,$$

and $\Delta(x, y; a, h) = 1 - 2a + 2x + 2y + h^2x^2y^2 - 6yxa + 2ya^2x - 4y^2ax - 2y^3ax - 4y^2ax^2 - 4yax^2 - 2x^3ya - 6hx^2y - 6kxy^2 - 10hx^2y^2 - 2hx^2y^3 - 2hx^3y^2 - 4hx^3y - 4kxy^3 + 4yx^2 + a^2 - 4xa + 4yx - 4ya - 2kxy + x^2 + y^2 + 4y^2x + 2xa^2 - 2kxy^2a - 2hx^2ya - 2kxya + y^2a^2 + 2x^2y^3 + 2xy^3 + 2ya^2 - 2x^2a + 2x^3y + x^2a^2 - 2y^2a + y^4x^2 + y^2x^4 + 2x^3y^2 + 5x^2y^2 + 2y^3x^3$.

By looking at the above coefficients it is easy to check that, if $x > 0$ and $y > 0$, $\alpha(x, y; a, h)^2 - \Delta > 0$ and hence $z_{\pm}(x, y; a, h) \neq 0$ on $Q^+ := \{(x, y) : x > 0, y > 0\}$. Thus, either $z_{\pm}(x, y; a, h) \in O^+$ for all $(x, y) \in Q^+$, or $z_{\pm}(x, y; a, h) \in O^- := \{(x, y, z) : x > 0, y > 0, z < 0\}$ for all $(x, y) \in Q^+$, that is each connected component of $\{V_2 = h\}$ with $x > 0$ and $y > 0$ is completely contained either in O^+ or in O^- for all $h > h_c$.

On the other hand observe that each level surface has an equator given by the equation $z_{\pm}(x, y; a, h)|_{\Delta(x,y;a,h)=0}$. The description of the planar algebraic curves, dropping the subindex a, $\Gamma_h := \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0, \Delta(x, y; a, h) = 0\}$ is again the key of the proof of Proposition 8 (b), see Figure 3. We will use the following lemma, proved in Appendix B.

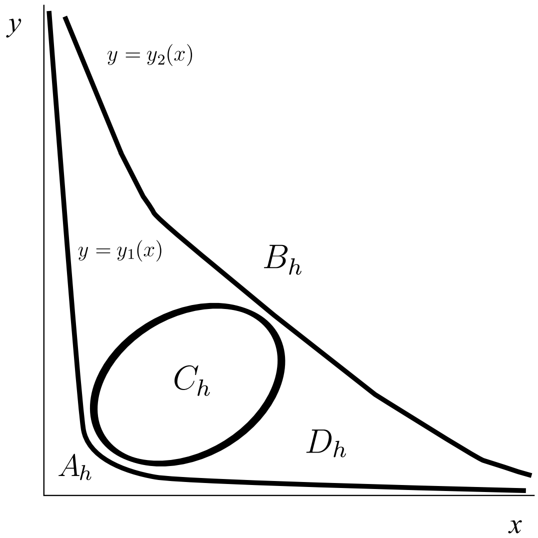

Figure 3: The curve Γ_h of Lemma 13.

Lemma 13. *For* $h \geq h_c$ *the planar algebraic curve* $\Gamma_h := \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0, \Delta(x,y;a,h) = 0\}$ *consists of*

(a) *Two branches* $y = y_i(x)$, $i = 1,2$, *such that* $y_1(x) < y_2(x)$, *for* $x > 0$. *Furthermore these two branches satisfy* $\lim_{x\to 0^+} y_i(x) = +\infty$ *and* $\lim_{x\to+\infty} y_i(x) = 0^+$.

(b) *An oval* γ_h, *contained between these to branches if* $h > h_c$, *and a single point if* $h = h_c$.

Proof of Proposition 8 (b). By using Lemma 13 we have that for any $h > h_c$, Q^+ splits in four regions A_h, B_h, C_h and D_h, as is shown in Figure 4, defined in the following way:

$$\begin{aligned}
A_h &= \{(x,y) \in Q^+ : y \leq y_1(x)\},\\
B_h &= \{(x,y) \in Q^+ : y \geq y_2(x)\},\\
C_h &= \gamma_h \cup \mathrm{Int}(\gamma_h),\\
D_h &= \{(x,y) \in Q^+ : y_1(x) \leq y \leq y_2(x)\} \setminus \{\gamma_h \cup \mathrm{Int}(\gamma_h)\}.
\end{aligned}$$

It is easy to check that for any $h > h_c$, $\Delta(x,y;a,h) > 0$ for all $(x,y) \in A_h \cup B_h \cup C_h$; and $\Delta(x,y;a,h) < 0$ for all $(x,y) \in D_h$. This means that the surface $\{V_2 = h\}$, is only defined by (9) on the region $A_h \cup B_h \cup C_h$.

We now observe that $\lim_{y\to 0^+}\alpha(x,y;a,h)=\alpha(x,0;a,h)=-1-a-(a+3)x-2x^2<0$, and $\lim\limits_{y\to+\infty}\alpha(x,y;a,h)=-\infty$, for all $x>0$ and h. Hence $\{V_2=h\}\subset O^-$, for all $(x,y)\in A_h\cup B_h$.

We observe that $z_{\pm}(x_c,x_c;a,h_c)=x_c>0$, hence by continuity $z_{\pm}(x_c,x_c;a,h)>0$ for $h\gtrsim h_c$. But, as seen before each connected component of $\{V_2=h\}$ with $x>0$ and $y>0$ is completely contained either in O^+ or in O^- for all $h>h_c$. So $z_{\pm}(x,y;a,h)>0$ for $(x,y)\in C_h$.

Therefore M_h is given by (9) for $(x,y)\in C_h$, hence M_h is indeed a topological sphere. Finally, let us see that the surface M_h is a differentiable manifold for $h>h_c$. It is enough to see that V_2 has no critical points on M_h. The partial derivatives of V_2 are:

$$\frac{\partial V_2}{\partial x}=-\frac{(y+z+1)\left(-x^2-x^2z+ya+y^2+yz+a+y+z\right)}{x^2yz},$$
$$\frac{\partial V_2}{\partial y}=-\frac{a+x+z+xza+xa+x^2+3\,xz-2\,y^2-xy^2z+za+z^2+2\,x^2z-2\,xy^2-y^2a-2\,y^2z+2\,xz^2+x^2z^2-2\,y^3}{xy^2z},$$
$$\frac{\partial V_2}{\partial z}=-\frac{(x+y+1)\left(-z^2-xz^2+ya+xy+y^2+a+x+y\right)}{xyz^2}.$$

The critical points on O^+ have to satisfy:

$$\begin{aligned}
p(x,y,z):=&\quad -x^2-x^2z+ya+y^2+yz+a+y+z=0\\
q(x,y,z):=&\quad a+x+z+xza+xa+x^2+3\,xz-2\,y^2-xy^2z+za+z^2+\\
&\quad +2\,x^2z-2\,xy^2-y^2a-2\,y^2z+2\,xz^2+x^2z^2-2\,y^3=0\\
r(x,y,z):=&\quad -z^2-xz^2+ya+xy+y^2+a+x+y=0.
\end{aligned}$$

Since $p(x,y)-r(x,y)=(z-x)(1+x+y+z+xz)$ we get $z=x$, and substituting this equality in $p(x,y,z)$ and $q(x,y,z)$ we have the system:

$$\begin{aligned}
&s(x,y)=-x^2-x^3+ya+y^2+xy+a+y+x=0,\\
&t(x,y)=(x+y+1)\left(x^3+3\,x^2-yx^2+xa+2\,x-2\,xy-2\,y^2+a-ya\right)=0.
\end{aligned}$$

Denoting

$$u(x,y)=x^3+3\,x^2-yx^2+xa+2\,x-2\,xy-2\,y^2+a-ya$$

we get

$$2s(x,y)+u(x,y)=-x^3+x^2-yx^2+xa+4x+3a+ya+2y=0$$

which let us to isolate y in terms of $x:$

$$y=\frac{x^3-x^2-(4+a)x-3a}{-x^2+a+2}. \tag{10}$$

Then

$$s\left(x,\frac{x^3-x^2-(a+4)x-3a}{(-x^2+a+2)}\right)=\frac{h(x)}{\left(-x^2+a+2\right)^2},$$

where $h(x) = -\left(a-1+x+x^2\right)\left(a+2\,x-x^2\right)\left(2\,a+xa-x^3-2-2\,x-2\,x^2\right)$, and hence we have to consider three cases depending on the zeros of $h(x)$.

If $a-1+x+x^2=0$ and $x>0$, then $x=x_c$ and from (10), $y=x_c$. Since $z=x$ we get the fixed point.

If $a+2\,x-x^2=0$ and $x>0$, then $x=(-1+\sqrt{5-4a})/2$ and substituting this value of x at (10) we see that the corresponding y is negative, so we do not need to study this case.

Now we have to consider the positive roots of $g(x) := x^3+2\,x^2+(2-a)\,x+2(1-a)$. We notice that when $a<1$ there are not changes on the signs on the coefficients of $g(x)$, and hence there are not positive roots of $g(x)=0$. When $a>1$, then there is a unique change of signs between the coefficients of $g(x)$, and hence, by the Descartes rule, and since $g(0)<0$ and $\lim_{x\to+\infty} g(x)=+\infty$, we get exactly one positive real root, say $\bar{x}$. We claim that for $x=\bar{x}$, the corresponding value of y given in (10) is negative. To prove this observe that $g(\sqrt{a+2})=4\sqrt{a+2}+4\,a+6>0$, which implies that $\bar{x}<\sqrt{a+2}$. So, $a+2-\bar{x}^2>0$, i. e., the denominator of (10) is positive. By evaluating the numerator of (10) at $\bar{x}$, we get:

$$\bar{x}^3-\bar{x}^2-(4+a)\bar{x}-3a=\bar{x}^3-\bar{x}^2-(4+a)\bar{x}-3a-g(\bar{x})=-(3\bar{x}^2+6\bar{x}+a+2)<0.$$

Hence, there are no critical points of V_2 in O^+ different from (x_c,x_c,x_c), and the result follows. □

2.3. **Proof of Proposition 11.** To prove Proposition 11 we need some technical results, stated below.

Lemma 14. (a) *$I_{k_c,h_c}=(x_c,x_c,x_c)$, and $I_{k,h_c}=\emptyset$ for $k>k_c$.*

(b) *Fix $h>h_c$. Set $k_1=\min_{M_h} V_1$ and $k_2=\max_{M_h} V_1$, then the following statements hold.*

(i) *$I_{k,h}\neq\emptyset$ if and only if $k\in[k_1,k_2]$.*

(ii) *$I_{k_i,h}\subset\mathcal{F}=\mathcal{G}\cup\mathcal{L}$, for each $i=1,2$.*

Proof. (a) For $k=k_c$ $L_{k_c}=M_{h_c}=(x_c,x_c,x_c)$, hence $I_{k_c,h_c}=(x_c,x_c,x_c)$. For $k>k_c$, since $(x_c,x_c,x_c)\notin L_k$, $I_{k,h_c}=\emptyset$.

(b) Since M_h is compact then there exists $k_1=\min_{M_h} V_1$ and $k_2=\max_{M_h} V_1$.

Observe that $k_2\neq k_1$ because $V_1(M_h)$ is not constant, since otherwise $\nabla V_1(M_h)\|\nabla V_2(M_h)$, but this only happens in $M_h\cap(\mathcal{L}\cup\mathcal{G})\neq M_h$ (to prove this last inequality just consider that if $h>h_c$, then $M_h\cap\mathcal{L}$ consists of two points which are not contained in $\mathcal{G}$, hence $M_h\cap(\mathcal{L}\cup\mathcal{G})$ is disconnected).

By definition of k_1 and k_2 it is obvious that $I_{k,h}=\emptyset$ if $k\notin[k_1,k_2]$.

Let $a,b\in\mathbb{R}^3$ be a points in M_h such that $V_1(a)=k_1$, and $V_1(b)=k_2$ (observe that this points exist because the absolute extrema of $V_{1|\{M_h\}}$ are reached).

Take now a continuous curve $\gamma : [0,1] \longrightarrow M_h$ such that $\gamma(0) = a$ and $\gamma(1) = b$. The function

$$\begin{array}{rcl} g : [0,1] & \longrightarrow & [k_1, k_2] \\ t & \longrightarrow & V_1(\gamma(t)) \end{array}$$

is continuous, that is for all $k \in (k_1, k_2)$ there exists at least t_k such that $g(t_k) = k$, and $\gamma(t_k) \in I_{k,h}$. Hence the result follows.

(ii) By the definition of k_1 and k_2, and using the theory of extrema with constraints, V_1 reaches these values at points where the gradient vectors of V_1 and V_2 are parallel, hence in $\mathcal{F}$, as we wanted to prove. $\square$

Lemma 15. *Fixed $k > k_c$. $V_2|_{\{L_k \cap \mathcal{L}\}} = c_1$ and $V_2|_{\{L_k \cap \mathcal{G}\}} = c_2$ where c_1 and c_2 are different constants.*

Proof. From Lemma 10 (i), $L_k \cap \mathcal{L} = p_1 \sqcup p_2$, but since $\{p_1, p_2\}$ is a 2–periodic orbit and V_2 is an invariant $V_2(p_1) = V_2(F(p_2)) = V_2(p_2) = c_1$.

Also V_2 is constant over $L_k \cap \mathcal{G}$. Indeed, by Lemma 10 (ii), $L_k \cap \mathcal{G} \cong \mathbb{S}^1$, hence we can consider a $\mathcal{C}^1$-parameterization of $L_k \cap \mathcal{G}$ given by $\gamma(t)$. By definition $\dfrac{d}{dt}V_1(\gamma(t)) = \nabla V_1(\gamma(t)) \cdot \gamma'(t) = 0$. But observe that on $\mathcal{G}$, $\nabla V_1 \| \nabla V_2$, hence $\nabla V_2(\gamma(t)) \cdot \gamma'(t) = \nabla V_1(\gamma(t)) \cdot \gamma'(t) = 0$, thus $V_2(\gamma(t))$ is constant. Hence $V_2(L_k \cap \mathcal{G}) = c_2$.

To prove that $c_1 \neq c_2$, just observe that by a same argument than the one used in the proof of Lemma 14, c_1 and c_2 both must be the extrema of $V_2(L_k)$, and V_2 is not constant over L_k. $\square$

Corollary 16. (i) *Either*

$$I_{k_1,h} = \mathcal{L} \cap L_k \cong p_1 \sqcup p_2, \text{ and } I_{k_2,h} = \mathcal{G} \cap L_k \cong \mathbb{S}^1, \quad \text{or}$$

$$I_{k_1,h} = \mathcal{G} \cap L_k \cong \mathbb{S}^1, \text{ and } I_{k_2,h} = \mathcal{L} \cap L_k \cong p_1 \sqcup p_2,$$

where p_1, p_2 is the two-periodic orbit, located in L_k.

(ii) *If $k \in (k_1, k_2)$ then $L_k \pitchfork M_h$. In particular $I_{k,h} \cong \sqcup_{\text{finite}} \mathbb{S}^1$.*

Proof. (i) Lemma 14 (ii) ensures that $I_{k_1,h} \subset \mathcal{F} = \mathcal{L} \cup \mathcal{G}$. Lemma 15 prevents that there exist points $a, b \in L_{k_1} \cap M_h$ such that $a \in \mathcal{L}$ and $b \in \mathcal{G}$. Thus either we have

- Case 1: $I_{k_1,h} \subset \mathcal{L}$ and hence $I_{k_1,h} = I_{k_1,h} \cap \mathcal{L}$, or
- Case 2: $I_{k_1,h} \subset \mathcal{G}$, (hence $I_{k_1,h} = I_{k_1,h} \cap \mathcal{G}$).

Observe that by Lemma 10 (i) $L_{k_1} \cap \mathcal{L} = p_1 \sqcup p_2$, but since, by Lemma 15, $V_2(p_1) = V_2(p_2)$, we have that in the Case 1, $I_{k_1,h} = L_{k_1} \cap M_h \cap \mathcal{L} = L_{k_1} \cap \mathcal{L} = p_1 \sqcup p_2$.

In the second case, we need to prove that $I_{k_1,h} = L_{k_1} \cap M_h \cap \mathcal{G} = L_{k_1} \cap \mathcal{G} \cong \mathbb{S}^1$. Observe that this is a consequence of the fact that by Lemma 15, V_2 is constant over $L_{k_1} \cap \mathcal{G}$.

The same argument holds if instead of $I_{k_1,h}$ we consider $I_{k_2,h}$. But now observe that if we are in the first case, then $I_{k_2,h} = L_k \cap \mathcal{G}$, because each point belongs only to one level set of V_1. The same happens in the second case.

(ii) Statement (i) implies that the locus of non transversal intersections of the foliation of O^+ given by $\{L_k\}_{\{k>k_c\}}$ with M_h are given only by $I_{k_i,h}$, $i = 1, 2$. On the other hand Lemma 14 (i) ensures that $I_{k,h} \neq \emptyset$ for $k \in (k_1, k_2)$, therefore L_k and M_h must intersect transversally in this case. Thus, see [9, page 30], $\mathrm{Codim}(L_k \cap M_h) = \mathrm{Codim}(L_k) + \mathrm{Codim}(M_h) = 1 + 1 = 2$ and $L_k \cap M_h$ is a submanifold of $\mathbb{R}^3$. This implies that $I_{k,h}$ is a union of curves. But since both L_k and M_h are compact, and each connected, compact 1–dimensional manifold is diffeomorphic to $\mathbb{S}^1$, see [9, page 208], then $I_{k,h} \cong \sqcup \mathbb{S}^1$. But these disjoint union of $\mathbb{S}^1$ lie in a compact region (say M_h) and are defined by analytic equations, therefore it must be a finite union. □

Next lemma shows that in (ii) of the above Corollary the finite union is exactly two $\mathbb{S}^1$.

Lemma 17. (Topology at the transversal intersections of L_k and M_h) *For all $k \in (k_1, k_2)$, $I_{k,h} \cong \mathbb{S}^1 \sqcup \mathbb{S}^1$.*

Proof. From Corollary 16 (ii) and Proposition 8 (b) we know that for all $k \in (k_1, k_2)$, $I_{k,h} \cong \sqcup_{\text{finite}} \mathbb{S}^1$.

Observe that $\mathcal{V}_1 = \{L_k\}_{\{k\in(k_1,k_2)\}}$, induce a foliation of closed curves on M_h nesting the 2–periodic points defined by $M_h \cap \mathcal{L}$. These two periodic points are the only ones in the foliation of M_h induced by $\mathcal{V}_2 = \{L_k\}_{\{k\in[k_1,k_2]\}}$, and since they are in the plane $z = x$, each of the closed curves of $\mathcal{V}_1$ must intersect the plane $z = x$. We want to prove there are only two of them.

Consider now the restriction of M_h to the plane $z = x$, given by the equation $V_2(x, y, x) = h$, which solutions are described by two functions $x \to y_\pm(x, h)$.

Let $v_1(x) := V_1(x, y_+(x, h), x)$ be the restriction of V_1 over the branch $y = y_+(x, h)$. We only need to prove that fixed $k \in (k_1, k_2)$, the equation

$$V_1(x, y_+(x, h), x) = k, \tag{11}$$

has only two solutions, that correspond to the two closed curves of the statement (observe that from expression (6) for any closed invariant curve γ_1 such that $\gamma_1 \cap \{G < 0\} \neq \emptyset$, we have $\gamma_1 \cap \{G > 0\} = \emptyset$). To see this we will prove that the singular points of $v_1(x)$ are located in $(\mathcal{G}\cup\mathcal{L})\cap\{z=x\}$, hence v_1 is monotonic for those x such that $(x,y_+(x,h),x)\in\{G>0\}\cap\{z=x\}$ or $\{G<0\}\cap\{z=x\}$, and therefore equation (11) has two solutions. Indeed, in $z=x$, $(V_i)_x=(V_i)_z$ for $i=1,2$,

$$v_1'(x)=\left((V_1)_x+(V_1)_y\frac{dy_+}{dx}+(V_1)_z\right)\Bigg|_{\{z=x,y=y_+\}}= \\ =\left(2(V_1)_x+(V_1)_y\frac{dy_+}{dx}\right)\Bigg|_{\{z=x,y=y_+\}}, \tag{12}$$

and from $V_2(x,y,x)=h$, we have

$$\frac{dy_+}{dx}=-\left(\frac{(V_2)_x+(V_2)_z}{(V_2)_y}\right)\Bigg|_{\{z=x,y=y_+\}}=-\left(\frac{2(V_2)_x}{(V_2)_y}\right)\Bigg|_{\{z=x,y=y_+\}}. \tag{13}$$

Using equations (12) and (13) we have $v_1'(x)=0$ if and only if

$$\frac{(V_1)_x}{(V_2)_x}=\frac{(V_1)_y}{(V_2)_y}.$$

Hence on the locus where the gradient vectors of V_1 and V_2 are parallel. This set is $(\mathcal{L}\cup\mathcal{G})\cap\{z=x\}$, as we wanted to prove. □

Proof of Proposition 11. Statements (a) and (e) are a direct consequence of Lemma 14 (i), statements (b) and (d) from Corollary 16 (i) and, finally, statement (c) follows from Lemma 17. □

3. Dynamics of F. Proof of Theorem 2.

Next result relates, under some hypotheses, the dynamics of an ordinary differential equation and a discrete dynamical system that share an one dimensional invariant set.

Theorem 18. *Let $f:U\to U$ be a $\mathcal{C}^1$ map where U is an open connected set $U\subset\mathbb{R}^n$, satisfying the following assumptions:*

(A1) *There exists a $\mathcal{C}^1$ vector field X in U such that*

$$X(f(q))=(Df)_q\,X(q)\quad\text{for all}\quad q\in U. \tag{14}$$

(A2) *For a fixed $p\in U$, the map f leaves invariant γ_p, where γ_p is the orbit of $\dot{x}=X(x)$ which passes trough p. In particular there exists $\tau\in\mathbb{R}$ such that $\varphi(\tau,p)=f(p)$, where $\varphi(t,p)$ is the flow of X.*

Then, if $\gamma_p\cong\mathbb{S}^1$, the restriction of f on γ_p is conjugated to a rotation on the circle with rotation number τ/T, where T is the period of γ_p.

Proof. By substituting $q=\varphi(t,p)$ in (14) we obtain:

$$X(f(\varphi(t,p)))=(Df)_{\varphi(t,p)}\,X(\varphi(t,p))\quad \text{for all}\quad t\in\mathbb{R}.$$

Notice that the above equality precisely says that the function $t\to f(\varphi(t,p))$ is also a solution of $\dot{x}=X(x)$. Since when $t=0$ it passes trough $f(p)$, by the theorem of uniqueness of solutions we have that

$$\varphi(t,f(p))=f(\varphi(t,p))\quad \text{for all}\quad t\in I_p\cap I_{f(p)}.$$

Hence, since $\varphi(\tau,p)=f(p)$, if $q=\varphi(t,p)$ we get that $\varphi(t,\varphi(\tau,p))=f(q)$ or, equivalently that

$$\varphi(\tau,q)=f(q)\qquad \text{for all}\quad q\in\gamma_p. \tag{15}$$

Let us prove by using this relation that the map $f:\gamma_p\to\gamma_p$ is conjugated to a rotation of the circle with rotation number $\rho:=\tau/T$.

Indeed we prove that the map $h:\mathbb{S}^1\to\gamma_p$ given by $h(\exp{(it)})=\varphi\left(\frac{T}{2\,\pi}\,t,p\right)$ is the desired conjugation. To see this it suffices to show that $f\circ h=h\circ r_\tau$, where r_τ is the rotation of angle $2\,\pi\tau/T$. The following chains of equalities give us the desired result.

$$(f\circ h)(\exp{(it)})\;=\;f\left(\varphi\left(\frac{T}{2\pi}t,p\right)\right)=\varphi\left(\tau+\frac{T}{2\pi}t,p\right),$$

$$\begin{aligned}(h\circ r_\tau)(\exp{(it)})\;&=\;h\left(\exp\left(i(t+\frac{2\pi\tau}{T})\right)\right)=\varphi\left(\frac{T}{2\pi}(t+\frac{2\pi\tau}{T}),p\right)=\\ &=\;\varphi\left(\tau+\frac{T}{2\pi}t,p\right),\end{aligned}$$

where we have used (15). □

Proof of Theorem 2. By using Theorem 7, only remains to prove that F or F^2 restricted to the invariant leaves given in this theorem are conjugated to rotations. This will be done by using Theorem 18. To apply it we need a vector field X having the same invariant leaves that in Theorem 7 and satisfying (14). We start with the vector field $\tilde{X}=\nabla V_1\times\nabla V_2$, where recall that V_1 and V_2 are the invariants of F. We obtain that

$$\begin{aligned}\tilde{X}_1(x,y,z)&:=(x+1)(1+y+z)(yz-x-y-a)G(x,y,z)/(x^2y^3z^3),\\ \tilde{X}_2(x,y,z)&:=(y+1)(z-x)(a+x+y+z+xz)G(x,y,z)/(x^3y^2z^3),\\ \tilde{X}_3(x,y,z)&:=(z+1)(1+x+y)(a+y+z-xy)G(x,y,z)/(x^3y^3z^2).\end{aligned}$$

Clearly it has V_1 and V_2 as first integrals, but unfortunately it does not satisfy (14). It is natural to try to remove the common factors of the components of the above vector field. We consider the new differential equation defined by the vector field $X(x,y,z) := \frac{(xyz)^2}{G(x,y,z)}(\nabla V_1(x,y,z) \times \nabla V_2(x,y,z))$:

$$\begin{array}{rcl} \dot{x} &=& X_1(x,y,z) := (x+1)(1+y+z)(yz-x-y-a)/(yz), \\ \dot{y} &=& X_2(x,y,z) := (y+1)(z-x)(a+x+y+z+xz)/(xz) \\ \dot{z} &=& X_3(x,y,z) := (z+1)(1+x+y)(a+y+z-xy)/(xy), \end{array} \quad (16)$$

A computation shows that it satisfies condition (14), i.e. $X(F(q)) = (DF)_q\, X(q)$ in O^+, and then also $X(F^2(q)) = (DF^2)_q\, X(q)$ in O^+.

Since X also has V_1 and V_2 as a first integrals, each connected component of $I_{k,h}$ will be an orbit of $\dot{x} = X(x)$. By Theorem 7, the sets $I_{k,h} \cap \{G > 0\} \cong \mathbb{S}^1$ and $I_{k,h} \cap \{G < 0\} \cong \mathbb{S}^1$, for $k \in (k_1, k_2)$, are periodic orbits of X and invariant by F^2. Since condition (14) is satisfied for F^2, Theorem 18 applies and F^2 is conjugated to a rotation of the circle. Hence, statement (i) follows.

(ii) Now consider $\mathcal{G} \cap \{V_1 = k\}$, which by Theorem 7 is also a periodic orbit of $\dot{x} = X(x)$ and is invariant by F. Since (14) is also satisfied, by using again Theorem 18 we get that on $\mathcal{G} \cap \{V_1 = k\}$, F is conjugated to a rotation of the circle, as we wanted to prove. □

4. Properties of the rotation numbers of F

The main result of this section proves the analyticity of the rotation number of F^2 in $O^+ \setminus \{\mathcal{L}\}$ and computes the limit of these rotation numbers when we tend in a certain way to the line of two periodic points $\mathcal{L}$.

Proposition 19. *(i) For each fixed $a > 0$ and $h > h_c$ consider the values $k_1 := k_1(a)$ and $k_2 := k_2(a)$ given in Theorem 11. Then the assignment $(k,h,a) \longrightarrow \rho(k,h,a)$, where $\rho(k,h,a)$ is the rotation number of F^2 restricted to $I_{k,h}$, is analytic for all (k,h,a) with $a > 0$, $h > h_c$ and $k \in (k_1,k_2)$.*

(ii) Fix $a > 0$ and $k > k_c$. Let $\rho_F(k)$ be the rotation number of F at each level curve $\{V_1 = k\} \cap \mathcal{G}$, and let $-1, \cos(\bar{\theta}) \pm i \sin(\bar{\theta})$ be the three eigenvalues of DF at the fix point of F. Then

$$\lim_{k \to k_c} \rho_F(k) = \frac{\bar{\theta}}{2\pi} = \frac{1}{2\pi} \arccos\left(\frac{a - 1 + \sqrt{1+a}}{2a} \right).$$

(iii) Fix $a > 0$ and consider the surface M_h with $h > h_c$ a fixed value. The set $M_h \cap \{G > 0\}$ is filled by closed curves given by $I^+_{k,h}$ and by a fix point of F^2 given by $p_h = M_h \cap \{V_1 = k^*\} \cap \{G > 0\} \in \mathcal{L}$ given by $p_h = (x_h, (a + x_h)/(x_h - 1), x_h)$, $x_h > 1$. Let $\rho_{F^2}(k)$ be the rotation number of F^2 on $I^+_{k,h}$ and $1, \cos(\theta_h) \pm i \sin(\theta_h)$ the three eigenvalues of DF^2 at p_h.

Then,

$$\lim_{k\to k^*}\rho_{F^2}(k)=\frac{\theta_h}{2\pi}=\frac{1}{2\pi}\arccos\left(\frac{(a-1)(1-x_h)}{2x_h(a+x_h)}\right).$$

To prove the above proposition we need some preliminary results.

Lemma 20. *(i) The only singular points of the vector field (16) in O^+ are on $\mathcal{L}$. Moreover, except on $\mathcal{L}$, the plane $\Sigma=\{z=x\}$ is a global transversal section for its flow.*

(ii) Set $a=a_0>0$, $h>h_c$, $k\in(k_1,k_2)$, and $p_0=(x_0,y_0,x_0)\in\Sigma\setminus\{\mathcal{L}\cup\mathcal{G}\}$. Then there exists a neighborhood of p_0 in Σ (namely $\Sigma_{loc}=\Sigma\cap B_\varepsilon(p_0)$), such that the points $p\in\Sigma_{loc}$ depend analytically on a,k and h.

Proof. A straightforward computation shows that X is orthogonal to Σ outside $\mathcal{L}$ and that $\mathcal{L}$ is filled by the only singular points of X in O^+. So, statement (i) follows.

It is important to notice that as we will see in the proof of Lemma 17, all the periodic orbits of the vector field X must intersect Σ.

To prove (ii) consider

$$\begin{array}{rlcl} V: & \mathbb{R}^5 & \longrightarrow & \mathbb{R}^2 \\ & (x,y,a,h,k) & \longrightarrow & (V_1(x,y,x)-k, V_2(x,y,x)-h), \end{array}$$

where the dependence of a is hidden in V_1 and V_2. On one hand $V(x_0,y_0,a_0,k_0,h_0)=0$, and on the other hand

$$\det\left(\begin{array}{cc} \frac{\partial V_1}{\partial x} & \frac{\partial V_1}{\partial y} \\ \frac{\partial V_2}{\partial x} & \frac{\partial V_2}{\partial y}\end{array}\right)\Bigg|_{z=x}=-\frac{(x+1)(x+y+1)(a+x-xy+y)\,G(x,y,x)}{x^5y^3},$$

which is nonvanishing in $\Sigma\setminus\{\mathcal{L}\cup\mathcal{G}\}$. From the implicit function theorem, x and y are analytic functions of a,k and h in a neighborhood of $p_0=(x_0,y_0,x_0)$, namely Σ_{loc}. □

Lemma 21. *Let $\varphi(t,p,a)$ be the flow of (16), where we explicitly write the dependence with respect to a. Fix $a>0$ and take $p=(x,y,x)\in\Sigma\setminus\{\mathcal{L}\cup\mathcal{G}\}$.*

- (i) *If $T(p,a)$ is the period of the periodic orbit of X passing through p, then $T(p,a)$ is analytic at (p,a).*
- (ii) *If $\tau(p,a)$ is defined by the equation $\varphi(\tau(p,a),p,a)=F^2(p)$, then $\tau(p,a)$ is also analytic at (p,a).*

Proof. Consider the system $\varphi(t,p,a)-p=0$. Obviously $\varphi(T(p,a),p,a)-p=0$, and

$$\frac{\partial}{\partial t}(\varphi(t,p,a)-p)=X(p)=(X_1(p),0,-X_1(p))\neq 0,$$

because $X_1(p) \neq 0$ in $\Sigma \setminus \mathcal{L}$. By applying the implicit function theorem to the first component of the above system we have that in a neighborhood of (p,a) in $\Sigma \setminus \{\mathcal{L} \cup \mathcal{G}\}$ the period function $T(p,a)$ is analytic. The proof of (ii) follows applying the same argument to equation $\varphi(t,p,a) - F^2(p) = 0$. $\square$

Lemma 22. *Suppose that we have a smooth vector field $\dot{q} = X(q)$ and a smooth map f in a neighborhood $\mathcal{U} \subseteq \mathbb{R}^n$, satisfying condition (14). If $p = h(q)$, and h is a diffeomorphism between $\mathcal{U}$ and $h(\mathcal{U})$, then the induced vector field $\tilde{X}(p) := (Dh)_q X(q)$ and the map $\tilde{f} = h \circ f \circ h^{-1}$ also satisfies condition (14).*

Proof. Indeed $\tilde{X}(p) := (Dh)_q X(q) = (Dh)_{h^{-1}(p)} X\left(h^{-1}(p)\right)$. Hence

$$\begin{aligned}
\tilde{X}\left(\tilde{f}(p)\right) &= (Dh)_{h^{-1}(h(f(h^{-1}(p))))} X\left(h^{-1}(h(f(h^{-1}(p))))\right) = \\
&= (Dh)_{f(h^{-1}(q))} (Df)_{h^{-1}(p)} X\left(h^{-1}(p)\right) = \\
&= \left[(Dh)_{f(h^{-1}(p))} (Df)_{h^{-1}(p)} (Dh^{-1})_p\right] \cdot \left[(Dh)_{h^{-1}(p)} X\left(h^{-1}(p)\right)\right] = \\
&= D\tilde{f}(p) \cdot \tilde{X}(p).
\end{aligned}$$

$\square$

Lemma 23. *Consider the planar vector field*

$$X(u,v) := -v\, g(u^2+v^2) \frac{\partial}{\partial u} + u\, g(u^2+v^2) \frac{\partial}{\partial v}, \tag{17}$$

with $g(0) \neq 0$ and f a differentiable map in a neighborhood of the origin, such that it leaves invariant the circles $\gamma_r := \{u^2+v^2 = r^2\}$ and condition (14) is satisfied, i.e. $X(f(u,v)) = (Df)_{(u,v)} X(u,v)$. Then f is conjugated on each γ_r to a rotation with rotation number $\rho(r)$ and $\lim_{r \to 0} \rho(r) = \theta$, where $\cos\theta \pm i \sin\theta$ are the eigenvalues of $Df|_{(0,0)}$.

Proof. By Theorem 18, and taking into account that the sets γ_r are also invariant under X, we obtain that on each γ_r, f is conjugated to a rotation and there exists $\tau(\sqrt{u^2+v^2})$, such that

$$f(u,v) = \varphi\left(\tau(\sqrt{u^2+v^2}), (u,v)\right) = \varphi\left(\tau(r), (u,v)\right),$$

where φ is the flow of X. By taking polar coordinates it is not difficult to obtain that

$$\varphi(t,(u,v)) = \begin{pmatrix} \cos\left(g(r^2)t\right) & -\sin\left(g(r^2)t\right) \\ \sin\left(g(r^2)t\right) & \cos\left(g(r^2)t\right) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

Hence on each set γ_r, f is indeed the rotation of angle $\alpha(r) = g(r^2)\tau(r)$, which has rotation number $\rho(r) = \alpha(r)/(2\pi)$. By using this fact and the differentiability of f at the origin we have

$$\begin{aligned} f(u,v) &= \begin{pmatrix} \cos\alpha(r) & -\sin\alpha(r) \\ \sin\alpha(r) & \cos\alpha(r) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \\ &= \begin{pmatrix} \cos\alpha(0) & -\sin\alpha(0) \\ \sin\alpha(0) & \cos\alpha(0) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + O_2(u,v). \end{aligned}$$

Hence $\alpha(0) = \theta$ and therefore $\lim_{r\to 0} \rho(r) = \lim_{r\to 0} \alpha(r)/(2\pi) = \theta/(2\pi)$, as we wanted to prove. □

Proof of Proposition 19. (i) From the above lemmas we know that the functions $\tau(k,h,a) := \tau(p(k,h),a)$, and $T(k,h,a) := T(p(k,h),a)$ are analytic functions in Σ_{loc}. Since by Theorem 18 we know that $\rho(k,h,a) = \tau(k,h,a)/T(k,h,a)$, then the rotation number is analytic as well.

(ii) Consider the map F restricted on $\mathcal{G}$ Since

$$G(x,y,z) = x(x+1)z^2 + (x(x+1) - y(y+1))z - y^3 - (1+a+x)y^2 - (a+x)y,$$

then equation $G = 0$ is equivalent to $z = z_\pm(x,y)$ where

$$z_\pm(x,y) = \frac{y(y+1) - x(x+1) \pm \sqrt{\Delta(x,y)}}{2x(x+1)}$$

and $\Delta(x,y) = (y(y+1) - x(x+1))^2 + 4x(x+1)(y^3 + (1+a+x)y^2 + (a+x)y)$. If $x > 0$ and $y > 0$ then $z_+(x,y) > 0$ and $z_-(x,y) < 0$. Consequently the surface $\mathcal{G}$ can be described as:

$$\mathcal{G} = \{(x, y, z_+(x,y)) : x > 0, y > 0\}.$$

Hence, in a neighborhood of the fix point, $F|_{\mathcal{G}}$ can be though as the planar map $\bar{F}(x,y) = (y, z_+(x,y))$ in $U = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}$.

Clearly the map $\bar{F}(x,y)$ has (x_c, x_c) as a fix point and the matrix $D\bar{F}\big|_{(x_c,y_c)}$ has the eigenvalues given by $\lambda = \cos\bar{\theta} \pm i\sin\bar{\theta}$ where

$$\bar{\theta} = \arccos\left(\frac{1+x_c}{2x_c}\right) = \arccos\left(\frac{a-1+\sqrt{1+a}}{2a}\right).$$

Let $X(x,y,z)$ be the vector field given by (16). Then the map $\bar{F}(x,y)$ has an associated vector field

$$X|_{\mathcal{G}} =: \bar{X}(x,y) = X_1(x,y,z_+(x,y))\frac{\partial}{\partial x} + X_2(x,y,z_+(x,y))\frac{\partial}{\partial y},$$

which is the restriction of the vector field (16) on $\mathcal{G}$. It can be checked that the vector field $\bar{X}$ has the point (x_c, x_c) as a singular point, which is a non-degenerated center and satisfies $\bar{X}(\bar{F}) = (D\bar{F})\bar{X}$. Via an analytic change of variables, $\bar{X}$ is conjugated with a vector field given in the normal form (17), say $\bar{X}_N$. Through this conjugation we also obtain that $\bar{F}$ is conjugated with a new map $\bar{F}_N$ (which by Lemma 22 satisfies condition (14) with $\bar{X}_N$). The maps $\bar{F}$ and $\bar{F}_N$ share the same eigenvalues at their respective fixed points. Using Lemma 23 we have that $\lim_{k\to k_c} \rho_F(k) = \bar{\theta}/(2\pi)$, as we wanted to prove.

(iii) The study of this case is similar to one of (ii), where here F and the surface $\mathcal{G}$ are replaced by F^2 and M_h, respectively.

A tedious computation shows that the characteristic polynomial of $(DX)_{p_h}$ is given by $P(\lambda) = \lambda\left(\lambda^2 + p(x_h, a)\right)$, where

$$p(x_h, a) = \\ = \frac{(x_h+1)(2x_h+a-1)\left(-a+3x_h a-x_h+1+2{x_h}^2\right)\left({x_h}^2+x_h+a-1\right)^2}{(a+x_h)^2{x_h}^2(x_h-1)^2}.$$

Since $x_h > 1$ and $a > 0$, we have that $p(x_h; a) > 0$, hence the eigenvalues of $(DX)_{p_h}$ are 0 (which corresponds to the tangential direction of $\mathcal{L}$), and a couple of conjugated pure imaginary ones. By the implicit function theorem, in a neighbourhood of p_h the set $M_h^+ := M_h \cap \{G > 0\}$ is a differentiable manifold of dimension 2, invariant by X. Thus X restricted to M_h^+ induces a two dimensional vector field having a non-degenerated center at p_h. At this point the proof follows in the same manner than in (ii). The computation of θ_h is straightforward. $\square$

The following result will be useful to study the odd periods of F. Although it seems natural that it is true for any $a \neq 1$, we have not been able to provide a general proof.

Proposition 24. *Consider $a = a^* := \frac{3-4c}{(2c-1)^2} \simeq 8.29590$, where $c = \cos(2\pi/7)$. Then, there exists $\varepsilon > 0$ such that for any value of a satisfying $|a - a^*| < \varepsilon$, the rotation number of F over the invariant curves $\{V_1 = k\}$ which foliate $\mathcal{G}$ is not constant.*

Proof. First we prove the result for $a = a^*$. We proceed by contradiction. If the set of rotation numbers were degenerated to a point, by Proposition 19 this value should be the value of the limit when we tend to the fix point, which is

$$\frac{\arccos\left(\frac{a^*-1+\sqrt{1+a^*}}{2a^*}\right)}{2\pi} = \frac{1}{7}.$$

Indeed we have chosen a^* to obtain this value. It gives the smallest denominator of all the rational numbers given by the expression $(\arccos\left(\frac{a-1+\sqrt{1+a}}{2a}\right))/(2\pi)$, where $a \in (0, \infty)$. By Theorem 2 we also would have that F^7 restricted to $\mathcal{G}$ would be the identity.

On the other hand, take another point in $\mathcal{G}$, for instance $q = (1, 1, \frac{\sqrt{8c^2-12c+5}}{2c-1}) \simeq (1, 1, 3.20872)$. To prove that $F^7(q) \neq q$ it is convenient for the moment to consider F with $a = \frac{3-4d}{(2d-1)^2}$ and $r = (1, 1, \frac{\sqrt{8d^2-12d+5}}{2d-1})$, being d an unknown parameter. The equation that forces that the first components of $F^7(r)$ and r coincide is

$$\frac{(51-272d+540d^2-464d^3+144d^4)\sqrt{8d^2-12d+5}-110+724d-1900d^2+2504d^3-1664d^4+448d^5}{(38-168d+292d^2-240d^3+80d^4)\sqrt{8d^2-12d+5}-75+470d-1156d^2+1400d^3-832d^4+192d^5}=1.$$

Working with the above equation we obtain that its solutions are included in the solutions of

$$64(d-1)^2(d^2-d/2-1/8)(d^2-3d/2+5/8)(128d^4-64d^3-128d^2+104d-19)=0.$$

Since the value $d = c$ is not a solution, we have got that $F^7(q) \neq q$, which is in contradiction with our initial assumption. Thus for $a = a^*$ we have proved that the set of rotation numbers on $\mathcal{G}$ is not degenerated to a point. Recall that in Proposition 19 (i) it is proved that the rotation number varies continuously with respect to initial conditions and the parameter a. From this result we obtain that the set of all the rotation numbers over $\mathcal{G}$ is not degenerated to a point for the values of a in some neighbourhood of a^*, as we wanted to prove. Notice that when $a = 1$ this rotation number over $\mathcal{G}$, and also over $O^+ \setminus \mathcal{L}$, is reduced to the value $1/8$. □

5. On the set of periods of F. Proof of Theorem 3

In this section we prove Theorem 3 and its consequences (Corollary 4 and Proposition 5). Firstly, we present a constructive way for obtaining the denominators of the irreducible fractions which belong to a given interval.

Theorem 25. *Fix a real open interval $I = (a, b)$ and denote by $p_1 = 2, p_2 = 3, p_3, \dots, p_n, \dots$ the set of all the prime numbers, ordered following the usual order. Associated to I we consider the following natural numbers:*

(i) *The smallest prime number p_{m+1} satisfying that $p_{m+1} > \max(3/(b-a), 2)$,*
(ii) *Given any prime number p_n, $1 \le n \le m$, the smallest natural number s_n such that $p_n^{s_n} > 4/(b-a)$.*

By using the above numbers, define the following finite subset of $\mathbb{N}$:

$$F_{s_1,s_2,\dots,s_m} := \{n \in \mathbb{N} : n = p_1^{t_1} p_2^{t_2} \cdots p_m^{t_m} \quad \text{with} \quad 0 \le t_i \le s_i - 1,\, i = 1, 2, \dots, m\}.$$

Then for any $r \in \mathbb{N} \setminus F_{s_1,s_2,\dots,s_m}$ there exists and irreducible fraction q/r such that $q/r \in I$.

Next result easily follows from the above Theorem:

Corollary 26. *Fix an open real interval (a, b). Following the notations of the above theorem consider the number $p := p_1^{s_1-1} p_2^{s_2-1} \cdots p_m^{s_m-1}$. Then, for any $r > p$ there exists an irreducible fraction q/r such that $q/r \in (a, b)$.*

Proof of Theorem 25. We prove the following two assertions:

(a) If p is a prime number and $p \geq p_{m+1}$ then for any natural number $k \geq 1$ there exists an irreducible fraction of the form $\dfrac{q}{kp} \in I$.

(b) If p_i is any prime number $p_i < p_{m+1}$ and s_i is the integer number given in the statement of the theorem, then for any natural number $k \geq 1$ there exists an irreducible fraction of the form $\dfrac{q}{kp_i^{s_i}} \in I$.

Clearly the theorem follows from them.

Let us prove the first one. From the fact that $p_{m+1} > 3/(b-a)$, we have that if p is a prime number and $p \geq p_{m+1}$ then there exists an ℓ such that

$$a < \frac{\ell-1}{p} < \frac{\ell}{p} < \frac{\ell+1}{p} < b, \tag{18}$$

where the three fractions are irreducible. Hence we have proved our assertion (a) for $k = 1$. Take now any $k > 1$. From the above inequalities we have that

$$a < \frac{k\ell-k}{kp} < \frac{k\ell-1}{kp} < \frac{k\ell}{kp} < \frac{k\ell+1}{kp} < \frac{k\ell+k}{kp} < b.$$

Note that either $\dfrac{k\ell-1}{kp}$ or $\dfrac{k\ell+1}{kp}$ have to be irreducible because the factors of k never divides their numerators and if both were reducible the number p should divide both numbers $k\ell \pm 1$. Taking their difference we would have that p divides 2, a contradiction. Thus assertion (a) is proved.

Let us prove assertion (b). Fix any prime number $p = p_n$, smaller that p_{m+1} and consider its associated number $s = s_n$. From the inequality $p_n^{s_n} > 4/(b-a)$ we have that

$$a < \frac{j-1}{p^s} < \frac{j}{p^s} < \frac{j+1}{p^s} < \frac{j+2}{p^s} < b,$$

Note that either $j+1$ or j have to be coprime with p hence taking ℓ either j or $j+1$ we have that

$$a < \frac{\ell-1}{p^s} < \frac{\ell}{p^s} < \frac{\ell+1}{p^s} < b,$$

being the fraction ℓ/p^s irreducible, like in (18). When $p > 2$ we can argue as in the previous case and assert that either $\dfrac{k\ell-1}{kp^s}$ or $\dfrac{k\ell+1}{kp^s}$ have to be irreducible, proving our result. When $p = 2$ we consider

$$a < \frac{j}{2^s} < \frac{j+1}{2^s} < \frac{j+2}{2^s} < b.$$

Taking $k>2$ we have that

$$a<\frac{kj}{k2^s}<\frac{kj+1}{k2^s}<\frac{kj+2}{k2^s}<\frac{kj+k}{k2^s}<b,$$

and again one of the fractions $\dfrac{kj+1}{k2^s}$, $\dfrac{kj+2}{k2^s}$, has to be irreducible, as we wanted to prove.

□

In the sequel we prove Theorem 3, Corollary 4 and Proposition 5.

Proof of Theorem 3. For each $a>0$, $a\neq 1$ and each $x>1$ consider the function

$$r(x)=\frac{1}{2\pi}\arccos\left(\frac{(a-1)(1-x)}{2(ax+x^2)}\right).$$

Recall that from Proposition 19 (iii), the function $r(x)$ gives the limit of $\rho_{F^2}(k)$ when k tend to $V_1(p_x)$, where p_x is the point on $\mathcal{L}$ given by $(x,\frac{a+x}{x-1},x)$. Observe that $r(x)$ has a unique critical point which is a maximum (resp. a minimum) at $x=x_c=1+\sqrt{1+a}$ when $a>1$ (resp. $0<a<1$). Furthermore $r(1)=1/4$ and $\lim_{x\to\infty}r(x)=1/4$. Now consider the value $r(x_c)$ and denote it by ρ_a :

$$\rho_a=\frac{1}{2\pi}\arccos\left(\frac{(1-a)\sqrt{1+a}}{2(1+\sqrt{1+a})(1+a+\sqrt{1+a})}\right). \tag{19}$$

Take $a>1$ and a number $\rho^*\in(1/4,\rho_a)$ (the case $a<1$ and $\rho^*\in(\rho_a,1/4)$ can be studied in a similar way). Let us see that there is a continuum of initial conditions in $\{G>0\}\setminus\mathcal{L}$ such that their rotation number is ρ^* (and notice that by using expression (6), the images by F of these initial conditions satisfy the same property and are in $\{G<0\}\setminus\mathcal{L}$). For $\varepsilon>0$ small enough, there are two periodic points of F in $\mathcal{L}$, say $p^\pm=(x^\pm,\frac{a+x^\pm}{x^\pm-1},x^\pm)$, such that $r(p^\pm)=\rho^*\pm\varepsilon$. By Proposition 19 (i) there exist initial conditions $r^\pm\in\{V_1=V_1(p^\pm)\}\cap\{G>0\}$ such that their respective rotation numbers, $\varrho^\pm$ satisfy $\rho^*-2\varepsilon<\varrho^-<\rho^*<\varrho^+<\rho^*+2\varepsilon$. Joining r^- and r^+ by a continuous path $\Gamma\subset\{G>0\}\setminus\mathcal{L}$, and by using again the continuous dependence of the rotation number with respect the initial conditions, we obtain the existence of a point $r\in\Gamma$ such that its rotation number is exactly ρ^*. By Theorem 2 the same happens with all the points in $\{G>0\}$ of $\{V_1=V_1(r)\}\cap\{V_2=V_2(r)\}\cong\mathbb{S}^1$, as we wanted to prove. □

Proof of Corollary 4. (i) By using Theorem 3 and Corollary 26 the result follows.

(ii) Observe that the function ρ_a given in (19) is an increasing function such that

$$\lim_{a\to 0^+} \rho_a = \frac{\pi - 2\,\arcsin(1/8)}{4\pi} \simeq 0.23005 \quad \text{and} \quad \lim_{a\to +\infty} \rho_a = \frac{1}{3}.$$

Therefore, by using again Theorem 3, for each number in $((\pi - 2\,\arcsin(1/8))/(4\pi), 1/4)$ there exists some $a \in (0,1)$ and some initial condition outside $\mathcal{G}$ with this rotation number for F^2. Similarly, for each number in $[1/4, 1/3)$ there exist some $a \geq 1$ and some initial condition, also outside $\mathcal{G}$, with this rotation number. In particular, for all the irreducible rational numbers p/q with the property

$$\frac{\pi - 2\,\arcsin(1/8)}{4\pi} < \frac{p}{q} < \frac{1}{3}$$

we can find a value of a such that F^2 has continua of periodic orbits of period q.

(iii) Setting $a = (\pi - 2\,\arcsin(1/8))/(4\pi)$, $b = 1/3$ and using the notation introduced in Theorem 25, we have that, $m = 10$, $p_{11} = 31$ and $p_1 = 2$ (with $s_1 = 5$), $p_2 = 3$ (with $s_2 = 4$), $p_3 = 5$ (with $s_3 = 3$), $p_4 = 7$, $p_5 = 11$, $p_6 = 13$, $p_7 = 17$, $p_8 = 19$, $p_9 = 23$ and $p_{10} = 29$ (where $s_i = 2$ for $i \in \{4, \ldots, 10\}$). From Theorem 25, we have that for all $q \in \mathbb{N}$, such that $q > q_0 := 2^4 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 = 2\,329\,089\,562\,800$ there exists some $a > 0$ and some q–periodic orbit for F_a^2. It is now easy to develop a finite algorithm in order to find which irreducible fractions p/q with $q \leq q_0$ are in I_{rot}. Implementing this algorithm we get that there appear irreducible fractions with all the denominators except $1, 2, 3, 5, 6, 8, 9, 12, 14$ and 20. Doubling these numbers, and taking into account that $\mathcal{L}$ is full of two periodic points of F, we obtain (iii). □

Proof of Proposition 5. (i) This result is a direct consequence of expression (6).

(ii) Fix a value of a of the ones given in Proposition 24. By this value we know that the set of all rotation numbers of all the points of $\mathcal{G}$ contains an open interval. By applying Corollary 26 to this interval the result follows.

(iii) Similarly that in the proof of (i) of Theorem 3, for each $a > 0$, we consider the function $s(a) = \frac{1}{2\pi} \arccos\left(\frac{a-1+\sqrt{1+a}}{2a}\right)$. Recall that from Proposition 19 (iii), this function gives the limit of the rotation numbers over $\mathcal{G}$ when we approach to the fix point. The range of this function when $a > 0$ is J_{rot}. Taking into account the continuity of the rotation number with respect to initial conditions and the parameter a, and arguing as in the last part of proof of Theorem 3, the result follows. □

6. Some numerical results

In this section we present some numerical explorations which lead us to establish the open questions stated in Section 1.2.

The following tables of rotation numbers have been obtained using the relation (15), by numerical integration of the vector field (16) using a 7-8^{th} order Runge–Kutta method. Table 1 has been obtained taking $a = 3$, and gives the rotation number associated to the orbit passing through some points of the surface $\mathcal{G}$. These points have been taken by considering the following path over $\mathcal{G}$:

$$p(t) = (x(t), y(t), z(t)) = \left(x_c + t, \frac{a + x(t)}{x(t) - 1}, z(x(t), y(t)) \right), \qquad (20)$$

where $z(x(t), y(t))$ is one of the two branches of solutions of equation $G(x(t), y(t), z) = 0$. Recall that $\tau(p) := \tau_F(p)$ is given by the relation $\varphi(\tau_F(p), p) = F(p)$, $T(p)$ is the period of the periodic orbit of (16) passing through p and the rotation number of F at the orbit starting at p is $\rho_F(p) = \tau_F(p)/T(p)$. Similarly we can define $\tau_{F^2}(p)$ and $\rho_{F^2}(p)$. Note also that when both numbers have sense $\rho_{F^2}(p) = 2\rho_F(p)$. In general we observe that the function $t \to \rho(x(t), y(t), z(t))$ seems to be decreasing.

t	Point p	$T(p)$	$\tau_F(p)$	$\rho_F(p)$	$\rho_{F^2}(p)$
0	$(3, 3, 3)$	$--$	$--$	0.13386	0.26772
1	$(4, 7/3, 1.62395)$	0.41781	0.05586	0.13369	0.26737
2	$(5, 2, 1.06969)$	0.36063	0.04810	0.13337	0.26674
3	$(6, 9/5, 0.78049)$	0.30622	0.04074	0.13305	0.26610
4	$(7, 5/3, 0.60637)$	0.26009	0.03452	0.13274	0.26549
5	$(8, 11/7, 0.49153)$	0.22226	0.02944	0.13247	0.26494
6	$(9, 3/2, 0.41083)$	0.19148	0.02532	0.13223	0.26446
7	$(10, 13/9, 0.35143)$	0.16635	0.02196	0.13201	0.26402
8	$(11, 7/5, 0.30610)$	0.14570	0.01921	0.13182	0.26364
9	$(12, 15/11, 0.27051)$	0.12860	0.01693	0.13164	0.26328
10	$(13, 4/3, 0.24191)$	0.11430	0.01503	0.13149	0.26298
11	$(14, 17/13, 0.21849)$	0.10225	0.01343	0.13134	0.26269
12	$(15, 9/7, 0.19899)$	0.09202	0.01207	0.13122	0.26244
13	$(16, 19/15, 0.18253)$	0.08325	0.01091	0.13110	0.26220

Table 1. Rotation number on $\mathcal{G}$ for $a = 3$.

Note that the results of Table 1 also give light to know which are the odd periods for F for a given value of a. For instance when $a = 3$ it seems clear that $0.1333\ldots = 2/15$ is one of the rotation numbers reached by F over $\mathcal{G}$. Hence for this value of a, F must have periodic points of period 15. By applying a three dimensional Newton method to the system $F^{15}(x,y,z) = (x,y,z)$ we have obtained the approximated solution $r \simeq (2.00557, 5.20647, 9.89389)$. Note that $|F^{15}(r)-r|_1 < 3.8\times 10^{-5}$, where as usual $|(x,y,z)|_1 = |x|+|y|+|z|$. Indeed there should exist infinitely many 15-periodic points, given by all the orbits starting at the periodic orbit of (16) passing through a given 15-periodic point. We also have checked that

$$\left|F^{15\times 10^n}(r) - r\right|_1 < 2.8\times 10^{n-6} \quad \text{for} \quad n = 1,2,3,4.$$

If r were a true 15-periodic point the above values should have been zero, but as we have already noticed in Remark 6 the dynamical system generated by F has sensible dependence with respect to initial conditions.

Table 2 is again obtained taking $a = 3$. Now the rotation number of F^2 is computed for some points in the curve given by

$$\begin{aligned} p(t) &= (x(t), y(t), z(t)) \\ &= \left((x_c+9)\cdot(1-t)+tx_1, \frac{a+x(t)}{x(t)-1}, z(x(t),y(t))\right),\ t\in[0,1], \end{aligned} \tag{21}$$

where $z(t)$ is a fixed branch of the two branches of solutions of equation (7) and $x(1) = x_1$ is the first coordinate of the two periodic point of F, $p(1) \simeq (1.11929, 34.53097, 1.11929)$. This curve joints the point $p(0) = (12, 15/11, 0.27050\ldots) \in \mathcal{G}$ with $p(1)$ running over the level surface

$$V_1 = k^* := \frac{28561}{43560}\,\frac{\left(19+3\sqrt{89}\right)\left(197+\sqrt{89}\right)}{-79+13\sqrt{89}} \simeq 146.70452.$$

t	Point p	$T(p)$	$\tau_{F^2}(p)$	$\rho_{F^2}(p)$
0	$(12, 1.36364, 0.27051)$	0.12860	0.03386	0.26328
0.1	$(10.91193, 1.40355, 0.24737)$	0.12857	0.03385	0.26327
0.2	$(9.82386, 1.45332, 0.22528)$	0.12848	0.03382	0.26325
0.3	$(8.73579, 1.51708, 0.20426)$	0.12829	0.03376	0.26320
0.4	$(7.64772, 1.60171, 0.18437)$	0.12796	0.03367	0.26310
0.5	$(6.55965, 1.71947, 0.16576)$	0.12743	0.03351	0.26295
0.6	$(5.47158, 1.89454, 0.14879)$	0.12655	0.03324	0.26270
0.7	$(4.38350, 2.18221, 0.13432)$	0.12501	0.03279	0.26226
0.8	$(3.29543, 2.74259, 0.12498)$	0.12205	0.03191	0.26145
0.9	$(2.20736, 4.31300, 0.13351)$	0.11498	0.02985	0.25962
0.95	$(1.66333, 7.03020, 0.17364)$	0.10648	0.02744	0.25768
0.99	$(1.22810, 18.53618, 0.43340)$	0.09080	0.02314	0.25484
0.999	$(1.13017, 31.72824, 0.95253)$	0.08589	0.02183	0.25414
0.9999	$(1.12038, 34.22789, 1.09981)$	0.08575	0.02179	0.25412
1	$(1.11929, 34.53097, 1.11929)$	$--$	$--$	0.25412

Table 2. Rotation number on $\{V_1 = k^* \simeq 146.70452\}$ when $a = 3$.

Table 3 has been obtained by repeating the first experiment but taking $a = 7/9$. So it gives the rotation number associated to the orbit passing through some points of the surface $\mathcal{G}$, by considering the path given by (20). Notice that in this case the rotation number seems an increasing function of t.

t	Point p	$T(p)$	$\tau_F(p)$	$\rho_F(p)$	$\rho_{F^2}(p)$
0	$(7/3, 7/3, 7/3)$	$--$	$--$	0.12338	0.24676
1	$(10/3, 37/21, 1.11361)$	0.48969	0.06043	0.12340	0.24681
2	$(13/3, 23/15, 0.71973)$	0.39978	0.04935	0.12345	0.24690
3	$(16/3, 55/39, 0.52965)$	0.32588	0.04025	0.12350	0.24700
4	$(19/3, 4/3, 0.41853)$	0.26908	0.03324	0.12354	0.24708
5	$(22/3, 73/57, 0.34583)$	0.22552	0.02787	0.12358	0.24716
6	$(25/3, 41/33, 0.29462)$	0.19171	0.02370	0.12361	0.24722
7	$(28/3, 91/75, 0.25662)$	0.16502	0.02040	0.12364	0.24729
8	$(31/3, 25/21, 0.22731)$	0.14363	0.01776	0.12367	0.24734
9	$(34/3, 109/93, 0.20401)$	0.12622	0.01561	0.12369	0.24739
10	$(37/3, 59/51, 0.18506)$	0.11187	0.01384	0.12372	0.24744
11	$(40/3, 127/111, 0.16933)$	0.09990	0.01236	0.12374	0.24748
12	$(43/3, 17/15, 0.15607)$	0.08980	0.01111	0.12376	0.24752

Table 3. Rotation number on $\mathcal{G}$ for $a = 7/9$.

Finally, Table 4 is obtained taking $a = 7/9$, considering the path (21), which runs from the point $p(0) = (34/3, 109/93, -296730/604469 + 1/1813407\sqrt{1587984839746}) \simeq (11.33333,\ 1.17204,\ 0.20401) \in \mathcal{G}$ to the fixed point of F^2, $p(1) \simeq (1.04794, 38.08255, 1.04794)$, over the level surface

$$\begin{aligned} V_1 &= \bar{k} := \tfrac{101}{272452149}\, \tfrac{\left(923217+\sqrt{1587984839746}\right)\left(69592724+3\,\sqrt{1587984839746}\right)}{-890190+\sqrt{1587984839746}} \\ &\simeq 0.24956. \end{aligned}$$

t	Point p	$T(p)$	$\tau_{F^2}(p)$	$\rho_{F^2}(p)$
0	$(11.33333, 1.17204, 0.20402)$	0.12622	0.03123	0.24738
0.1	$(10.30479, 1.19106, 0.18573)$	0.12620	0.03122	0.24739
0.2	$(9.27625, 1.21480, 0.16809)$	0.12612	0.03120	0.24740
0.3	$(8.24772, 1.24529, 0.15108)$	0.12596	0.03116	0.24742
0.4	$(7.21918, 1.28585, 0.13473)$	0.12569	0.03110	0.24744
0.5	$(6.19064, 1.34250, 0.11911)$	0.12526	0.03010	0.24748
0.6	$(5.16210, 1.42714, 0.10435)$	0.12455	0.03083	0.24755
0.7	$(4.13356, 1.56733, 0.09081)$	0.12335	0.03055	0.24765
0.8	$(3.10502, 1.84454, 0.07963)$	0.12107	0.03001	0.24784
0.9	$(2.07648, 2.65147, 0.07637)$	0.11551	0.02867	0.24824
0.95	$(1.56221, 4.16212, 0.09117)$	0.10807	0.02687	0.24866
0.99	$(1.15080, 12.78937, 0.23101)$	0.08942	0.02229	0.24932
0.999	$(1.05822, 31.53212, 0.74344)$	0.07887	0.01968	0.24955
0.9999	$(1.04897, 37.30369, 1.00598)$	0.07829	0.01954	0.24956
1	$(1.04794, 38.08255, 1.04794)$	$--$	$--$	0.24957

Table 4. Rotation number on $\{V_1 = \bar{k} \simeq 0.24956\}$ when $a = 7/9$.

Appendices

Appendix A. Proof of Lemma 12

To describe the foliation of Q^+, induced by $\Delta(x, y; a, k) = 0$ obtained for a fixed value of $a > 0$, and varying $k \geq k_c$, we solve the quadratic equation (with respect k): $\Delta(x, y; a, k) = x^2y^2\, k^2 + p_1(x, y; a)\, k + p_0(x, y; a) = 0$. Thus the curve $\Delta(x, y; a, k) = 0$ in Q^+ can also be described by two equations

$$k = m_\pm(x, y; a) = \frac{(x + y + a + 1 \pm 2\sqrt{x + y + a})\,(x + 1)\,(y + 1)}{xy}.$$

We make the following claims:

Claim 1: *For any fixed $k \geq k_c$ the following statements hold*

(i) If $a > 1$, there exist two values $x_{1,k} < x_{2,k}$ such that equation (with unknown y)

$$k = m_-(\bar{x}, y; a), \tag{22}$$

has two solutions if $\bar{x} \in (x_{1,k}, x_{2,k})$, one solution if $\bar{x} = x_{i,k}$ $i = 1, 2$, and none solution if $\bar{x} \notin [x_{1,k}, x_{2,k}]$. This means that varying $x > 0$, equation (22) describes an oval ζ_k.

(ii) If $a < 1$, then there exist a value $x_k > 1 - a$ such that equation (22) has two solutions if $\bar{x} < x_k$, one solution if $\bar{x} = x_k$, and none solution if $\bar{x} > x_k$. This means that varying $x > 0$, equation (22) describes a curve consisting of a point (x_k, y_k) and (from right to left) two positive branches $y_1(x) < y_2(x)$ defined only for $x \in (0, x_k)$. A more accurate analysis will

show that these two branches meet at the point $(x,y)=(0,1-a)$*. Therefore they describe an oval, namely* ζ_k*.*

Claim 2: *(i) For* $k>k_c$*, there exist two values* $x_{1,k}<x_c<x_{2,k}$ *such that equation*

$$k=m_+(\bar{x},y;a), \tag{23}$$

has two solutions if $\bar{x}\in(x_{1,k},x_{2,k})$*, one solution if* $\bar{x}=x_{i,k}$ $i=1,2$*, and none solution if* $\bar{x}\notin[x_{1,k},x_{2,k}]$*. This means that varying* $x>0$*, equation (23) describes an oval* γ_k*.*

(ii) The equation $k_c=m_+(\bar{x},y;a)$ *has a unique solution if* $\bar{x}=x_c$ *and none solution if* $\bar{x}\neq x_c$*.*

Since $m_+(x,y;a)>m_-(x,y;a)$ it is easy to see that each oval γ_k surrounds the corresponding oval ζ_k. From this fact and the above claims the proof of the lemma follows.

Before proving both claims we establish some common facts. We fix $\bar{x}>0$ and we use the following notation:

$$\frac{\partial m\pm}{\partial y}(\bar{x},y;a)=-(x+1)\left[\frac{\pm f(\bar{x},y;a)+g(\bar{x},y;a)\sqrt{h(\bar{x},y;a)}}{\bar{x}y^2\sqrt{h(\bar{x},y;a)}}\right],$$

where $f(\bar{x},y;a)=-y^2+y+2\bar{x}+2a$, $g(\bar{x},y;a)=-y^2+\bar{x}+a+1$ and $h(\bar{x},y;a)=\bar{x}+y+a$. The solutions in Q^+ of $\frac{\partial m\pm}{\partial y}(\bar{x},y;a)=0$ are described by

$$\begin{aligned}&\left(f^2-g^2\,h\right)(\bar{x},y;a)=\\&\qquad(y+\bar{x}+a-1)(-y^2-y+\bar{x}+a-1)(-y^2+y+\bar{x}+a).\end{aligned} \tag{24}$$

So this equation gives the local extrema of $y\to m_\pm(\bar{x},y;a)$.

It can be easily proved that

$$\lim_{y\to0^+}m_\pm(\bar{x},y;a)=+\infty \text{ and } \lim_{y\to+\infty}m_\pm(\bar{x},y;a)=+\infty, \tag{25}$$

for all $\bar{x}>0$.

Let us now proceed with the proof of both claims.

Proof of Claim 1. (i) If $a>1$, and taking into account that $\bar{x}>0$, it is easy to see that $\frac{\partial m_-}{\partial y}(\bar{x},y;a)=0$ if and only if

$$Q(\bar{x},y)=-y^2-y+\bar{x}+a-1=0 \tag{26}$$

which has the unique positive solution $y=y_{\min}(\bar{x})=(-1+\sqrt{-3+4a+4\bar{x}})/2$.

We point out that this solution is well defined for all $\bar{x} > 0$ and $a > 1$, and that $y_{\min}(\bar{x}) > 0$. Hence, taking into account equation (25), the function $y \to m_-(\bar{x}, y; a)$ takes a minimum at the point $y_{\min}(\bar{x})$. So we have that for each $\bar{x} > 0$ the functions $y \to m_-(\bar{x}, y; a)$ are decreasing in the interval $y \in (0, y_{\min})$ and increasing in $y \in (y_{\min}, +\infty)$.

Now we study the function $x \to m_-(x, y_{\min}(x); a)$. We have the following facts:

(I) $\lim_{x\to 0^+} m_-(x, y_{\min}(x); a) = +\infty$,

(II) It is easy to see that at infinity $m_-(x, y_{\min}(x); a) \sim x$, thus

$$\lim_{x\to+\infty} m_-(x, y_{\min}(x); a) = +\infty.$$

(III) Since the only positive solutions of $\dfrac{\partial m_-}{\partial x}(x, y; a) = 0$ are given by the solutions of the equation $Q(y, x) = -x^2 - x + y + a - 1 = 0$, where $Q(x, y)$ is defined in (26), and $Q(x, y) = Q(y, x) = 0$ if and only if $x = y = x_* := \sqrt{a-1}$, we obtain that $\dfrac{\partial m_-}{\partial x}(x, y_{\min}(x); a) \neq 0$ for $x \neq x_*$, and $\dfrac{\partial m_-}{\partial x}(x_*, x_*; a) = 0$. This means that x_* is the unique critical point of the function $x \to m_-(x, y_{\min}(x); a)$ when $x > 0$.

(IV) It can be easily checked that $k_* = m_-(x_*, x_*; a) = (1 + \sqrt{a-1})^2$ and $k_* < k_c$.

Collecting the results in items (I)–(IV) we obtain that the function $x \to m_-(x, y_{\min}(x); a)$, which gives the minimum values of each function $y \to m_-(x, y; a)$, has a unique minimum at $x = x_*$, decreases from $+\infty$ to k_* for $x \in (0, x_*)$, and increases from k_* to $+\infty$ for $x \in (x_*, +\infty)$.

This proves that for any fixed $k \geq k_c$ there always exist two solutions $x_{i,k}$, $i = 1, 2$ of equation $k = m_-(x, y_{\min}(x); a)$, see also Figure 3.

For these two values, the minimum of the functions $y \to m_-(x_{i,k}, y; a)$ is k. This implies that equation (22) only has one solution for $\bar{x} = x_{i,k}$, $i = 1, 2$. Now observe that for all $\bar{x} \in (x_{1,k}, x_{2,k})$, since the minimum values of the functions $y \to m_-(\bar{x}, y; a)$ are below k, we can conclude that for these values of $\bar{x}$ equation (22) only has two solutions. Finally for $\bar{x} \notin (x_{1,k}, x_{2,k})$, since the minimum values of the functions $y \to m(\bar{x}, y; a)$ are greater than k, equation (22) has no solutions. In summary, equation (22) describes one and only one oval ζ_k.

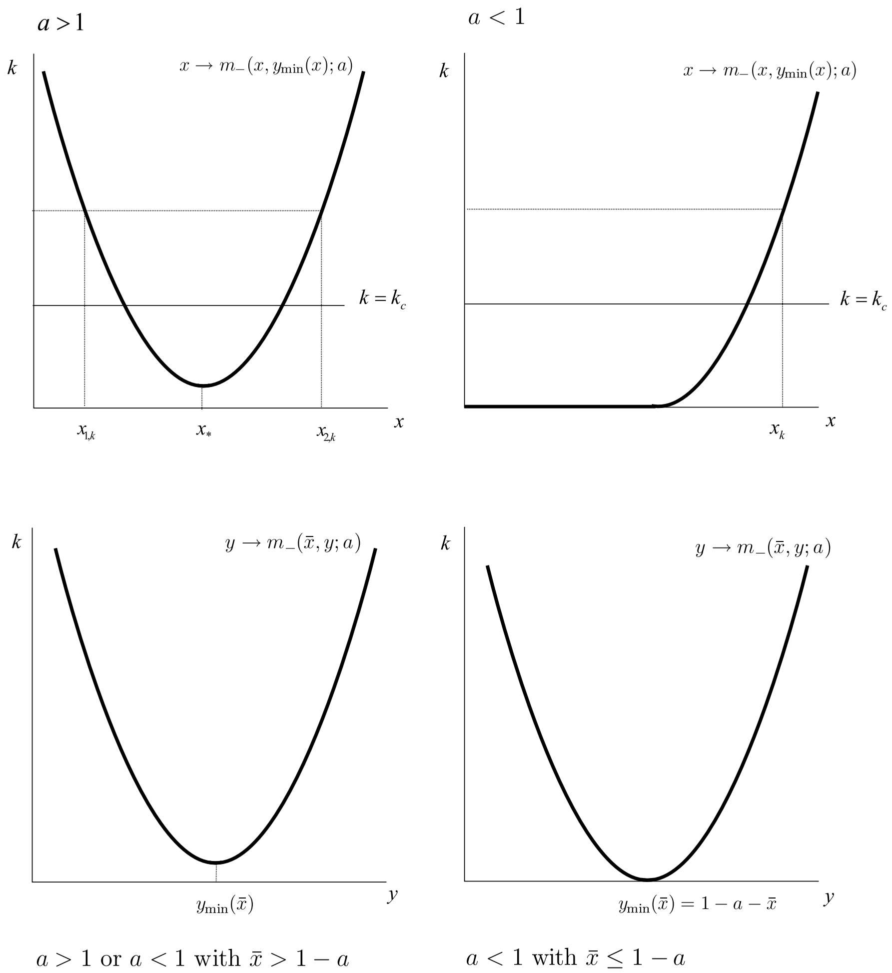

Figure 3. Plot of some functions involved in the proof of Lemma 12.

(ii) If $a < 1$, $\dfrac{\partial m_-}{\partial y}(\bar{x}, y; a) = 0$ at the curve $y = y_{\min}(\bar{x})$, described by the unique positive root of $(y + \bar{x} + a - 1)(-y^2 - y + \bar{x} + a - 1)$. Thus

$$y_{\min}(\bar{x}) = \begin{cases} 1 - a - \bar{x} & \text{if } \bar{x} \leq 1 - a, \\ (-1 + \sqrt{-3 + 4a + 4\bar{x}})/2 & \text{if } \bar{x} > 1 - a, \end{cases}$$

which is well defined for all $\bar{x} > 0$. Taking into account equation (25), $y_{\min}(x)$ gives a minimum for $y \to m_-(\bar{x}, y; a)$.

Now, we look at the function $x \to m_-(x, y_{\min}(x); a)$. First notice that for $x \le 1-a$, $m_-(x, y_{\min}(x); a) = m_-(x, x+a-1; a) = 0$. In the region $x > 1-a$, we have

$$m_-(x, y_{\min}(x); a) = \frac{n(x)(x+1)(1+\sqrt{-3+4a+4x})^2}{8x(x-1+a)},$$

where $n(x) = \tilde{l}(s(x))$ with $s(x) = 2(x+a) + \sqrt{-3+4x+4a}$, and $\tilde{l}(s) = s+1+2\sqrt{2}\sqrt{s-1}$.

Observe that $x > 1-a$ if and only if $s > 3$, and that the function $\tilde{l}(s)$ is monotonic increasing from 0 to $+\infty$ for $s > 3$. Hence for $x > 1-a$, we have $m_-(x, y_{\min}(x); a)$ is monotonic increasing from 0 to $+\infty$.

Therefore, for $\bar{x} < x_k$, the minimum of the functions $y \to m_-(\bar{x}, y; a)$ is always below k, hence we can conclude that for these values of $\bar{x}$ equation (22) has exactly two solutions, giving rise to two positive branches $y_1(\bar{x})$ and $y_2(\bar{x})$. Since $\Delta(0, y; a, k) = (y+1)^2(a-1+y)^2$ we can conclude that these two branches meet at the point $(0, 1-a)$, which is an order two contact point of Γ_k with $\{x = 0\}$. The symmetry of Δ with respect the line $\{y = x\}$, gives the other contact point $(1-a, 0)$.

For $\bar{x} = x_k$ the minimum of the functions $y \to m_-(\bar{x}, y; a)$ is k. This implies that equation (22) has only one solution, giving the point where the previous mentioned two branches meet.

Finally, observe that for all $\bar{x} > x_k$, since the minimum values of the functions $y \to m_-(\bar{x}, y; a)$ are over k, equation (22) has no solutions.

In summary equation (22) describes one and only one oval ζ_k, which has two contact points with the boundary of Q^+ at $(0, 1-a)$ and $(1-a, 0)$. This ends the proof of Claim 1.

Proof of Claim 2. We have to study the function m_+. It is easy to see that $y \to m_+(\bar{x}, y; a)$ has a unique minimum at $y = y_{\min}(\bar{x}) = \frac{1+\sqrt{1+4a+4\bar{x}}}{2}$, the unique positive solution of the equation $C(x, y) := -y^2 + y + \bar{x} + a = 0$, and that $m_+(\bar{x}, y; a)$ is decreasing from infinity to $m_+(\bar{x}, y_{\min}(\bar{x}); a)$ for $y < y_{\min}(\bar{x})$ and increasing to infinity for $y > y_{\min}(\bar{x})$.

We need now to study the function $x \to m_+(x, y_{\min}(x); a)$. Observe that:
(I) It holds that $y_{\min}(x_c) = x_c$.
(II) Since $\lim_{x \to 0^+} y_{\min}(x) = (1+\sqrt{1+4a})/2 > 0$, we get $\lim_{x \to 0^+} m_+(x, y_{\min}(x); a) = +\infty$.
(III) It is easy to check that at infinity $m_+(x, y_{\min}(x); a) \sim x$, thus

$$\lim_{x \to +\infty} m_+(x, y_{\min}(x); a) = +\infty.$$

(IV) Since the only positive solutions of $\dfrac{\partial m_+}{\partial x}(x,y;a)=0$ are given by the equation $C(y,x)=-x^2+x+y+a=0$, and $C(x,y)=C(y,x)=0$ if and only if $x=y=x_c=1+\sqrt{1+a}$, we have that $\dfrac{\partial m_+}{\partial x}(x,y_{\min}(x);a)\neq 0$ for $x\neq x_c$, and $\dfrac{\partial m_+}{\partial x}(x_c,y_{\min}(x_c);a)=0$.

Collecting the information summarized in (I)–(IV), we obtain that the function $x\to m_+(x,y_{\min}(x);a)$, which gives the minimum values of each function $y\to m_+(x,y;a)$, has a unique minimum at $x=x_c$, and decreases from $+\infty$ to k_c for $x\in(0,x_c)$, and increases from k_c to $+\infty$ for $x\in(x_c,+\infty)$. A simple computation, omitted here, shows that $m_+(x_c,y_{\min}(x_c);a)=m_+(x_c,x_c;a)=k_c$.

This proves that for any fixed $k>k_c$ there exists only two solutions $x_{i,k}$, $i=1,2$ of equation

$$k=m_+(x,y_{\min}(x);a), \tag{27}$$

such that $x_c\in(x_{1,k},x_{2,k})$, see again Figure 3. For these two values, the minimum of the functions $y\to m_+(x_{i,k},y;a)$ is k. This means that equation (23) has only one solution for $\bar{x}=x_{i,k}$ $i=1,2$. We note that for all $\bar{x}\in(x_{1,k},x_{2,k})$, since the minimum values of the functions $y\to m_+(\bar{x},y;a)$ are below k, we can conclude that for these values of $\bar{x}$ equation (23) has only two solutions. Finally for $\bar{x}\notin(x_{1,k},x_{2,k})$, since the minimum values of the functions $y\to m_+(\bar{x},y;a)$ are greater than k, equation (23) has no solutions. In summary equation (23) describes one and only one oval γ_k. This ends the proof of (i).

The above analysis of $x\to m_+(x,y_{\min}(x);a)$ shows that $k_c=m(\bar{x},y;a)$ if and only if $\bar{x}=y=x_c$. So the branch of Γ_{k_c} described by equation (23) collapses to the point (x_c,x_c). This ends the proof of (i), and so the proof of Claim 2. □

Appendix B. Proof of Lemma 13

To describe the foliation of Q^+, induced by $\Delta(x,y;a,h)=0$ obtained for a fixed value of $a>0$, and varying $h>h_c$, we can rewrite $\Delta(x,y;a,h)=x^2y^2\,h^2+p_1(x,y;a)\,h+p_0(x,y;a)$ where

$$\begin{aligned}
p_0(x,y;a)= {}& x^2y^4+\big(2x^3+2x^2+(-2a+2)\,x\big)\,y^3\\
&+\big(x^4+2x^3+(-4a+5)\,x^2+(-4a+4)\,x+a^2-2a+1\big)\,y^2\\
&+\big((-2a+2)\,x^3+(-4a+4)\,x^2+\\
&+\big(4+2a^2-6a\big)\,x+2+2a^2-4a\big)\,y\\
&+\big(a^2-2a+1\big)\,x^2+\big(2+2a^2-4a\big)\,x+a^2-2a+1,\ \text{and}\\
p_1(x,y;a)= {}& ((-2x^2-4x))y^3+(-2x^3-10x^2+(-2a-6)x)y^2\\
&+(-4x^3+(-2a-6)\,x^2+(-2-2a)\,x)y.
\end{aligned}$$

Thus the curve $\Delta(x,y;a,h)=0$ in Q^+ can also be described by two functions

$$h = m_{\pm}(x,y;a) = \frac{\left(yx + 2x + 2y + a + 1 \pm 2\sqrt{d(x,y;a)}\right)(1+x+y)}{yx},$$

where

$$d(x,y;a) = x^2y + xy^2 + x^2 + y^2 + (a+2)xy + (a+1)x + (a+1)y + a.$$

As in the previous appendix, to prove the lemma we make two claims:

Claim 1: *For $h \geq h_c$ and for all fixed $\bar{x} > 0$, we will see that there exist two solutions $y_1(\bar{x}) < y_2(\bar{x})$ of the equation*

$$h = m_-(\bar{x}, y; a), \tag{28}$$

that give rise to the two branches of Γ_h, $y_1(x)$ and $y_2(x)$ given in the statement of the lemma. Moreover, $\lim_{x\to 0^+} y_i(x) = +\infty$ and $\lim_{x\to+\infty} y_i(x) = 0^+$.

Claim 2: *(i) For $h > h_c$, if we consider the equation*

$$h = m_+(\bar{x}, y; a), \tag{29}$$

then there exist two values $x_{1,h} < x_{2,h}$ such that equation (29) has two solutions if $\bar{x} \in (x_{1,h}, x_{2,h})$, one solution if $\bar{x} = x_{i,h}$ $i = 1,2$, and none solution if $\bar{x} \notin [x_{1,h}, x_{2,h}]$. This means that varying $x > 0$, equation (29) describes the oval γ_h.

(ii) The equation $h_c = m_+(\bar{x}, y; a)$ has a unique solution if $\bar{x} = x_c$ and none solution if $\bar{x} \neq x_c$.

Note that, since $m_+(x,y;a) > m_-(x,y;a)$, the solutions of the equation (29) (whenever they exist) are contained in the interval $(y_1(x), y_2(x))$ defined by equation (28). This implies that the oval γ_h is contained between the two branches $y_1(x)$ and $y_2(x)$. So, by using the above two claims, the lemma follows.

Before giving the proof of the above claims we establish some common facts. We fix $\bar{x} > 0$ and we use the following notation:

$$\frac{\partial m\pm}{\partial y}(\bar{x}, y; a) = \frac{\mp f(\bar{x},y;a) + g(\bar{x},y;a)\sqrt{h(\bar{x},y;a)}}{\bar{x}y^2\sqrt{h(\bar{x},y;a)}},$$

where

$$\begin{aligned} f(\bar{x},y;a) = {} & (-2\bar{x}-2)\,y^3 + \left(-\bar{x}^2 + (-2-a)\,\bar{x} - 1 - a\right) y^2 \\ & + \left(\bar{x}^3 + (3+a)\,\bar{x}^2 + (2a+3)\,\bar{x} + 1 + a\right) y \\ & + 2\bar{x}^3 + (2a+4)\,\bar{x}^2 + (2+4a)\,\bar{x} + 2a, \\ g(\bar{x},y;a) = {} & (\bar{x}+2)\,y^2 - 2\bar{x}^2 + (-a-3)\,\bar{x} - 1 - a, \\ h(\bar{x},y;a) = {} & (y+1)\,(\bar{x}+1)\,(\bar{x}+a+y)\,. \end{aligned}$$

So $\dfrac{\partial m\pm}{\partial y}(\bar{x},y;a) = 0$ if and only if

$$\begin{aligned} & f^2(\bar{x},y;a) - g^2(\bar{x},y;a)\,h(\bar{x},y;a) = \\ & \quad = (\bar{x}+1)\left(-y^3 - y^2 + (1+\bar{x})y + (1+\bar{x})(a+\bar{x})\right) \cdot \\ & \quad \cdot (\bar{x}y - a + 1)\left(\bar{x}y^2 + (1+\bar{x})(a-1+\bar{x})y + (1+\bar{x})(a-1)\right) = 0. \end{aligned}$$

This equation gives the local extrema of $y \to m_-(\bar{x},y;a)$ and $y \to m_+(\bar{x},y;a)$.

It is not difficult to see that

$$\lim_{y\to 0^+} m_\pm(\bar{x},y;a) = +\infty \quad \text{and} \quad \lim_{y\to +\infty} m_\pm(\bar{x},y;a) = +\infty, \tag{30}$$

for all $\bar{x} > 0$.

Proof of Claim 1. If $a > 1$, taking into account that $\bar{x} > 0$, it is easy to see that $\dfrac{\partial m_-}{\partial y}(\bar{x},y;a) = 0$ if and only if $y_0 = (a-1)/\bar{x}$. Furthermore $m_-(\bar{x},(a-1)/\bar{x};a) = 0$ for all $x > 0$, which is a minimum of $y \to m_-(\bar{x},y;a)$. Taking into account equation (30), we have that for each $\bar{x} > 0$ the functions $y \to m_-(\bar{x},y;a)$ are decreasing from $+\infty$ to 0 for $y \in (0,y_0)$ and increasing from 0 to $+\infty$ if $y \in (y_0,+\infty)$. This proves that in this case equation (28) always has two solutions for any $h > 0$, in particular for any $h \geq h_c$.

If $0 < a < 1$, $\dfrac{\partial m_-}{\partial y}(\bar{x},y;a) = 0$ if and only if $y = y_q(\bar{x})$, where $y_q(\bar{x})$ is the only positive solution of the quadratic equation $\bar{x}y^2 + (1+\bar{x})(a-1+\bar{x})y + (1+\bar{x})(a-1) = 0$ which taking into account (30) gives a minimum of $y \to m_-(\bar{x},y;a)$. To see that equation (28) always has two solutions for $h \geq h_c$, we only have to see that $m_-(x,y_q(x);a) < h_c$ for all $x > 0$. Since $m_-(x,y;a) = 1-a$ if and only if $xy^2+(1+x)(a-1+x)y+(1+x)(a-1) = 0$, we have that $m_-(x,y_q(x);a) = 1-a$. On the other hand it is not difficult to check that $1-a < h_c$. So the first part of the claim is proved.

To end the proof it remains to see that $\lim_{x\to 0^+} y_i(x) = +\infty$ and $\lim_{x\to +\infty} y_i(x) = 0^+$, where $y_1(x)$ and $y_2(x)$ are the two branches of $\Delta = 0$. We observe that each curve $\Delta(x,y;a,h) = 0$ is symmetric with respect the axis $y = x$. So it is equivalent to see that $\lim_{x\to +\infty} y_i(x) = 0$, $i = 1,2$ or

$\lim_{x\to 0^+} y_i(x) = +\infty$ $i = 1, 2$. So we will prove the first equality. To do this we will study if there arrive any branch of Γ_h to the "infinity line" in the projective space $\mathbf{P}\mathbb{R}^2$ in the direction $y = 0$.

For each affine curve $\Delta(x, y; a, h) = 0$ we can consider the projectivized curve in $\mathbf{P}\mathbb{R}^2$ (given in homogeneous coordinates (x, y, u)) $\tilde{\Gamma}_h = \{\tilde{\Delta}(x, y, u; a, h) = 0\}$, where
$\tilde{\Delta}(x, y, u; a, h) = \big(a^2 - 2a + 1\big)\, u^6 + \big(2+2a^2-4a\big)\, xu^5 + \big(a^2 - 2a+1\big)\, x^2u^4 + \big(2 + 2a^2 - 4a\big)\, yu^5 + \big(4 + 2a^2-6a-2h-2ka\big)\, xyu^4 + (-4a - 2ka + 4 - 6h)\cdot$ $\cdot x^2yu^3 + (2 - 4h - 2a)\, x^3yu^2 + \big(a^2 - 2a + 1\big)\, y^2u^4 + (-4a - 2ka + 4 - 6h)\cdot$ $\cdot xy^2u^3 + \big(h^2 - 4a - 10h + 5\big)\, x^2y^2u^2 + (-2h + 2)\, x^3y^2u + y^2x^4 + (2 - 4h - 2a)\cdot$ $\cdot xy^3u^2 + (-2h + 2)\, x^2y^3u + 2y^3x^3 + y^4x^2$. In the local chart $\{x \neq 0\}$ this curve is given by (just taking $x = 1$) $\tilde{\Gamma}_h = \{\tilde{\Delta}(1, y, u; a, h) = 0\}$, where:
$\tilde{\Delta}(1, y, u; a, h) = \big(a^2 - 2a + 1\big)\, u^6 + \big(2 + 2a^2 - 4a\big)\, u^5 + \big(a^2 - 2a + 1\big)\, u^4 + \big(2 + 2a^2 - 4a\big)\, yu^5 + \big(4 + 2a^2-6a-2h-2ka\big)\, yu^4 + (-4a - 2ka + 4 - 6h)\cdot$ $\cdot yu^3 + (2 - 4h - 2a)\, yu^2 + \big(a^2 - 2a + 1\big)\, y^2u^4 + (-4a - 2ka + 4 - 6h)\, y^2u^3 + \big(h^2-4a-10h+5\big)\, y^2u^2 + (-2h+2)\, y^2u + y^2 + (2 - 4h - 2a)\, y^3u^2 + (-2h + 2)\cdot$ $\cdot y^3u + 2y^3 + y^4$.

We want to prove is that there are two "affine" branches of $\tilde{\Gamma}_h$ arriving at the point of the infinity line with coordinates $(y, u) = (0, 0) \in \tilde{\Gamma}_h$. Since $\tilde{\Delta}(1, y, u; a, h) = y^2 + 2y^3 + (2 - 2h)y^2u + 2(1 - a - 2h)yu^2 + O((x, y)^4)$, we need to perform the blow–up: $(y, u) = (v\, u, u)$, which after removing the factor u^2 transforms $\tilde{\Gamma}_h$ into
$\tilde{\Gamma}_h^* = \{\big[\big(a^2-2\,a+1\big)\, v^2 + \big(2+2\,a^2-4\,a\big)\, v + a^2-2\,a+1\big]\, u^4$ $+\ \big[(2-4\,h - 2\,a)\, v^3 + (-4\,a - 2\,ka + 4 - 6\,h)\, v^2 + \big(4+2\,a^2-6\,a-2\,h-2\,ka\big)\cdot$ $\cdot v + 2 + 2\,a^2 - 4\,a\big]\, u^3 + \big[v^4 + (-2\,h + 2)\, v^3 + \big(h^2 - 4\,a - 10\,h + 5\big)\, v^2 +$ $(-4\,a - 2\,ka + 4 - 6\,h)\, v + a^2 - 2\,a + 1\big]\, u^2 + \big[2\,v^3 + (-2\,h + 2)\, v^2 +$ $(2 - 4\,h - 2\,a)\, v\big]\, u + v^2 = 0\}$.

The intersection $\tilde{\Gamma}_h^*$ with $\{v = 0\}$ are the points $(v, u) = (0, 0)$, and $(v, u) = (0, -1)$. But the last point is not interesting for us since, the affine region $\{(x, y) : x > 0, y \geq 0\}$ in this local coordinates corresponds with $\{(v, u) : v \geq 0, u > 0\}$. The directions of approach of $\tilde{\Gamma}_h^*$ to $(v, u) = (0, 0)$ are given by: $u = \lambda_\pm\, v$, where

$$\lambda_\pm = \frac{(2h + a - 1) \pm 2\sqrt{h(h + a - 1)}}{(a - 1)^2}.$$

It is easy to check that for $a \neq 1$, both $\lambda_\pm$ are positive, therefore there exist two branches of $\tilde{\Gamma}_h^*$ arriving at the singular point $(0, 0)$ in $\{(v, u) : v \geq 0, u > 0\}$. As only the two branches of Γ_h described by $y_i(x)$, $i = 1, 2$ are defined when $x \to +\infty$ these ones are the two branches described by the blow–up procedure. This ends the proof of the claim.

Proof of Claim 2. It is easy to see that $y \to m_+(\bar{x}, y; a)$ has a unique minimum at $y = y_c(\bar{x})$, where, by Descartes' rule, $y_c(\bar{x})$ is the only positive solution of the equation $C(x, y) := -y^3 - y^2 + (1 + \bar{x})y + (1 + \bar{x})(a + \bar{x}) = 0$, and its decreasing at $y < y_c(\bar{x})$ at increasing for $y > y_c(\bar{x})$.

Now we state three facts concerning the curve $y = y_c(x)$, which are relevant to the study of $x \to m_+(x, y_c(x); a)$.

(a) An straightforward computation shows that $y_c(x_c) = x_c$.
(b) Once again, applying Descartes's Rule on the cubic $C(x, y) = 0$, we have that $\lim_{x\to 0^+} y_c(x) > 0$, which implies that $\lim_{x\to 0^+} m_+(x, y_c(x); a) = +\infty$.
(c) A detailed analysis of the asymptotic expansion of $y_c(x)$ at infinity gives that $y_c(x) \sim (\sqrt[3]{216}/6)\sqrt[3]{x^2}$, hence $\lim_{x\to+\infty} y_c(x) = +\infty$, and as a consequence $\lim_{x\to+\infty} m_+(x, y_c(x); a) = +\infty$.

Since the only positive solutions of $\dfrac{\partial m_+}{\partial x}(x, y; a)$ are given by the cubic equation $C(y, x) = -x^3 - x^2 + (1 + y)x + (1 + y)(a + y) = 0$, and $C(x, y) = C(y, x) = 0$ if and only if $x = x_c$ and $y = x_c$, we get that $\dfrac{\partial m_+}{\partial x}(x, y_c(x); a) \neq 0$ for $x \neq x_c$, and $\dfrac{\partial m_+}{\partial x}(x_x, y_c(x_c); a) = 0$.

This means that the function $x \to m_+(x, y_c(x); a)$, which gives the minimum values of each function $y \to m_+(x, y; a)$, has a unique minimum at $x = x_c$, decreases from $+\infty$ to h_c for $x \in (0, x_c)$, and increases from h_c to $+\infty$ for $x \in (x_c, +\infty)$. A simple computation shows that $m_+(x_c, y_c(x_c); a) = m_+(x_c, x_c; a) = h_c$.

The above results prove (see Figure 5), that for any fixed $h > h_c$ there exists only two solutions $x_{i,h}$, $i = 1, 2$ of equation

$$h = m_+(x, y_c(x); a), \tag{31}$$

such that $x_c \in (x_{1,h}, x_{2,h})$. For these two values, the minimum of the functions $y \to m(x_{i,h}, y; a)$ is h. This means that equation (29) has only one solution for $\bar{x} = x_{i,h}$ $i = 1, 2$. Now we observe that for all $\bar{x} \in (x_{1,h}, x_{2,h})$, since the minimum values of the functions $y \to m(\bar{x}, y; a)$ are below h, we can conclude that for these values of $\bar{x}$ equation (29) only has two solutions. Finally for $\bar{x} \notin (x_{1,h}, x_{2,h})$, since the minimum values of the functions $y \to m(\bar{x}, y; a)$ are greater than h, equation (29) has no solutions. In summary equation (29) describes one and only one oval γ_h.

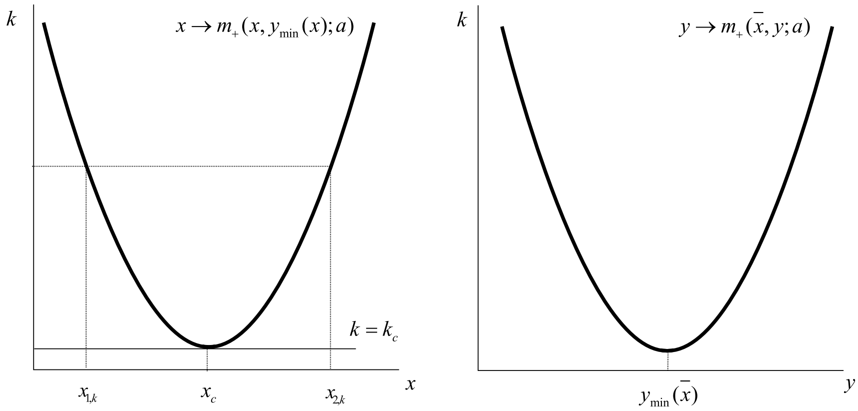

Figure 5. Plot of some functions involved in the proof of Lemma 13.

The above analysis of $x \to m_+(x, y_c(x); a)$ shows that $h_c = m(\bar{x}, y; a)$ if and only if $\bar{x} = y = x_c$. So the branch of Γ_{h_c} described by equation (29) is only the point (x_c, x_c). This ends the proof of the claim. □

Appendix C. Proof of Proposition 10

(i) Set

$$h(x) := V_1|_{\mathcal{L}} = V_1\left(x, \frac{x+a}{x-1}, x\right) = \frac{(2x+a-1)^2(x+1)^2}{x(x+a)(x-1)}, \text{ for } x > 1.$$

Trivial but tedious computations show that the unique solution of $h'(x) = 0$ such that $x > 1$ is $x = x_c$ which is a minimum, and that $\lim_{x\to 1^+} h(x) = \lim_{x\to+\infty} h(x) = +\infty$. Hence for any $k > k_c$, the equation $V_1(x, (x+a)/(x-1), x) = k$ has a unique solution in $(1, x_c)$ and a unique solution in $(x_c, +\infty)$. Hence $L_k \cap \mathcal{L}$ consists of two points, say p_1 and p_2. Recall that $\mathcal{L}$ is the curve of 2–periodic points of F, hence $\{p_1, p_2\}$ is a 2–periodic orbit and (i) holds.

To prove statement (ii) we will see that if $k > k_c$ the locus of non transversal intersections of L_k with $\mathcal{G}$ is the empty set. Consider the system $G = 0$ and

$$(V_1)_x/(V_1)_y = G_x/G_y, \tag{32}$$
$$(V_1)_x/(V_1)_z = G_x/G_z, \tag{33}$$

obtained by imposing $\{\nabla V_1 \| \nabla G\} \cap \{G = 0\}$.

The only positive solutions of equation (32) are given by the zeroes of $m_1 := xaz + xaz^2 + 2ay^3x + 3x^2az + 5y^4 + 3y^5 + 2y^3 + 3x^2az^2 + 2x^3az +$

$2x^3az^2+xy^2z-y^2xz^2+6ay^2z+4ay^3z+2xy^3z+2ay^2x-3y^2x^2z^2+x^2z+xz^2+xz^3+3x^3z-2y^2x^3z^2-4y^2x^3+2x^3z^3+2x^4z^2+2x^4z-2y^4x^2-2y^3x^3+3x^2z^3+4x^2z^2+5x^3z^2-4y^2x^2a-2y^3x^2a-2y^2x^3z-2yx^2a+2ayz-2y^3x^2z-2x^2yz-2yx^3+3y^4x+2y^3z^2+5ay^4+3y^2z^2+3y^2z+3ay^2+8y^3z+4y^3x+8ay^3+3a^2y^2-4y^3x^2-3y^2x^2-yx^2+a^2y+2a^2y^3+y^2x+yz^2+5\,y^4z-7\,y^2x^2z$, and its zeroes over $\mathcal{G}$ are given by the the zeroes of r_1, where it satisfies $m_1=q_1\,G\,+r_1$, and it is given by $r_1:=\left(2xz-2x^3z+2xz^2-2x^3z^2\right)y^2+\left(4xz^2+2x^2z+2xaz^2-2x^3z\,+4x^2z^2+2xz^3+2x^2az^2-2x^3z^2-2x^4z^2\right.$ $\left.-2x^4z+2x^2z^3+2xaz+2xz+2x^2az\right)y\,+2x^3z^3+2xaz+2xz^2+2xz^3+4x^2az^2+2x^3az^2+4x^2az+2x^3z^2+4x^2z^3+2x^3az+2xaz^2+4x^2z^2$.

The only positive solutions of equation (33) are given by the zeroes of $m_2:=x-z$ and $m_3:=2xaz+2xaz^2+2x^2az-y^3+2x^2az^2-ayx+2xyz^2-2xy^2z-ay^2z-ay^2x+4x^2z+4xz^2+3xz^3+3x^3z+2x^3z^3+5x^2z^3+9x^2z^2+5x^3z^2-ya-ayz+2x^2yz+2yz^2x^2-xy-yz-y^2-y^2z^2-2y^2z-ay^2-y^3z-y^3x-y^2x^2-yx^2-2y^2x-yz^2+xz$.

Suppose that $m_2=0$ so $z=x$ and $r_1(x,y,x)=-2x^2(x+1)^2(1+x+y)(-a-x-y+xy)$, hence the positive solutions of $r_1(x,y,x)=0$ are given by $y=(x+a)/(x-1)$, thus the points in $\mathcal{L}$. But as mentioned above $\mathcal{L}\cap\mathcal{G}=(x_c,x_c,x_c)$.

To study the zeroes of m_3 in $\mathcal{G}$, we consider the zeroes of r_2, satisfying $m_2=q_2\,G\,+r_2$. But it is given by $r_2:=2xz(x+1)(z+1)(a+x+y+z+xz)$, and therefore $r_2=0$ has not positive solutions.

In summary, if $k>k_c$, then ∇V_1 is never parallel to ∇G over $\mathcal{G}$ and hence $L_k \pitchfork \mathcal{G}$.

(iii) Recall that $\mathcal{G}$ is defined by the equation

$$G=-y^3-(x+z+a+1)y^2-(x+z+a)y+xz(x+1)(z+1)=0.$$

By applying Descartes' Rule on $\mathcal{G}$ we obtain that for all $x>0$ and $z>0$ there exist a unique $y(x,z)>0$ solution of $G=0$. Consider the function $v(x,z):=V_1(x,y(x,z),z)$. Now the proof is done in two steps: (I) The only singular point of v is (x_c,x_c), which is a minimum. (II) Each level curve $v(x,z)=k>k_c$ is a closed curve surrounding (x_c,x_c).

Step I: To find the singular points of $h(x,z)$ we look for the solutions of system

$$\begin{cases} v_x=(V_1)_x+(V_1)_y\dfrac{\partial y}{\partial x}=0,\\ v_z=(V_1)_z+(V_1)_y\dfrac{\partial y}{\partial z}=0,\end{cases}$$

such that $x>0$ and $z>0$. The only factors in v_x and v_z giving rise to such solutions are

$m:=xaz+xaz^2+2ay^3x+3x^2az+5y^4+3y^5+2y^3+3x^2az^2+2x^3az+2x^3az^2+xy^2z-y^2xz^2+6ay^2z+4ay^3z+2xy^3z+2ay^2x-3y^2x^2z^2+x^2z+$

$xz^2+xz^3-2y^3x^2a-4y^2x^2a+3x^3z+2x^3z^3+2x^4z^2+2x^4z+3x^2z^3+4x^2z^2+5x^3z^2-2y^2x^3z^2-4y^2x^3+3y^4x+2y^3z^2+5ay^4+3y^2z^2+3y^2z+3ay^2+8y^3z+4y^3x+8ay^3+3a^2y^2-4y^3x^2-3y^2x^2-2yx^3-yx^2+a^2y+2a^2y^3+y^2x+yz^2+5y^4z-7y^2x^2z+2ayz-2yx^2a-2y^2x^3z-2y^3x^2z-2x^2yz-2y^4x^2-2y^3x^3$ and $n:=xaz+3xaz^2+4ay^3x+x^2az+5y^4+3y^5+2y^3+3x^2az^2+2xaz^3+2x^2az^3+2x^2z^4+2xz^4-2y^2x^2z^3-2y^2xz^3-2yz^2a-4y^2z^2a-2y^3z^2a-2y^3xz^2+2ayx-2xyz^2+xy^2z-7y^2xz^2+2ay^2z+2ay^3z+2xy^3z+6ay^2x-3y^2x^2z^2+x^2z+xz^2+3xz^3+x^3z+2x^3z^3+5x^2z^3+4x^2z^2+3x^3z^2+5y^4x-4y^3z^2+5ay^4-3y^2z^2+y^2z+3ay^2+4y^3z+8y^3x+8ay^3+3a^2y^2+2y^3x^2+3y^2x^2+yx^2+a^2y-4y^2z^3-2yz^3-2y^4z^2-2y^3z^3+2a^2y^3+3y^2x-yz^2+3y^4z-y^2x^2z$, respectively. Here y denotes $y(x,z)$.

The common zeroes of m and n in $\mathcal{G}$ are given by the zeroes of the functions r and s respectively, where $m=pG+r$ and $n=qG+s$ for some polynomials p and q. These functions are

$$\begin{aligned} r &= 2xz\,(x+1)\,(1+x+y)\,(z+1)\,(a+y+z-xy),\\ s &= -2xz\,(x+1)\,(1+y+z)\,(z+1)\,(-a-x-y+yz). \end{aligned}$$

The only positive solutions of $r=0$ and $s=0$ are given by $(x,(x+a)/(x-1),x)$, which are the points of $\mathcal{L}$. Since $\mathcal{L}\cap\mathcal{G}=(x_c,x_c,x_c)$, the proof of (I) is finished.

Step II: Note the following facts: from statement (ii), $\mathcal{G}\pitchfork L_k$; the level curves $v(x,z)=k$, are defined by $L_k\cap\mathcal{G}$ and thus for analytic equations and for $k>k_c$ they have no critical points (since otherwise the hamiltonian vector field $-v_z\partial_x+v_x\partial_z$ would have another critical point than (x_c,x_c) in contradiction with Step I); The sets L_k are compact. From all them we conclude that for a fixed $k>k_c$, each level set of $v(x,z)=k$ is diffeomorphic to a finite union of closed curves. To prove that indeed it is formed by an unique closed curve it suffices to show that the function $x\to v(x,x_c)$ is monotonic in $(x_c,+\infty)$. To see this we will prove that it has the unique critical point $x=x_c$. To this end note that the only positive solutions of $r=0$ are given by the factor $a+y+z-xy$, hence $x=(a+y+z)/y$. Now $G((a+y+x_c)/y,y,x_c)=(y-x_c)\,q_4(y)/y^2$, where q_4 is a degree four polynomial in y without positive solutions. Thus, as we wanted to see, $x=x_c$ is the unique critical point of $x\to v(x,x_c)$ and (iii) follows. $\square$

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