MULTIPLICITY OF LIMIT CYCLES AND ANALYTIC m-SOLUTIONS FOR PLANAR DIFFERENTIAL SYSTEMS

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ABSTRACT. This work deals with limit cycles of real planar analytic vector fields. It is well known that given any limit cycle Γ of an analytic vector field it always exists a real analytic function $f_0(x, y)$, defined in a neighbourhood of Γ, and such that Γ is contained in its zero level set. In this work we introduce the notion of $f_0(x, y)$ being an *m-solution*, which is a merely analytic concept. Our main result is that a limit cycle Γ is of multiplicity m if and only if $f_0(x, y)$ is an m-solution of the vector field. We apply it to study in some examples the stability and the bifurcation of periodic orbits from some non hyperbolic limit cycles.

1. Introduction and statement of the results

Consider a planar differential system of the form:

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

where $P(x, y)$ and $Q(x, y)$ are real analytic functions with isolated singularities and defined in some nonempty open set $\mathcal{U} \subseteq \mathbb{R}^2$. The main goal of this work is to exhibit the equivalence between the existence of a limit cycle of multiplicity m with the existence of a function, which we will call m-solution, defined and characterized below.

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Recall that a limit cycle for system (1) is an isolated periodic orbit in the set of all periodic orbits. We assume that system (1) has a limit cycle and we denote by $(x, y) = \gamma(t)$ the equations corresponding to this closed trajectory and by Γ the set of points in $\mathbb{R}^2$ which form it, that is, $\Gamma := \{\gamma(t) \,|\, 0 \leq t < T\}$, where $T > 0$ is the minimal period of the limit cycle. Γ is said to be hyperbolic, or of multiplicity one, if the first derivative of the Poincaré map associated to it is different from one. Otherwise, the limit cycle Γ is said to be of multiplicity m, with $m \geq 2$, if the first derivative of the Poincaré map is equal to 1, all its derivatives from order 2 to order $m-1$ are zero and its m^{th} derivative is different from zero. When the vector field is analytic we can always find such an integer m. We give a precise definition of the Poincaré map and the results related with this notion of multiplicity in Section 2.

In the sequel we give some preliminaries to introduce the notion of $m-$solution. Given a planar differential system (1) the explicit expression of its limit cycles is usually not known. Nevertheless, as we will recall in Lemma 4 of Section 2, for any given limit cycle Γ of system (1), it always exists a real analytic curve given by $f_0(x, y) = 0$ whose graphic has an oval and such oval is Γ. Since Γ is an orbit for system (1), the curve $f_0(x, y) = 0$ is indeed an invariant curve of the vector field. Recall that an *invariant curve* for system (1) is described by a real analytic function $f_0 : \mathcal{U} \to \mathbb{R}$, where $\mathcal{U} \subseteq \mathbb{R}^2$ is a non empty open set, such that there exists an analytic function $k_0 : \mathcal{U} \to \mathbb{R}$ satisfying the following equation:

$$P(x,y) \, \frac{\partial f_0}{\partial x}(x,y) \, + \, Q(x,y) \, \frac{\partial f_0}{\partial y}(x,y) \; = \; k_0(x,y) \, f_0(x,y). \tag{2}$$

The function $k_0(x, y)$ corresponds to the invariant curve $f_0(x, y) = 0$ and it is called its *cofactor*. As we will see, in our case the curve $f_0(x, y) = 0$ can always be chosen such that the vector ∇f_0 is different from zero in all the points of Γ.

Associated to invariant algebraic curves there is the notion of exponential factors, see [5]. We extend this concept to analytic invariant curves as follows: a *generalized exponential factor of order* d associated to the invariant curve $f_0(x, y) = 0$ is a function of the form: $F_d(x, y) = \exp\left\{g_d(x, y)/f_0(x, y)^d\right\}$ such that:

- d is a positive integer number, $d \geq 1$,
- $g_d : \mathcal{U} \to \mathbb{R}$ is an analytic function in the open set $\mathcal{U}$ of definition of $f_0(x, y)$,
- if $p \in \mathcal{U}$ is a point such that $f_0(p) = 0$ but $\nabla f_0(p) \neq 0$, then $g_d(p) \neq 0$,

- there exists an analytic function $k_d : \mathcal{U} \to \mathbb{R}$ such that the following identity is satisfied:

$$P(x,y)\frac{\partial F_d}{\partial x}(x,y) \;+\; Q(x,y)\frac{\partial F_d}{\partial y}(x,y) \;=\; k_d(x,y)\, F_d(x,y).$$

This function $k_d(x,y)$ is related to the generalized exponential factor $F_d(x,y)$ and it is called its *cofactor*. We note that the existence of such a cofactor satisfying the last identity is equivalent to the existence of such a function $k_d(x,y)$ satisfying:

$$P(x,y)\frac{\partial g_d}{\partial x}(x,y) + Q(x,y)\frac{\partial g_d}{\partial y}(x,y) \;=\; dk_0(x,y)\, g_d(x,y) + k_d(x,y) f_0(x,y)^d.$$

Classically, an exponential factor is given for *polynomial* systems (1) having invariant *algebraic* curves. Moreover, the function $g_d(x,y)$ of an exponential factor needs to be a polynomial. This notion is widely studied in the works [4, 5] and the references therein. One of the main results stated and proved in [5] is that a polynomial system (1) exhibits an exponential factor of order d associated to an invariant algebraic curve $f_0(x,y)=0$ when this curve is the result of the coalescence of $d+1$ invariant algebraic curves of nearby systems. As a consequence of the theorems given in our paper, we are going to recover several results given in previous works but taking into account analytic systems (1) and invariant curves and generalized exponential factors, which do not need to be algebraic.

Let us consider a system (1) with a limit cycle $\Gamma = \{\gamma(t) \,|\, 0 \le t < T\}$ and with an invariant curve $f_0(x,y)=0$ defined in a neighborhood $\mathcal{U}$ of Γ and with associated cofactor $k_0(x,y)$. Assume also that $\Gamma \subseteq \{(x,y) \in \mathcal{U} \,:\, f_0(x,y) = 0\}$. Given a positive integer $m \ge 1$, we say that $f_0(x,y)=0$ is an analytic *m-solution* of system (1) if there exist $m-1$ generalized exponential factors $F_1(x,y)$, $F_2(x,y)$, $F_3(x,y)$, $\ldots$, $F_{m-1}(x,y)$ of consecutive orders $d = 1,2,3\ldots,m-1$ and with associated cofactors $k_1(x,y)$, $k_2(x,y)$, $k_3(x,y)$, $\ldots$, $k_{m-1}(x,y)$, and such that:

$$\int_0^T k_j(\gamma(t))dt \;=\; 0 \text{ for } j = 0,1,2,\ldots,m-2 \quad\text{and}\quad \int_0^T k_{m-1}(\gamma(t))dt \neq 0.$$

In Section 3, we ensure that this is a good definition.

Our main result is:

Theorem 1. *Let Γ be a limit cycle of system* (1) *and let $f_0(x,y)=0$ be an analytic invariant curve of* (1) *defined in a neighbourhood U of Γ and such that $\Gamma \subseteq \{(x,y) \in \mathcal{U} \,:\, f_0(x,y)=0\}$. Then Γ has multiplicity m if and only if $f_0(x,y)=0$ is an analytic m-solution of* (1).

In our proof of the above theorem, we use a local set of coordinates associated to the limit cycle Γ, the so called curvilinear coordinates. However, the "only if" part of the thesis of this theorem, could also be proved without using them. When we want to show that the existence of $m-1$ generalized exponential factors of subsequent orders $1, 2, 3, \ldots, m-1$ associated to $f_0(x, y) = 0$ implies that Γ has multiplicity m, we can use a similar argument as the one given in [6]. In that work, the authors define the notion of infinitessimal multiplicity by means of the so-called *generalized invariant algebraic curve of order m*. However, only polynomial systems with invariant *algebraic* curves are considered there, so their definition should be extended to the analytic case to be used in our context. The advantage of their proof is that no change of coordinates is needed. We have not included their arguments since the part of our proof corresponding to this direction does not differ so much from theirs. On the other hand it seems no easy to extrapolate, by the arguments given in [6], the implication that Γ being of multiplicity m gives $m-1$ exponential factors.

We also remark that in the same work [6], five notions of multiplicity for invariant algebraic curves are defined and shown to be equivalent under several assumptions. In particular in that work all the invariant curves, and the functions involved in the exponential factors are algebraic, while as we will see in Section 4 the functions involved in our computations are not. Their work is related to ours when the curve $f_0(x, y) = 0$ is algebraic and satisfies all of the assumptions described in [6].

As an example of practical application of Theorem 1 we get the following result, proved in Section 4:

Proposition 2. *Consider system*

$$\begin{aligned}\dot{x} &= -y + (x^2 + y^2 - 1)\left(\alpha_0 x + \alpha_1 y^2 + \alpha_2 y + \alpha_3(1 - x^2 - 3y^2) + \alpha_4 y\right), \\ \dot{y} &= x + (x^2 + y^2 - 1)\left(-1/2 - \alpha_1 xy + \alpha_2(-1 - x + x^2 + y^2) + \right. \\ &\qquad \left. + 2\alpha_3 xy - \alpha_4 x\right),\end{aligned}$$

where $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ are real parameters. It has the unit circumference Γ as hyperbolic limit cycle if and only if $\alpha_0 \neq 0$. If $\alpha_0 = 0$ and $\alpha_1 \neq 0$, then Γ is a limit cycle of multiplicity 2. If $\alpha_0 = \alpha_1 = 0$ and $\alpha_2\,\alpha_3 \neq 0$, then Γ is a limit cycle of multiplicity 3. And, if $\alpha_0 = \alpha_1 = 0$ and $\alpha_2\,\alpha_3 = 0$, then Γ belongs to a continuum of periodic orbits. Furthermore when $|\alpha_0| \ll |\alpha_1| \ll |\alpha_2\,\alpha_3|$ and α_0, α_1 and $\alpha_2\,\alpha_3$ alternate signs, Γ is a hyperbolic limit cycle and two more hyperbolic limit cycles appear in a neighborhood of Γ.

Note that the above result can be interpreted as the solution of the "center-focus" problem for the periodic orbit Γ.

This paper is organized as follows. Section 2 contains a set of preliminary definitions and results related with the proof of Theorem 1, which is given in Section 3. The last section is formed by several applications of this result, including the previous example, as well as the computation of the Lyapunov constants for quadratic systems by using Theorem 1 and one example of analytic system having an analytic, non-algebraic, multiple limit cycle.

2. Preliminary definitions and results

In this section, we first summarize some classical definitions and notions related with limit cycles. Afterwards, we give some results concerning m-solutions of a system (1).

Consider a limit cycle $\Gamma = \{\gamma(t) : 0 \leq t < T\}$ of system (1). It is well known that Γ can be either stable, unstable or semi-stable, see for instance [10]. Its character is given by its associated *Poincaré map*. We consider a point $p_0 \in \Gamma$ and a *section* Σ through it. A section through a point is an arc of a curve containing the point, such that the considered vector field is not tangent to any point of the arc of the curve. Since Γ is a periodic orbit, for each point q of Σ, the solution of system (1) starting at q cuts Σ again in another point for some positive time. We denote by $\Pi(q)$ the point corresponding to the first intersection of the solution of system (1) starting in q with Σ. We notice that since Γ is a periodic orbit and $p_0 \in \Gamma$, we have that $\Pi(p_0) = p_0$. The function $\Pi : \Sigma \to \Sigma$ defined in this way is called the *Poincaré map* for Γ at p_0. It can be shown that $\Pi : \Sigma \to \Sigma$ is a diffeomorphism with the same regularity than system (1). Clearly, from its definition, Π controls the stability of Γ. Assume that Π is the identity, then Γ belongs to a continuous band of periodic orbits. Assume that $\Pi'(p_0) \neq 1$, then if $\Pi'(p_0) > 1$, Γ is an unstable limit cycle and if $\Pi'(p_0) < 1$, then Γ is a stable limit cycle. If $\Pi'(p_0) \neq 1$, we say that Γ is a *hyperbolic*, or of multiplicity 1, limit cycle. In case that $\Pi'(p_0) = 1$, but Π is not the identity, there exists an integer m, with $m > 1$, such that the m^{th} derivative of Π evaluated in p_0 is different from zero and m is the lowest value with this property. We say that Γ is a limit cycle of multiplicity m. In this case, if m is odd and $\Pi^{(m)}(p_0) > 0$ then Γ is an unstable limit cycle, and if $\Pi^{(m)}(p_0) < 0$ then Γ is a stable limit cycle. If m is even, then Γ is semi-stable. It is also well known that at most m limit cycles, taking into account their multiplicities, can bifurcate from a limit cycle of multiplicity m.

A classical known result is that $\Pi'(p_0) = \exp\left\{\int_0^T \mathrm{div}(\gamma(t))\,dt\right\}$, where as usual $\mathrm{div}(x,y) = \partial P/\partial x + \partial Q/\partial y$ is the divergence of system (1). As we will see in Lemma 4, for any periodic orbit Γ of system (1) there exists an invariant curve $f_0(x,y) = 0$ which contains it. If $k_0(x,y)$ is the cofactor of this curve it is proved in [8] that

$$\Pi'(p_0) = \int_0^T \mathrm{div}(\gamma(t))\,dt = \int_0^T k_0(\gamma(t))\,dt. \tag{3}$$

In the sequel we will see how to extend the idea of [8] to obtain more derivatives of the Poincaré map from the generalized cofactors associated to $f_0(x,y) = 0$.

With this aim we introduce the *curvilinear coordinates* (s,n) near a periodic orbit, as described in [10, p. 27] or in [1, pp. 110–118]. Let $\Gamma := \{\gamma(t)\,|\,0 \leq t < T\}$, where $\gamma(t)$ is a periodic orbit of system (1) with minimal period $T > 0$. Let us consider a neighborhood $\mathcal{U}$ of Γ sufficiently small so that there is no singular point of system (1) contained in $\mathcal{U}$. Let us fix a point p_0 in Γ and we denote by s the arc length of Γ at each of its points measured from p_0 and the direction of increasing s coincides with the direction of increasing t. We denote by n the length of the normal to Γ whose outward direction is taken positive if Γ is oriented clockwise or whose inward direction is taken positive if Γ is oriented counterclockwise. Suppose that the equations of Γ with s as parameter are $x = \varphi(s)$ and $y = \psi(s)$ and that the complete length of Γ is $L > 0$. It is clear that $\varphi(s)$ and $\psi(s)$ are L-periodic functions. With these definitions, the formulas connecting the rectangular coordinates (x,y) of a point in a neighborhood of Γ and its curvilinear coordinates (s,n) are: $x = \varphi(s) - n\,\psi'(s)$ and $y = \psi(s) + n\,\varphi'(s)$. It can be shown, provided that $|n|$ is sufficiently small, that if $\mathcal{U}$ is small enough, we always have the jacobian $|\partial(x,y)/\partial(s,n)| > 0$ on $\mathcal{U}$. Hence, the formulas $(x,y) = (\varphi(s) - n\,\psi'(s), \psi(s) + n\,\varphi'(s))$ represent a coordinate transformation preserving the orientation of Γ.

By applying the above change of coordinates $(x,y) \to (s,n)$ to system (1), we get an analytic, non-autonomous differential equation of the form:

$$\frac{dn}{ds} = F(s,n) = \sum_{j \geq 1} F_j(s)\,n^j, \tag{4}$$

defined for $|n|$ small enough. Since $\varphi(s)$ and $\psi(s)$ are L-periodic functions, we have that the functions $F_j(s)$ are also L-periodic functions in the variable s. Moreover, we have that $n = 0$ is a solution of (4), which corresponds to the prior closed periodic trajectory Γ of (1).

Our interest is to study the stability and multiplicity of the solution $n = 0$ for (4), which coincides with the stability and multiplicity of Γ as periodic orbit of (1). By using our notations the following theorem, see [10, Thm 2.5], relates the values of $\Psi_j(L)$ with the stability of Γ.

Theorem 3. *Let equation* (4) *be the expression of system* (1) *in the local curvilinear coordinates associated to a given periodic orbit* Γ. *Let*

$$\Psi(s;n_0) \;=\; \sum_{j\geq 1} \Psi_j(s)\, n_0^j. \tag{5}$$

be the flow of equation (4) *such that* $\Psi(0;n_0)=n_0$. *Then*

(i) Γ *is a hyperbolic limit cycle if and only if* $\Psi_1(L)\neq 1$,

(ii) Γ *is limit cycle of multiplicity exactly* $m\geq 2$ *if and only if* $\Psi_1(L)=1$ *and* $\Psi_j(L)=0$ *for* $j=2,3,\ldots,m-1$ *but* $\Psi_m(L)\neq 0$.

The values $\Psi_j(L)$ can be determined in a recursive way, although many computations are involved. To do so, we only need to impose that (5) is a solution of equation (4) and equate the same powers of n_0. We note that, in this way, we get a set of recursive linear differential equations for each $\Psi_j(s)$ whose coefficients involve $F_1(s), F_2(s), \ldots, F_j(s)$ and $\Psi_1(s), \Psi_2(s), \ldots,$ $\Psi_{j-1}(s)$. Each of the functions $\Psi_j(s)$ is uniquely determined from the initial condition $\Psi(0;n_0)=n_0$, which implies that $\Psi_1(0)=1$ and $\Psi_j(0)=0$ for $j>1$. In the first step of this recursion, we get $\Psi_1(s)$ as the following expression:

$$\Psi_1(s) \;=\; \exp\left\{\int_0^s F_1(\sigma)\,d\sigma\right\}. \tag{6}$$

Following the recursive method, we get:

$$\begin{aligned}
\Psi_2(s) &= \Psi_1(s)\left[\int_0^s \Psi_1(\sigma)\,F_2(\sigma)\,d\sigma\right],\\
\Psi_3(s) &= \Psi_1(s)\left[\left(\int_0^s \Psi_1(\sigma)\,F_2(\sigma)\,d\sigma\right)^2 + \int_0^s \Psi_1(\sigma)^2\,F_3(\sigma)\,d\sigma\right],
\end{aligned}$$

and the following values of $\Psi_j(s)$ can also be computed but their expressions are much more complicated. By undoing the change to curvilinear coordinates, we could display formulas for $\Psi_j(L)$ in terms of (x,y). For instance, it can be shown that:

$$\Psi_1(L) \;=\; \exp\left\{\int_0^T \operatorname{div}(\gamma(t))\,dt\right\}. \tag{7}$$

This is a way to prove that Γ is hyperbolic if, and only if, $\Psi_1(L)\neq 1$ or, equivalently, if, and only if, $\int_0^T \operatorname{div}(\gamma(t))\,dt \neq 0$.

Let us recall also how the use of the curvilinear coordinates ensures the existence of an invariant curve containing any periodic orbit Γ, whenever it exists. The following lemma is equivalent to Lemma 1 appearing in [1, p. 124]. We include here a proof for the sake of completeness.

Lemma 4. *If system* (1) *has a limit cycle* Γ*, then there exists an analytic invariant curve* $f_0(x,y)=0$ *for system* (1) *defined in a neighborhood* $\mathcal{U}$ *of* Γ *and such that* $\Gamma \subseteq \{(x,y) \in \mathcal{U} \,:\, f_0(x,y)=0\}$*. Moreover, the curve* $f_0(x,y)=0$ *can always be chosen such that the vector* ∇f_0 *is different from zero in all the points of* Γ*.*

Proof. In the curvilinear coordinates (s,n) defined above, the periodic orbit Γ corresponds to the equation $n=0$. The inverse of change of variables between coordinates (x,y) and (s,n) is analytic and can be written as $n = f_0(x,y)$, $s=g_0(x,y)$. Thus we have that $f_0(x,y)=0$ is an invariant curve of system (1) which contains the periodic orbit Γ.

Moreover, since by definition n is the length of the normal to Γ, we deduce that the vector ∇f_0 is different form zero in all the points of Γ. ∎

We consider a system (1) with a periodic orbit $\Gamma := \{\gamma(t) \,|\, 0 \leq t < T\}$ of minimal period $T>0$. By the above lemma, we know that there exists an invariant curve $f_0(x,y)=0$ such that $\Gamma \subseteq \{(x,y) \in \mathcal{U} \,|\, f_0(x,y)=0\}$, where $\mathcal{U}$ is a neighborhood of Γ. We denote by $k_0(x,y)$ the cofactor associated to the invariant curve $f_0(x,y)=0$. As we have already stated, see (3), in [8] it is proved that:

$$\int_0^T \mathrm{div}(\gamma(t))\, dt \;=\; \int_0^T k_0(\gamma(t))\, dt.$$

In this way we are able to determine the hyperbolicity of Γ by using two different integrands.

We remark that if we consider Γ in curvilinear coordinates (s,n), the associated invariant curve reads for $n=0$ and its cofactor in relation with equation (4) is

$$k_0(s,n) \;=\; \frac{F(s,n)}{n} \;=\; \sum_{j\geq 1} F_j(s)\, n^{j-1}.$$

The cofactor of $f_0(x,y)=0$ in cartesian coordinates (x,y) is just the transformation of this function by $(s,n) \to (x,y)$. By (6) we have that

$$\Psi_1(L) \;=\; \exp\left\{ \int_0^L F_1(s)\, ds \right\} \;=\; \exp\left\{ \int_0^L k_0(s,0)\, ds \right\}.$$

Therefore, undoing the change of variables to cartesian coordinates and parameterizing again by t instead than by s, we deduce that:

$$\Psi_1(L) \;=\; \exp\left\{ \int_0^T k_0(\gamma(t))\, dt \right\},$$

and, from (7), we recover relation (3). We have given an alternative proof to relation (3) to the one described in [8] by using curvilinear coordinates.

Our purpose is to give a generalization of this relation (3) but related to generalized exponential factors. If $F_i(x,y) = \exp\left\{g_i(x,y)/f_0(x,y)^i\right\}$ is a generalized exponential factor associated to $f_0(x,y) = 0$ of order i, we denote by $k_i(x,y)$ its cofactor.

Theorem 5. *Assume that we have a system* (1) *with a periodic orbit* Γ *and let* $f_0(x,y) = 0$ *be the invariant curve such that* $\Gamma \subseteq \{(x,y) \in \mathcal{U} \mid f_0(x,y) = 0\}$.

Γ *has multiplicity at least* m *if, and only if, there exist* $m-1$ *generalized exponential factors of subsequent orders* $1, 2, 3, \ldots, m-1$ *associated to* $f_0(x,y) = 0$.

Moreover, in such a case

$$\Psi_{i+1}(L) \;=\; -i\, g_i(\gamma(0)) \int_0^T k_i(\gamma(t))\, dt, \tag{8}$$

for $i = 1, 2, \ldots, m-1$, *where the values* $\Psi_{i+1}(L)$ *are the ones introduced in Theorem 3.*

We note that this Theorem 5 is equivalent to Theorem 1 but composed by using the defined values $\Psi_j(L)$. The following Section 3 contains its proof.

By the definition of generalized exponential factor, we have that $g_i(\gamma(0)) \neq 0$ because the point $\gamma(0)$ is such that $f_0(\gamma(0)) = 0$ and $\nabla f_0(\gamma(0)) \neq 0$. In fact, we can always multiply $g_i(x,y)$ by a non-negative constant so that we get a generalized exponential factor $F_i(x,y)$ with $g_i(\gamma(0)) = -1/i$.

We remark that relation (8) is more than a generalization of relation (3) because, although the values of $\Psi_j(L)$ can be computed in a recursive way, there is no explicit formula for $\Psi_j(L)$ for a high value of j, due to its computational difficulty. In this way, relation (8) gives us an explicit way to determine the values $\Psi_j(L)$.

We conclude that if Γ is of multiplicity m, we can always find a set of $m-1$ generalized exponential factors associated to $f_0(x,y) = 0$ with a correlative sequence of orders up to $m-1$. Although Theorem 5 only states the existence of such generalized exponential factors, in the proof of this

Theorem we will show a constructive way to give them, provided that we use curvilinear coordinates, see Remark 9. Reciprocally, if we have any set of $m-1$ generalized exponential factors associated to $f_0(x,y)=0$ with a correlative sequence of orders up to $m-1$, we can ensure that Γ has at least multiplicity m and we can decide whether its multiplicity is higher or not by computing $\int_0^T k_m(\gamma(t))\,dt$.

We want to end this section stating the following theorem because it describes the creation of limit cycles from a multiple limit cycle Γ, by using its implicit expression $f_0(x,y)=0$.

Theorem 6. ([1, Ch. X, §27.1]) *Consider an analytic system* (1) *with a limit cycle* Γ *of multiplicity exactly* m $(m \geq 1)$*, then, by perturbing this system in the world of analytic systems, at most* m *limit cycles bifurcate from* Γ*, taking into account their multiplicities. Furthermore this upper bound is sharp.*

The first part of proof of the above result uses the Weierstrass' Preparation Theorem, while the second one considers the following perturbation of system (1):

$$\begin{aligned} \dot{x} &= P(x,y) + \left(\varepsilon_1 f_0 + \varepsilon_2 f_0^2 + \ldots + \varepsilon_{m-1} f_0^{m-1}\right) \partial f_0/\partial x, \\ \dot{y} &= Q(x,y) + \left(\varepsilon_1 f_0 + \varepsilon_2 f_0^2 + \ldots + \varepsilon_{m-1} f_0^{m-1}\right) \partial f_0/\partial y, \end{aligned} \tag{9}$$

where ε_1, ε_2, ε_3, ..., ε_{m-1} are suitable real parameters.

3. Proof of the Main Theorem

Since we have introduced the values $\Psi_j(L)$ in the previous section and these values characterize the stability of Γ, to prove Theorem 1 it suffices to prove Theorem 5.

Let us consider a system (1) with a periodic orbit Γ. We transform the system to equation (4) by using curvilinear coordinates (s,n). We know that these curvilinear coordinates only have sense for $|n|$ sufficiently small, so we restrict ourselves to these neighborhood of Γ in the rest of this section.

The equation (4) has $n=0$ as invariant curve with cofactor $k_0(s,n) = F(s,n)/n$. We note that this cofactor is analytic and L-periodic in s. We may consider functions analytic in (s,n) which are not L-periodic. In such a case, when we undo the change of variables to cartesian coordinates (x,y) we do not get an analytic function well-defined in a neighborhood $\mathcal{U}$ of Γ. Any analytic function in (x,y) well-defined in a neighborhood $\mathcal{U}$ of Γ is transformed to an analytic function which is L-periodic in the variable s in curvilinear coordinates. And the other way round, any analytic function which is L-periodic in the variable s in curvilinear coordinates transforms to an analytic function in (x, y) well-defined in a neighborhood $\mathcal{U}$ of Γ.

Let us consider the solution $\Psi(s; n_0)$ of equation (4) such that $\Psi(0; n_0) = n_0$. We define the following function:

$$G(s, n; n_0) \ := \ \ln |n - \Psi(s; n_0)| ,$$

where $|\cdot|$ denotes the absolute value. For each positive integer j, with $j \geq 1$, we also define:

$$g_j(s, n) \ := \ \frac{n^j}{j!} \, \frac{\partial^j G(s, n; n_0)}{\partial n_0^j} \bigg|_{\{n_0 = 0\}} . \tag{10}$$

If we develop the function $G(s, n; n_0)$ in powers of n_0, we note that $g_j(s, n)$ is the product of n^j by the coefficient of $G(s, n; n_0)$ corresponding to n_0^j. Hence, for $|n|$ sufficiently small, each $g_j(s, n)$ is also an analytic function.

Proposition 7. (i) *For any $j \geq 1$, there exists an analytic function $k_j(s, n)$ such that:*

$$\frac{\partial g_j(s, n)}{\partial s} + \frac{\partial g_j(s, n)}{\partial n} F(s, n) \ = \ j \, k_0(s, n) \, g_j(s, n) \, + \, k_j(s, n) \, n^j. \tag{11}$$

(ii) *For any $j \geq 1$, we have $g_j(s, 0) \neq 0$ for any value of s.*

(iii) *There exists an integer m, $m > 1$, such that $\Psi_1(L) = 1$ and $\Psi_j(L) = 0$ for $j = 2, 3, \ldots, m$ if, and only if, all the functions $g_1(s, n)$, $g_2(s, n)$, $\ldots$, $g_m(s, n)$ are periodic in s of period L.*

(iv) *There exists an integer m, $m > 1$, such that $\Psi_1(L) = 1$, $\Psi_j(L) = 0$ for $j = 2, 3, \ldots, m-1$ and $\Psi_m(L) \neq 0$ if, and only if,*

$$\int_0^L k_j(s, 0) \, ds \ = \ 0 \quad \text{for } j = 0, 1, 2, \ldots m - 2$$

and

$$\int_0^L k_{m-1}(s, 0) \, ds \ \neq \ 0.$$

Moreover $\Psi_m(L) \ = \ \displaystyle\int_0^L k_{m-1}(s, 0) \, ds.$

It is clear that the statements (i) and (ii) imply that $F_j(s, n) := \exp\left\{ g_j(s, n)/n^j \right\}$ is a generalized exponential factor of equation (4) associated to the curve $n = 0$ of order j and $k_j(s, n)$ is its associated cofactor. Although each $g_j(s, n)$ gives rise to an exponential factor for the equation in curvilinear coordinates (4), each $g_j(s, n)$ does not need to be transformed into a well-defined generalized exponential factor for the system in cartesian coordinates (1). When applying the change from curvilinear coordinates to cartesian coordinates $(s, n) \to (x, y)$, we only get as much well-defined generalized exponential factors, in increasing order, for system (1) as the multiplicity of Γ minus one. This is the conclusion extracted from statement (iii). The last statement gives the relation between the values of $\Psi_j(L)$ and the cofactors of the defined generalized exponential factors.

Proof of Proposition 7. Define:

$$K(s,n;n_0) \;=\; \frac{F(s,n)-F(s,\Psi(s;n_0))}{n-\Psi(s;n_0)},$$

which is an analytic function in a neighborhood of $n=0$ and also in a neighborhood of $n_0=0$. Using the previously defined function $G(s,n;n_0) = \ln|n-\Psi(s;n_0)|$, an easy computation shows that:

$$\frac{\partial G(s,n;n_0)}{\partial s} + \frac{\partial G(s,n;n_0)}{\partial n}\,F(s,n) \;=\; K(s,n;n_0), \tag{12}$$

where we have that used that $\partial\Psi(s;n_0)/\partial s = F(s,\Psi(s;n_0))$.

Now we introduce the functions $k_j(s,n)$ as the ones satisfying the following relation:

$$K(s,n;n_0) \;=\; \sum_{j\geq 0} k_j(s,n)\,n_0^j.$$

Equating the coefficients of n_0^j in the previous identity (12), we have that:

$$\frac{1}{n^j}\left(\frac{\partial g_j(s,n)}{\partial s}+\frac{\partial g_j(s,n)}{\partial n}F(s,n)-j\,g_j(s,n)\,\frac{F(s,n)}{n}\right) \;=\; k_j(s,n),$$

where we have used the definition (10). Since $k_0(s,n)=F(s,n)/n$, we get exactly (11) and, hence, statement (i).

We have just seen that the cofactor associated to the generalized exponential factor $F_j(s,n) = \exp\left\{g_j(s,n)/n^j\right\}$ is $k_j(s,n)$, which is an analytic function in coordinates (s,n). We note that, although we are considering $j\geq 1$, this formula also has sense for $j=0$ and it gives the value of the cofactor of $n=0$ for equation (4): $k_0(s,n)=F(s,n)/n$.

In the definition of $g_j(s,n)$ given in (10), we only need to consider values of $(s,n;n_0)$ such that $|\Psi(s;n_0)/n|$ is small. We have that:

$$\begin{aligned} G(s,n;n_0) \;&=\; \ln|n-\Psi(s;n_0)| \;=\; \ln|n| \,+\, \ln\left|1-\frac{\Psi(s;n_0)}{n}\right| \\ &=\; \ln|n| \,-\, \sum_{i\geq 1}\frac{1}{i}\left(\left|\frac{\Psi(s;n_0)}{n}\right|\right)^i, \end{aligned}$$

where we have used the development of the logarithm $\ln(1-x) = -\sum_{i\geq 1} x^i/i$ for small $|x|$. If we develop $G(s,n;n_0)$ in powers of n_0, we note that $g_j(s,n)$ is defined from the coefficient of degree j in n_0 of this development. This fact implies that to compute $g_j(s,n)$ we only need to consider the sum of terms until $i=j$ in the last expression of $G(s,n;n_0)$. We have that $g_j(s,n)$ is the product of the coefficient of $G(s,n;n_0)$ of degree j in n_0 by n^j, that is,

$$g_j(s,n) \ = \ -\sum_{i=1}^{j} \frac{n^{j-i}}{i} \ \left[n_0^j\right] \left(\Psi(s,n_0)^i\right). \tag{13}$$

Given $\psi(n_0)$ an analytic function of n_0 in a neighborhood of $n_0=0$, we denote by $\left[n_0^j\right](\psi(n_0))$ the coefficient of the term corresponding to n_0^j in the Taylor development of $\psi(n_0)$ in a neighborhood of $n_0=0$.

When taking $n=0$, the only term that it is not cancelled is the one which corresponds to $i=j$, and since $\Psi(s;n_0)=\sum_{i\geq 1}\Psi_i(s)\,n_0^i$, we deduce:

$$g_j(s,0) \ = \ -\frac{\Psi_1(s)^j}{j}.$$

We recall that, by (6), $\Psi_1(s)\neq 0$ for any value of s. Therefore, $g_j(s,0)$ is different from zero for any value of s.

To prove the third statement, we remark that the condition $\Psi_1(L)=1$ and $\Psi_j(L)=0$ for $j=2,3,\ldots,m$ is equivalent to say that all the functions $\Psi_j(s)$ for $1\leq j\leq m$ are L-periodic. We first assume that there exists an integer m, $m>1$, such that $\Psi_1(L)=1$ and $\Psi_j(L)=0$ for $j=2,3,\ldots,m$. From identity (13), we have that the dependence in s of all the functions $g_1(s,n)$, $g_2(s,n)$, …, $g_m(s,n)$ is via $\Psi_j(s)$ for $j=1,2,\ldots,m$, which are periodic functions of period L. Therefore, given any integer j with $1\leq j\leq m$, the function $g_j(s,n)$ is L-periodic in s.

Reciprocally, assume that all the functions $g_1(s,n)$, $g_2(s,n)$, …, $g_m(s,n)$ are periodic in s of period L. By (13), we have that if $g_1(s,n)$ is L-periodic in s, then $\Psi_1(s)$ must be L-periodic in s. Since $\Psi_1(0)=1$, we deduce that $\Psi_1(L)=1$. Then, if $g_2(s,n)$ is L-periodic in s and since $\Psi_1(L)=1$, we deduce that $\Psi_2(s)$ must be L-periodic. We can continue the same inductive argument and we obtain that all the functions $\Psi_j(s)$ for $1\leq j\leq m$ are L-periodic.

In order to prove the last statement, we note that the equation (5) can also be written in integral form as:

$$\Psi(s;n_0) \ = \ n_0 \exp\left\{\int_0^s \frac{F(\sigma;\Psi(\sigma;n_0))}{\Psi(\sigma;n_0)}d\sigma\right\}.$$

Therefore, we have that:

$$\Psi(s;n_0) \;=\; n_0 \exp\left\{ \int_0^s K(\sigma,0;n_0) d\sigma \right\}, \tag{14}$$

where $K(s,n;n_0)$ is the one defined in the proof of statement (i). If we develop the last formula (14) in powers of n_0 and we equate the coefficients of same degree in both members, we can relate the values of $\Psi_j(s)$ and $\int_0^s k_{j-1}(\sigma;0)\, d\sigma$. Note that given any integer number m, $m \geq 1$, and any real constant a_m, we have:

$$n_0 \, \exp\left\{ a_m n_0^m + \mathcal{O}\left(n_0^{m+1}\right) \right\} \;=\; n_0 \,+\, a_m n_0^{m+1} \,+\, \mathcal{O}\left(n_0^{m+2}\right). \tag{15}$$

We first equate the coefficients of degree 1 in n_0 of equation (14) and we have that $\Psi_1(s) = \exp\left\{\int_0^s k_0(\sigma,0)\, d\sigma\right\}$. We recover, in this way, the conclusion extracted from (7) and (3). Therefore, $\Psi_1(L) = 1$ if, and only if, $\int_0^L k_0(\sigma,0)\, d\sigma = 0$. We are going to assume that $\Psi_1(L) = 1$ for the rest of the proof.

We are going to use induction over m. The basic step is for $m = 2$ and we equate the coefficient of n_0 of degree 2 in the expression (14) once evaluated in $s = L$. Using (15), we get that $\Psi_2(L) = \int_0^L k_1(\sigma,0)\, d\sigma$. Hence, $\Psi_2(L) = 0$ if, and only if, $\int_0^L k_1(\sigma,0)\, d\sigma = 0$.

The inductive hypothesis states that $\Psi_1(L) = 1$, $\Psi_j(L) = 0$ for $j = 2,3,\ldots,m-1$ and $\Psi_m(L) \neq 0$ if, and only if, $\int_0^L k_j(s,0)\, ds = 0$ for $j = 0,1,2,\ldots m-2$ and $\int_0^L k_{m-1}(s,0)\, ds \neq 0$. Moreover, $\Psi_m(L) = \int_0^L k_{m-1}(s,0)\, ds$. We first assume that $\Psi_1(L) = 1$, $\Psi_j(L) = 0$ for $j = 2,3,\ldots,m$ and by induction hypothesis we have that $\int_0^L k_j(s,0)\, ds = 0$ for $j = 0,1,2,\ldots m-1$. Now, we equate the coefficients of degree $m+1$ in n_0 of (14) once evaluated in $s = L$ and, using the expansion given in (15), we deduce that $\Psi_{m+1}(L) = \int_0^L k_m(s,0)\, ds$. Therefore, we get that $\Psi_{m+1}(L) = 0$ if, and only if, $\int_0^L k_m(s,0)\, ds = 0$. ∎

The following proposition reads that the set of obtained generalized exponential factors is not unique but the corresponding functions $g_j(s,n)$ are determined up to order n^j and the value of the integral of its cofactor does not vary. We note that, as a corollary, we have that the definition of m-solution is a good definition. If $g_d(x,y)$ and $\tilde{g}_d(x,y)$ give rise to two different exponential factors of order d associated to the same invariant curve $f_0(x,y) = 0$, then the integrals of their cofactors over Γ have the same value up to a multiplicative constant.

Proposition 8. *Let $\tilde{g}_m(s,n)$ be an analytic function such that m is an integer number with $m \geq 1$, $\tilde{g}_m(s,0) \not\equiv 0$, $\tilde{g}_m(s,n)$ is L-periodic in s and* *there exists an analytic function* $\tilde{k}_m(s,n)$ *satisfying that:*

$$\frac{\partial \tilde{g}_m(s,n)}{\partial s} + \frac{\partial \tilde{g}_m(s,n)}{\partial n} F(s,n) \;=\; m\, k_0(s,n)\, \tilde{g}_m(s,n) + \tilde{k}_m(s,n)\, n^m. \quad (16)$$

Then, there exists a set of real constants $\{w_i \in \mathbb{R} \,|\, i = 1,2,\ldots,m\}$ *with* $w_m = -m\, \tilde{g}_m(0,0) \neq 0$ *such that* $\tilde{g}_m(s,n) \;=\; \sum_{i=1}^{m} w_i\, g_i(s,n)\, n^{m-i} + \mathcal{O}(n^m)$*, where* $g_i(s,n)$ *are the ones given in the previous Proposition* 7*, and*

$$\int_0^L \tilde{k}_m(s,0)\, ds \;=\; \sum_{i=1}^{m} w_i \int_0^L k_i(s,0)\, ds\,. \quad (17)$$

We note that Theorem 5 is a consequence of Propositions 7 and 8 just changing from curvilinear coordinates (s,n) to cartesian coordinates.

Proof of Proposition 8. We are going to use three steps. We first show that any function $\tilde{g}_m(s,n)$ for which there exists an analytic function $\tilde{k}_m(s,n)$ satisfying (16) writes as $\tilde{g}_m(s,n) \;=\; \sum_{i=1}^{m} w_i\, g_i(s,n)\, n^{m-i} + \mathcal{O}(n^m)$ for a certain set of real constants w_i, $i = 1,2,3,\ldots,m$. The second step is to show that, in fact, $\tilde{g}_m(s,0) \neq 0$ for all values of s and that $w_m = -m\, \tilde{g}_m(0,0)$. The last step of the proof is to use that $\tilde{g}_m(s,n)$ is L-periodic in s so as to show the identity (17).

In order to prove that any function $\tilde{g}_m(s,n)$ for which there exists an analytic function $\tilde{k}_m(s,n)$ satisfying (16) writes as $\tilde{g}_m(s,n) = \sum_{i=1}^{m} w_i\, g_i(s,n)\, n^{m-i} + \mathcal{O}(n^m)$ for a certain set of real constants w_i, $i = 1,2,3,\ldots,m$, we will use induction over m. But, we first encounter a previous result which is going to be useful for both the basic step and the induction step in the inductive process.

Let us consider an integer number $m \geq 1$ and a function $\tilde{g}_m(s,n)$ for which there exists an analytic function $\tilde{k}_m(s,n)$ satisfying (16) and we define the quotient $q_m(s,n) := \tilde{g}_m(s,n)/g_m(s,n)$. Since both $g_m(s,n)$ and $\tilde{g}_m(s,n)$ are analytic functions in a neighborhood of $n=0$ and $g_m(s,0) \neq 0$ for all s by the proved statement (ii), we have that $q_m(s,n)$ is an analytic function in a neighborhood of $n=0$. By (11) and (16), we have that:

$$\frac{\partial q_m(s,n)}{\partial s} + \frac{\partial q_m(s,n)}{\partial n} F(s,n) \;=\; \frac{1}{g_m(s,n)^2} \Big(\Big[m\, k_0(s,n) \tilde{g}_m(s,n) + $$

$$+ \tilde{k}_m(s,n) n^m \Big] g_m(s,n) \;-\; \Big[m\, k_0(s,n) g_m(s,n) + k_m(s,n) n^m \Big] \tilde{g}_m(s,n) \Big)\,.$$

Therefore,

$$\frac{\partial q_m(s,n)}{\partial s}+\frac{\partial q_m(s,n)}{\partial n}F(s,n)=\left(\frac{\tilde{k}_m(s,n)g_m(s,n)-k_m(s,n)\tilde{g}_m(s,n)}{g_m(s,n)^2}\right)n^m. \tag{18}$$

We have that $m \geq 1$ and, by statement (ii), that $g_m(s,0) \neq 0$ for any value of s, so the right hand side of identity (18) is identically null for $n = 0$. Let us consider $q_m(s,0)$ and evaluating (18) in $n = 0$, we deduce that $(\partial q_m(s,0)/\partial s) \equiv 0$ Therefore, $q_m(s,0) \equiv w_m$, where w_m is a real constant.

Let us start with the inductive process. The basic step corresponds to $m = 1$ and we consider a function $\tilde{g}_1(s,n)$ for which there exists an analytic function $\tilde{k}_1(s,n)$ satisfying (16). We define $q_1(s,n) := \tilde{g}_1(s,n)/g_1(s,n)$ and by the reasoning given in the previous paragraph, we deduce that $q_1(s,n) = w_1 + \mathcal{O}(n)$, for a certain real constant w_1. Hence, $\tilde{g}_1(s,n) = w_1\, g_1(s,n) + \mathcal{O}(n)$.

We consider the step in which we prove the statement for $m+1$ once the statement for m is assumed. We consider a function $\tilde{g}_{m+1}(s,n)$ for which (16) is satisfied with a certain analytic function $\tilde{k}_{m+1}(s,n)$. We define the quotient function $q_{m+1}(s,n) := \tilde{g}_{m+1}(s,n)/g_{m+1}(s,n)$. We have proved that there exists a real constant w_{m+1} such that $q_{m+1}(s,n) = w_{m+1} + \mathcal{O}(n)$, therefore we deduce that $\tilde{g}_{m+1}(s,n) = w_{m+1}\, g_{m+1}(s,n) + \mathcal{O}(n)$. Let us define the following function: $\tilde{g}_m(s,n) := (\tilde{g}_{m+1}(s,n) - w_{m+1}g_{m+1}(s,n))/n$. We have that $\tilde{g}_m(s,n)$ is an analytic function in a neighborhood of $n = 0$. We compute the derivative of $\tilde{g}_m(s,n)$ with respect to equation (4), applying (11), (16) and that $F(s,n) = k_0(s,n)\, n$:

$$\frac{\partial \tilde{g}_m(s,n)}{\partial s} + \frac{\partial \tilde{g}_m(s,n)}{\partial n}\, F(s,n) = \frac{1}{n^2}\Big\{\Big(\Big[(m+1)k_0(s,n)\tilde{g}_{m+1}(s,n) + $$

$$+\tilde{k}_{m+1}(s,n)n^{m+1}\Big] - w_{m+1}\Big[(m+1)k_0(s,n)g_{m+1}(s,n) + k_{m+1}(s,n)n^{m+1}\Big]\Big)n$$

$$-\Big(\tilde{g}_{m+1}(s,n) - w_{m+1}\, g_{m+1}(s,n)\Big)k_0(s,n)n\Big\}.$$

We reorder the terms to get that:

$$\begin{aligned}\frac{\partial \tilde{g}_m(s,n)}{\partial s} + \frac{\partial \tilde{g}_m(s,n)}{\partial n}\, F(s,n) \; &= \\ &= m\, k_0(s,n)\left(\frac{\tilde{g}_{m+1}(s,n) - w_{m+1}\, g_{m+1}(s,n)}{n}\right) + \\ &+ \Big(\tilde{k}_{m+1}(s,n) - w_{m+1}\, k_{m+1}(s,n)\Big)\, n^m.\end{aligned}$$

We define $\tilde{k}_m(s,n) := \tilde{k}_{m+1}(s,n) - w_{m+1}\, k_{m+1}(s,n)$, which is an analytic function, and we deduce that:

$$\frac{\partial \tilde{g}_m(s,n)}{\partial s} + \frac{\partial \tilde{g}_m(s,n)}{\partial n}\, F(s,n) \;=\; m\, k_0(s,n)\, \tilde{g}_m(s,n) \,+\, \tilde{k}_m(s,n)\, n^m.$$

We conclude that $\tilde{g}_m(s,n)$ is an analytic function that satisfies (16) for the order m and, hence, we can apply the induction hypothesis. There exists a set of real constants $\{w_i \,|\, i = 1,2,3,\ldots m\}$ such that $\tilde{g}_m(s,n) = \sum_{i=1}^{m} w_i\, g_i(s,n)\, n^{m-i} \,+\, \mathcal{O}(n^m)$. Since $\tilde{g}_m(s,n) = (\tilde{g}_{m+1}(s,n) - w_{m+1}\, g_{m+1}(s,n))\,/n$, we deduce that:

$$\tilde{g}_{m+1}(s,n) \;=\; \sum_{i=1}^{m+1} w_i\, g_i(s,n)\, n^{m+1-i} \,+\, \mathcal{O}(n^{m+1}).$$

As we have seen $w_m = q_m(s,0)$ for all values of s. Therefore, $\tilde{g}_m(s,0) = w_m g_m(s,0)$. By the proof of statement (ii), we have $g_m(s,0) = -\Psi_1(s)^m/m$, which is different from zero for all values of s. By hypothesis $\tilde{g}_m(s,0)$ is not equivalent to zero, therefore, we deduce that $w_m \neq 0$ and that $\tilde{g}_m(s,0)$ is different from zero for all values of s. In order to give the value of w_m we evaluate the identity $\tilde{g}_m(s,0) = w_m\, g_m(s,0)$ in $s=0$. We recall that $\Psi_1(0) = 1$ and we conclude that $g_m(0,0) = -1/m$ and, hence, $w_m = -m\, \tilde{g}_m(0,0)$.

Let us consider the function $\tilde{g}_m(s,n)$ given by the hypothesis and we define the function $\phi_m(s)$ such that $\tilde{g}_m(s,n)$ writes as:

$$\tilde{g}_m(s,n) \;=\; \sum_{i=1}^{m} w_i\, g_i(s,n)\, n^{m-i} \,+\, \phi_m(s) n^m \,+\, \mathcal{O}(n^{m+1})$$

where w_i, $i = 1,2,3,\ldots,m$, are the real constants whose existence has been proved in previous paragraphs. We note that, by (13), the functions $g_i(s,n)$ are polynomials in n of degree at most $i-1$ and whose coefficients are functions of $\Psi_j(s)$ for $1 \leq j \leq i$. Therefore, the sum $\sum_{i=1}^{m} w_i\, g_i(s,n)\, n^{m-i}$ is the expression of the terms of order n^i with $i = 0,1,2,3,\ldots,m-1$ in the development of $\tilde{g}_m(s,n)$ in powers of n. The function $\phi_m(s)$ is the coefficient of n^m in this development.

We compute the derivative of $\tilde{g}_m(s,n)$ with respect to equation (4) using equation (11) and that $F(s,n) = k_0(s,n)\, n$:

$$\frac{\partial \tilde{g}_m(s,n)}{\partial s} + \frac{\partial \tilde{g}_m(s,n)}{\partial n}\, F(s,n) =$$

$$= \sum_{i=1}^{m} w_i \Bigg[\left(\frac{\partial g_i(s,n)}{\partial s} + \frac{\partial g_i(s,n)}{\partial n} F(s,n) \right) n^{m-i} +$$

$$+ g_i(s,n)(m-i)k_0(s,n)n^{m-i} \Bigg] + \phi_m'(s)n^m + \phi_m(s)mk_0(s,n)n^m + \mathcal{O}(n^{m+1}) =$$

$$= \sum_{i=1}^{m} w_i \Bigg[\Big(i\, g_i(s,n)k_0(s,n) + k_i(s,n)n^i \Big) n^{m-i} +$$

$$+ g_i(s,n)(m-i)k_0(s,n)n^{m-i} \Bigg] +$$

$$+ \phi_m'(s)n^m + \phi_m(s)m\, k_0(s,n)n^m + \mathcal{O}(n^{m+1}).$$

Reordering the terms we get that:

$$\frac{\partial \tilde{g}_m(s,n)}{\partial s} + \frac{\partial \tilde{g}_m(s,n)}{\partial n}\, F(s,n) =$$

$$= m\, k_0(s,n) \left(\sum_{i=1}^{m} w_i\, g_i(s,n)\, n^{m-i} + \phi_m(s)n^m \right) +$$

$$+ \left(\sum_{i=1}^{m} w_i\, k_i(s,n) + \phi_m'(s) \right) n^m + \mathcal{O}(n^{m+1}) =$$

$$= m\, k_0(s,n)\, \tilde{g}_m(s,n) + \left(\sum_{i=1}^{m} w_i\, k_i(s,n) + \phi_m'(s) \right) n^m + \mathcal{O}(n^{m+1}).$$

By (16) we deduce that:

$$\tilde{k}_m(s,n)\, n^m = \left(\sum_{i=1}^{m} w_i\, k_i(s,n) + \phi_m'(s) \right) n^m + \mathcal{O}(n^{m+1}).$$

We divide this last identity by n^m and we evaluate in $n = 0$ to conclude that:

$$\tilde{k}_m(s,0) = \sum_{i=1}^{m} w_i\, k_i(s,0) + \phi_m'(s). \tag{19}$$

Since $\tilde{g}_m(s,n)$ is L-periodic in s, we deduce that each of its coefficients in n is L-periodic in s. In particular $\phi_m(s)$ in L-periodic in s and, hence, $\int_0^L \phi_m'(s)\,ds = \phi_m(L) - \phi_m(0) = 0$. Therefore, integrating in s from 0 to L both members of relation (24), we exactly get identity (17). ∎

Remark 9. As a corollary of this proof, we note that $g_1(s,n) = -\Psi_1(s)$, see equations (10) and (13). Using equation (6) we have $g_1(s,n) = -\exp\left\{\int_0^s k_0(\sigma,0)\,d\sigma\right\}$. We can traduce this expression to cartesian coordinates in the following way: if we have a periodic orbit $\Gamma = \{\gamma(t) : 0 \leq t < T\}$ whose corresponding invariant curve $f_0(x,y) = 0$ is such that $\int_0^T k_0(\gamma(t))\,dt = 0$, then a suitable function $g_1(x,y)$ can be given by computing the function $\int_0^t k_0(\gamma(\sigma))\,d\sigma$ and undoing the parameterization $t \mapsto (x,y) = \gamma(t)$. We note that the resulting function $g_1(x,y)$ will only be well defined in rectangular coordinates (x,y) when $\int_0^T k_0(\gamma(t))\,dt = 0$. This function gives an exponential factor of order 1 for $f_0(x,y) = 0$: $F_1(x,y) = \exp\{g_1(x,y)/f_0(x,y)\}$.

In the same way, if we know generalized exponential factors of consecutive orders from 1 to $j-1$ whose cofactors $k_i(x,y)$ all verify $\int_0^T k_i(\gamma(t))\,dt = 0$ for $i = 0, 1, \ldots, j-1$, we can construct a generalized exponential factor of the next order j using the expressions given in (10) and (14). With the integrals of the cofactors, we give the functions:

$$\tilde{\Psi}(t;n_0) = n_0 \exp\left\{ \sum_{i=0}^{j-1} \left(\int_0^t k_i(\gamma(\sigma))\,d\sigma \right) n_0^i \right\}$$

and $\tilde{G}(s,n;n_0) = \ln|n - \tilde{\Psi}(t;n_0)|$, which coincide with their homonymous without tilde up to order j in n_0 and the relation (13) ensures that no other terms are needed. Hence, the expression given in (10) can also be used to define $g_j(t,n)$ which gives $g_j(x,y)$ undoing the parameterization $t \mapsto (x,y) = \gamma(t)$ and $n \mapsto f_0(x,y)$. We get an exponential factor of order j for $f_0(x,y) = 0$: $F_j(x,y) = \exp\left\{g_j(x,y)/f_0(x,y)^j\right\}$.

4. Examples and applications.

In most text books only simple examples of hyperbolic algebraic limit cycles appear. In this section we present several examples, including multiple, algebraic and non-algebraic limit cycles. We study them by using Theorem 1.

We start with an easy example:

Example 10. *Consider system*

$$\begin{aligned} \dot{x} &= -y + \left(\frac{1}{2} - \alpha\, x\, y\right)\left(x^2+y^2-1\right), \\ \dot{y} &= x + \alpha\, x^2 \left(x^2+y^2-1\right), \end{aligned} \tag{20}$$

being α a real parameter. When $\alpha \neq 0$ the unit circumference $x^2+y^2 = 1$ is a limit cycle of multiplicity 2. When $\alpha = 0$ the unit circumference belongs to a continuum of periodic orbits.

Proof. It is clear that the curve $\Gamma = \{\gamma(t) = (\cos(t), \sin(t)) : 0 \leq t < 2\pi\}$ is a periodic orbit of system (20) and that the curve $f_0(x,y) = x^2+y^2-1 = 0$ is an invariant curve for system (20), with cofactor $k_0(x,y) = x$. Since

$$\int_0^{2\pi} k_0(\gamma(t))\, dt \;=\; \int_0^{2\pi} \cos(t)\, dt \;=\; 0,$$

by Theorem 1 we have that Γ is a periodic orbit of multiplicity at least two. Furthermore there exists a generalized exponential factor of order 1 associated to the Γ. By using Remark 9 we can find it, obtaining

$$F_1(x,y) \;=\; \exp\left\{ \frac{e^y}{x^2+y^2-1} \right\},$$

with cofactor $k_1(x,y) \;=\; \alpha\, e^y\, x^2$. When $\alpha \neq 0$, the integral of the cofactor $k_1(x,y)$ over $\gamma(t)$ is:

$$\int_0^{2\pi} k_1(\gamma(t))\, dt \;=\; \alpha \int_0^{2\pi} e^{\sin(t)} \cos^2(t)\, dt \;\neq\; 0.$$

Hence, again by Theorem 1, we deduce that when $\alpha \neq 0$, Γ is a limit cycle of system (20) of multiplicity exactly two. When $\alpha = 0$ it is clear that $H = (x^2+y^2-1)\, e^{-y}$ is a first integral of system (1) which is well-defined over all the plane and, hence, the periodic orbit Γ belongs to a continuum of periodic orbits, as we wanted to prove. ∎

Proof of Proposition 2. For the sake of clarity we repeat here the equations of the system

$$\begin{aligned} \dot{x} &= -y+(x^2+y^2-1)\left(\alpha_0 x + \alpha_1 y^2 + \alpha_2 y + \alpha_3(1-x^2-3y^2) + \alpha_4 y\right), \\ \dot{y} &= x + (x^2+y^2-1)\left(-1/2 - \alpha_1 xy + \alpha_2(-1-x+x^2+y^2) + \right. \\ &\qquad \left. +2\alpha_3 xy - \alpha_4 x\right). \end{aligned} \tag{21}$$

It is clear that the unit circumference $\Gamma \;=\; \{\gamma(t) \,:\, 0 \leq t < 2\pi\}$, with $\gamma(t) = (\cos(t), \sin(t))$, is a periodic orbit of system (21). To study its stability we use the invariant curve $f_0(x,y) = x^2+y^2-1 = 0$ and Theorem 1. Indeed, we will show that $f_0(x, y) = 0$ is a m-solution of system (21) with $m = 1, 2, 3$ depending on the values of the parameters α_i, $i = 0, 1, 2, 3$.

We note that the cofactor associated to $f_0(x, y) = 0$ is $k_0(x, y) = -y + 2\alpha_0 x^2 - 2(x^2 + y^2 - 1)(\alpha_3 x - \alpha_2 y)$. Therefore,

$$\int_0^{2\pi} k_0(\gamma(t))\, dt \;=\; 2\alpha_0 \int_0^{2\pi} \cos^2(t)\, dt.$$

Hence this value is zero if and only if $\alpha_0 = 0$. Therefore, if $\alpha_0 \neq 0$, we have that the curve $f_0(x, y) = 0$ is a 1-solution of system (21) and thus Γ is a hyperbolic limit cycle.

We assume that $\alpha_0 = 0$ from now on. We know that there exists a generalized exponential factor of order 1 associated to $f_0(x, y) = 0$ for system (21). By using Remark 9 we obtain the following generalized exponential factor:

$$F_1(x, y) \;=\; \exp\left\{ \frac{e^x}{f_0(x, y)} \right\},$$

with cofactor

$$k_1(x, y) \;=\; e^x \left(\alpha_1 y^2 + (\alpha_4 - \alpha_2) y + \alpha_3 (1 + 2x - x^2 - 3y^2) \right).$$

An easy computation exhibits that:

$$\int k_1(\gamma(t))\, dt \;=\; e^{\cos(t)} \left(\alpha_2 - \alpha_4 + 2\alpha_3 \sin(t) \right) \;+\; \alpha_1 \int e^{\cos(t)} \sin^2(t)\, dt.$$

Therefore,

$$\int_0^{2\pi} k_1(\gamma(t))\, dt \;=\; \alpha_1 \int_0^{2\pi} e^{\cos(t)} \sin^2(t)\, dt.$$

It is clear that this value is zero if and only if $\alpha_1 = 0$. Hence, if $\alpha_1 \neq 0$, we deduce that the curve $f_0(x, y)$ is a 2-solution of system (21) and Γ is a limit cycle of multiplicity exactly 2.

We assume that $\alpha_1 = 0$ from now on. Since $f_0(x, y) = 0$ is a m-solution, with $m \geq 3$, of system (21) in this case, we can ensure the existence of a generalized exponential factor of order 2 associated to it. Using Remark 9, we find the function:

$$F_2(x, y) \;=\; \exp\left\{ \frac{e^{2x} - 2e^{2x} \left((\alpha_2 - \alpha_4) + 2\alpha_3 y \right) (x^2 + y^2 - 1)}{(x^2 + y^2 - 1)^2} \right\}$$

which is a generalized exponential factor for system (21) (with $\alpha_0 = \alpha_1 = 0$) associated to $f_0(x, y) = 0$ and of order 2. Its cofactor is:

$$k_2(x, y) \;=\; 4e^{2x} \left(\alpha_4 - 2\alpha_3 y \right) \left(\alpha_3 (1 + 2x - x^2 - 3y^2) + (\alpha_4 - \alpha_2) y \right).$$

Some computations prove that:

$$\int k_2(\gamma(t))dt \;=\; 2e^{2\cos(t)}\left(\alpha_2\alpha_4 - 2\alpha_3^2 - \alpha_4^2 + 4\alpha_3\alpha_4\sin(t) + 2\alpha_3^2\cos(2t)\right) + 8\alpha_2\alpha_3 \int e^{2\cos(t)}\sin^2(t)\,dt,$$

which implies that:

$$\int_0^{2\pi} k_2(\gamma(t))\,dt \;=\; 8\alpha_2\alpha_3 \int_0^{2\pi} e^{2\cos(t)}\sin^2(t)\,dt.$$

We conclude that this value is zero if and only if $\alpha_2\alpha_3 = 0$. We have just proved that when $\alpha_0 = \alpha_1 = 0$ and $\alpha_2\alpha_3 \neq 0$, the curve $f_0(x, y)$ is a 3-solution of system (21) and Γ is a limit cycle of multiplicity exactly 3.

To finish with, we are going to show that when $\alpha_0 = \alpha_1 = 0$ and $\alpha_2\alpha_3 = 0$, the periodic orbit Γ belongs to a continuum of periodic orbits. We encounter a first integral well-defined over all $\mathbb{R}^2$, which implies this fact.

When $\alpha_0 = \alpha_1 = \alpha_2 = 0$, the function

$$H \;=\; (x^2 + y^2 - 1)\, e^{-x + (2\alpha_3 y - \alpha_4)(x^2+y^2-1)},$$

is a first integral of system (21).

When $\alpha_2 \neq 0$ and $\alpha_0 = \alpha_1 = \alpha_3 = 0$, the function

$$H \;=\; e^{-x}\,(x^2 + y^2 - 1)\,\left(-1 + 2\alpha_2(x^2 + y^2 - 1)\right)^{\frac{\alpha_4 - \alpha_2}{2\,\alpha_2}},$$

is a first integral of system (21).

It is well known that at most three limit cycles, taking into account their multiplicities, can bifurcate from a triple limit cycle, see Theorem 6. Let us see that this number is achieved in our system. By using the standard tools utilized to study degenerated Hopf bifurcations, see for instance [3, 7], we obtain that when $|\alpha_0| \ll |\alpha_1| \ll |\alpha_2\,\alpha_3|$ and α_0, α_1 and $\alpha_2\,\alpha_3$ alternate signs, at least two other limit cycles born from Γ due to its changes of stability moving the parameters. Since the system has three limit cycles near Γ they have to be hyperbolic, as we wanted to prove. ∎

Notice that in the above proof the values α_0, α_1 and $\alpha_2\,\alpha_3$ associated to the periodic orbit $x^2 + y^2 - 1 = 0$ of system (20) plays a similar role to the Lyapunov constants associated to a critical point of focus type of a planar system.

Next, we include an extension of an example given in [8] that shows a multiple, analytic and non-algebraic limit cycle.

Example 11. *Consider the planar system*

$$\begin{aligned}
\dot{x} &= -y(m-1)(x^2+y^2)+ \\
&\quad + \left(y^2-\cos(x)\right)^{m-1}\left((x+y)\cos(x)-y(x^2+xy+2y^2)\right), \\
\dot{y} &= \frac{\sin(x)}{2}(m-1)(x^2+y^2)+ \\
&\quad + \left(y^2-\cos(x)\right)^{m-1}\left((x-y)(y^2-\cos(x))+(x^2+y^2)\frac{\sin(x)}{2}\right),
\end{aligned} \tag{22}$$

with $m \geq 1$ *being an integer number. Then the oval of the curve* $y^2 - \cos(x) = 0$ *that surrounds the origin is a limit cycle of multiplicity* m *of* (22).

Proof. First of all, notice that there is no singular point of the system on the curve $y^2 - \cos(x) = 0$. Moreover, this curve is an invariant curve as straightforward computations show. Its cofactor is:

$$k_0(x,y) \,=\, \left(y^2-\cos(x)\right)^{m-1}\left(2y(x-y)-(x+y)\sin(x)\right).$$

Hence, we have that the oval surrounding the origin of this curve is a periodic orbit of system (22). We denote by $\Gamma_0 = \{\gamma_0(t) \,:\, 0 \leq t < T_0\}$ this periodic orbit where $T_0 > 0$ is its minimal positive period.

In order to show that Γ_0 is a limit cycle of multiplicity exactly m we use Theorem 1. Consider the following generalized exponential factors: $F_\ell(x,y) = \exp\left\{1/\left(y^2-\cos(x)\right)^\ell\right\}$, for $\ell = 1,2,\ldots,m-1$. It is easy to see that the cofactor associated to each one of these generalized exponential factors is:

$$k_\ell(x,y) \,=\, -\ell\left(y^2-\cos(x)\right)^{m-1-\ell}\left(2y(x-y)-(x+y)\sin(x)\right).$$

This cofactor is analytic in a neighborhood of Γ_0 for $\ell = 1,2,\ldots,m-1$. Fixed a natural number $m \geq 1$, we note that:

$$\int_0^{T_0} k_i(\gamma_0(t))\,dt \,=\, 0, \quad \text{for } i = 0,1,2,\ldots,m-2,$$

since these cofactors identically vanish over Γ_0. Hence Γ_0 has multiplicity at least m.

In order to prove that its multiplicity is exactly m, we are going to show that:

$$\int_0^{T_0} k_{m-1}(\gamma_0(t))\,dt \,\neq\, 0.$$

We do not explicitly know the parameterization $\gamma_0(t)$ of this periodic orbit, but we know that it is counterclockwise and we can parameterize this oval in two parts by using the variable x: $y = \pm\sqrt{\cos(x)}$, $-\pi/2 \leq x \leq \pi/2$.

We omit a multiplicative factor $m-1$ and use that over Γ_0, $dx/dt = -y(m-1)(x^2+y^2)$ (when $m=1$, $dx/dt = -y(x^2+y^2)$). Therefore, the value of the integral is:

$$\int_{-\pi/2}^{\pi/2} \left.\frac{2y(x-y)-(x+y)\sin(x)}{-y(x^2+y^2)}\right|_{y=-\sqrt{\cos(x)}} dx \; +$$

$$+ \int_{\pi/2}^{-\pi/2} \left.\frac{2y(x-y)-(x+y)\sin(x)}{-y(x^2+y^2)}\right|_{y=\sqrt{\cos(x)}} dx \; =$$

$$= -2 \int_{-\pi/2}^{\pi/2} \frac{2\cos(x)+x\sin(x)}{\sqrt{\cos(x)}\,(x^2+\cos(x))}\, dx \; =$$

$$= -4 \arctan\left(\frac{x}{\sqrt{\cos(x)}}\right)\Bigg|_{x=-\pi/2}^{x=\pi/2} = -4\pi.$$

Hence, when $m>1$ (resp. $m=1$) Γ_0 is a multiple (resp. hyperbolic) limit cycle of system (22) of multiplicity exactly m. ∎

Recovering Lyapunov constants. In the work [9], the quadratic systems of the form:

$$\begin{array}{rcl} \dot{x} & = & \lambda x - y - bx^2 - Cxy - dy^2, \\ \dot{y} & = & x + \lambda y + ax^2 + Axy - ay^2, \end{array} \qquad (23)$$

where $\lambda, a, b, d, A, B, C$ are real parameters, are studied, see also [2]. These quadratic systems are said to be in the *Kapteyn canonical form*. The origin of this system is a focus whose order depends on the values of the parameters. We are going to see how to induce the order of the fine focus from generalized exponential factors. To do so, let us consider the change to polar coordinates of system (1):

$$\begin{array}{rcl} \dot{r} & = & r\left[\lambda + ar\left(\cos^2(\theta)\sin(\theta) - \sin^3(\theta)\right) - br\cos^3(\theta) - dr\cos(\theta)\sin^2(\theta) + \right. \\ & & \left. + Ar\cos(\theta)\sin^2(\theta) + Cr\cos^2(\theta)\sin(\theta)\right], \\ \dot{\theta} & = & 1 + ar\left(\cos^3(\theta) - \cos(\theta)\sin^2(\theta)\right) + br\cos^2(\theta)\sin(\theta) + dr\sin^3(\theta) + \\ & & + Ar\cos^2(\theta)\sin(\theta) + Cr\cos(\theta)\sin^2(\theta). \end{array} \qquad (24)$$

We take the invariant curve $r = 0$ and $0 \le \theta < 2\pi$, which corresponds to the origin of system (23). This curve has as cofactor:

$$\begin{aligned} k_0(\theta, r) &= \lambda + ar\left[\cos^2(\theta)\sin(\theta) - \sin^3(\theta)\right] - br\cos^3(\theta) - \\ &\quad -dr\cos(\theta)\sin^2(\theta) + Ar\cos(\theta)\sin^2(\theta) + Cr\cos^2(\theta)\sin(\theta). \end{aligned}$$

As we have seen, the invariant curve $r = 0$ is hyperbolic, and therefore the origin of system (23) is a strong focus, if and only if, the following integral is different from zero:

$$\int_0^{2\pi} k_0(\theta, 0)\, d\theta \;=\; 2\pi\lambda.$$

From our results, when $\lambda = 0$ we are able to encounter a generalized exponential factor of order 1 associated to $r = 0$. We assume that $\lambda = 0$ from now on, and we have that $F_1(\theta, r) \;=\; \exp\{1/r\}$ is a generalized exponential factor of system (24) with cofactor:

$$\begin{aligned} k_1(\theta, r) &= a\left(\sin^3(\theta) - \cos^2(\theta)\sin(\theta)\right) + b\cos^3(\theta) + d\cos(\theta)\sin^2(\theta) \\ &\quad -A\cos(\theta)\sin^2(\theta) + C\cos^2(\theta)\sin(\theta). \end{aligned}$$

The value of the integral of this cofactor[1] is:

$$\int_0^{2\pi} k_1(\theta, 0)\, d\theta \;=\; 0,$$

which implies that there exists a generalized exponential factor of order 2 associated to $r = 0$: $F_2(\theta, r) \;=\; \exp\{g_2(\theta, r)/r^2\}$. Some computations show that we can define:

$$\begin{aligned} g_2(\theta, r) &= 1 + \frac{r}{6}\left(3(C - a - 3c)\cos(\theta) + (C + c - a)\cos(3\theta)\right. \\ &\quad \left. -2(d + 5b - A + (A + b - d)\cos(2\theta))\sin(\theta)\right), \end{aligned}$$

and $F_2(\theta, r)$ is a generalized exponential factor with cofactor $k_2(\theta, r)$. We do not give the explicit expression of $k_2(\theta, r)$ due to its length. The value of the integral of its cofactor is:

$$\int_0^{2\pi} k_2(\theta, 0)\, d\theta \;=\; -\frac{1}{2}\,(2a + C)\,(b + d)\,\pi.$$

Although we do not explicitly give the expressions and computations, it can be shown that if $(2a + C)\,(b + d) = 0$ then there exists a generalized exponential factor of order 3 associated to $r = 0$ whose integral over the cofactor on $r = 0$ and from $\theta = 0$ to $\theta = 2\pi$ is 0 and that this process can be continued until solving totally the center-focus problem for this case.

[1] That the integral of this cofactor vanishes is not a casuality. It is well known that after a polar blow up the multiplicity of a weak focus is always an odd number.

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