STABLE RANK OF LEAVITT PATH ALGEBRAS

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Abstract. We characterize the values of the stable rank for Leavitt path algebras, by giving concrete criteria in terms of properties of the underlying graph.

Introduction and background

Leavitt path algebras have been recently introduced in [1] and [4]. Given an arbitrary (but fixed) field K and a row-finite graph E, the Leavitt path algebra $L_K(E)$ is the algebraic analogue of the Cuntz-Krieger algebra $C^*(E)$ described in [10]. Several interesting ring-theoretic properties have been characterized for this class of algebras. For instance, the Leavitt path algebras which are purely infinite simple have been characterized in [2], and [6] contains a characterization of the Leavitt path algebras which are exchange rings in terms of condition (K), a purely graph-theoretic condition defined below.

In this paper, we show that the only possible values of the (Bass) stable rank of a Leavitt path algebra are 1, 2 and ∞. Moreover a precise characterization in terms of properties of the graph of the value of the stable rank is provided (Theorem 2.8). A similar result was obtained in [6, Theorem 7.6] under the additional hypothesis that the graph satisfies condition (K) (equivalently, $L(E)$ is an exchange ring). Many tools of the proof of that result must be re-worked in our general situation. We have obtained in several situations simpler arguments that work without the additional hypothesis of condition (K) on the graph. Another new feature of our approach is a detailed analysis of the stable rank in extensions of Leavitt path algebras of stable rank 2, in order to show that the stable rank of these extensions

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cannot shift to 3. Our main tool for this study is the well-known concept of elementary rank; see for example [9, Chapter 11].

The (topological) stable rank of the Cuntz-Krieger algebras $C^*(E)$ was computed in [7]. This paper has been the inspiration for the general strategy of the proof here. Note that, by a result of Herman and Vaserstein [8], the topological and the Bass stable ranks coincide for C^*-algebras. For the sake of comparison, let us mention that, although the possible values of the stable rank of $C^*(E)$ are also 1, 2 and ∞, it turns out that there are graphs E such that the stable rank of $C^*(E)$ is 1 while the stable rank of $L(E)$ is 2. Concretely, if E is a graph such that no cycle has an exit and E contains some cycle, then the stable rank of $C^*(E)$ is 1 by [7, Theorem 3.4], but the stable rank of $L(E)$ is 2 by Theorem 2.8.

Along this paper, we describe the Leavitt path algebras following the presentation of [4, Sections 2 and 4], but using the notation of [1] for the elements.

A *(directed) graph* $E = (E^0, E^1, r, s)$ consists of two countable sets E^0, E^1 and maps $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*.

A vertex which emits no edges is called a *sink*. A graph E is *finite* if E^0 and E^1 are finite sets. If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called *row-finite*. A *path* μ in a graph E is a sequence of edges $\mu = (\mu_1, \dots, \mu_n)$ such that $r(\mu_i) = s(\mu_{i+1})$ for $i = 1, \dots, n-1$. In such a case, $s(\mu) := s(\mu_1)$ is the *source* of μ and $r(\mu) := r(\mu_n)$ is the *range* of μ. If $s(\mu) = r(\mu)$ and $s(\mu_i) \neq s(\mu_j)$ for every $i \neq j$, then μ is a called a *cycle*. If $v = s(\mu) = r(\mu)$ and $s(\mu_i) \neq v$ for every $i > 1$, then μ is a called a *closed simple path based at* v. We denote by $CSP_E(v)$ the set of closed simple paths in E based at v. For a path μ we denote by μ^0 the set of its vertices, i.e., $\{s(\mu_1), r(\mu_i) \mid i = 1, \dots, n\}$. For $n \geq 2$ we define E^n to be the set of paths of length n, and $E^* = \bigcup_{n \geq 0} E^n$ the set of all paths. We define a relation $\geq$ on E^0 by setting $v \geq w$ if there is a path $\mu \in E^*$ with $s(\mu) = v$ and $r(\mu) = w$. A subset H of E^0 is called *hereditary* if $v \geq w$ and $v \in H$ imply $w \in H$. A set is *saturated* if every vertex which feeds into H and only into H is again in H, that is, if $s^{-1}(v) \neq \emptyset$ and $r(s^{-1}(v)) \subseteq H$ imply $v \in H$. Denote by $\mathcal{H}$ (or by $\mathcal{H}_E$ when it is necessary to emphasize the dependence on E) the set of hereditary saturated subsets of E^0.

Let $E = (E^0, E^1, r, s)$ be a graph, and let K be a field. We define the *Leavitt path algebra* $L_K(E)$ associated with E as the K-algebra generated by a set $\{v \mid v \in E^0\}$ of pairwise orthogonal idempotents, together with a set of variables $\{e, e^* \mid e \in E^1\}$, which satisfy the following relations:

(1) $s(e)e = er(e) = e$ for all $e \in E^1$.

(2) $r(e)e^* = e^*s(e) = e^*$ for all $e \in E^1$.

(3) $e^*e' = \delta_{e,e'}r(e)$ for all $e, e' \in E^1$.
(4) $v = \sum_{\{e\in E^1 | s(e)=v\}} ee^*$ for every $v \in E^0$ that emits edges.

Note that the relations above imply that $\{ee^* \mid e \in E^1\}$ is a set of pairwise orthogonal idempotents in $L_K(E)$. Note also that if E is a finite graph then we have $\sum_{v\in E^0} v = 1$. In general the algebra $L_K(E)$ is not unital, but it can be written as a direct limit of unital Leavitt path algebras (with non-unital transition maps), so that it is an algebra with local units. Along this paper, we will be concerned only with row-finite graphs E and we will work with Leavitt path algebras over an arbitrary but fixed field K. We will usually suppress the field from the notation.

Recall that $L(E)$ has a $\mathbb{Z}$-grading. For every $e \in E^1$, set the degree of e as 1, the degree of e^* as -1, and the degree of every element in E^0 as 0. Then we obtain a well-defined degree on the Leavitt path K-algebra $L(E)$, thus, $L(E)$ is a $\mathbb{Z}$-graded algebra:

$$L(E) = \bigoplus_{n\in\mathbb{Z}} L(E)_n, \quad L(E)_n L(E)_m \subseteq L(E)_{n+m}, \text{ for all } \; n, m \in \mathbb{Z}.$$

An ideal I of a $\mathbb{Z}$-graded algebra $A = \oplus_{n\in\mathbb{Z}} A_n$ is a *graded ideal* in case $I = \oplus_{n\in\mathbb{Z}}(I \cap A_n)$. By [4, Proposition 4.2 and Theorem 4.3], an ideal J of $L(E)$ is graded if and only if it is generated by idempotents; in fact, J is a graded ideal if and only if J coincides with the ideal $I(H)$ of $L(E)$ generated by H, where $H = J \cap E^0 \in \mathcal{H}_E$. Indeed, the map $H \mapsto I(H)$ defines a lattice isomorphism between $\mathcal{H}_E$ and $\mathcal{L}_{\mathrm{gr}}(L(E))$.

Recall that a graph E satisfies condition (L) if every closed simple path has an exit, and satisfies condition (K) if for each vertex v on a closed simple path there exists at least two distinct closed simple paths α, β based at v.

Section 1 contains some basic information on the structure of Leavitt path algebras, which will be very useful for the computations in Section 2 of the stable rank of such algebras.

1. Basic facts

For a graph E and a hereditary subset H of E^0, we denote by E_H the *restriction graph*

$$(H, \{e \in E^1 \mid s(e) \in H\}, r|_{(E_H)^1}, s|_{(E_H)^1}).$$

Observe that if H is finite then $L(E_H) = p_H L(E) p_H$, where $p_H = \sum_{v\in H} v \in L(E)$. On the other hand, for $X \in \mathcal{H}_E$, we denote by E/X the *quotient graph*

$$(E^0 \setminus X, \{e \in E^1 \mid r(e) \notin X\}, r|_{(E/X)^1}, s|_{(E/X)^1})$$

By [6, Lemma 2.3(i)] we have a natural isomorphism $L(E)/I(X) \cong L(E/X)$ for $X \in \mathcal{H}_E$. Our next result shows that $I(X)$ is also a Leavitt path algebra.

Definition 1.1. ([7, Definition 1.3]) Let E be a graph, and let $\emptyset \neq X \in \mathcal{H}_E$. Define

$$F_E(X) = \{\alpha = (\alpha_1 \dots \alpha_n) \in E^n \mid n \geq 1, s(\alpha_1) \in E^0 \setminus X, r(\alpha_i) \in E^0 \setminus X \text{ for every } i < n, r(\alpha_n) \in X\}.$$

Let $\overline{F_E(X)} = \{\overline{\alpha} \mid \alpha \in F_E(X)\}$. Then, we define the graph ${}_XE = ({}_XE^0, {}_XE^1, s', r')$ as follows:

1. ${}_XE^0 = X \cup F_E(X)$.
2. ${}_XE^1 = \{e \in E^1 \mid s(e) \in X\} \cup \overline{F_E(X)}$.
3. For every $e \in E^1$ with $s(e) \in X$, $s'(e) = s(e)$ and $r'(e) = r(e)$.
4. For every $\overline{\alpha} \in \overline{F_E(X)}$, $s'(\overline{\alpha}) = \alpha$ and $r'(\overline{\alpha}) = r(\alpha)$.

Lemma 1.2. *Let E be a graph, and let $\emptyset \neq X \in \mathcal{H}_E$. Then, $I(X) \cong L({}_XE)$ (as nonunital rings).*

Proof. We define a map $\phi : L({}_XE) \to I(X)$ by the following rule: (i) For every $v \in X$, $\phi(v) = v$; (ii) For every $\alpha \in F_E(X)$, $\phi(\alpha) = \alpha\alpha^*$; (iii) For every $e \in E^1$ with $s(e) \in X$, $\phi(e) = e$ and $\phi(e^*) = e^*$; (iv) For every $\overline{\alpha} \in \overline{F_E(X)}$, $\phi(\overline{\alpha}) = \alpha$ and $\phi(\overline{\alpha}^*) = \alpha^*$.

By definition, it is clear that the images of the generators of $L({}_XE)$ satisfy the relations defining $L({}_XE)$. Thus, ϕ is a well-defined K-algebra morphism.

To see that ϕ is onto, it is enough to show that every vertex of X and every finite path α of E which ranges in X are in the image of ϕ. For any $v \in X$, $\phi(v) = v$, so that this case is clear. Now, let $\alpha = (\alpha_1 \dots \alpha_n)$ with $\alpha_i \in E^1$. If $s(\alpha_1) \in X$, then $\alpha = \phi(\alpha_1) \cdots \phi(\alpha_n)$. Suppose that $s(\alpha_1) \in E^0 \setminus X$ and $r(\alpha_n) \in X$. Then, there exists $1 \leq j \leq n-1$ such that $r(\alpha_j) \in E^0 \setminus X$ and $r(\alpha_{j+1}) \in X$. Thus, $\alpha = (\alpha_1, \dots \alpha_{j+1})(\alpha_{j+2}, \dots \alpha_n)$, where $\beta = (\alpha_1, \dots \alpha_{j+1}) \in F_E(X)$. Hence, $\alpha = \phi(\overline{\beta})\phi(\alpha_{j+2}) \cdots \phi(\alpha_n)$.

To show injectivity, notice that, for every $\alpha \in F_E(X)$, $\alpha = \overline{\alpha}\overline{\alpha}^*$. Hence, every element $t \in L({}_XE)$ can be written as

$$t = \sum_{\alpha,\beta \in F_E(X)} \overline{\alpha} a_{\alpha,\beta} \overline{\beta}^*, \tag{1}$$

where $a_{\alpha,\beta} \in L(E_X)$. Suppose that $0 \neq \text{Ker}(\phi)$, and let $0 \neq t \in \text{Ker}(\phi)$ written as in (1). By definition of the map ϕ,

$$0 = \phi(t) = \sum_{\alpha,\beta \in F_E(X)} \alpha a_{\alpha,\beta} \beta^*. \tag{2}$$

Let $\alpha_0 \in F_E(X)$ with maximal length among those appearing (with a nonzero coefficient) in the expression (2). Then, for any other $\alpha \in F_E(X)$

appearing in the same expression, $\alpha_0^* \cdot \alpha$ is 0 if $\alpha \neq \alpha_0$ or $r(\alpha_0)$ if $\alpha = \alpha_0$. Thus,

$$0 = \sum_{\alpha,\beta \in F_E(X)} \alpha_0^* \alpha a_{\alpha,\beta}\beta^* = \sum_{\beta \in F_E(X)} a_{\alpha_0,\beta}\beta^*. \tag{3}$$

Now, let $\beta_0 \in F_E(X)$ with maximal length among those appearing in the expression (3). The same argument as above shows that

$$0 = \sum_{\beta \in F_E(X)} a_{\alpha_0,\beta}\beta^*\beta_0 = a_{\alpha_0,\beta_0}. \tag{4}$$

But $0 \neq a_{\alpha_0,\beta_0}$ by hypothesis, and we reach a contradiction. Thus, we conclude that ϕ is injective. □

Lemma 1.3. *Let R be a ring, and let $I \lhd R$ an ideal with local unit. If there exists an ideal $J \lhd I$ such that I/J is a unital simple ring, then there exists an ideal $K \lhd R$ such that $R/K \cong I/J$.*

Proof. Given $a \in J$, there exists $x \in I$ such that $a = ax = xa$. Thus, $J \subseteq JI$, and $J \subseteq IJ$. Hence, $J \lhd R$.

By hypothesis, there exists an element $e \in I$ such that $\overline{e} \in I/J$ is the unit. Consider the set $\mathcal{C}$ of ideals L of R such that $J \subseteq L$ and $e \notin L$. If we order $\mathcal{C}$ by inclusion, it is easy to see that it is inductive. Thus, by Zorn's Lemma, there exists a maximal element of $\mathcal{C}$, say K. Then, $J \subseteq K \cap I \subsetneqq I$, whence $J = K \cap I$ by the maximality of J in I. Thus,

$$I/J = I/(K \cap I) \cong I + K/K \lhd R/K.$$

Suppose that $R \neq I + K$. Clearly, $\overline{e} \in I + K/K$ is a unit. Thus, $\overline{e}$ is a central idempotent of R/K generating $I + K/K$. So, $L = \{a - a\overline{e} \mid a \in R/K\}$ is an ideal of R/K, while

$$R/K = \overline{e}(R/K) + L,$$

being the sum an internal direct sum. If $\pi \colon R \twoheadrightarrow R/K$ is the natural projection map, then $\pi^{-1}(L) = K + \{a - ae \mid a \in R\}$ is an ideal of R containing K (and so J). If $e \in \pi^{-1}(L)$, then $L = R/K$, which is impossible. Hence, $\pi^{-1}(L) \in \mathcal{C}$, and contains strictly K, contradicting the maximality of K in $\mathcal{C}$. Thus, $I + K = R$, and so $R/K \cong I/J$, as desired. □

Corollary 1.4. *Let E be a graph, and let $H \in \mathcal{H}_E$. If there exists $J \lhd I(H)$ such that $I(H)/J$ is a unital simple ring, then there exists an ideal $K \lhd L(E)$ such that $L(E)/K \cong I(H)/J$.*

Proof. By Lemma 1.2, $I(H) \cong L({}_HE)$, whence $I(H)$ has a local unit. Thus, the result holds by Lemma 1.3. □

Recall that an idempotent e in a ring R is called *infinite* if eR is isomorphic as a right R-module to a proper direct summand of itself. A simple ring R is called *purely infinite* in case every nonzero right ideal of R contains an infinite idempotent. See [3] for some basic properties of purely infinite simple rings and [2, Theorem 11] for a characterization of purely infinite simple Leavitt path algebras in terms of properties of the graph.

Proposition 1.5. *Let E be a row-finite graph, and let J be a maximal two-sided ideal of $L(E)$. If $L(E)/J$ is a unital purely infinite simple ring, then $J \in \mathcal{L}_{gr}(L(E))$.*

Proof. Let a be an element of $L(E)$ such that $a+J$ is the unit in $L(E)/J$. There are $v_1, \cdots, v_n \in E^0$ such that $a \in pL(E)p$, where $p = v_1 + \cdots + v_n \in L(E)$. Since $av = va = 0$ for all $v \in E^0 \setminus \{v_1, \dots, v_n\}$, it follows that the hereditary saturated set $X = \{v \in E^0 \mid v \in J\}$ is cofinite in E^0 and thus passing to $L(E)/I(X) \cong L(E/X)$, we can assume that E is a finite graph and that $E^0 \cap J = \emptyset$.

Since E is finite, the lattice $\mathcal{L}_{\rm gr}(L(E))$ of graded ideals (equivalently, idempotent-generated ideals) of $L(E)$ is finite by [4, Theorem 4.3], so that there exists a nonempty $H \in \mathcal{H}_E$ such that $I = I(H)$ is minimal as a graded ideal. Since $I + J = L(E)$ by our assumption that $J \cap E^0 = \emptyset$, we have

$$I/(I \cap J) \cong L(E)/J,$$

so that I has a unital purely infinite simple quotient. Since $I \cong L({}_HE)$ and $J \cap I$ does not contain nonzero idempotents, it follows from our previous argument that ${}_HE$ is finite and so I is unital. So $I = eL(E)$ for a central idempotent e in $L(E)$. Since I is graded-simple, [4, Remark 5.7] and [2, Theorem 11] imply that I is either $M_n(K)$ or $M_n(K[x, x^{-1}])$ for some $n \geq 1$ or it is simple purely infinite. Since I has a quotient algebra which is simple purely infinite, it follows that $I \cap J = 0$ and $J = (1-e)L(E)$ is a graded ideal. Indeed we get $e = 1$ because we are assuming that J does not contain nonzero idempotents. □

Notice that, as a consequence of Proposition 1.5 and [2, Theorem 11], we get the following generalization of [6, Lemma 7.2]

Lemma 1.6. *Let E be a row-finite graph. Then, $L(E)$ has a unital purely infinite simple quotient if and only if there exists $H \in \mathcal{H}_E$ such that the quotient graph E/H is nonempty, finite, cofinal, contains no sinks and each cycle has an exit.*

2. Stable rank for Leavitt path algebras

Let S be any unital ring containing an associative ring R as a two-sided ideal. The following definitions can be found in [11]. A column vector

$b = (b_i)_{i=1}^n$ is called *R-unimodular* if $b_1 - 1, b_i \in R$ for $i > 1$ and there exist $a_1 - 1, a_i \in R$ $(i > 1)$ such that $\sum_{i=1}^n a_i b_i = 1$. The *stable rank* of R (denoted by $\mathrm{sr}(R)$) is the least natural number m for which for any R-unimodular vector $b = (b_i)_{i=1}^{m+1}$ there exist $v_i \in R$ such that the vector $(b_i + v_i b_{m+1})_{i=1}^m$ is R-unimodular. If such a natural m does not exist we say that the stable rank of R is infinite.

Lemma 2.1. (cf. [6, Lemma 7.1]) *Let E be an acyclic graph. Then, the stable rank of $L(E)$ is* 1.

Lemma 2.2. *Let E be a graph. If there exists a unital purely infinite simple quotient of $L(E)$, then the stable rank of $L(E)$ is ∞.*

Proof. If there exists a maximal ideal $M \lhd L(E)$ such that $L(E)/M$ is a unital purely infinite simple ring, then $\mathrm{sr}(L(E)/M) = \infty$ (see e.g. [3]). Since $\mathrm{sr}(L(E)/M) \leq \mathrm{sr}(L(E))$ (see [11, Theorem 4]), we conclude that $\mathrm{sr}(L(E)) = \infty$. □

We adapt the following terminology from [7]: we say that a graph E has isolated cycles if whenever $(a_1, \dots, a_n)$ and $(b_1, \dots, b_m)$ are closed simple paths such that $s(a_i) = s(b_j)$ for some i, j, then $a_i = b_j$. Notice that, in particular, if E has isolated cycles, the only closed simple paths it can contain are cycles.

Lemma 2.3. (cf. [7, Lemma 3.2], [6, Lemma 7.4]) *Let E be a graph. If $L(E)$ does not have any unital purely infinite simple quotient, then there exists a graded ideal $J \lhd L(E)$ with $sr(J) \leq 2$ such that $L(E)/J$ is isomorphic to the Leavitt path algebra of a graph with isolated cycles. Moreover $sr(J) = 1$ if and only if $J = 0$.*

Proof. Set

$$X_0 = \{v \in E^0 \mid \exists e \neq f \in E^1 \text{ with } s(e) = s(f) = v,\ r(e) \geq v,\ r(f) \geq v\},$$

and let X be the hereditary saturated closure of X_0. Consider $J = I(X)$. Then J is a graded ideal of $L(E)$ and $L(E)/J \cong L(E/X)$ by [6, Lemma 2.3(1)]. It is clear from the definition of X_0 that E/X is a graph with isolated cycles.

It remains to show that $\mathrm{sr}(J) \leq 2$ and that $\mathrm{sr}(J) = 2$ if $J \neq 0$. The proof of these facts follows the lines of the proof of [6, Lemma 7.4], using Corollary 1.4 instead of [6, Proposition 5.4] and Lemma 1.2 instead of [6, Lemma 5.2]. □

Definition 2.4. *Let A be a unital ring with stable rank n. We say that A has stable rank closed by extensions in case for any unital ring extension*

$$0 \longrightarrow I \longrightarrow B \longrightarrow A \longrightarrow 0$$

of A with $sr(I) \leq n$ we have $sr(B) = n$.

Recall that a unital ring R has *elementary rank* n in case that, for every $t \geq n+1$, the elementary group $E_t(R)$ acts transitively of the set $U_c(t, R)$ of t-unimodular columns with coefficients in R, see [9, 11.3.9].

In the next lemma, we collect some properties that we will need in the sequel.

Lemma 2.5. *Let A be a unital ring. Assume that $sr(A) = n < \infty$.*

(1) *If $er(A) < n$ then $M_m(A)$ has stable rank closed by extensions for every $m \geq 1$.*
(2) *Let D be any (commutative) euclidean domain such that $sr(D) > 1$ and let m be a positive integer. Then $sr(M_m(D)) = 2$ and $er(M_m(D)) = 1$. In particular $M_m(D)$ has stable rank closed by extensions.*
(3) *Let*

$$0 \longrightarrow I \longrightarrow B \longrightarrow A \longrightarrow 0$$

be a unital extension of A. If $er(A) < n$ and I has a local unit (g_i) such that $sr(g_i I g_i) \leq n$ and $er(g_i I g_i) < n$ for all i, then $sr(B) = n$ and $er(B) < n$.

Proof. (1) This is essentially contained in [11]. We include a sketch of the proof for the convenience of the reader. Assume that we have a unital extension B of A with $\mathrm{sr}(I) \leq n$. Let $\mathbf{a} = (a_1, \ldots, a_{n+1})^t \in U_c(n+1, B)$. Then $\overline{\mathbf{a}} = (\overline{a_1}, \ldots, \overline{a_{n+1}})^t \in U_c(n+1, A)$. Since $\mathrm{sr}(A) = n$, there exists $b_1, \ldots, b_n \in B$ such that $(\overline{a_1} + \overline{b_1}\,\overline{a_{n+1}}, \ldots, \overline{a_n} + \overline{b_n}\,\overline{a_{n+1}})^t \in U_c(n, A)$. Replacing $\mathbf{a}$ with $(a_1 + b_1 a_{n+1}, \ldots, a_n + b_n a_{n+1}, a_{n+1})$, we can assume that $(\overline{a_1}, \ldots, \overline{a_n})^t \in U_c(n, A)$.

Since $\mathrm{er}(A) \leq n-1$, there exists $E \in E(n, B)$ such that $\overline{E} \cdot (\overline{a_1}, \ldots, \overline{a_n})^t = (1, 0, \ldots, 0)^t$. Since $\mathbf{a}$ is reducible if and only if $\mathrm{diag}(E, 1) \cdot \mathbf{a}$ is reducible, we can assume that $(\overline{a_1}, \ldots, \overline{a_n})^t = (1, 0, \ldots, 0)^t$. Finally, replacing a_{n+1} with $a_{n+1} - a_1 a_{n+1}$, we can assume that $\overline{\mathbf{a}} = (1, 0, \ldots, 0)^t$, that is, $\mathbf{a} \in U_c(n+1, I)$. Now, as $\mathrm{sr}(I) \leq n$, $\mathbf{a}$ is reducible in I, and so in B, as desired.

Given any positive integer $m \geq 1$, $\mathrm{sr}(M_m(A)) = \lceil (\mathrm{sr}(A) - 1)/m \rceil + 1$ by [11, Theorem 3] and $\mathrm{er}(M_m(A)) \leq \lceil \mathrm{er}(A)/m \rceil$ by [9, Theorem 11.5.15]. So, it is clear that $\mathrm{er}(A) < \mathrm{sr}(A)$ implies $\mathrm{er}(M_m(A)) < \mathrm{sr}(M_m(A))$. Hence, by the first part of the proof, $M_m(A)$ has stable rank closed by extensions, as desired.

(2) It is well known that an Euclidean domain has stable rank less than or equal to 2, and that it has elementary rank equal to 1, see e.g. [9, Proposition 11.5.3]. So, the result follows from part (1).

(3) Since $\mathrm{sr}(I) \leq n$, the fact that $\mathrm{sr}(B) = n$ follows from part (1). Now, take $m \geq n$, and set $\mathbf{a} = (a_1, \dots, a_m)^t \in U_c(m, B)$. Since $\mathrm{er}(A) < n$, there exists $E \in E(m, B)$ such that $\overline{E} \cdot \overline{\mathbf{a}} = (1, 0, \dots, 0)^t$. So, $\mathbf{b} := E \cdot \mathbf{a} \equiv (1, 0, \dots, 0)^t (\mathrm{mod}\ I)$. Let $g \in I$ an idempotent in the local unit such that $b_1 - 1, b_2, \dots, b_m \in gIg$. Since $\mathrm{er}(gIg) < n$ by hypothesis, there exists $G \in E(m, gIg)$ such that $(G + \mathrm{diag}(1-g, \dots, 1-g)) \cdot \mathbf{b} = (1, 0, \dots, 0)^t$. □

Corollary 2.6. *Let A be a unital K-algebra with sr$(A) = n \geq 2$ and er$(A) <$ sr(A). Then, for any non necessarily unital K-algebra B and two-sided ideal I of B such that $B/I \cong A$ and sr$(I) \leq n$, we have sr$(B) = n$.*

Proof. Given any K-algebra R, we define the unitization $R^1 = R \times K$, with the product
$$(r, a) \cdot (s, b) = (rs + as + rb, ab).$$
Consider the unital extension
$$0 \longrightarrow I \longrightarrow B^1 \longrightarrow A^1 \longrightarrow 0.$$
Notice that $A^1 \cong A \times K$, because A is unital. So, $\mathrm{sr}(A^1) = \mathrm{sr}(A)$ and $\mathrm{er}(A^1) = \mathrm{er}(A)$. Now, by Lemma 2.5(1), $\mathrm{sr}(B^1) \leq n$. Since $n \leq \mathrm{sr}(B) \leq \mathrm{sr}(B^1) \leq n$, the result holds. □

Proposition 2.7. *Let E be a finite graph with isolated cycles. Then sr$(L(E)) \leq 2$ and er$(L(E)) = 1$. Moreover, sr$(L(E)) = 1$ if and only if E is acyclic.*

Proof. We proceed by induction on the number of cycles of E. If E has no cycles then clearly $\mathrm{sr}(L(E)) = 1$, so that $\mathrm{er}(L(E)) = 1$ by [9, Proposition 11.3.11]. Assume that E has cycles $C_1, \dots, C_n$. Define a binary relation on the set of cycles by setting $C_i \geq C_j$ iff there exists a finite path α such that $s(\alpha) \in C_i^0$ and $r(\alpha) \in C_j^0$. Since E is a graph with isolated cycles, $\geq$ turns out to be a partial order on the set of cycles. Since the set of cycles is finite, there exists a maximal one, say C_1. Set $A = \{e \in E^1 \mid s(e) \in C_1 \text{ and } r(e) \notin C_1\}$, let $S(E)$ denote the set of sinks of E, and define $B = \{r(e) \mid e \in A\} \cup S(E) \cup \bigcup_{i=2}^n C_i^0$. Let H be the hereditary and saturated closure of B. By construction of H, C_1 is the unique cycle in E/H, and it has no exits. Moreover, E/H coincides with the hereditary and saturated closure of C_1. Then, $L(E/H) \cong M_k(K[x, x^{-1}])$ for some $k \geq 1$. Consider the extension
$$0 \longrightarrow I(H) \longrightarrow L(E) \longrightarrow L(E/H) \longrightarrow 0.$$
Now, by Lemma 2.5(2), $\mathrm{sr}(L(E/H)) = 2$ and $\mathrm{er}(L(E/H)) = 1$. Consider the local unit (p_X) of $L({}_HE) \cong I(H)$ consisting of idempotents $p_X = \sum_{v \in X} v$ where X ranges on the set of vertices of ${}_HE$ containing H. Since these sets are hereditary in $({}_HE)^0$, we get that $p_X I(H) p_X = p_X L({}_HE) p_X =$

$L(({}_HE)_X)$ is a path algebra of a graph with isolated cycles, containing exactly $n-1$ cycles. By induction hypothesis, $\mathrm{sr}(p_X I(H) p_X) \leq 2$ and $\mathrm{er}(p_X I(H) p_X) = 1$. So, by Lemma 2.5(3), we conclude that $\mathrm{sr}(L(E)) = 2$ and $\mathrm{er}(L(E)) = 1$. Hence, the induction step works, so we are done. □

We are now ready to obtain our main result.

Theorem 2.8. *Let E be a row-finite graph. Then the values of the stable rank of $L(E)$ are:*

(1) *sr*$(L(E)) = 1$ *if E is acyclic.*
(2) *sr*$(L(E)) = \infty$ *if there exists $H \in \mathcal{H}_E$ such that the quotient graph E/H is nonempty, finite, cofinal, contains no sinks and each cycle has an exit.*
(3) *sr*$(L(E)) = 2$ *otherwise.*

Proof. (1) derives from Lemma 2.1, while (2) derives from Lemma 2.2 and Lemma 1.6. We can thus assume that E contains cycles and, using Lemma 1.6, that $L(E)$ does not have any unital purely infinite simple quotient.

By Lemma 2.3, there exists a hereditary saturated set X of E^0 such that $\mathrm{sr}(I(X)) \leq 2$, while E/X is a graph having isolated cycles. By [4, Lemma 2.1], there is an ascending sequence (E_i) of complete finite subgraphs of E/X such that $E/X = \bigcup_{i\geq 1} E_i$. So, by [4, Lemma 2.2], $L(E/X) \cong \varinjlim L(E_i)$. For each $i \geq 1$, there is a natural graded K-algebra homomorphism $\phi_i \colon L(E_i) \to L(E/X)$. The kernel of ϕ_i is a graded ideal of $L(E_i)$ whose intersection with E_i^0 is empty, so ϕ_i is injective and the image L_i of $L(E_i)$ through ϕ_i is isomorphic with $L(E_i)$. It follows from Proposition 2.7 that, for every $i \geq 1$, $\mathrm{sr}(L_i) \leq 2$ and $\mathrm{er}(L_i) = 1$. If $\pi : L(E) \to L(E/X)$ denotes the natural epimorphism (see [6, Lemma 2.3(1)]), then given any $i \geq 1$, we have

$$0 \longrightarrow I(X) \longrightarrow \pi^{-1}(L_i) \longrightarrow L_i \longrightarrow 0.$$

If $\mathrm{sr}(L_i) = 1$, then $\mathrm{sr}(\pi^{-1}(L_i)) \leq 2$ by [11, Theorem 4]. If $\mathrm{sr}(L_i) = 2$ then it follows from Corollary 2.6 that $\mathrm{sr}(\pi^{-1}(L_i)) = 2$. Since $L(E) = \bigcup_{i\geq 1} \pi^{-1}(L_i)$ we get that $\mathrm{sr}(L(E)) \leq 2$. Since E contains cycles we have that either $I(X) \neq 0$ or E/X contains cycles. If $I(X) \neq 0$ then $\mathrm{sr}(I(X)) = 2$ by Lemma 2.3 and so $\mathrm{sr}(L(E)) = 2$ by [11, Theorem 4]. If $I(X) = 0$, then E has isolated cycles. Take a vertex v in a cycle C of E and let H be the hereditary subset of E generated by v. Observe that $L(E_H) = pL(E)p$ for the idempotent $p = \sum_{w\in H^0} w \in \mathcal{M}(L(E))$, where $\mathcal{M}(L(E))$ denotes the multiplier algebra of $L(E)$; see [5]. Let I be the ideal of $pL(E)p$ generated by all the basic idempotents $r(e)$ where $e \in E^1$ is such that $s(e) \in C$ and $r(e) \notin C$. Since E has isolated cycles it follows that I is a proper ideal of

$pL(E)p$ and moreover $pL(E)p/I \cong M_k(K[x, x^{-1}])$, where k is the number of vertices in C. We get

$$\mathrm{sr}(pL(E)p) \geq \mathrm{sr}(pL(E)p/I) = 2.$$

It follows that $1 < \mathrm{sr}(L(E)) \leq 2$ and thus $\mathrm{sr}(L(E)) = 2$, as desired. □

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