

# ON THE STABILITY OF PERIODIC ORBITS FOR DIFFERENTIAL SYSTEMS IN $\mathbb{R}^n$

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ABSTRACT. We consider an autonomous differential system in  $\mathbb{R}^n$  with a periodic orbit and we give a new method for computing the characteristic multipliers associated to it. Our method works when the periodic orbit is given by the transversal intersection of  $n - 1$  codimension one hypersurfaces and is an alternative to the use of the first order variational equations. We apply it to study the stability of the periodic orbits in several examples, including a periodic solution found by Steklov studying the rigid body dynamics.

## 1. INTRODUCTION AND STATEMENT OF THE RESULTS.

Consider a differential system in  $\mathbb{R}^n$ , with  $n \geq 2$ , given by:

$$\frac{d\mathbf{x}}{dt} = \mathbf{X}(\mathbf{x}), \quad (1)$$

where  $\mathbf{X} : \mathcal{U} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a  $C^1$  function in some non-null open set  $\mathcal{U} \subseteq \mathbb{R}^n$  and  $t$  is a real independent variable. We assume that system (1) exhibits a periodic orbit  $\Gamma := \{\gamma(t) \mid 0 \leq t < T\} \subset \mathcal{U}$  with period  $T > 0$ . As usual we will denote by  $\phi(t, p)$  the flow solution of (1) such that  $\phi(0, p) = p$ , by  $x_i$  the  $i^{\text{th}}$  component of the point  $\mathbf{x}$ , that is,  $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$  and analogously  $\mathbf{X} = (X_1, X_2, \dots, X_n)^T$ , where  $T$  denotes transposition.

It is well known that to determine the behavior of the flow near  $\Gamma$ , a first step is to get the characteristic multipliers associated to this orbit. These multipliers are usually obtained through the study of the first order variational equations. In this paper we propose an alternative way for obtaining

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them which works when  $\Gamma$  is given by the transversal intersection of  $n - 1$  codimension one hypersurfaces.

Recall that the behavior near  $\Gamma$  is given by the Poincaré map, which is defined in a section  $\Sigma$  through a point  $p \in \Gamma$ , where  $\Sigma$  passes through  $p$  and is a local smooth manifold of dimension  $n - 1$  transversal to  $\Gamma$ . Given a section  $\Sigma$ , the *Poincaré map* is defined as:

$$\begin{aligned} \Pi: \Sigma &\longrightarrow \Sigma \\ q &\longmapsto \phi(\tau(q), q), \end{aligned}$$

where  $\tau(q)$  is the unique real function such that  $\phi(\tau(q), q) \in \Sigma$  and  $\lim_{q \rightarrow p} \tau(q) = T$ .

The  $n - 1$  eigenvalues of  $D\Pi(p)$  are independent of  $p$  and  $\Sigma$  and are called the *characteristic multipliers* of  $\Gamma$ . It is well known that the stable (respectively, unstable) manifold associated to  $\Gamma$  has dimension the number of characteristic multipliers with modulus smaller than 1 (respectively, bigger than 1). It is also well known, see [8], that if all the characteristic multipliers have modulus lower than or equal to 1, then  $\Gamma$  is Liapunov stable and if all the characteristic multipliers have modulus strictly lower than 1, then it is asymptotically stable.

Our result is motivated from a previous result given in [5] for planar differential systems. Recall that in the planar case there is only one characteristic multiplier which is given by:

$$\Pi'(p) = \exp \left\{ \int_0^T \operatorname{div}(\mathbf{X})(\gamma(t)) dt \right\},$$

where  $\operatorname{div}(\mathbf{X}) = \partial X_1 / \partial x_1 + \partial X_2 / \partial x_2$  is the *divergence* of the system, see for instance [10, p. 214]. In [5], the authors give an alternative expression for the same value  $\Pi'(p)$ . Let us consider an *invariant curve*  $f(x_1, x_2) = 0$  for a planar system (1), that is a curve defined by a real  $\mathcal{C}^1(\mathcal{U})$ -function  $f(x_1, x_2)$ , for which there exists a function  $k(x_1, x_2) : \mathcal{U} \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  of class  $\mathcal{C}^1$  satisfying that:

$$Df(\mathbf{x}) \mathbf{X}(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{X}(\mathbf{x})^\top = k(\mathbf{x}) f(\mathbf{x}),$$

where  $\nabla f = (\partial f / \partial x_1, \partial f / \partial x_2)$ . Under these assumptions, the function  $k(x_1, x_2)$  is called the *cofactor* associated to the invariant curve  $f(x_1, x_2) = 0$ .

**Proposition 1.** [5] *Consider system (1) with  $n = 2$  and a periodic orbit  $\Gamma = \{\gamma(t) \mid 0 \leq t < T\}$  contained in a planar invariant curve  $f(x_1, x_2) = 0$ .*

If  $\nabla f$  does not vanish on  $\Gamma$ , then

$$\Pi'(p) = \exp \left\{ \int_0^T k(\gamma(t)) dt \right\}.$$

The above result allows to compute the characteristic multiplier associated to  $\Gamma$  through the integration over the cofactor associated to an invariant curve containing the periodic orbit. In Theorem 2, we extend this result to periodic orbits in  $\mathbb{R}^n$  given as the transversal intersection of  $n - 1$  codimension one hypersurfaces.

**Theorem 2.** *Let  $\Gamma = \{\gamma(t) \mid 0 \leq t < T\}$  be a  $T$ -periodic solution of (1). Consider a smooth function  $\mathbf{f} : \mathcal{U} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ ,  $\mathbf{f} = (f_1, f_2, \dots, f_{n-1})^T$ , such that:*

- $\Gamma$  is contained in  $\bigcap_{i=1,2,\dots,n-1} \{f_i(\mathbf{x}) = 0\}$ ,
- the crossings of all the manifolds  $\{f_i(\mathbf{x}) = 0\}$  for  $i = 1, 2, \dots, n - 1$  are transversal over  $\Gamma$ ,
- there exists a  $(n - 1) \times (n - 1)$  matrix  $\mathbf{k}(\mathbf{x})$  of real functions satisfying:

$$D\mathbf{f}(\mathbf{x}) \mathbf{X}(\mathbf{x}) = \mathbf{k}(\mathbf{x}) \mathbf{f}(\mathbf{x}). \quad (2)$$

Let  $\mathbf{v}(t)$  be the  $(n - 1) \times (n - 1)$  fundamental matrix solution of

$$\frac{d\mathbf{v}(t)}{dt} = \mathbf{k}(\gamma(t)) \mathbf{v}(t), \quad \mathbf{v}(0) = \text{Id}. \quad (3)$$

Then the characteristic multipliers of  $\Gamma$  are the eigenvalues of  $\mathbf{v}(T)$ .

Indeed in the proof of this theorem, we will deduce that the matrices  $\mathbf{v}(T)$  and  $D\Pi(p)$  are conjugated. On the other hand, recall that the usual way of computing the characteristic multipliers of  $\Gamma$  lies in finding the fundamental matrix  $\mathbf{u}(T)$ , solution of the variational equation:

$$\frac{d\mathbf{u}(t)}{dt} = D\mathbf{X}(\gamma(t)) \mathbf{u}(t), \quad \mathbf{u}(0) = \text{Id}, \quad (4)$$

which is a  $n \times n$  matrix having eigenvalues 1 and the  $n - 1$  characteristic multipliers of  $\Gamma$ .

Note that the assumptions stated in our theorem are straightforward generalizations of the notion of invariant curve containing the periodic orbit  $\Gamma$  for a system in dimension  $n$  and the integration of the cofactor, which in

$\mathbb{R}^n$  consists on solving the linear differential system  $\mathbf{v}'(t) = \mathbf{k}(\gamma(t)) \mathbf{v}(t)$ .

Next section is devoted to prove the above theorem. Finally, in Section 3 we apply it to study the stability of the periodic orbits of several differential equations. Example 1 is devoted to study a 4-dimensional polynomial system, that includes two systems studied in [4], exhibiting an explicit periodic orbit. Example 2 shows a 3-dimensional polynomial system for which we prove that the stability of the given periodic orbit is equivalent to the study of the stability of the Mathieu's equation. The last example deals with a more involved case, a 6-dimensional system that controls the dynamics of a rigid body. In our approach, and by using the existence of three independent first integrals, we reduce the study of the stability of the Steklov periodic orbit to the study of a second order linear differential equation. This Steklov orbit is introduced in [11]. It is worth to say that in all the above examples we have also tried to use the usual approach, namely the variational equations, for studying the stability of the given periodic orbit and we have found that the result of Theorem 2 makes the computations easier.

## 2. PROOF OF THEOREM 2

Fix a point  $p \in \Gamma$ . To prove the result we consider the Poincaré section given by the  $(n - 1)$ -dimensional orthogonal hyperplane to  $\mathbf{X}(p)$ ,

$$\Sigma = \langle \mathbf{X}(p) \rangle^\perp = \langle \mathbf{e}_1, \dots, \mathbf{e}_{n-1} \rangle.$$

Take a new system of coordinates centered at  $p$  with basis  $\mathbf{e}_1, \dots, \mathbf{e}_{n-1}, \mathbf{X}(p)$ . These new coordinates write as  $\mathbf{y} = A(\mathbf{x} - p)$ , for some invertible matrix  $A$ . By using them the differential equation (1) is converted into

$$\frac{d\mathbf{y}}{dt} = \mathbf{Y}(\mathbf{y}) := A\mathbf{X}(A^{-1}\mathbf{y} + p), \quad (5)$$

and the manifold containing  $\Gamma$  is given by  $\mathbf{g}(\mathbf{y}) := \mathbf{f}(A^{-1}\mathbf{y} + p) = \mathbf{0}$ , which satisfies

$$D\mathbf{g}(\mathbf{y}) \mathbf{Y}(\mathbf{y}) = \tilde{\mathbf{k}}(\mathbf{y}) \mathbf{g}(\mathbf{y}), \quad (6)$$

where  $\tilde{\mathbf{k}}(\mathbf{y}) := \mathbf{k}(A^{-1}\mathbf{y} + p)$ , because by using (2), the following equality holds:

$$D\mathbf{f}(A^{-1}\mathbf{y} + p) A^{-1} A \mathbf{X}(A^{-1}\mathbf{y} + p) = \mathbf{k}(A^{-1}\mathbf{y} + p) \mathbf{f}(A^{-1}\mathbf{y} + p).$$

Recall that  $\phi(t, p)$  is the solution of (1) such that when  $t = 0$  passes through  $p$ . Let  $\psi(t, q)$  be the solution of (5) such that when  $t = 0$  passes through  $q$ . Then  $\psi(t, q) = A[\phi(t, A^{-1}q + p) - p]$ .

In these new coordinates note that  $\Sigma = \{y_n = 0\}$  and if  $q = (y_1, y_2, \dots, y_{n-1}, 0)^T \in \Sigma$  then the Poincaré map  $\Pi : \Sigma \rightarrow \Sigma$  writes as

$$\Pi(q) = (\psi_1(\tau(q), q), \psi_2(\tau(q), q), \dots, \psi_{n-1}(\tau(q), q), 0)^T,$$

where  $\tau(q)$  is precisely the time such that  $\psi_n((\tau(q), q)) = 0$ , which is known to be a smooth function. Indeed we can identify  $\Pi$  with a map  $\tilde{\Pi}$  from  $\mathbb{R}^{n-1}$  into itself with variables  $y_1, \dots, y_{n-1}$ , defined as

$$\tilde{\Pi}(y_1, \dots, y_{n-1}) = (\Pi_1(y_1, \dots, y_{n-1}, 0), \dots, \Pi_{n-1}(y_1, \dots, y_{n-1}, 0))^T.$$

Observe also that the hypotheses on the transversal cutting of  $\Gamma$  and the hypersurfaces  $f_i(\mathbf{x}) = 0$  imply that extended map,  $\hat{\mathbf{f}} = (f_1, f_2, \dots, f_{n-1}, f_n)$ , where  $f_n(\mathbf{x}) := X_1(p)x_1 + X_2(p)x_2 + \dots + X_n(p)x_n$  is such that  $\det(D\hat{\mathbf{f}}(p)) \neq 0$ . This information translated to the function  $\mathbf{g}(\mathbf{y})$  implies that the matrix  $D\mathbf{g}(0)$  has rank  $n - 1$  and that a minor with determinant different from zero is the one corresponding to the partial derivatives with respect to  $y_1, y_2, \dots, y_{n-1}$ .

On the other hand, by using (6) we have that:

$$\begin{aligned} \frac{\partial \mathbf{g}(\psi(t, q))}{\partial t} &= D\mathbf{g}(\psi(t, q)) \frac{\partial (\psi(t, q))}{\partial t} = D\mathbf{g}(\psi(t, q)) \mathbf{Y}(\psi(t, q)) \\ &= \tilde{\mathbf{k}}(\psi(t, q)) \mathbf{g}(\psi(t, q)), \end{aligned}$$

for any point  $q$  in the domain of definition of (5).

Let  $\mathbf{v}(t; q)$  be the fundamental matrix solution of

$$\frac{d\mathbf{v}(t)}{dt} = \tilde{\mathbf{k}}(\psi(t, q)) \mathbf{v}(t). \quad (7)$$

Thus  $\mathbf{g}(\psi(t, q)) = \mathbf{v}(t; q) \mathbf{g}(q)$ , because the function  $\mathbf{g}(\psi(t, q))$  is a solution of the same linear differential system and satisfies that  $\mathbf{g}(\psi(0, q)) = \mathbf{g}(q)$ .

Consider a point  $q \in \Sigma$ . In these coordinates, we have  $q = (q_1, q_2, \dots, q_{n-1}, 0)^T$ . We define  $z = (q_1, q_2, \dots, q_{n-1})^T$  and  $h(z) = (z^T, 0)^T = q$  like the inclusion of  $z$  as a point in the domain of definition of (5). Since  $\Pi(q) = \psi(\tau(q), q)$ , by the above result we have that  $\mathbf{g}(\Pi(q)) = \mathbf{v}(\tau(q), q) \mathbf{g}(z)$ , or equivalently that

$$\mathbf{g}(h(\tilde{\Pi}(z))) = \mathbf{v}(\tau(h(z)); h(z)) \mathbf{g}(h(z)).$$

We differentiate the previous identity with respect to  $z$ :

$$\begin{aligned} D\mathbf{g}(h(\tilde{\Pi}(z))) Dh(\tilde{\Pi}(z)) D\tilde{\Pi}(z) &= D[\mathbf{v}(\tau(h(z)); h(z))] \mathbf{g}(h(z)) + \\ &+ \mathbf{v}(\tau(h(z)); h(z)) D\mathbf{g}(h(z)) Dh(z). \end{aligned}$$

By evaluating at  $z = \mathbf{0}$ , which corresponds exactly to the point  $p \in \Gamma$ , and using that  $\mathbf{g}(h(\mathbf{0})) = \mathbf{0}$ ,  $h(\mathbf{0}) = (\mathbf{0}^T, 0)^T$ ,  $\tau(\mathbf{0}) = T$ ,  $\tilde{\Pi}(\mathbf{0}) = \mathbf{0}$  and the expression of  $h$ , we obtain that:

$$D_z \mathbf{g}(\mathbf{0}) D\tilde{\Pi}(\mathbf{0}) = \mathbf{v}(T; h(\mathbf{0})) D_z \mathbf{g}(\mathbf{0}),$$

where  $D_z \mathbf{g}(\mathbf{0}) := D\mathbf{g}(h(\mathbf{0})) Dh(\mathbf{0})$  is precisely the invertible squared matrix formed by the derivative of  $\mathbf{g}$  only with respect to  $q_1, \dots, q_{n-1}$ . Hence we have that the characteristic multipliers associated to  $\Gamma$  coincide with the eigenvalues of  $\mathbf{v}(T; \mathbf{0})$  because the matrices  $D\tilde{\Pi}(\mathbf{0})$  and  $\mathbf{v}(T; h(\mathbf{0}))$  are similar.

Finally note that  $\tilde{\mathbf{k}}(\psi(t, h(\mathbf{0}))) = \mathbf{k}(A^{-1}\psi(t, h(\mathbf{0})) + p) = \mathbf{k}(\phi(t, p))$ . Thus equation (7) is the same that

$$\frac{d\mathbf{v}(t)}{dt} = \mathbf{k}(\gamma(t)) \mathbf{v}(t),$$

and  $\mathbf{v}(T; h(\mathbf{0}))$  coincides with the matrix  $\mathbf{v}(T)$  as defined in (3), as we wanted to prove.

### 3. SOME EXAMPLES AND APPLICATIONS.

**Example 1.** The following example is an extension of two systems extracted from [4]. The goal of the examples given in [4] is to illustrate that the asymptotic stability of a periodic orbit of a system (1) is not determined by the eigenvalues of the matrix defining the first variational equation.

We consider the following differential system in  $\mathbb{R}^4$ :

$$\begin{aligned} \dot{x} &= -y - x(x^2 + y^2 - 1), \\ \dot{y} &= x - y(x^2 + y^2 - 1), \\ \dot{z} &= -w - sz(z^2 + w^2 - 1) - sk(x - z), \\ \dot{w} &= z - sw(z^2 + w^2 - 1) - sk(y - w), \end{aligned} \tag{8}$$

with  $s, k$  real parameters. This system coincides with the first example given in [4] when  $s = 1$  and with the second example when  $s = -1$ . We note that this system always exhibits the periodic orbit  $\Gamma = \{\gamma(t) : 0 \leq t < 2\pi\}$ , where  $\gamma(t) = (\cos(t), \sin(t), \cos(t), \sin(t))$ . We are able to compute all the characteristic multipliers associated to  $\Gamma$  for any real value of the parameters  $s$  and  $k$ . We consider the hypersurfaces  $f_i(x, y, z, w) = 0$ ,  $i = 1, 2, 3$ , given by:  $f_1(x, y, z, w) = x^2 + y^2 - 1$ ,  $f_2(x, y, z, w) = x - z$  and  $f_3(x, y, z, w) = y - w$ . We denote by  $\mathbf{f} = (f_1, f_2, f_3)$ . It is easy to see that  $\Gamma$  is contained in the intersection of these three hypersurfaces and that the crossings of these hypersurfaces are normal over the periodic orbit  $\Gamma$ , as the computation of

the following determinant shows:

$$\begin{vmatrix} \nabla f_1(\gamma(t)) \\ \nabla f_2(\gamma(t)) \\ \nabla f_3(\gamma(t)) \\ \mathbf{X}(\gamma(t)) \end{vmatrix} = \begin{vmatrix} 2 \cos(t) & 2 \sin(t) & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -\sin(t) & \cos(t) & -\sin(t) & \cos(t) \end{vmatrix} = 4 \neq 0.$$

Straightforward computations show that  $D\mathbf{f}(\mathbf{x})\mathbf{X}(\mathbf{x}) = \mathbf{k}(\mathbf{x})\mathbf{f}(\mathbf{x})$ , with the following matrix of cofactors:

$$\mathbf{k}(x, y, z, w) := \begin{pmatrix} -2(x^2 + y^2) & 0 & 0 \\ sz - x & sk - sz(x + z) & -1 - sz(y + w) \\ sw - y & 1 - sw(x + z) & sk - sw(y + w) \end{pmatrix},$$

which evaluated on the periodic orbit  $\Gamma$  reads for:

$$\mathbf{k}(\gamma(t)) = \begin{pmatrix} -2 & 0 & 0 \\ (s-1)\cos(t) & s(k-2\cos^2(t)) & -1-2s\cos(t)\sin(t) \\ (s-1)\sin(t) & 1-2s\cos(t)\sin(t) & s(k-2\sin^2(t)) \end{pmatrix}.$$

The fundamental matrix solution of the linear equation  $\mathbf{v}'(t) = \mathbf{k}(\gamma(t))\mathbf{v}(t)$  is:

- when  $2 - 2s + sk \neq 0$ ,

$$\mathbf{v}(t) = \begin{pmatrix} e^{-2t} & 0 & 0 \\ (s-1)e^{-2t} \left( \frac{e^{(2-2s+ks)t} - 1}{2-2s+ks} \right) \cos(t) & e^{(k-2)st} \cos(t) & -e^{kst} \sin(t) \\ (s-1)e^{-2t} \left( \frac{e^{(2-2s+ks)t} - 1}{2-2s+ks} \right) \sin(t) & e^{(k-2)st} \sin(t) & e^{kst} \cos(t) \end{pmatrix},$$

- when  $2 - 2s + sk = 0$ , we put  $k = 2(s-1)/s$ :

$$\mathbf{v}(t) = \begin{pmatrix} e^{-2t} & 0 & 0 \\ (s-1)e^{-2t} t \cos(t) & e^{-2t} \cos(t) & -e^{2(s-1)t} \sin(t) \\ (s-1)e^{-2t} t \sin(t) & e^{-2t} \sin(t) & e^{2(s-1)t} \cos(t) \end{pmatrix}.$$

We can compute the eigenvalues of the matrix  $\mathbf{v}(2\pi)$ , which by Theorem 2, correspond to the characteristic multipliers associated to  $\Gamma$ . These eigenvalues are:

- when  $2 - 2s + sk \neq 0$ :  $e^{-4\pi}$ ,  $e^{2s(k-2)\pi}$  and  $e^{2ks\pi}$ ,
- when  $k = 2(s-1)/s$ :  $e^{-4\pi}$ ,  $e^{-4\pi}$  and  $e^{4\pi(s-1)}$ .

Therefore, for instance in the first case ( $2 - 2s + sk \neq 0$ ) when  $sk > 0$  or  $s(k-2) > 0$  we have that  $\Gamma$  is unstable and, in the second case ( $2 - 2s + sk = 0$ ) when  $s < 1$  then  $\Gamma$  is Liapunov unstable.

**Example 2.** We give an example related to the Mathieu's equation. The Mathieu's equation is a particular case of the Hill's equation and it has the form:

$$v''(t) + (a + 2q \cos(2t))v(t) = 0,$$

where  $a, q$  are real parameters. See, for instance, the book [6, pp. 121–131] for further information about the Hill's equation.

We consider the following differential system in  $\mathbb{R}^3$ :

$$\begin{aligned} \dot{x} &= -y + z x/2, \\ \dot{y} &= x + z y/2, \\ \dot{z} &= (-2q(x^2 - y^2) - a)(x^2 + y^2 - 1) + z^2, \end{aligned} \tag{9}$$

where  $a, q \in \mathbb{R}$ . This system has the  $2\pi$ -periodic orbit  $\Gamma := \{\gamma(t) \mid 0 \leq t < 2\pi\}$  with  $\gamma(t) = (\cos t, \sin t, 0)$ .

We consider the surfaces given by  $f_i(x, y, z) = 0$ ,  $i = 1, 2$ , with  $\mathbf{f}(x, y, z) = (x^2 + y^2 - 1, z)^T$ . Their intersection gives the periodic orbit  $\Gamma$  and the crossings over it are transversal, as the following computation shows:

$$\begin{vmatrix} \nabla f_1(\gamma(t)) \\ \nabla f_2(\gamma(t)) \\ \mathbf{X}(\gamma(t)) \end{vmatrix} = \begin{vmatrix} 2 \cos(t) & 2 \sin(t) & 0 \\ 0 & 0 & 1 \\ -\sin(t) & \cos(t) & 0 \end{vmatrix} = -2 \neq 0.$$

We have that  $D\mathbf{f}(\mathbf{x})\mathbf{X}(\mathbf{x}) = \mathbf{k}(\mathbf{x})\mathbf{f}(\mathbf{x})$ , with the following matrix of cofactors:

$$\mathbf{k}(x, y, z) := \begin{pmatrix} 0 & x^2 + y^2 \\ -2q(x^2 - y^2) - a & z \end{pmatrix}.$$

Therefore, the cofactor matrix over the periodic orbit reads for:

$$\mathbf{k}(\gamma(t)) = \begin{pmatrix} 0 & 1 \\ M(t) & 0 \end{pmatrix},$$

where we denote  $M(t) := -2q \cos(2t) - a$ .

By Theorem 2, the stability of  $\Gamma$  is given by the eigenvalues of  $\mathbf{v}(2\pi)$ , where  $\mathbf{v}(t)$  is the fundamental matrix solution of

$$\begin{pmatrix} v'_{11}(t) & v'_{12}(t) \\ v'_{21}(t) & v'_{22}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ M(t) & 0 \end{pmatrix} \begin{pmatrix} v_{11}(t) & v_{12}(t) \\ v_{21}(t) & v_{22}(t) \end{pmatrix}.$$

Hence,



- $v''_{11}(t) = M(t)v_{11}(t)$  which is the Mathieu's equation with initial conditions  $v_{11}(0) = 1$  and  $v'_{11}(0) = 0$ , and
- $v''_{12}(t) = M(t)v_{12}(t)$  which is the Mathieu's equation with initial conditions  $v_{12}(0) = 0$  and  $v'_{12}(0) = 1$ .

We have that the system  $\mathbf{v}'(t) = \mathbf{k}(\gamma(t))\mathbf{v}(t)$  coincides with the characteristic system associated to the Mathieu's equation. Then the stability of the  $2\pi$ -periodic orbit  $\Gamma := \{\gamma(t) \mid 0 \leq t < 2\pi\}$  with  $\gamma(t) = (\cos t, \sin t, 0)$  of system (9) coincides with the stability of the Mathieu's equation, which is studied in [3] and [6, pp. 128–130].

**Example 3.** Our third example consists on the study of a periodic solution related to rigid body dynamics encountered by Steklov [11]. We consider the motion of a rigid body around a fixed point in a uniform gravity field. We denote the weight of the body by  $W$  and the distance between the center of gravity and the fixed point by  $\ell$ . As described in the work [9], we can consider two frames of reference both with the origin at the moving body. The first frame  $OXYZ$  is fixed and has axis  $OZ$  vertical and upward directed. The second frame of reference is moving solidary with the body and its axes  $Ox$ ,  $Oy$  and  $Oz$  are directed along the major axes of inertia for the point  $O$ , with corresponding moments of inertia denoted by  $a$ ,  $b$  and  $c$ . We denote by  $p$ ,  $q$  and  $r$  the components of the angular velocity vector of the movement and by  $\gamma_1$ ,  $\gamma_2$  and  $\gamma_3$ , the components of the unit vector in the direction  $OZ$  written in coordinates  $Oxyz$ . Steklov considered the case in which the center of mass is located on the major axis of inertia, which we assume to be the  $Ox$ -axis. The Euler–Poisson equations describe the motion of the rigid body:

$$\begin{aligned} \dot{p} &= \frac{(b-c)}{a}qr, & \dot{q} &= \frac{(c-a)}{b}pr + \frac{W\ell}{b}\gamma_3, & \dot{r} &= \frac{(a-b)}{c}pq - \frac{W\ell}{c}\gamma_2, \\ \dot{\gamma}_1 &= r\gamma_2 - q\gamma_3, & \dot{\gamma}_2 &= p\gamma_3 - r\gamma_1, & \dot{\gamma}_3 &= q\gamma_1 - p\gamma_2. \end{aligned} \quad (10)$$

We are going to assume, without loss of generality and following [9], that:

$$b > c, \quad a + b > c, \quad b + c > a, \quad c + a > b, \quad b > a > 2c. \quad (11)$$

The general Euler–Poisson equations exhibit three first integrals, whose expressions are described in [2]. We are going to rewrite them for the particular case (10). The first one corresponds to the projection of the angular momentum onto the vertical:

$$H_1 := ap\gamma_1 + bq\gamma_2 + cr\gamma_3.$$

The second first integral that we encounter is the geometric property of the vector  $(\gamma_1, \gamma_2, \gamma_3)$  to be of constant modulus:

$$H_2 := \gamma_1^2 + \gamma_2^2 + \gamma_3^2 - 1.$$

We also find the full energy (sum of kinetic and potential energies) of the body as first integral:

$$H_3 := \frac{1}{2}(ap^2 + bq^2 + cr^2) + W\ell\gamma_1 + \frac{(a^2 - 2ab - 2ac + 2bc)W\ell}{2(b-a)(a-c)}.$$

Straightforward computations show that, if we denote by  $\mathbf{X}$  the vector defined by system (10), we have  $\nabla H_i \cdot \mathbf{X}^T \equiv 0$ , for  $i = 1, 2, 3$ .

As described in [9], Steklov in [11] looked for real particular solutions of system (10) satisfying the two relations  $\gamma_2 = \beta_2 pq$  and  $\gamma_3 = \beta_3 rp$ , with  $\beta_2$  and  $\beta_3$  suitable constants, and he found the following particular periodic solution for system (10). Let us define the constant  $\mu := \sqrt{W\ell/a}$ , the dimensionless “time”  $\nu := \mu(t + t_0)$ , with  $t_0$  an arbitrary constant, and the following constants and variable:

$$\beta_0 := \frac{a(a-2c)}{(b-a)(a-c)}, \quad \beta_1 := \frac{a(2b-a)}{(b-a)(a-c)},$$

$$k^2 := \frac{b-a}{b-c}, \quad z := \frac{1}{k} \sqrt{\frac{a}{a-c}} \nu.$$

Then, the periodic orbit found by Steklov can be written as:

$$\begin{aligned} p(t) &:= -\mu \sqrt{\frac{\beta_0(2b-a)}{(a-c)}} \operatorname{cn}(z; k), & \gamma_1(t) &:= 1 - \frac{a}{a-c} \operatorname{cn}^2(z; k), \\ q(t) &:= \mu \sqrt{\frac{\beta_0 a}{(b-c)}} \operatorname{sn}(z; k), & \gamma_2(t) &:= \sqrt{\beta_1 k} \operatorname{sn}(z; k) \operatorname{cn}(z; k), \\ r(t) &:= \mu \sqrt{\frac{\beta_1 a}{(a-c)}} \operatorname{dn}(z; k), & \gamma_3(t) &:= -\sqrt{\frac{\beta_0(b-a)}{(a-c)}} \operatorname{cn}(z; k) \operatorname{dn}(z; k), \end{aligned} \tag{12}$$

where  $\operatorname{cn}(z; k)$ ,  $\operatorname{sn}(z; k)$  and  $\operatorname{dn}(z; k)$  are elliptic Jacobi functions of variable  $z$  and module  $k$ .

We recall that the module  $k$  must always satisfy  $0 \leq k \leq 1$  and that these elliptic Jacobi functions are defined as:  $\operatorname{cn}(z; k) = \cos(\operatorname{am}(z; k))$ ,  $\operatorname{sn}(z; k) = \sin(\operatorname{am}(z; k))$  and  $\operatorname{dn}(z; k) = \sqrt{1 - k^2 \sin^2(\operatorname{am}(z; k))}$ , where  $\operatorname{am}(z; k)$  is the Jacobi amplitude defined as the inverse function of the elliptic integral of first kind  $F(w; k)$ , that is,  $\operatorname{am}(z; k) = w$  if and only if  $F(w; k) = z$ .

As stated in [9], the minimal positive period  $T$  of this Steklov solution with respect to the time  $t$  is:

$$T = 4kK(k)\sqrt{\frac{a-c}{W\ell}},$$

where  $K(k)$  is a complete elliptic integral of the first kind. See [1, 12] for further information about Jacobi elliptic functions and integrals. We only recall that:

$$F(z; k) := \int_0^z \frac{d\theta}{\sqrt{1-k^2\sin^2(\theta)}}, \quad K(k) := F\left(\frac{\pi}{2}; k\right),$$

and

$$\operatorname{cn}^2(z; k) + \operatorname{sn}^2(z; k) = 1, \quad \operatorname{dn}^2(z; k) + k^2 \operatorname{sn}^2(z; k) = 1. \quad (13)$$

Our goal is to study the characteristic multipliers associated to the periodic orbit given by Steklov. Since the system has three functionally independent first integrals, it is well known that the  $6 \times 6$  monodromy matrix computed from the variational equations associated to the periodic orbit needs to have at least 4 eigenvalues equal to 1. Several works target to the aim of obtaining the remaining two characteristic multipliers following the classical first variational analysis. In [2], a numerical study of this particular periodic solution is done and a visualization of it is provided. In the work [9], the problem of orbital stability of the Steklov solution is numerically examined. In [7], it is proved that if  $a > 1$ , then this Steklov periodic orbit is orbitally unstable. In this paper we will see how our method allows to reduce the computation of these characteristic multipliers to the study of a linear second order differential equation.

We consider the following five hypersurfaces which contain the periodic orbit given by Steklov:

$$\begin{aligned} f_1 &:= \frac{1}{2}(ap^2 + bq^2 + cr^2) + W\ell\gamma_1 + \frac{(a^2 - 2ab - 2ac + 2bc)W\ell}{2(b-a)(a-c)} = 0, \\ f_2 &:= \gamma_2 - \frac{(a-b)(a-c)}{W\ell(a-2c)}pq = 0, \\ f_3 &:= \gamma_3 - \frac{(a-b)(a-c)}{W\ell(a-2b)}pr = 0, \\ f_4 &:= \frac{(a-b)}{(a-2c)}p^2 + \frac{(b-c)}{a}r^2 - \frac{W\ell(a-2b)}{(a-b)(a-c)} = 0, \\ f_5 &:= -\frac{(a-c)}{(a-2b)}p^2 + \frac{(b-c)}{a}q^2 + \frac{W\ell(a-2c)}{(a-b)(a-c)} = 0. \end{aligned}$$

The first hypersurface corresponds to one of the described first integrals. The second and third hypersurfaces correspond to the ones looked for by Steklov and the last two hypersurfaces also contain the periodic orbit and are independent from the previous ones. Note that only the first one of the above hypersurfaces is invariant by the flow of (10). When we substitute the expressions of the parameterization (12), and using (13), we get that each  $f_i$  values identically zero for any  $t$ .

The expressions of the other two first integrals in relation with the five polynomials  $f_i$  are:

$$\begin{aligned}
H_1 &= \frac{ap}{W\ell} f_1 + bq f_2 + cr f_3 + \frac{ac(a(a-2c) + 2bc)p}{2(a-2b)(b-c)W\ell} f_4 + \\
&+ \frac{ab(a(a-2b) + 2bc)p}{2(a-2c)(b-c)W\ell} f_5, \\
H_2 &= \left( \frac{2\gamma_1}{W\ell} - \frac{f_1}{W^2\ell^2} \right) f_1 + \left( \frac{2\delta pq}{(a-2c)} + f_2 \right) f_2 + \left( \frac{2\delta pr}{(a-2b)} + f_3 \right) f_3 + \\
&+ \left( \frac{-ac}{(b-c)W\ell} + \frac{a\delta(a^3 - a^2b - 2a^2c + abc + 2ac^2 - 2bc^2)p^2}{(a-2b)^2(a-2c)(b-c)W\ell} \right) + \\
&+ \left( \frac{a^2c^2}{4(b-c)^2W^2\ell^2} f_4 + \frac{ac(4ab - 3ac - 2bc + 2c^2)}{4(b-c)^2W^2\ell^2} f_5 \right) f_4 + \\
&+ \left( \frac{-ab}{(b-c)W\ell} + \frac{a\delta(a^3 - 2a^2b + 2ab^2 - a^2c + abc - 2b^2c)p^2}{(a-2b)(a-2c)^2(b-c)W\ell} \right) + \\
&+ \left( \frac{a^2b^2}{4(b-c)^2W^2\ell^2} f_5 - \frac{ac(2ab - 3ac - 2bc + 2c^2)}{4(b-c)^2W^2\ell^2} f_4 \right) f_5,
\end{aligned} \tag{14}$$

where  $\delta = (a-b)(a-c)/(W\ell)$ . We note that over the periodic orbit, each one of these first integrals does not conform a unique leaf since its coefficients over the hypersurfaces  $f_i = 0$  are not constants. This assertion means that the intersection of any combination of four of the five hypersurfaces  $f_i = 0$ ,  $i = 1, 2, 3, 4, 5$ , and the set  $H_1 = 0$  (equivalently  $H_2 = 0$ ) contains more points than  $\Gamma$ . This is the reason why we do not directly use these first integrals in the computations. We have computed the previous expressions because they will lead us to show the existence of two characteristic multipliers equal to 1.

The crossings of the five hypersurfaces  $f_i = 0$ ,  $i = 1, 2, 3, 4, 5$  are normal over the periodic orbit because the value of the determinant of the matrix formed by the gradients of each of the  $f_i$  in the first five rows and the vector

field in the last row, all of them evaluated over the periodic orbit (12) is:

$$\begin{aligned} \det = & \frac{2(b-c)W^2\ell^2}{a(a-c)^2} \left( \frac{2a(a-2c)(a-c)}{(b-a)^2} + \frac{2a(a-c)^2}{(2b-a)(b-a)} p(t)^2 + \right. \\ & + \frac{2(2b-a)(a-c)(b-c)}{a(b-a)} q(t)^2 + \frac{2(2b+2c-3a)(a-c)^2(b-c)}{(2b-a)(b-a)W\ell} r(t)^2 + \\ & \left. + \frac{8(a-c)(b-c)W\ell}{a(2b-a)} \gamma_2(t)^2 \right), \end{aligned}$$

which is positive for any  $t$ . This assertion is true because the conditions (11) imply that all the coefficients are positive except the term  $(2b+2c-3a)$ , which can be positive, negative or zero. If it is positive or zero, we already have that  $\det > 0$ . If it is negative, we are going to show that:

$$\frac{2a(a-2c)(a-c)}{(b-a)^2} - \frac{2(3a-2b-2c)(a-c)^2(b-c)}{(2b-a)(b-a)W\ell} r(t)^2 > 0$$

which ensures that  $\det > 0$  for any value of  $t$ . We note that  $-1 \leq \operatorname{dn}(z; k) \leq 1$ , so we consider any value of  $t$  for which  $\operatorname{dn}(z; k)^2$  is equal to 1 and, using (12) and some computations, we can bound the previous expression by:

$$\begin{aligned} & \frac{2a(a-2c)(a-c)}{(b-a)^2} - \frac{2(3a-2b-2c)(a-c)^2(b-c)}{(2b-a)(b-a)W\ell} r(t)^2 \geq \\ & \geq \frac{2a(a-2c)(a-c)}{(b-a)^2} - \frac{2(3a-2b-2c)(a-c)^2(b-c)}{(2b-a)(b-a)W\ell} \frac{W\ell(2b-a)a}{(b-a)(a-c)^2} = \\ & = \frac{2a(2b-a)}{(b-a)} > 0. \end{aligned}$$

The matrix of cofactors  $\mathbf{k} = (k_{ij})$  associated to the five previous hypersurfaces, that is,

$$D\mathbf{f} \mathbf{X} = \begin{pmatrix} \nabla f_1 \\ \nabla f_2 \\ \nabla f_3 \\ \nabla f_4 \\ \nabla f_5 \end{pmatrix} \mathbf{X} = \mathbf{k} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix},$$

reads for:

$$\begin{aligned}
k_{1j} &:= 0 \quad \text{for } i = 1, 2, \dots, 5, \\
k_{21} &:= \frac{-r}{W\ell}, \quad k_{22} := 0, \quad k_{23} := -\frac{(a^2 - 2ab - ac + 3bc)p}{b(a-2c)}, \\
k_{24} &:= \frac{acr}{2(b-c)W\ell}, \\
k_{25} &:= -\frac{(a^2b - 2ab^2 - 2a^2c + 2abc + 2b^2c + 2ac^2 - 2bc^2)r}{2(a-2c)(b-c)W\ell}, \\
k_{31} &:= \frac{q}{W\ell}, \quad k_{32} := \frac{(a^2 - ab - 2ac + 3bc)p}{(a-2b)c}, \quad k_{33} := 0, \\
k_{34} &:= -\frac{(2a^2b - 2ab^2 - a^2c - 2abc + 2b^2c + 2ac^2 - 2bc^2)q}{2(a-2b)(b-c)W\ell}, \\
k_{35} &:= \frac{-abq}{2(b-c)W\ell}, \\
k_{41} &:= k_{43} := k_{44} := k_{45} := 0, \quad k_{42} := \frac{-2(b-c)W\ell r}{ac}, \\
k_{51} &:= k_{52} := k_{54} := k_{55} := 0, \quad k_{53} := \frac{2(b-c)W\ell q}{ab}.
\end{aligned}$$

The fundamental matrix of solutions, evaluated in  $T$ , of the linear differential system of equations

$$\mathbf{v}'(t) = \mathbf{k}(\gamma(t)) \mathbf{v}(t), \quad (15)$$

with,

$$\mathbf{k}(\gamma(t)) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ k_{21}(t) & 0 & k_{23}(t) & k_{24}(t) & k_{25}(t) \\ k_{31}(t) & k_{32}(t) & 0 & k_{34}(t) & k_{35}(t) \\ 0 & k_{42}(t) & 0 & 0 & 0 \\ 0 & 0 & k_{53}(t) & 0 & 0 \end{pmatrix}$$

the matrix of cofactors evaluated over the periodic orbit (12), gives the characteristic multipliers associated to the periodic orbit encountered by Steklov. We note that the first equation of this system is  $v_1'(t) = 0$  which corresponds to the fact that  $f_1$  is a first integral of the system and gives us that  $v_1(t)$  needs to be a constant. Analogously, the other two first integrals give us relations among the values of the solutions of system (15). In

particular, from (14),  $H_1$  gives us that the function:

$$h_1(t) := \frac{ap(t)}{W\ell} v_1(t) + bq(t)v_2(t) + cr(t)v_3(t) + \frac{ac(a(a-2c)+2bc)p(t)}{2(a-2b)(b-c)W\ell} v_4(t) + \frac{ab(a(a-2b)+2bc)p(t)}{2(a-2c)(b-c)W\ell} v_5(t),$$

satisfies  $h_1'(t) \equiv 0$  when considered over the solutions of system (15). And the first integral  $H_2$  gives the function:

$$h_2(t) := \left( \frac{2\gamma_1(t)}{W\ell} \right) v_1(t) + \left( \frac{2\delta p(t)q(t)}{(a-2c)} \right) v_2(t) + \left( \frac{2\delta p(t)r(t)}{(a-2b)} \right) v_3(t) + \left( \frac{-ac}{(b-c)W\ell} + \frac{a\delta(a^3 - a^2b - 2a^2c + abc + 2ac^2 - 2bc^2)p(t)^2}{(a-2b)^2(a-2c)(b-c)W\ell} \right) v_4(t) + \left( \frac{-ab}{(b-c)W\ell} + \frac{a\delta(a^3 - 2a^2b + 2ab^2 - a^2c + abc - 2b^2c)p(t)^2}{(a-2b)(a-2c)^2(b-c)W\ell} \right) v_5(t),$$

where  $\delta = (a-b)(a-c)/(W\ell)$ , which is also constant over the solutions of (15).

We obtain these constant functions over the solutions of (15) because each function  $H_i(\gamma(t))$ ,  $i = 1, 2$ , is 0 for any  $t$ . We can then derive  $H_i(\gamma(t))$  with respect to  $t$  and we deduce the functions  $h_i(t)$  taking into account that  $v_i(t)$  is related with  $\nabla f_i(\gamma(t))$ ,  $i = 1, 2, \dots, 5$ .

We first note that the following constants conform a solution of system (15):

$$v_1(t) = 1, \quad v_2(t) = 0, \quad v_3(t) = 0, \\ v_4(t) = \frac{2(a-2b)}{a^2 - 2ab - 2ac + 2bc}, \quad v_5(t) = \frac{-2(a-2c)}{a^2 - 2ab - 2ac + 2bc}.$$

The functions  $h_1(t)$  and  $h_2(t)$  take the constant values 0 and  $4\delta/(a^2 - 2ab - 2ac + 2bc)$ , with  $\delta = (a-b)(a-c)/(W\ell)$ , over this solution, respectively. Since this solution takes a nonzero value on  $v_1(t)$ , which needs to be constant, we may get the first column of the fundamental matrix solution of (15) from the last four columns and this particular constant solution. Hence, we only need to take into account the last four columns of the fundamental matrix solution of (15).

Let us now consider the fundamental matrix  $\mathbf{v}(t)$  solution of (15) whose initial condition is the identity matrix. We define  $v_{ij}(t)$  as the function corresponding to row  $i$  column  $j$  of this fundamental matrix of solutions.

Using that the three functions  $v_1(t)$ ,  $h_1(t)$  and  $h_2(t)$  are constants, we can obtain certain relations among the rows of this matrix at the value  $T$ .

We define the constants:

$$\rho_1 := \frac{2(2b-a)(b-c)W\ell}{\sqrt{a}(b-a)(a-2c)^{3/2}}, \quad \rho_2 := \frac{2\sqrt{a}(b-c)W\ell}{(b-a)\sqrt{a-2c}(a^2-2ab+2bc)},$$

and we get that  $\mathbf{v}(T)$  equals

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ v_{21}(T) & v_{22}(T) & v_{23}(T) & v_{24}(T) & v_{25}(T) \\ v_{31}(T) & v_{32}(T) & v_{33}(T) & \frac{v_{44}(T)-1}{\rho_1} & \frac{v_{55}(T)-1}{\rho_2} \\ \rho_1 v_{31}(T) & \rho_1 v_{32}(T) & \rho_1 (v_{33}(T)-1) & v_{44}(T) & \rho_1 \left( \frac{v_{55}(T)-1}{\rho_2} \right) \\ \rho_2 v_{31}(T) & \rho_2 v_{32}(T) & \rho_2 (v_{33}(T)-1) & \rho_2 \left( \frac{v_{44}(T)-1}{\rho_1} \right) & v_{55}(T) \end{pmatrix}.$$

The computation of the characteristic polynomial associated to this matrix gives:

$$\det(v(T) - \lambda I) = -(\lambda - 1)^3 (\lambda^2 + B\lambda + C),$$

where

$$B := 2 - v_{22}(T) - v_{33}(T) - v_{44}(T) - v_{55}(T),$$

and

$$C := v_{22}(T)(v_{33}(T) + v_{44}(T) + v_{55}(T) - 2) + v_{32}(T)(v_{23}(T) + \rho_1 v_{24}(T) + \rho_2 v_{25}(T)).$$

We note that each one of the first integrals gives an eigenvalue equal to one. Moreover, using Liouville's formula, we know that the product of all the eigenvalues of  $\mathbf{v}(T)$  is equal to one because the trace of matrix  $\mathbf{k}(\gamma(t))$  is identically zero and  $\mathbf{v}(t)$  is a fundamental solution of (15). Hence, we get that  $C = 1$ . Then, we have that the roots of  $\lambda^2 + B\lambda + 1 = 0$  are:

- if  $B^2 > 4$ , then the roots are both real, and one of modulus greater than 1 and the other with modulus lower than 1: so the Steklov solution is unstable,
- if  $B^2 \leq 4$ , then the roots are both of modulus equal to one and the characteristic multipliers do not decide if the Steklov solution is unstable or not.

Let us go back to the differential linear system (15). We know that  $v_1(t)$  is constant, and we denote its value by  $v_{10} := v_1(t)$ . Analogously, we denote by  $h_{10} := h_1(t)$  and  $h_{20} := h_2(t)$ , the values of the other two constants. We



can determine  $v_2(t)$  and  $v_3(t)$  from the last two equations of (15) and also from the two relations  $h_1(t) = h_{10}$  and  $h_2(t) = h_{20}$ . We equate the two expressions of  $v_2(t)$  and  $v_3(t)$  obtained in these two different ways and we get a linear differential system of equations for  $v_4(t)$  and  $v_5(t)$  which reads for:

$$\begin{aligned} & \frac{\partial \operatorname{dn}(z; k)}{\partial t} v_4'(t) - \frac{W\ell(2b-a)\operatorname{dn}(z; k)}{a(a-c)(b-c)} \left[ \left( c - \frac{ab-2ac+2bc}{2b-a} \operatorname{cn}(z; k)^2 \right) v_4(t) + \right. \\ & \quad \left. + \left( b - \frac{(2b-a)(ab-ac-bc)}{(a-2c)c} \operatorname{cn}(z; k)^2 \right) v_5(t) + \right. \\ & \quad \left. + \frac{2(b-c)\sqrt{W\ell(b-a)(a-2c)}}{ac\sqrt{2b-a}} \operatorname{cn}(z; k) h_{10} + \frac{2(b-c)}{c} v_{10} \operatorname{cn}(z; k)^2 + \right. \\ & \quad \left. + \frac{b-c}{a} (W\ell h_{20} - 2v_{10}) \right] = 0, \\ & \frac{\partial \operatorname{sn}(z; k)}{\partial t} v_5'(t) + \frac{W\ell(a-2c)\operatorname{sn}(z; k)}{a(b-a)(b-c)} \left[ \left( c - \frac{r_0(ab-ac+bc)}{(2b-a)} \operatorname{cn}(z; k)^2 \right) v_4(t) + \right. \\ & \quad \left. + \left( b + \frac{(b-a)(2ab-ac-2bc)}{(a-2c)(a-c)} \operatorname{cn}(z; k)^2 \right) v_5(t) + \right. \\ & \quad \left. + \frac{-2(b-c)\sqrt{W\ell(b-a)(2b-a)}}{ab\sqrt{a-2c}} \operatorname{cn}(z; k) h_{10} - \right. \\ & \quad \left. - \frac{2(b-a)(b-c)}{b(a-c)} \operatorname{cn}(z; k)^2 v_{10} + \frac{b-c}{a} (W\ell h_{20} - 2v_{10}) \right] = 0, \end{aligned}$$

where  $r_0 = (b-a)(a-2c)/(b(a-c))$ . Equating  $v_5(t)$  from the first equation and substituting its value in the second one, we get a second order linear differential equation for  $v_4(t)$  whose fundamental set of solutions would let us to the computation of the characteristic multipliers associated to the Steklov periodic orbit. We define

$$\omega(t) := b - \frac{(2b-a)\delta_1}{(a-2c)c} \operatorname{cn}(z; k)^2, \quad \delta_1 := ab - ac - bc$$

and this second order linear differential equation for  $v_4(t)$  is:

$$v_4''(t) - \left( \frac{\omega'(t)}{\omega(t)} + \frac{2}{\operatorname{dn}(z; k)} \frac{\partial \operatorname{dn}(z; k)}{\partial t} \right) v_4'(t) + A_0(t) v_4(t) + A_{nh}(t) = 0, \quad (16)$$

where

$$\frac{A_0(t)}{W\ell} :=$$

$$\begin{aligned}
&:= \frac{2(b-c)(a^2-4b(a-c))}{a(a-2c)(a-c)} \frac{\operatorname{dn}(z;k)^2}{\omega(t)} + \frac{4}{a} + \frac{a^2-2(\delta_1+2ac)}{bc(a-c)} \operatorname{cn}(z;k)^2, \\
A_{nh}(t) &:= \\
&:= \frac{-2(2b-a)^2}{a\delta(a-2c)c^2\omega(t)} \left( \alpha_3(t)h_{10} + \alpha_4(t)v_{10} + \alpha_5(t)(2v_{10}-W\ell h_{20}) \right), \\
&\text{with } \delta := (a-b)(a-c)/(W\ell) \text{ and} \\
\alpha_3(t) &:= \sqrt{\frac{W\ell(b-a)}{(2b-a)(a-2c)}} \left( \frac{(a-2c)(3ac(b+c-a)+5bc(b-c))}{(2b-a)} + \right. \\
&\quad \left. + 2ab(2c-a) + \frac{\delta_1((b-a)^2+\delta_1)}{b} \operatorname{cn}(z;k)^2 \right) \operatorname{cn}(z;k), \\
\alpha_4(t) &:= -a(a-2c) \left( \frac{bc}{2b-a} + (b-2c)\operatorname{cn}(z;k)^2 + \frac{(b-a)\delta_1}{b(a-2c)} \operatorname{cn}(z;k)^4 \right), \\
\alpha_5(t) &:= \frac{abc}{2} + \frac{c^2(a^2-ab-b^2-ac+bc)}{2b-a} - \frac{(a-2b+2c)\delta_1}{2} \operatorname{cn}(z;k)^2.
\end{aligned}$$

In short, we have reduced the problem of computing the characteristic multipliers for the periodic orbit (12) to the study of the second-order linear differential equation (16). The classical approach to this problem is to compute the characteristic multipliers via the first variational equations, thus involving a linear differential system of order 6. Our method starts with a linear differential system of order 5, see (15). In this particular problem, we know three first integrals which let us reduce the order by 3, getting the second-order linear differential equation (16). We note that the use of the first integrals is not trivial, since we need to write them in terms of the considered hypersurfaces, see (14), and then relate them to the variables of the linear differential system (15).

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