RANDOM GRAPHS ON SURFACES

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Abstract. Counting labelled planar graphs, and typical properties of random labelled planar graphs, have received much attention recently. We start the process here of extending these investigations to graphs embeddable on any fixed surface S. In particular we show that the labelled graphs embeddable on S have the same growth constant as for planar graphs, and the same holds for unlabelled graphs. Also, if we pick a graph uniformly at random from the graphs embeddable on S which have vertex set $\{1, \ldots, n\}$, then with probability tending to 1 as $n \to \infty$, this random graph either is connected or consists of one giant component together with a few nodes in small planar components.

1. Introduction

For any surface S, let $\mathcal{G}^S$ be the class of graphs which can be embedded in S, and let $\mathcal{G}_n^S$ be the set of graphs in $\mathcal{G}_S$ on the vertex set $\{1, \ldots, n\}$. (See [21] for a discussion of embeddings in a surface.)

We consider two related questions. Firstly, how large is $\mathcal{G}_n^S$? Secondly, let $R_n \in_U \mathcal{G}_n^S$, that is let R_n be a graph picked uniformly at random from $\mathcal{G}_n^S$. What are typical properties of R_n for large n? Does R_n behave similarly to the planar case? For example, does R_n usually have a giant component, does it have many vertices of degree 1, and so on? To proceed with the second question we need to consider the first one.

Such questions, together with that of how to generate R_n quickly, have received much attention recently for the case when S is the sphere (or the plane), see for example [2, 4, 5, 6, 7, 10, 11, 12, 13, 15, 16, 18, 19, 20, 23]. Let us write $\mathcal{P}$ for $\mathcal{G}^S$ in this case. A key part of the investigations involve estimating $|\mathcal{P}_n|$. It is shown in [19] that

$$(|\mathcal{P}_n|/n!)^{1/n} \to \gamma_\ell \quad \text{as } n \to \infty,$$

where γ_ℓ is the *planar graph growth constant*, with bounds known on γ_ℓ. Giménez and Noy [16] improve greatly on this: they give an explicit analytic

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expression for γ_ℓ, showing that $\gamma_\ell \approx 27.2269$ (correct to 6 significant figures), and further show that

$$|\mathcal{P}_n| \sim g \cdot n^{-\frac{7}{2}} \gamma_\ell^n \, n! \tag{1}$$

where the constant $g \approx 4.2609 \cdot 10^{-6}$ (correct to 5 significant figures) also has an explicit analytic expression. They also give a corresponding expression for the number of connected graphs in $\mathcal{P}_n$ which differs only in that the leading constant is not g but $c \approx 4.1044 \cdot 10^{-6}$ (correct to 5 significant figures),

$$\mathbb{P}[R_n \text{ is connected }] \to c/g \approx 0.96325 \tag{2}$$

(correct to 5 significant figures).

The plan of the paper is as follows. In the next section we introduce our new general results. Then we give results assuming 'smoothness': for example the class of planar graphs is known to have this property, and some of these results are new even when specialised to planar graphs. In the next two sections we prove first the general results and then the results assuming smoothness, and finally we make some concluding remarks.

2. General results

The crucial step to get started on investigating $R_n \in_U \mathcal{G}^S_n$ is to estimate $|\mathcal{G}^S_n|$. Clearly $|\mathcal{G}^S_n|$ is in general bigger than $|\mathcal{P}_n|$, but how much? Since $\mathcal{G}^S$ is minor-closed it follows [22] that it is 'small', that is for some constant c we have $|\mathcal{G}^S_n| \leq c^n n!$ for all n. The first new result shows that $\mathcal{G}^S$ has a growth constant, and indeed it is the planar graph growth constant γ_ℓ.

Theorem 2.1. *For any fixed surface S, as $n \to \infty$*

$$(|\mathcal{G}^S_n|/n!)^{1/n} \to \gamma_\ell \quad \textit{as } n \to \infty;$$

that is, $\mathcal{G}^S$ has growth constant γ_ℓ.

The same result holds for connected graphs (since the number of connected graphs in $\mathcal{G}^S_n$ is clearly at least $|\mathcal{G}^S_{n-1}|$). We do not approach the accuracy of the Giménez and Noy result (1).

Let us briefly consider unlabelled graphs. Let $\mathcal{UG}^S_n$ denote the set of unlabelled n-vertex graphs embeddable in S; that is, the set of isomorphism classes of graphs in $\mathcal{G}^S_n$. When S is the sphere (or the plane), let us write $\mathcal{UP}_n$ for $\mathcal{UG}^S_n$. It was shown in [10] by a supermultiplicativity argument that there is a constant γ_u, the *unlabelled planar graph growth constant*, such that if u_n is the number of connected unlabelled planar graphs on n vertices, then $u_n^{1/n} \to \gamma_u$ as $n \to \infty$. Since $u_{n-1} \leq |\mathcal{UP}_{n-1}| \leq n u_n$ (by considering adding a distinguished vertex joined to each component) it follows that $|\mathcal{UP}_n|^{1/n} \to \gamma_u$ as $n \to \infty$. It is known also that $\gamma_\ell < \gamma_u \leq 30.061$, see [8, 19].

Theorem 2.2. *For any fixed surface S, as $n \to \infty$*

$$(|\mathcal{UG}_n^S|)^{1/n} \to \gamma_u \quad \text{as } n \to \infty.$$

The same result holds for connected unlabelled graphs.

Now let us return to labelled graphs. Since $\mathcal{G}^S$ has a growth constant there are many results which we can read off from [19] or [20]. In particular there is an 'appearances' theorem - see Theorem 4.1 in [19].

Let H be a graph with vertex set $\{1, \ldots, h\}$, and let G be a graph on the vertex set $\{1, \ldots, n\}$ where $n > h$. Let $W \subset V(G)$ with $|W| = h$, and let the 'root' r_W denote the least element in W. We say that H *appears* at W in G if (a) the increasing bijection from $\{1, \ldots, h\}$ to W is an isomorphism from H to the induced subgraph $G[W]$ of G; and (b) there is exactly one edge in G between W and the rest of G, and this edge is incident with the root r_W.

Theorem 2.3. *Let S be a fixed surface and let $R_n \in_U \mathcal{G}_n^S$. Let H be a fixed connected planar graph on vertices $1, \ldots, h$. Then there exists a constant $\alpha > 0$ such that, with probability $1 - e^{-\Omega(n)}$, there are at least αn pairwise node-disjoint appearances of H in R_n.*

By applying Theorem 2.3 to appropriately chosen graphs H, for example to a star or cycle on k vertices, we can deduce from it various results about vertex degrees, face sizes and numbers of automorphisms in a random graph R_n, arguing as in [19, 20].

Corollary 2.4. *(a) For each positive integer k, there is a constant $\alpha > 0$ such that, with probability $1 - e^{-\Omega(n)}$, there are at least αn nodes of degree k in R_n.*

(b) For each integer $k \geq 3$, there is a constant $\alpha > 0$ such that, with probability $1 - e^{-\Omega(n)}$, in each embedding of R_n in S there are at least αn facial walks of length k.

(c) There are positive constants α and β such, with probability $1 - e^{-\Omega(n)}$, the number $aut(R_n)$ of automorphisms of R_n satisfies $2^{\alpha n} \leq aut(R_n) \leq 2^{\beta n}$.

Let us briefly consider again the unlabelled random graph $U_n \in_U \mathcal{UG}_n^S$. It is known [2] that $aut(U_n)$ stochastically dominates $aut(R_n)$ (we give a full proof later for completeness, see Lemma 5.3 below). Thus, with the same $\alpha > 0$ as above, with probability $1 - e^{-\Omega(n)}$ we have $aut(U_n) \geq 2^{\alpha n}$.

The behaviour of the maximum degree in a random planar graph was an open problem until recently, see [20], and similarly for the maximum size of a face. However, it was very recently shown [18] that for $R_n \in_U \mathcal{G}_n^S$ the maximum degree $\Delta(R_n)$ is $\Theta(\ln n)$ whp; and similarly, whp in each embedding the maximum length of a facial walk is $\Theta(\ln n)$.

Our last general result here concerns connectedness and components. We need some definitions and notation. The *big component* Big(G) of a graph G is the (lexicographically first) component with the most vertices, and Miss(G) is the subgraph induced on the vertices not in (missed by) the big component. We denote the numbers of vertices in Big(G) and Miss(G) by big(G) and miss(G) respectively, so big(G) + miss(G) equals the number of vertices of G. (We allow Miss(G) to be empty, with miss(G) = 0.)

Given $\lambda > 0$ we let Po(λ) denote the Poisson distribution with mean λ, or a random variable with this distribution. Also, we say that S' is a *simpler* surface than S if S can be obtained from S' by adding one or more handles or crosscaps. In the theorem below, part (a) follows immediately from Theorem 2.2 of [19] or [20], parts (b) and (c) are similar to and extend Theorems 6.2 and 6.4 respectively of [20], and part (d) is new.

Theorem 2.5. *Let S be a fixed surface and let $R_n \in_U \mathcal{G}^S_n$. Then*

: *(a) the number of components $\kappa(R_n)$ is stochastically at most* $1 + \mathrm{Po}(1)$*, and thus the probability that R_n is connected is at least* $1/e$*;*

: *(b) for any fixed planar graph H,*

$$\liminf_{n\to\infty} \Pr\left[\mathrm{Miss(R_n)} \approx \mathrm{H}\right] > 0$$

and thus $\limsup_{n\to\infty} \Pr\left[R_n \textit{ is connected }\right] < 1$*;*

: *(c)* $\mathbb{E}[\mathrm{miss(R_n)}] \leq 6 + \mathrm{o}(1)$ *(so the big component* $\mathrm{Big(R_n)}$ *is 'giant'); and*

: *(d) whp* $\mathrm{Miss(R_n)}$ *is planar and* $\mathrm{Big(R_n)}$ *is not embeddable on any simpler surface.*

Let us close this section by trying to set the basic result Theorem 2.1 in relief, by giving a contrasting result. Consider two proper minor-closed classes of graphs $\mathcal{A}$ and $\mathcal{B}$ for which each excluded minor is 2-edge-connected. [For example, the class of planar graphs has this property, as the excluded minors are the complete graph K_5 and the complete bipartite graph $K_{3,3}$; but this is not true for the class $\mathcal{G}^S$ of graphs embeddable on any other surface S, as there will be some disconnected excluded minors.] By [22] $\mathcal{A}$ and $\mathcal{B}$ are 'small', and since they are also 'addable' it follows from Theorem 3.3 of [19] that they have growth constants $\gamma_{\mathcal{A}}$ and $\gamma_{\mathcal{B}}$ respectively. In contrast to Theorem 2.1, if $\mathcal{A} \subset \mathcal{B}$ then $\gamma_{\mathcal{A}} < \gamma_{\mathcal{B}}$. For if H is a connected graph in $\mathcal{B} \setminus \mathcal{A}$ and $R_n \in_U \mathcal{B}$, then by the 'appearances' theorem in [20] the probability that R_n has no subgraph isomorphic to H is $e^{-\Omega(n)}$.

3. Results assuming smoothness

Consider the ratio $r_n = n|\mathcal{G}^S_{n-1}|/|\mathcal{G}^S_n|$. It is straightforward to see that r_n is the expected number of isolated vertices in R_n, see [19], and that

$$\liminf_{n\to\infty} r_n \leq \rho \leq \limsup_{n\to\infty} r_n$$

where $\rho = \gamma_\ell^{-1} = 0.0367$ to four decimal places. It follows from the asymptotic result (1) that for planar graphs, we have $r_n \to \rho$ as $n \to \infty$. For surfaces S other than the sphere we do not know if r_n tends to a limit (which would have to be ρ). If this happens for a class $\mathcal{A}$ of graphs (that is, if the ratio $n|\mathcal{A}_{n-1}|/|\mathcal{A}_n|$ tends to a limit as $n \to \infty$) we say that the class of graphs is *smooth*. Some other classes of graphs known to be smooth include forests and trees, outerplanar graphs [4], series parallel graphs [4], and cubic planar graphs (if we consider only even n) [7]. (In each case this is because we know a precise asymptotic counting formula.)

Now let S be any fixed surface. It seems reasonable to conjecture that the class $\mathcal{G}^S$ is smooth. If we assume that this is the case then we can say much more about $R_n \in_U \mathcal{G}^S_n$, and we find much behaviour like that for planar graphs. To show this we give four theorems below, some of which extend what was previously known for the planar case.

Given a graph G we let $v(G)$ denote the number of vertices and $\mathrm{aut}(G)$ denote the number of automorphisms of G. For the following result, we shall need little work to extract it from section 5 of [19].

Theorem 3.1. *Let S be a fixed surface, assume that the class $\mathcal{G}^S$ is smooth, and let $R_n \in_U \mathcal{G}^S_n$. For each graph H let $\lambda(H) = \rho^{v(H)}/aut(H)$, and let $X_n(H)$ be the number of components of R_n isomorphic to H. Let $H_1, \ldots, H_k$ be a fixed family of pairwise non-isomorphic connected planar graphs. Then as $n \to \infty$ the joint distribution of $X_n(H_1), \ldots, X_n(H_k)$ converges to the product distribution $\mathrm{Po}(\lambda(H_1)) \otimes \cdots \otimes \mathrm{Po}(\lambda(H_k))$ in total variation distance. Thus in particular for each graph H*

$$\Pr[R_n \textit{ has no component isomorphic to } H] \;\to\; e^{-\lambda(H)} \quad \textit{as } n \to \infty.$$

Further, we also have convergence for all moments; that is, for each positive integer j we have $\mathbb{E}[X_n(H)^j] \to \lambda(H)^j$ as $n \to \infty$.

[Recall that the *total variation distance* between discrete distributions (p_i) and (q_i) is $\frac{1}{2}\sum_i |p_i - q_i|$.]

For the next two results, we need to introduce the exponential generating function $A(z)$ for the class $\mathcal{P}$ of planar graphs, and $C(z)$ for the class $\mathcal{C}$ of connected planar graphs. Thus $A(z) = \sum_{n\geq 0} |\mathcal{P}_n|\ z^n/n!$ (where $|\mathcal{P}_0| = 1$); and $C(z) = \sum_{n\geq 0} |\mathcal{C}_n|\ z^n/n!$ where $\mathcal{C}_n$ is the set of connected graphs $G \in \mathcal{P}_n$ (and $\mathcal{C}_0 = \emptyset$). It is well known that $A(z) = e^{C(z)}$. The quantity

$\rho = \gamma_\ell^{-1}$ which we met earlier is the radius of convergence of these generating functions. Two related important constants which we shall meet below are $\lambda = C(\rho)$ which equals 0.03744 to four significant figures, and $e^{-\lambda} = e^{-C(\rho)} = A(\rho)^{-1}$ which equals 0.9633 to four significant figures. Observe from (1) that $A(\rho), A'(\rho)$ and $A''(\rho)$ are finite but $A'''(\rho)$ is infinite, and using also (2) that the corresponding result holds for $C(z)$ and its derivatives at $z = \rho$.

Theorem 3.2. *Let S be a fixed surface, assume that the class $\mathcal{G}^S$ is smooth, and let $R_n \in_U \mathcal{G}^S_n$.*
(a) As $n \to \infty$, $\kappa(R_n)$ converges to $1 + \mathrm{Po}(\lambda)$ in total variation distance and for all moments, where $\lambda = C(\rho) \approx 0.03744$; and in particular

$$\Pr[R_n \textit{ is connected}] \;\to\; e^{-\lambda} \approx 0.9633$$

and

$$\mathbb{E}[\kappa(R_n)] \to 1 + \lambda \approx 1.03744.$$

(b) More generally, let $\mathcal{D} \subseteq \mathcal{C}$ be a non-empty class of connected planar graphs, and let $D(z)$ be the exponential generating function for $\mathcal{D}$. Then as $n \to \infty$ the number of components of $\mathrm{Miss(R_n)}$ in $\mathcal{D}$ tends to $\mathrm{Po}(D(\rho))$ in total variation distance and for all moments.

The planar case of part (a) of Theorem 3.2 above is essentially Theorem 6 of [16]; and the planar case of part (b) is a slight extension of Theorem 7 in that paper. In some cases it is easy to consider $\mathrm{Big(R_n)}$ too in part (b). For example, if S is any surface other than the sphere, then whp $\mathrm{Big(R_n)}$ is not planar (by part (d) of Theorem 2.5): and so the number of components of R_n in $\mathcal{D}$ tends to $\mathrm{Po}(D(\rho))$ in distribution.

Next we consider limiting distributions related to the random graph $\mathrm{Miss(R_n)}$. We have already seen in Theorem 2.5 that whp $\mathrm{Miss(R_n)}$ is planar. It is convenient to deal with $\mathcal{U}\mathrm{Miss(R_n)}$, the unlabelled graph corresponding to $\mathrm{Miss(R_n)}$. In the next theorem we shall meet what we call the *Miss* distribution on the class $\mathcal{UP}$ of unlabelled planar graphs, and the *miss* distribution on the non-negative integers.

Theorem 3.3. *Let S be a fixed surface, assume that the class $\mathcal{G}^S$ is smooth, and let $R_n \in_U \mathcal{G}^S_n$. Then the random unlabelled graph $\mathcal{U}\mathrm{Miss(R_n)}$ converges in total variation distance to the Miss distribution (p_M) on $\mathcal{UP}$, where for $H \in \mathcal{UP}$*

$$p_M(H) = \frac{1}{A(\rho)} \frac{\rho^{v_H}}{aut(H)} \;;$$

and $\mathrm{miss(R_n)}$ *converges in total variation distance to the miss distribution* (q_m) *on the non-negative integers, where for* $n \geq 0$

$$q_m(n) = \frac{1}{A(\rho)} |\mathcal{P}_n| \frac{\rho^n}{n!} \ .$$

Further the miss distribution has probability generating function $G(x) = A(\rho x)/A(\rho) = e^{C(\rho x) - C(\rho)}$*, it has mean equal to the radius of convergence* R *of the exponential generating function for 2-connected planar graphs (where* $R = 0.03819$ *to four significant figures) it has variance* 0.03979 *to four significant figures, and it is a compound Poisson distribution.*

Let us make some observations concerning this last result. There seems no obvious reason why the expected value of the miss distribution should be R. Under the Miss distribution, the expected number of isolated vertices is ρ, so the expected number of non-isolated vertices is $R - \rho \approx 0.001463$ to four significant figures. We saw in Theorem 3.2 that the probability that R_n is connected (and so $\mathrm{Miss(R_n)}$ is empty) tends to $e^{-C(\rho)} = A(\rho)^{-1}$ as $n \to \infty$. From the last theorem we may see for example that the probability that $\mathrm{Miss(R_n)}$ has no edges tends to $e^{\rho - C(\rho)} = 0.99929$ to five significant figures as $n \to \infty$. [To see this, note that for a random H from the Miss distribution, the probability that H has no edges equals

$$\sum_{k \geq 0} \mathbb{P}[H \cong \overline{K_k}] = e^{-C(\rho)} \sum_{k \geq 0} \frac{\rho^k}{k!} = e^{\rho - C(\rho)},$$

where $\overline{K_k}$ denotes the k-vertex graph with no edges.] Similarly the probability that $\mathrm{Miss(R_n)}$ has exactly one edge tends to $\frac{1}{2}\rho^2 e^{\rho - C(\rho)} = 0.00067$ to two significant figures; and so the probability that $\mathrm{Miss(R_n)}$ has more than one edge is about 4×10^{-5}.

The above results on $\mathrm{Miss(R_n)}$ are new even for planar graphs (which form a smooth class), but for planar graphs we can say more, for example that the mean and variance of $\mathrm{miss(R_n)}$ converge to those of the limiting distribution. Indeed, in this case, for $t < \frac{5}{2}$ the t-th moment of $\mathrm{miss(R_n)}$ converges to the tth moment of the limiting miss distribution (which is finite), and for $t \geq \frac{5}{2}$ it tends to ∞ – see Proposition 5.2 below. In particular, for a random planar graph the expected number of vertices not in the big component tends to R as $n \to \infty$.

Finally here let us go back to appearances, and give one last result.

Theorem 3.4. *Let* S *be a fixed surface, assume that the class* $\mathcal{G}^S$ *is smooth, and let* $R_n \in_U \mathcal{G}^S_n$*. Let* H *be a connected planar graph on the vertex set* $\{1, \ldots, h\}$*, and let* $X_n(H)$ *be the number of appearances of* H *in* R_n*. Then* $X_n(H)/n \to \rho^h/h!$ *in probability as* $n \to \infty$*.*

In the planar case much fuller results are known, for example that $X_n(H)$ is asymptotically normally distributed with a given mean and variance, see Theorem 4 in [16].

4. Proofs for general results

First we give proofs of Theorems 2.1 and 2.2. We need some lemmas to prove these results. The first is the key one.

Lemma 4.1. *([9]) For any surface S and any graph $G \in \mathcal{G}_S(n)$ embedded in S, there is a noncontractible cycle C in S which meets the graph in at most $k = \lfloor \sqrt{2n} \rfloor + 2$ vertices (and nowhere else).*

The orientable case of this lemma was proved in [1] (with $k = \lfloor \sqrt{2n} \rfloor$), with 'noncontractible' strengthened to 'non-surface-separating'. It is shown in [9] that if each noncontractible cycle meets the graph G at least k times then there is a family of at least $\lfloor \frac{k-1}{2} \rfloor$ pairwise disjoint (homotopic) noncontractible cycles in G, and so G must have at least $k\lfloor \frac{k-1}{2} \rfloor \geq k(k/2-1)$ vertices. But this number is $> n$ if $k \geq \sqrt{2n}+2$, which yields the lemma.

For each non-negative integer g, let $\mathcal{A}^{(g)}$ denote the class of graphs embeddable on a surface of Euler genus g, and let $\mathcal{B}^{(g)}$ denote the class of graphs G such that either $G \in \mathcal{A}^{(g)}$, or $G \in \mathcal{A}^{(g+1)}$ and G has a component H such that both H and $G - H$ are in $\mathcal{A}^{(g)}$.

Lemma 4.2. *Let g be a non-negative integer, let n be a positive integer and let $k = k(n)$ be as in Lemma 4.1. Let S be the set of k-tuples $x = (x_1, \ldots, x_k)$ of distinct vertices in $\{1, \ldots, n\}$. Given a graph $G \in \mathcal{B}^{(g)}_{n+k}$ and a list $x \in S$, let $\psi(G, x)$ denote the (multi-) graph obtained by starting with G and identifying vertices x_j and $n + j$ for each $j = 1, \ldots, k$. Then for each graph $G \in \mathcal{A}^{(g+1)}_n$ there is a graph $\tilde{G} \in \mathcal{B}^{(g)}_{n+k}$ and a list $x \in S$ such that $\psi(\tilde{G}, x) = G$; and thus*

$$|\mathcal{A}^{(g+1)}_n| \leq |\mathcal{B}^{(g)}_{n+k}| \cdot n^k.$$

Proof Let $G \in \mathcal{A}^{(g+1)}$. Let G be embedded in a surface S of Euler genus at most $g+1$. By Lemma 4.1, there is a non-contractible cycle C in S meeting G in k' vertices, for some $0 \leq k' \leq k$.

List the vertices along C as $v_1, \ldots, v_{k'}$. The cycle may be one-sided or two-sided, but in either case we form a graph G' by cutting the surface along C, splitting v_i into two vertices v_i and $n + i$, with edges incident to the original v_i set incident to either the new v_i or to $n + i$ (see for example section 4.2 of [21]). (If $k' = 0$ then G' is just G.) Observe that, from the graph G' together with the list $x' = (v_1, \ldots, v_{k'})$ of vertices, we recover G

when we identify v_i and $n+i$ for each $i = 1, \ldots, k$; that is, $\psi(G', x') = G$. If C is non-surface-separating then $G' \in \mathcal{A}^{(g)}$, and otherwise $G' \in \mathcal{B}^{(g)}$. Thus in either case we have $G' \in \mathcal{B}^{(g)}_{n+k'}$. By adding isolated vertices if necessary we can construct $\tilde{G} \in \mathcal{B}^{(g)}_{n+k}$ together with a list x of exactly k distinct nodes in G such that $\psi(\tilde{G}, x) = G$. □

Proof of Theorem 2.1 We wish to show that $\mathcal{A}^{(g)}$ has growth constant γ_ℓ for each integer $g \geq 0$. From [19] we know the result for $g = 0$. Let $g \geq 0$ be an integer and suppose that we know $\mathcal{A}^{(g)}$ has growth constant γ_ℓ. We must show that $\mathcal{A}^{(g+1)}$ also has growth constant γ_ℓ.

Let us show first that $\mathcal{B}^{(g)}$ has growth constant γ_ℓ. Let $\epsilon > 0$. Let c be such that $|\mathcal{A}^{(g)}_n| \leq c(1+\epsilon)^n \gamma_\ell^n n!$ for each n. Then

$$\begin{aligned} |\mathcal{B}^{(g)}_n| &\leq \sum_{k=0}^{n-1} \binom{n}{k} |\mathcal{A}^{(g)}_k| \cdot |\mathcal{A}^{(g)}_{n-k}| \\ &= n! \sum_{k=0}^{n-1} \frac{|\mathcal{A}^{(g)}_k|}{k!} \cdot \frac{|\mathcal{A}^{(g)}_{n-k}|}{(n-k)!} \\ &\leq n!\; c^2 n (1+\epsilon)^n \gamma_\ell^n, \end{aligned}$$

and since $\mathcal{A}^{(g)} \subseteq \mathcal{B}^{(g)}$ it follows that $\mathcal{B}^{(g)}$ has growth constant γ_ℓ, as desired. (Indeed the class of all graphs such that each component is in $\mathcal{A}^{(g)}$ has growth constant γ_ℓ.)

Let $\epsilon > 0$. Since $\mathcal{A}^{(g)} \subseteq \mathcal{A}^{(g+1)}$, it suffices to show that for n sufficiently large we have

$$|\mathcal{A}^{(g+1)}_n|/n! \leq \gamma_\ell^n \cdot (1+\epsilon)^{2n}. \tag{3}$$

Since $\mathcal{B}^{(g)}$ has growth constant γ_ℓ, there exists n_0 such for all $n \geq n_0$ we have

$$|\mathcal{B}^{(g)}_n|/n! \leq \gamma_\ell^n \cdot (1+\epsilon)^n.$$

Let $n_1 \geq n_0$ be sufficiently large that $(\gamma_\ell(1+\epsilon)(n+k)n)^k \leq (1+\epsilon)^n$ for all $n \geq n_1$. For $n \geq n_1$, by Lemma 4.2

$$\begin{aligned} |\mathcal{A}^{(g+1)}_n|/n! &\leq |\mathcal{B}^{(g)}_{n+k}|/(n+k)!\; (n+k)_k\; n^k \\ &\leq \gamma_\ell^{n+k}(1+\epsilon)^{n+k}\; (n+k)^k n^k \\ &= \gamma_\ell^n (1+\epsilon)^n\; (\gamma_\ell(1+\epsilon)(n+k)n)^k \\ &\leq \gamma_\ell^n (1+\epsilon)^{2n}. \end{aligned}$$

Thus (3) holds, and we have established the induction step. This completes the proof. □

The next lemma will be useful for proving Theorem 2.2 on unlabelled graphs.

Lemma 4.3. *For each integer $g \geq 0$, and each positive integer n*

$$|\mathcal{UA}_n^{(g+1)}| \leq |\mathcal{UB}_{n+k}^{(g)}| \cdot (n+k)_{2k}$$

where $k = k(n)$ is as in Lemma 4.1.

Proof Let $t = |\mathcal{UA}_n^{(g+1)}|$ and list these unlabelled graphs as $\mathcal{U}^1, \ldots, \mathcal{U}^t$. Thus we have partitioned $\mathcal{A}_n^{(g+1)}$ into the t equivalence classes $\mathcal{U}^1, \ldots, \mathcal{U}^t$ (where equivalence corresponds to isomorphism). For each $i = 1, \ldots, t$ fix one graph G^i in $\mathcal{U}^i$; and for each graph $H \in \mathcal{U}^i$ fix an isomorphism ϕ_H from H to G^i.

Given an unlabelled graph $\mathcal{U}$ in $\mathcal{UB}_{n+k}^{(g)}$, and k-tuples y and z formed from $2k$ distinct elements in $\{1, \ldots, n+k\}$, let

$$T(\mathcal{U}, y, z) = \{(H, x) \in \mathcal{U} \times S : \phi_H((n+1, \ldots, n+k)) = y, \phi_H(x) = z\}.$$

[We use notation from the last lemma, and we use the natural convention that $\phi_H(x)$ denotes the k-tuple with jth co-ordinate $\phi_H(x_j)$.] Fix such a $\mathcal{U}$, y and z, and let (H, x) and (H', x') be in $T(\mathcal{U}, y, z)$. Then the graphs $\psi(H, x)$ and $\psi(H', x')$ are isomorphic. To see this, observe that the permutation $\phi = \phi_{H'}^{-1} \circ \phi_H$ is an isomorphism from H to H'; ϕ fixes each of $n+1, \ldots, n+k$; and $\phi(x) = x'$.

By Lemma 4.2, for each $i \in \{1, \ldots, t\}$, there is a graph $\tilde{G}^i \in \mathcal{B}_{n+k}^{(g)}$ and a list $x^i \in S$ such that $\psi(\tilde{G}^i, x^i) = G^i$. But if $i \neq j$ then by the above, the pairs $(\tilde{G}^i, x^i)$ and $(\tilde{G}^j, x^j)$ cannot be in the same set $T(\mathcal{U}, y, z)$. Thus t is at most the number of triples $\mathcal{U}, y, z$; and the lemma follows. □

Proof of Theorem 2.2 We must show that for each integer $g \geq 0$ we have

$$|\mathcal{UA}_n^{(g)}|^{1/n} \to \gamma_u \quad \text{as } n \to \infty. \tag{4}$$

From [19] we know the result for $g = 0$. Let $g \geq 0$ be an integer and suppose that we know (4) for g: we must prove it for $g+1$. Let us show first that $\mathcal{UB}^{(g)}$ has growth constant γ_u. Let $\epsilon > 0$ and let c be such that $|\mathcal{UA}_n^{(g)}| \leq c\gamma_u^n(1+\epsilon)^n$ for all n. Then

$$\begin{aligned} |\mathcal{UB}_n^{(g)}| &\leq \sum_{k=0}^{n-1} |\mathcal{UA}_k^{(g)}| \cdot |\mathcal{UA}_{n-k}^{(g)}| \\ &\leq n\ c^2 \gamma_u^n (1+\epsilon)^n, \end{aligned}$$

and it follows that $\mathcal{UB}^{(g)}$ has growth constant γ_u.

Let $\epsilon > 0$. Since $\mathcal{UA}^{(g)} \subseteq \mathcal{UA}^{(g+1)}$ it suffices to show that for n sufficiently large we have

$$|\mathcal{UA}_n^{(g+1)}| \leq \gamma_u^n \cdot (1+\epsilon)^{2n}. \tag{5}$$

Since $\mathcal{UB}^{(g)}$ has growth constant γ_u, there exists n_0 such for all $n \geq n_0$ we have

$$|\mathcal{UB}_n^{(g)}| \leq \gamma_u^n \cdot (1+\epsilon)^n.$$

Let $n_1 \geq n_0$ be sufficiently large that $2\left(\gamma_u(1+\epsilon)n^2\right)^k \leq (1+\epsilon)^n$ for all $n \geq n_1$. For $n \geq n_1$, by Lemma 4.3

$$\begin{aligned} |\mathcal{UA}_n^{(g+1)}| &\leq |\mathcal{UB}_{n+k}^{(g)}|\ (n+k)_{2k} \\ &\leq \gamma_u^{n+k}(1+\epsilon)^{n+k}\ 2n^{2k} \\ &= \gamma_u^n(1+\epsilon)^n\ 2\left(\gamma_u(1+\epsilon)n^2\right)^k \\ &\leq \gamma_u^n(1+\epsilon)^{2n}. \end{aligned}$$

Thus (5) holds, and the theorem follows. □

The next result that needs proof here is Theorem 2.5. Recall that part (a) follows directly from Theorem 2.2 of [19] or [20]. For part (b) we can follow the lines of the proof of Theorem 5.2 of [19], see also Theorem 6.2 of [20].

Proof of Theorem 2.5 part (b)

Let H be any fixed planar graph, on vertices $1,\ldots,h$. By Theorem 2.3 there is an $\alpha > 0$ such that whp R_n has at least αn pendant vertices. Let $\mathcal{B}_n$ be the set of connected graphs $G \in \mathcal{G}_n^S$ with at least αn pendant vertices. Then using also part (a) of this theorem, we see that $|\mathcal{B}_n| \geq (\frac{1}{e}+o(1))|\mathcal{G}_n^S|$.

For each graph $G \in \mathcal{B}_n$ and each set W of h pendant vertices of G, we delete the edges incident with the vertices in W and put a copy of H on W, where (for definiteness) we insist that the increasing bijection between $\{1,\ldots,h\}$ and W is an isomorphism. Clearly each graph G' constructed is in $\mathcal{G}_n^S$ and satisfies $\mathrm{Miss(G')} \approx \mathrm{H}$. But for $n > 2h$ each graph G' can be constructed at most n^h times (since that bounds the number of ways to reattach the vertices in W), and then

$$|\{G \in \mathcal{G}_n^S : \mathrm{Miss(G)} \approx \mathrm{H}\}| \geq |\mathcal{B}_\mathrm{n}| \binom{\lceil \alpha \mathrm{n} \rceil}{\mathrm{h}} / \mathrm{n^h} = \Omega(|\mathcal{G}_\mathrm{n}^\mathrm{S}|),$$

which completes the proof. □

To prove part (c) of Theorem 2.5 we first give a general lemma. We call a hereditary class $\mathcal{A}$ of graphs *weakly addable* if whenever a graph $G \in \mathcal{A}$ and e is an edge between different components of G then $G + e \in \mathcal{A}$.

Lemma 4.4. *Let the class $\mathcal{A}$ of graphs be weakly addable, and let $R_n \in_u \mathcal{A}_n$. Then*

$$\mathbb{E}[\mathrm{miss(R_n)}] \leq (2/\mathrm{n})\ \mathbb{E}[|\mathrm{E(R_n)}|].$$

Proof An easy convexity argument shows that if $x, x_1, x_2, \ldots$ are positive integers such that each $x_i \leq x$ and $\sum_i x_i = n$ then $\sum_i \binom{x_i}{2} \leq \frac{1}{2}n(x-1)$. For if $n = ax + y$ where $0 \leq y \leq x - 1$ then

$$\sum_i \binom{x_i}{2} \leq a\binom{x}{2} + \binom{y}{2} \leq a\binom{x}{2} + \frac{y(x-1)}{2} = \frac{1}{2}n(x-1).$$

Hence if $G \in \mathcal{A}_n$ has maximum component order x and thus $\mathrm{miss(G) = n - x}$, then the number of edges between components is at least

$$\binom{n}{2} - \frac{1}{2}n(x-1) = \frac{1}{2}n(n-x) = \frac{1}{2}n \ \mathrm{miss(G)}.$$

For each graph $G \in \mathcal{A}$ let $\mathrm{add}(G)$ be the number of edges $e \notin G$ such that $G + e \in \mathcal{A}$. By counting the pairs $(G, G+e)$ such that $G + e \in \mathcal{A}_n$ we see that

$$\sum_{G \in \mathcal{A}_n} \mathrm{add}(G) = \sum_{G \in \mathcal{A}_n} |E(G)|,$$

and so $\mathbb{E}[\mathrm{add}(R_n)] = \mathbb{E}[|E(R_n)|]$. Hence

$$\frac{1}{2}n\mathbb{E}[\mathrm{miss(R_n)}] \leq \mathbb{E}[\mathrm{add(R_n)}] = \mathbb{E}[|\mathrm{E(R_n)}|],$$

and the lemma follows. □

The next lemma follows immediately from the last lemma, since if $G \in \mathcal{G}_n^S$ (and $n \geq 2$) then G has at most $3n + 6g - 6$ edges.

Lemma 4.5. *Let the surface S have Euler genus g, and let $R_n \in_U \mathcal{G}_n^S$. Then*

$$\mathbb{E}[\mathrm{miss(R_n)}] \leq 6 + 12\mathrm{(g-1)/n}.$$

The above lemma gives part (c) of Theorem 2.5, and we may use it also to show that it is unlikely that there will be small non-planar components, which is needed for part (d).

Lemma 4.6. *For $R_n \in_U \mathcal{G}_n^S$, the probability that R_n has a non-planar component of order at most $\frac{n}{2}$ is $O(\frac{\ln n}{n})$.*

Proof Let g be the Euler genus of the surface S. Let $\mathcal{G}_n$ be the set of graphs in $\mathcal{G}_n^S$ such that there is a (giant) component of order $> n/2$. For positive integers $k \leq n/2$, let $\mathcal{B}_n^{(k)}$ be the set of graphs in $\mathcal{G}_n$ which have a non-planar component of order k. We claim that

$$|\mathcal{B}_n^{(k)}| \leq \frac{2g}{nk}|\mathcal{G}_n|. \tag{6}$$

To see this note that given a graph $G \in \mathcal{B}_n^{(k)}$ we can construct at least $kn/2$ graphs $G' \in \mathcal{G}_n$ by adding an edge between a non-planar component of order

k and the giant component. How often can a graph $G' \in \mathcal{G}_n$ be constructed? In the giant component of G' there must be a bridge e such that deleting e cuts off a set W of exactly k nodes where the induced subgraph on W is nonplanar. Any two such sets W must be disjoint, and so there can be at most g such sets W. Thus G' can be constructed at most g times. The claim (6) follows.

By Lemma 4.5 we have $\mathbb{E}[\mathrm{miss}(\mathrm{R_n})] \leq 7$ for n sufficiently large, and then $\mathbb{P}[R_n \notin \mathcal{G}_n] \leq 14/n$. Thus by (6) the probability that R_n has a non-planar component of order at most $\frac{n}{2}$ is at most

$$\sum_{k=1}^{\lfloor n/2 \rfloor} \frac{2g}{nk} + \mathbb{P}[R_n \notin \mathcal{G}_n] = O(\frac{\ln n}{n}).$$

□

In order to complete the proof of part (d) of Theorem 2.5 we need one more lemma, which shows for example that if S is any surface other than the sphere then $|\mathcal{G}_n^S|$ is much larger than $|\mathcal{P}_n|$.

Lemma 4.7. *If S is simpler than S' then*

$$|\mathcal{G}_n^{S'}| = \Omega(n) \cdot |\mathcal{G}_n^S|.$$

Proof Let $\mathcal{B}_n$ be the set of graphs $G \in \mathcal{G}_n^S$ such that Miss(G) consists of 5 isolated nodes. For $R_n \in_U \mathcal{G}_n^S$, let $\delta = \liminf_{n\to\infty} \mathbb{P}[R_n \in \mathcal{B}_n]$. Then by Theorem 2.5 (b) we have $\delta > 0$. Thus $|\mathcal{B}_n| \geq (\delta + o(1))|\mathcal{G}_n^S|$.

From each graph $G \in \mathcal{B}_n$ we can construct at least $n - 5$ graphs G' by forming a complete graph K_5 on the 5 isolated nodes, letting the root vertex be the smallest of these vertices, and adding an edge between the root vertex and the rest of the graph, thus building an appearance of K_5. Note that each graph G' constructed is in $\mathcal{G}_n^{S'}$.

How often can a graph $G' \in \mathcal{G}_n^{S'}$ be constructed? If S' has Euler genus g' then G' can have at most g' appearances of K_5, so G' can be constructed at most g' times. Hence

$$|\mathcal{G}_n^{S'}| \geq (n-5)|\mathcal{B}_n|/g' = \Omega(n) \cdot |\mathcal{G}_n^S|,$$

as required. □

Results for maps might suggest that the 'right' bound above is not $\Omega(n)$ but $\Omega(n^{1+\delta})$ where δ is $\frac{5}{4}$ or $\frac{3}{2}$. Also, adding a handle should lead to a factor of order $n^{2+2\delta}$?

Proof of Theorem 2.5 part (d) Observe first that the probability that $\mathrm{Miss(R_n)}$ is non-planar is at most the probability that R_n has a non-planar component with at most $n/2$ vertices, which is $O(\ln n/n)$ by Lemma 4.6.

Let S' be any surface simpler than S. Then the probability that $\mathrm{Big(R_n)}$ is embeddable in S' and $\mathrm{Miss(R_n)}$ is planar is at most the probability that R_n is embeddable in S', which is $O(1/n)$ by Lemma 4.7. Hence the probability that $\mathrm{Big(R_n)}$ is embeddable in S' is $O(\ln n/n)$. □

5. Proofs for results assuming smoothness

We start with a general lemma taken from Lemma 5.3 of [19] and its proof, see also the discussion in the last section of [20]. Let the non-empty classes $\mathcal{A}$ and $\mathcal{B}$ of graphs be such that, given any two disjoint graphs G and H with H in $\mathcal{B}$, the union of G and H is in $\mathcal{A}$ if and only if G is in $\mathcal{A}$. (Clearly this holds if $\mathcal{A}$ is $\mathcal{G}^S$ for some surface S and $\mathcal{B}$ is $\mathcal{P}$.) Let $r_n = na_{n-1}/a_n$. Recall that given a graph H we let $v(H) = |V(H)|$; let $\mathrm{aut}(H)$ be the number of automorphisms of H; let $\lambda(H) = \rho^{v(H)}/\mathrm{aut}(H)$; and let $X_n(H)$ be the number of components isomorphic to H in the random graph $R_n \in_u \mathcal{A}_n$.

Lemma 5.1. *Let $H_1, \ldots, H_m$ be a fixed collection of pairwise non-isomophic connected graphs in $\mathcal{B}$. Let $k_1, \cdots, k_m$ be non-negative integers, and let $K = \sum_{i=1}^m k_i v(H_i)$. Then for $R_n \in_u \mathcal{A}_n$,*

$$\mathbb{E}[\prod_{i=1}^m X_n(H_i)_{k_i}] = \prod_{i=1}^m \lambda(H_i)^{k_i} \prod_{j=1}^K (r_{n-j+1}/\rho).$$

Proof Let $v_i = v(H_i)$ for $i = 1, \ldots, m$; and let $a_n = |\mathcal{A}_n|$. We may construct a graph G in $\mathcal{A}_n$ with at least k_i components isomorphic to H_i as follows: choose the vertices of the different components, then insert appropriate copies of H_i on the vertices of each component; and finally choose any graph H of order $n-K$ in $\mathcal{A}$ on the remaining $n-K$ vertices. The number of such constructions is

$$\prod_{i=1}^m \prod_{j=1}^{k_i} \left(\binom{n - \sum_{s=1}^{i-1} k_s v_s - (j-1)v_i}{v_i} \cdot \frac{v_i!}{aut(H_i)} \right) \cdot a_{n-K}.$$

How often is a specific $G \in \mathcal{A}_n$ constructed? This depends on the number of components of G that are isomorphic to some H_i. If G contains exactly t_i components isomorphic to H_i for each i, then R_n is constructed exactly $\prod_{i=1}^m (t_i)_{k_i}$ times. Denote by $a(n; t_1, \ldots, t_m)$ the number of graphs in $\mathcal{A}_n$ with exactly t_i components isomorphic to H_i. Then the definition of the expectation implies:

$$
\begin{aligned}
&\mathbb{E}[\prod_{i=1}^{m} X_n(H_i)_{k_i}] \\
= & \sum_{t_1,\dots,t_m\geq 0} \prod_{i=1}^{m} (t_i)_{k_i} \frac{a(n;t_1,\dots,t_m)}{a_n} \\
= & \prod_{i=1}^{m}\prod_{j=1}^{k_i} \left(\binom{n-\sum_{s=1}^{i-1} k_s v_s - (j-1)v_i}{v_i} \cdot \frac{v_i!}{aut(H_i)} \right) \cdot \frac{a_{n-K}}{a_n} \\
= & \prod_{i=1}^{m} aut(H_i)^{-k_i} \cdot \prod_{j=1}^{K} (n-j+1) \frac{a_{n-j}}{a_{n-j+1}} \\
= & \prod_{i=1}^{m} \lambda(H_i)^{k_i} \cdot \prod_{j=1}^{K} (r_{n-j+1}/\rho).
\end{aligned}
$$

□

Proof of Theorem 3.1 Since $r_n \to \infty$ as $n \to \infty$, by the last lemma

$$\mathbb{E}[\prod_{i=1}^{m} X_n(H_i)_{k_i}] \to \prod_{i=1}^{m} \lambda(H_i)^{k_i}$$

as $n \to \infty$, for all non-negative integers $k_1, \dots, k_m$. A standard result on the Poisson distribution now shows that the joint distribution of the random variables $X_n(H_1), \dots, X_n(H_m)$ tends to that of independent random variables $\mathrm{Po}(\lambda(H_1)), \dots, \mathrm{Po}(\lambda(H_m))$, see for example Lemma 5.4 of [19] or see [17]. Finally note that, since $0 \leq X_n(H) \leq \kappa(R_n) \leq Y$ in distribution, where $Y \sim 1 + \mathrm{Po}(1)$ and so Y has finite jth moment, convergence in total variation distance and for the jth moment follow from convergence in distribution. □

Proof of Theorem 3.2 We prove part (b). Let X_n be the number of components of $\mathrm{Miss(R_n)}$ in $\mathcal{D}$. We consider convergence in distribution first. Let k be a fixed positive integer and let $\epsilon > 0$. We want to show that for n sufficiently large we have

$$|\mathbb{P}(X_n = k) - \mathbb{P}(\mathrm{Po}(\lambda) = k)| < \epsilon. \tag{7}$$

By Lemma 4.4 there is an n_0 such that

$$\mathbb{P}[\mathrm{miss(R_n) > n_0}] < \epsilon/3. \tag{8}$$

List the unlabelled graphs in $\mathcal{D}$ in non-decreasing order of the number of nodes as $H_1, H_2, \ldots$. For each positive integer m let $\lambda^{(m)} = \sum_{i=1}^{m} \lambda(H_i)$. Let $n_1 \geq n_0$ be sufficiently large that, if m is the largest index such that H_m has at most n_1 nodes, then

$$|\mathbb{P}[\mathrm{Po}(\lambda) = k] - \mathbb{P}[\mathrm{Po}(\lambda^{(m)}) = k]| < \epsilon/3. \tag{9}$$

Let $X_n^{(m)}$ denote the number of components of R_n isomorphic to one of $H_1, \ldots, H_m$, that is, with order at most n_1. Let $n > 2n_1$. Then

$$|\mathbb{P}[X_n = k] - \mathbb{P}[X_n^{(m)} = k]| \leq \mathbb{P}[\mathrm{miss(R_n) > n_1}] < \epsilon/3. \tag{10}$$

But by Theorem 3.1, for n sufficiently large,

$$|\mathbb{P}[X_n^{(m)} = k] - \mathbb{P}[\mathrm{Po}(\lambda^{(m)}) = k]| < \epsilon/3,$$

and then by (9) and (10) the inequality (7) follows.

Finally note that the convergence in total variation distance and for any moment follow as in the proof of Theorem 3.1. □

Proof of Theorem 3.3 We have already seen in Theorem 2.5 that whp $\mathrm{Miss(R_n)}$ is planar. Let $a_n = |\mathcal{G}_n^S|$ and let c_n be the number of connected graphs in $\mathcal{G}_n^S$. By Theorem 3.2,

$$c_n/a_n \to 1/A(\rho) = e^{-C(\rho)} \text{ as } n \to \infty. \tag{11}$$

Given a graph G on a finite subset V of the positive integers let $\phi(G)$ be the natural copy of G moved down on to $\{1, \ldots, |V|\}$; that is, let $\phi(G)$ be the graph on $\{1, \ldots, |V|\}$ such that the increasing bijection between V and $\{1, \ldots, |V|\}$ is an isomorphism between G and $\phi(G)$.

Let H be any planar graph on $\{1, \ldots, h\}$. Then

$$\begin{aligned}
\mathbb{P}[\phi(\mathrm{Miss(R_n)}) = \mathrm{H}] &= \binom{n}{h} \frac{c_{n-h}}{a_n} \\
&= \frac{c_{n-h}}{a_{n-h}} \frac{1}{h!} \frac{(n)_h a_{n-h}}{a_n} \\
&= \frac{c_{n-h}}{a_{n-h}} \frac{1}{h!} \prod_{i=0}^{h-1} r_{n-i} \\
&\to e^{-C(\rho)} \frac{\rho^h}{h!}
\end{aligned}$$

as $n \to \infty$ by (11) and the assumption of smoothness. Now by symmetry

$$\mathbb{P}[\mathrm{Miss(R_n)} \cong \mathrm{H}] = \frac{\mathrm{h!}}{\mathrm{aut(H)}} \mathbb{P}[\phi(\mathrm{Miss(R_n)}) = \mathrm{H}]$$

and hence as $n \to \infty$

$$\mathrm{\mathbb{P}[Miss(R_n) \cong H] \to e^{-C(\rho)} \frac{\rho^h}{aut(H)} = p_M(H).}$$

Observe that

$$\sum_{H \in \mathcal{UP}} p_M(H) = \frac{1}{A(\rho)} \sum_{n \geq 0} \sum_{H \in \mathcal{UP}_n} \frac{\rho^n}{\mathrm{aut}(H)} = \frac{1}{A(\rho)} \sum_{n \geq 0} \sum_{G \in \mathcal{P}_n} \frac{\rho^n}{n!} = 1,$$

so that we do indeed have a distribution. Note that we are including the empty graph $\emptyset$ with $p_M(\emptyset) = \frac{1}{A(\rho)}$. Further, as $n \to \infty$

$$\mathrm{\mathbb{P}[miss(R_n) = h] = \sum_{H \in \mathcal{G}^S_h} \mathbb{P}[\phi(Miss(R_n)) = H] \to e^{-C(\rho)} \frac{\rho^h}{h!} a_h = q_m(h).}$$

Since $\mathrm{\mathbb{E}[miss(R_n)] = O(1)}$ by part (c) of Theorem 2.5, convergence in total variation follows from convergence in distribution. By definition, the miss distribution has probability generating function

$$G(x) = \sum_{h \geq 0} q_m(h) x^h = e^{-C(\rho)} \sum_{h \geq 0} \frac{\rho^h x^h}{h!} = e^{-C(\rho)} A(\rho x)$$

and since $A(x) = e^{C(x)}$ we have

$$G(x) = A(\rho x)/A(\rho) = e^{C(\rho x) - C(\rho)}.$$

From the probability generating function $G(x)$ we may obtain the moments of the miss distribution: the mean is $\rho C'(\rho)$ and the variance is $\rho^2 C''(\rho) + \rho C'(\rho)$. From equation (4.5) in [16] we see that $\rho C'(\rho)$ equals the radius of convergence R of the exponential generating function for 2-connected planar graphs, which equals 0.03819 to four significant figures. Also from that same equation, $\rho^2 C''(\rho) = 2C_4$ where $2C_4 = -R - F_2$ and $F_2 = R^2/(2B_4 - R)$, and from the value for B_4 in the appendix of [16] $\rho^2 C''(\rho) + \rho C'(\rho) = 0.03979$ to four significant figures. □

Next we restrict our attention to the planar case, and consider the moments of $\mathrm{miss(R_n)}$.

Proposition 5.2. *Consider the planar case, and let $R_n \in_U \mathcal{P}_n$. For $k = 0, 1, \ldots$ let*

$$p_k = e^{-C(\rho)} \frac{a_k \rho^k}{k!}.$$

For any $\epsilon > 0$ there is an n_0 and $\delta > 0$ such that for all $n \geq n_0$ and all $0 \leq k \leq \delta n$ we have

$$(1 - \epsilon) p_k \leq \mathrm{\mathbb{P}[miss(R_n) = k] \leq (1 + \epsilon) p_k}. \tag{12}$$

Also,

$$\mathbb{P}[\mathrm{miss(R_n)} \geq \delta \mathrm{n}] = \mathrm{O}(\mathrm{n}^{-5/2}).$$

Let the random variable X have the miss distribution. Then as $n \to \infty$, $\mathbb{E}[\mathrm{miss(R_n)^t}] \to \mathbb{E}[\mathrm{X^t}] < \infty$ for $0 \leq t < \frac{5}{2}$, and $\mathbb{E}[\mathrm{miss(R_n)^t}] \to \infty$ for $t \geq \frac{5}{2}$.

Note that we already know from Theorem 3.3 that $\mathrm{miss(R_n)}$ tends to X in total variation distance.

Proof We may prove (12) by arguing as in the proof of the last theorem. Let $\epsilon > 0$. Let $\delta > 0$ be such that $(1-\delta)^{-7/2} \leq 1 - \epsilon/2$. For $n \to \infty$, uniformly over $0 \leq k \leq n/2$ we have

$$\begin{aligned}
\mathbb{P}[\mathrm{miss(R_n) = k}] &= \binom{n}{k} \frac{a_k c_{n-k}}{a_n} \\
&= \frac{c_{n-k}}{a_{n-k}} \, \frac{a_k}{k!} \, \frac{a_{n-k}/(n-k)!}{a_n/n!} \\
&= (1+o(1)) \, (\frac{n}{n-k})^{7/2} \; p_k.
\end{aligned}$$

But for $0 \leq k \leq \delta n$ the term $(\frac{n}{n-k})^{7/2}$ is at least $1 + o(1)$ and at most $1 + \epsilon/2 + o(1)$, and the result (12) follows.

For larger k we shall be less precise. First note that, in much the same way as above, we may show that there is a constant c_0 such that for all n and all $k \leq n/2$ the probability that some union of components of R_n has order k is at most $c_0 k^{-7/2}$. Thus the probability that some union of components of R_n has order k such that $\delta n/2 \leq k \leq n/2$ is $O(n^{-5/2})$.

Now we need a result on graphs. We claim that, given a graph $G = (V, E)$ of order n and with $\mathrm{miss(G) = m}$, there is a union of components which has order k for some k with $m/2 \leq k \leq n/2$. Let $s = n - m$, so that s is the largest order of a component. Note that there are at least $s - 1$ integers in $\lceil \frac{n-s+1}{2} \rceil, .., \lfloor \frac{n+s-1}{2} \rfloor$. Thus by adding components one at a time we see that there is a union of components, with vertex set W say, such that $|W|$ is in this set. Then $|W| \geq \lceil \frac{n-s+1}{2} \rceil \geq m/2$, and $n - |W| \geq n - \lfloor \frac{n+s-1}{2} \rfloor \geq m/2$. Thus W or $V \setminus W$ is as required.

It follows that if $\mathrm{miss(G)} \geq \delta \mathrm{n}$ then there is a union of components with order k such that $\delta n/2 \leq k \leq n/2$. Hence by the earlier bound,

$$\mathbb{P}(\mathrm{miss(R_n)} \geq \delta \mathrm{n}) = \mathrm{O}(\mathrm{n}^{-5/2}) \tag{13}$$

as required.

Now consider expected values. Since $p_k \sim g k^{-7/2}$ as $k \to \infty$, $\mathbb{E}[X^t] < \infty$ for $0 \leq t < 5/2$ and $\mathbb{E}[X^t] = \infty$ for $t \geq 5/2$.

First let $0 \leq t < 5/2$. By (12) and (13), for $n \geq n_0$

$$\mathrm{\mathbb{E}[miss(R_n)^t] \leq (1+\epsilon) \sum_{k=1}^{\lfloor \delta n \rfloor} k^t p_k + n^t \mathbb{P}[miss(R_n) \geq \delta n] \leq (1+\epsilon)\mathbb{E}[X^t] + o(1)},$$

and

$$\mathrm{\mathbb{E}[miss(R_n)^t] \geq (1-\epsilon) \sum_{k=1}^{\lfloor \delta n \rfloor} k^t p_k \geq (1-\epsilon)\mathbb{E}[X^t] + o(1)};$$

and so $\mathrm{\mathbb{E}[miss(R_n)^t] \to \mathbb{E}[X^t]}$ as $n \to \infty$. For $t \geq 5/2$, as above we find that for $n \geq n_0$

$$\mathrm{\mathbb{E}[miss(R_n)^t] \geq (1-\epsilon) \sum_{k=1}^{\lfloor \delta n \rfloor} k^t p_k \to \infty \text{ as } n \to \infty}.$$

□

The following result essentially appears in [2]: we give a full proof here for completeness.

Lemma 5.3. *Let $\mathcal{U}_n$ be a set of unlabelled n-node graphs, and let $\mathcal{A}_n$ be the set of graphs on nodes $1, \ldots, n$ which are isomorphic to some graph in $\mathcal{U}_n$. Let $U_n \in_U \mathcal{U}_n$ and $R_n \in_U \mathcal{A}_n$. Then aut$(R_n) \leq_s$ aut(U_n).*

Proof Let $m = |\mathcal{U}_n|$ and $a_n = |\mathcal{A}_n|$. List the graphs in $\mathcal{U}_n$ as $H_1, \ldots, H_m$ where $\text{aut}(H_1) \leq \cdots \leq \text{aut}(H_m)$. Let $p_i = \frac{n!}{\text{aut}(H_i) a_n}$. Let $t \geq 0$ and let $x_i = 1$ if $\text{aut}(H_i) \geq t$ and $= 0$ otherwise. Then $p_1 \geq \cdots \geq p_m$ and $x_1 \leq \cdots \leq x_m$, and so

$$0 \geq \sum_i \sum_j (p_i - p_j)(x_i - x_j) = 2m \sum_i p_i x_i - 2 \sum_i x_i$$

yielding the standard inequality

$$\sum_i p_i x_i \leq \frac{1}{m} \sum_i x_i.$$

But now

$$\mathbb{P}[\text{aut}(R_n) \geq t] = \sum_i p_i x_i \leq \frac{1}{m} \sum_i x_i = \mathbb{P}[\text{aut}(U_n) \geq t].$$

□

Proof of Theorem 3.4 We have

$$\mathbb{E}[X_n(H)] = \binom{n}{h}(n-h)|\mathcal{G}^S_{n-h}|/|\mathcal{G}^S_n| \sim n\rho^h/h!,$$

and similarly

$$\mathbb{E}[X_n(H)(X_n(H)-1)] = \binom{n}{h}\binom{n-h}{h}(n-2h)^2|\mathcal{G}^S_{n-2h}|/|\mathcal{G}^S_n| \sim (n\rho^h/h!)^2.$$

The result now follows by Chebyshev's inequality. □

6. Concluding remarks

We have seen that for each surface S, the labelled graphs embeddable in S have the same growth constant as for the planar case, and the same holds for unlabelled graphs. The same proof idea also works for two other cases of interest (as will be spelled out elsewhere), concerning 2-connected graphs embeddable in S and concerning graphs embeddable in S and with a given average degree.

We have found various properties of the random graph $R_n \in_U \mathcal{G}^S_n$, but many questions are left open. For example, for $R_n \in_U \mathcal{P}_n$, the expected value of the maximum order of a block is $\Omega(n)$, and whp there is at most one block of order $n^{\frac{2}{5}}\omega(n)$ for any function $\omega(n) \to \infty$ as $n \to \infty$ (see [**?**]). Are there similar results for any surface S?

Now suppose that S is not the sphere. Is R_n usually 4-colourable? We know that whp just the one component $\mathrm{Big(R_n)}$ is non-planar. What is the least order of a non-planar subgraph? Is it true that whp just one block is non-planar? How far is R_n from being planar: in particular, how large is the minimum face-width of R_n over all embeddings in S (see [3] concerning the face-width of maps)? If T denotes the torus and K the Klein bottle, the orientable and non-orientable surfaces of Euler genus 2 respectively, how do $|\mathcal{G}^T_n|$ and $|\mathcal{G}^K_n|$ compare?

When we made the assumption that the class $\mathcal{G}^S$ was smooth we obtained more refined results concerning the random graph R_n. It is plausible that even more may be true, and that some result like the precise asymptotic counting formula (1) holds for $\mathcal{G}^S$.

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