ON THE INJECTIVITY OF THE KUDLA-MILLSON LIFT AND SURJECTIVITY OF THE BORCHERDS LIFT

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Abstract. We consider the Kudla-Millson lift from elliptic modular forms of weight $(p+q)/2$ to closed q-forms on locally symmetric spaces corresponding to the orthogonal group $\mathrm{O}(p,q)$. We study the L^2-norm of the lift following the Rallis inner product formula. We compute the contribution at the Archimedean place. For locally symmetric spaces associated to even unimodular lattices, we obtain an explicit formula for the L^2-norm of the lift, which often implies that the lift is injective. For $\mathrm{O}(p,2)$ we discuss how such injectivity results imply the surjectivity of the Borcherds lift.

1. Introduction

In previous work [8], we studied the Kudla-Millson theta lift (see e.g. [19]) and Borcherds' singular theta lift (e.g. [3, 6]) and established a duality statement between these two lifts. Both of these lifts have played a significant role in the study of certain cycles in locally symmetric spaces and Shimura varieties of orthogonal type. In this paper, we study the injectivitiy of the Kudla-Millson theta lift, and revisit part of the material of [6] from the viewpoint of [8], to obtain surjectivity results for the Borcherds lift. Moreover, we provide evidence for the following principle: The vanishing of the standard L-function of a cusp form of weight $1+p/2$ at $s_0 = p/2$ corresponds to the existence of a certain "exceptional automorphic product" on $\mathrm{O}(p,2)$ (see Theorem 1.8).

We now describe the content of this paper in more detail. We begin by recalling the Kudla-Millson lift in a setting which is convenient for the application to the Borcherds lift. Let (V,Q) be a non-degenerate rational quadratic space of signature (p,q). We write $(\cdot,\cdot)$ for the bilinear form corresponding to the quadratic form Q. We write r for the Witt index of V, i.e., the dimension of a rational maximal isotropic subspace. Throughout

Date: June 5, 2006.
2000 *Mathematics Subject Classification.* 11F27, 11F55, 11F67.
* Partially supported by NSF grant DMS-0305448.

we assume that the dimension $m = p + q$ of V is even. Let $H = \mathrm{O}(V)$ be the orthogonal group of V. We let D be the bounded domain associated to $H(\mathbb{R})$, which we realize as the Grassmannian of oriented negative q-planes in $V(\mathbb{R})$.

Let $L \subset V$ be an even lattice of level N, and write $L^\#$ for the dual lattice. The quadratic form on L induces a non-degenerate $\mathbb{Q}/\mathbb{Z}$-valued quadratic form on the discriminant group $L^\#/L$. Recall that the Weil representation ρ_L of the quadratic module $(L^\#/L, Q)$ is a unitary representation of $\mathrm{SL}_2(\mathbb{Z})$ on the group ring $\mathbb{C}[L^\#/L]$, which can be defined as follows [3], [6]. If $(\mathfrak{e}_\gamma)_{\gamma \in L^\#/L}$ denotes the standard basis of $\mathbb{C}[L^\#/L]$, then ρ_L is given by the action of the generators $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ of $\mathrm{SL}_2(\mathbb{Z})$ by

$$\rho_L(T)(\mathfrak{e}_\gamma) = e(\gamma^2/2)\mathfrak{e}_\gamma,$$
$$\rho_L(S)(\mathfrak{e}_\gamma) = \frac{e(-(p-q)/8)}{\sqrt{|L^\#/L|}} \sum_{\delta \in L^\#/L} e(-(\gamma, \delta))\mathfrak{e}_\delta,$$

where $e(w) := e^{2\pi i w}$. This representation factors through the group $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$.

Let $\Gamma \subset \mathrm{O}(L)$ be a torsion-free subgroup of finite index which acts trivially on $L^\#/L$. Then

$$X = \Gamma \backslash D$$

is a real analytic manifold. For $x \in L^\#$ with $Q(x) > 0$, we let

$$D_x = \{z \in D;\ z \perp x\}.$$

Note that D_x is a subsymmetric space attached to the orthogonal group H_x, the stabilizer of x in H. Put $\Gamma_x = \Gamma \cap H_x$. The quotient

$$Z(x) = \Gamma_x \backslash D_x \longrightarrow X$$

defines a (in general relative) cycle in X. For $h \in L^\#/L$ and $n \in \mathbb{Q}$, the group Γ acts on $L_{h,n} = \{x \in L + h;\ Q(x) = n\}$ with finitely many orbits, and we define the composite cycle

$$Z(h, n) = \sum_{x \in \Gamma \backslash L_{h,n}} Z(x).$$

Kudla and Millson constructed Poincaré dual forms for such cycles by means of the Weil representation, see e.g. [19]. They constructed a Schwartz form $\varphi_{KM} \in [\mathcal{S}(V(\mathbb{R})) \otimes \mathcal{Z}^q(D)]^{H(\mathbb{R})}$ on $V(\mathbb{R})$ taking values in $\mathcal{Z}^q(D)$, the closed differential q-forms on D. Let ω_∞ be the Schrödinger model of the Weil representation of $\mathrm{SL}_2(\mathbb{R})$ acting on the space of Schwartz functions $\mathcal{S}(V(\mathbb{R}))$, associated to the standard additive character. We obtain a

$\mathbb{C}[L^\#/L]$-valued theta function on the upper half plane $\mathbb{H}$ by putting

$$\Theta(\tau, z, \varphi_{KM}) = v^{-m/4} \sum_{h \in L^\#/L} \sum_{x \in L+h} (\omega_\infty(g_\tau)\varphi_{KM})(x, z)\mathfrak{e}_h.$$

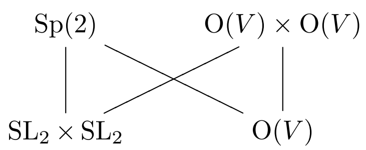
Here $\tau = u + iv \in \mathbb{H}$ and $g_\tau = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)\begin{pmatrix} \sqrt{v} & 0 \\ 0 & \sqrt{v}^{-1} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ is the standard element moving the base point $i \in \mathbb{H}$ to τ. In the variable τ, this theta function transforms as a (non-holomorphic) modular form of weight $\kappa = m/2$ for $\mathrm{SL}_2(\mathbb{Z})$ of type ρ_L. In the variable z, it defines a closed q-form on X. Kudla and Millson showed that the Fourier coefficient at $e^{2\pi i n\tau}\mathfrak{e}_h$ is a Poincaré dual form for the cycle $Z(h, n)$.

Let $S_{\kappa,L}$ denote the space of $\mathbb{C}[L^\#/L]$-valued cusp forms of weight κ and type ρ_L for the group $\mathrm{SL}_2(\mathbb{Z})$. We define a lifting $\Lambda : S_{\kappa,L} \to \mathcal{Z}^q(X)$ by the theta integral

$$f \mapsto \Lambda(f) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \langle f(\tau), \Theta(\tau, z, \varphi_{KM})\rangle \frac{du\,dv}{v^2}, \tag{1.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard scalar product on $\mathbb{C}[L^\#/L]$.

In the present paper, we consider the question whether Λ is injective. We compute the L^2-norm of the differential form $\Lambda(f)$ in the sense of Riemann geometry by means of the Rallis inner product formula [26]. First, using the see-saw



and the Siegel-Weil formula (see e.g. [20], [21], [23], [29]), the inner product can be expressed as a convolution integral of f against the restriction of a genus 2 Eisenstein series to the diagonal (see Proposition 4.8).

Such convolution integrals can be evaluated by means of the doubling method, see e.g. [5], [13], [25], [26]. If f is a Hecke eigenform of level N, one obtains a special value of the partial standard L-function of f (where the Euler factors corresponding to the primes dividing the level N and ∞ are ommited) times a product of "bad" local factors corresponding to the primes dividing N and ∞. If $m > 4$, then, by the Euler product expansion, the special value of the partial standard L-function is non-zero. Therefore the lift $\Lambda(f)$ vanishes precisely if at least one of the "bad" local factors vanishes. By the analysis of the present paper we determine the local factor at infinity.

In the special case where L is even and unimodular, the level of L is $N = 1$, so that ∞ is the only "bad" place. The space $S_{\kappa,L}$ is equal to the space $S_\kappa(\Gamma(1))$ of scalar valued cusp forms of weight κ for $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$. We obtain the following explicit formula for the L^2-norm of the lift (see Theorem 4.10):

Theorem 1.1. *Assume that $m > 3+r$, where r is the Witt index of V. Let $f \in S_\kappa(\Gamma(1))$ be a Hecke eigenform, and write $\|f\|_2^2$ for its Petersson norm, and $D_f(s)$ for its standard L-function. Then $\Lambda(f)$ is square integrable and*

$$\frac{\|\Lambda(f)\|_2^2}{\|f\|_2^2} = C \cdot \frac{D_f(m/2-1)}{\zeta(m/2)\zeta(m-2)},$$

where $C = C(p,q)$ is an explicit real constant, which does not depend on f.The constant C vanishes if and only if $p = 1$.

Corollary 1.2. *Assume that $m > \max(4, 3+r)$ and that L is even unimodular. When $p \neq 1$, the theta lift Λ is injective. When $p = 1$, the lift vanishes identically.*

It would be interesting to compute the bad local factors at finite primes (or at least to show their non-vanishing) as well. However, in our setting, this requires first a suitable Hecke theory for vector valued modular forms in $S_{\kappa,L}$, which is not yet available. It seems conceivable that one could prove more general injectivity results along these lines. For the relationship between the vector-valued modular forms in $S_{\kappa,L}$ and the adelic language, see [17].

Note that in this context, J.-S. Li [15] has used the theta correspondence and the doubling method for automorphic representations in great generality to obtain non-vanishing results for cohomology when passing to a sufficiently large level.

In the body of the paper, we actually consider the generalization of the Kudla-Millson lift due to Funke and Millson [12]. It maps cusp forms in $S_{\kappa,L}$ to closed differential q-forms with values in certain local coefficient systems. Moreover, we use an adelic set-up for the theta and Eisenstein series in question.

1.1. **Surjectivity of the Borcherds lift.** We briefly discuss how the injectivity results on the Kudla-Millson lift imply surjectivity results for the Borcherds lift. We revisit part of the material of [6] in the light of the adjointness result of [8] between the regularized theta lift and the Kudla Millson lift. We restrict ourselves to the Hermitean case of signature $(p, 2)$ where X is a p-dimensional complex algebraic manifold. The special cycles $Z(h, n)$ are algebraic divisors on X, also called Heegner divisors or rational quadratic divisors.

We say that a meromorphic modular form for Γ has a Heegner divisor, if its divisor on X is a linear combination of the $Z(h, n)$. A large supply of modular forms with Heegner divisor is provided by the Borcherds lift, see [2], [3]. We briefly recall its construction.

A meromorphic modular form for a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ is called *weakly holomorphic*, if its poles are supported on the cusps. If $k \in \mathbb{Z}$, we write $W_{k,L}$ for the space of weakly holomorphic modular forms of weight k for $\mathrm{SL}_2(\mathbb{Z})$ of type ρ_L. Any $f \in W_{k,L}$ has a Fourier expansion of the form

$$f(\tau) = \sum_{h \in L^\#/L} \sum_{n \in \mathbb{Z}+Q(h)} c(h,n) e(n\tau) \mathfrak{e}_h,$$

where only finitely many coefficients $c(h,n)$ with $n < 0$ are non-zero. We write V^- for the quadratic space $(V, -Q)$ of signature $(2,p)$ and L^- for the lattice $(L, -Q)$ in V^-.

Theorem 1.3 (Borcherds [3], Theorem 13.3)**.** *Let $f \in W_{1-p/2,L^-}$ be a weakly holomorphic modular form with Fourier coefficients $c(h,n)$. Assume that $c(h,n) \in \mathbb{Z}$ for $n < 0$. Then there exists a meromorphic modular form $\Psi(z,f)$ for Γ (with some multiplier system of finite order) such that:*

(i) *The weight of Ψ is equal to $c(0,0)/2$.*
(ii) *The divisor $Z(f)$ of Ψ is determined by the principal part of f at the cusp ∞. It equals*

$$Z(f) = \sum_{h \in L^\#/L} \sum_{n<0} c(h,n) Z(h,n).$$

(iii) *In a neighborhood of a cusp of Γ the function Ψ has an infinite product expansion analogous to the Dedekind eta function, see* [3], *Theorem 13.3 (5.).*

The proof of this result uses a regularized theta lift. Let $\varphi_0^{p,2} \in \mathcal{S}(V(\mathbb{R}))$ be the Gaussian for signature $(p,2)$. The corresponding Siegel theta function

$$\Theta(\tau, z, \varphi_0) = v \sum_{h \in L^\#/L} \sum_{x \in L+h} (\omega_\infty(g_\tau)\varphi_0)(x,z) \mathfrak{e}_h$$

transforms like a non-holomorphic modular form of weight $p/2 - 1$ of type ρ_L in the variable τ. Hence the theta integral

$$\Phi(z,f) = \int_{\Gamma(1)\backslash\mathbb{H}} \langle f(\tau), \overline{\Theta(\tau,z,\varphi_0^{p,2})} \rangle \, d\mu \tag{1.2}$$

formally defines a Γ-invariant function on D. Because of the singularities of f at the cusps, the integral diverges. However, Harvey and Moore discovered that it can be regularized essentially by viewing it as the limit $T \to \infty$ of the integral over the standard fundamental domain truncated at $\Im(\tau) = T$, see [3], [14]. It turns out that $\Phi(z,f)$ defines a smooth function on $X \setminus Z(f)$ which has a logarithmic singularity along $Z(f)$. Moreover,

$$\Phi(z,f) = -2\log \|\Psi(z,f)\|_{\mathrm{Pet}} + \text{constant},$$

where $\|\cdot\|_{\mathrm{Pet}}$ denotes the Petersson metric on the line bundle of modular forms of weight $c(0,0)/2$ over X. From this identity, the claimed properties of $\Psi(z,f)$ can be derived.

Modular forms for the group $\Gamma \subset \mathrm{O}(L)$ arising via this lift are called automorphic products or Borcherds products. By (ii) they have a Heegner divisor.

Here we consider the question whether the Borcherds lift is surjective. More precisely we ask whether every meromorphic modular form for Γ with Heegner divisor is the lift $\Psi(z,f)$ of a weakly holomorphic form $f \in W_{1-p/2,L^-}$?

An affirmative answer to this question was given in [6] in the special case that the lattice L splits two hyperbolic planes over $\mathbb{Z}$. In the (more restrictive) case that L is unimodular, a different proof was given in [7] using local Borcherds products and a theorem of Waldspurger on theta series with harmonic polynomials [28].

The approach of [6] was to first simplify the problem and to consider the regularized theta lift for a larger space of "input" modular forms. Namely, we let $H_{k,L}$ be the space of *weak Maass forms* of weight k and type ρ_L. This space consists of the smooth functions $f : \mathbb{H} \to \mathbb{C}[L^\#/L]$ that transform with ρ_L in weight k under $\mathrm{SL}_2(\mathbb{Z})$, are annihilated by the weight k Laplacian, and satisfy $f(\tau) = O(e^{Cv})$ as $\tau = u + iv \to i\infty$ for some constant $C > 0$ (see [8] Section 3).

Any $f \in H_{k,L}$ has a Fourier expansion of the form

$$(1.3)\qquad f(\tau) = \sum_{h\in L^\#/L}\sum_{n\in\mathbb{Q}} c^+(h,n)e(n\tau)\mathfrak{e}_h + \sum_{h\in L^\#/L} c^-(h,0)v^{1-k}\mathfrak{e}_h + \sum_{\substack{n\in\mathbb{Q}\\ n\neq 0}} c^-(h,n)H(2\pi nv)e(nu)\mathfrak{e}_h,$$

where only finitely many of the coefficients $c^+(h,n)$ (respectively $c^-(h,n)$) with negative (respectively positive) index n are non-zero. The function $H(w)$ is a Whittaker type function.

For $f \in H_{k,L}$, put $\xi_k(f) = R_{-k}(v^k\bar{f})$, where R_{-k} is the standard raising operator for modular forms of weight $-k$. This defines an antilinear map $\xi_k : H_{k,L} \to W_{2-k,L^-}$ to the space of weakly holomorphic modular forms in weight $2-k$. It is easily checked that $W_{k,L}$ is the kernel of ξ_k. According to [8], Corollary 3.8, the sequence

$$0 \longrightarrow W_{k,L} \longrightarrow H_{k,L} \xrightarrow{\xi_k} W_{2-k,L^-} \longrightarrow 0$$

is exact. We let $H^+_{k,L}$ be the preimage under ξ_k of the space of cusp forms S_{2-k,L^-} of weight $2-k$ with type ρ_{L^-}. Hence we have the exact sequence

$$0 \longrightarrow W_{k,L} \longrightarrow H^+_{k,L} \xrightarrow{\xi_k} S_{2-k,L^-} \longrightarrow 0\ .$$

The space $H^+_{k,L}$ can also be characterized as the subspace of those $f \in H_{k,L}$ whose Fourier coefficients $c^-(h,n)$ with non-negative index n vanish. This implies that

$$f(\tau) = \sum_{h\in L^\#/L} \sum_{n<0} c^+(h,n)e(n\tau)\mathfrak{e}_h + O(1), \qquad \Im(\tau) \to \infty,$$

i.e., the singularity at ∞, called the *principal part* of f, looks like the singularity of a weakly holomorphic form.

For $f \in H_{1-p/2,L^-}$, we can define the regularized theta lift $\Phi(z,f)$ as in (1.2), see [6], [8]. This generalized lift is related to the Kudla-Millson lift Λ defined in (1.1) in the following way (see [8], Theorem 6.1).

Theorem 1.4. *Let $f \in H^+_{1-p/2,L^-}$ and denote its Fourier expansion as in* (1.3)*. The* $(1,1)$*-form $dd^c\Phi(z,f)$ can be continued to a smooth form on X. It satisfies*

$$dd^c\Phi(z,f) = \Lambda(\xi_{1-p/2}(f))(z) + c^+(0,0)\Omega.$$

Here Ω denotes the invariant Kähler form on D normalized as in [8].

On the other hand, the following "weak converse theorem" is proved in [6], Theorem 4.23.

Theorem 1.5. *Assume that $p > r$. Let F be a meromorphic modular form for the group Γ with Heegner divisor*

$$\operatorname{div}(F) = \sum_h \sum_{n<0} c^+(h,n)Z(h,n)$$

(where $c^+(h,n) = c^+(-h,n)$ without loss of generality). Then there is a weak Maass form $f \in H^+_{1-p/2,L^-}$ with principal part $\sum_h \sum_{n<0} c^+(h,n)e(n\tau)\mathfrak{e}_h$ whose regularized theta lift satisfies

$$\Phi(z,f) = -2\log\|F\|_{\mathrm{Pet}} + \text{constant}. \tag{1.4}$$

Note that the proof in [6] is only given in the case that $p \geq 3$ (where the assumption on the Witt index is automatically fulfilled). However, the argument extends to the low dimensional cases. It is likely that the hypothesis on the Witt index can be dropped as well, but we have not checked this.

Corollary 1.6. *Assume that $p > r$. Let F be a meromorphic modular form for the group Γ with Heegner divisor as in Theorem 1.5. Let $f \in H^+_{1-p/2,L^-}$ be a weak Maass form whose regularized theta lift satisfies* (1.4)*. Then*

$$\Lambda(\xi_{1-p/2}(f)) = 0.$$

Proof. The assumption on f implies that

$$dd^c\Phi(z, f) = -2dd^c \log \|F\|_{\mathrm{Pet}} = c^+(0, 0)\Omega.$$

On the other hand, according to Theorem 1.4, we have

$$dd^c\Phi(z, f) = \Lambda(\xi_{1-p/2}(f))(z) + c^+(0, 0)\Omega.$$

If we combine these identities, we obtain the claim. □

Corollary 1.7. *Assume the hypotheses of Corollary 1.6. If Λ is injective, then f is weakly holomorphic, and F is a constant multiple of the Borcherds lift $\Psi(z, f)$ of f in the sense of Theorem 1.3.*

Proof. By Corollary 1.6 we have $\Lambda(\xi_{1-p/2}(f)) = 0$. Since Λ is injective, we find that $\xi_{1-p/2}(f) = 0$. But this means that f is weakly holomorphic. □

When the lattice L splits two hyperbolic planes over $\mathbb{Z}$, it was proved in [6] that Λ is injective by considering the Fourier expansion of the lift. In Section 4 of the present paper we show (for even unimodular lattices) how such injectivity results can be obtained by the Rallis inner products formula.

We end this section by stating a converse of Corollary 1.6. If $r > 0$, we let $\ell \in L$ be a primitive isotropic vector, and let $\ell' \in L^\#$ be a vector with $(\ell, \ell') = 1$. We let L_0 be the singular lattice $L \cap \ell^\perp$ and let K be the Lorentzian lattice $L_0/\mathbb{Z}\ell$.

Theorem 1.8. *Assume that $p \geq 2$ and $p > r$. Let $f \in H^+_{k,L^-}$ and assume that the Fourier coefficients $c^+(h, n)$ $(n < 0)$ of the principal part of f are integral. If $\xi_{1-p/2}(f) \in \ker(\Lambda)$, then there exists a meromorphic modular form F for Γ (with some multiplier system of finite order) such that:*

(i) *The weight of F is equal to $c^+(0, 0)/2$.*
(ii) *The divisor of F is equal to*

$$Z(f) = \sum_{h\in L^\#/L} \sum_{n<0} c^+(h, n)Z(h, n).$$

(iii) *In a neighborhood of a cusp of* Γ*, given by a primitive isotropic vector* $\ell \in L$*, the function* F *has an automorphic product expansion*

$$F(z) = Ce((\rho, z)) \prod_{\substack{\lambda \in K' \\ (\lambda, W) > 0}} \prod_{\substack{\delta \in L^{\#}/L \\ \delta|L_0 = \lambda}} \left(1 - e((\lambda, z)) + (\delta, \ell'))\right)^{c^+(\lambda, Q(\lambda))}.$$

Here C *is a non-zero constant, and we have used the notation of* [3].

Proof. Theorem 1.4 and the fact that $\Lambda\left(\xi_{1-p/2}(f)\right) = 0$ imply that

$$dd^c\Phi(z, f) = c^+(0,0)\Omega.$$

(In particular, if $c^+(0,0) = 0$, then f is pluriharmonic.) Now we can argue as in [6], Lemma 3.13 and Theorem 3.16 to prove the claim. □

We note that the assumption on r and p is needed to guarantee that the multiplier system of F has finite order. (When f is not weakly holomorphic, we cannot argue with the embedding trick as in [4], Correction).

If f is weakly holomorphic, then $\xi_{1-p/2}(f) = 0$ and the Theorem reduces to Theorem 1.3. However, if Λ is not injective, and f is a weak Maass form such that $\xi_{1-p/2}(f)$ is a non-trivial element of the kernel, then Theorem 1.8 leads to *exceptional automorphic products*. If there are any cases where Λ is not injective, it would be very interesting to construct examples of such exceptional automorphic products.

Remark 1.9. If $p \geq 4$, the existence of the meromorphic modular form F with divisor (ii) is related to the fact that $H^1(X, \mathcal{O}_X) = 0$ in this case, which can be proved following the argument of [10] §3.1. Therefore the Chern class map $\mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ is injective.

We thank S. Böcherer, E. Freitag, W. T. Gan, S. Kudla, and J. Millson for very helpful conversations on the content of this paper. Substantial work on this paper was done while the second named author visited the Max Planck Institut für Mathematik in Bonn/Germany during the summer 2005 and the Centre de Recerca Matemàtica in Bellaterra/Spain in the fall 2005 as part of the special year in Arakelov geometry and Shimura varieties. He thanks both institutions for their hospitality and for providing excellent working conditions.

2. Theta functions and the Siegel-Weil formula

Let V be a vector space over $\mathbb{Q}$ of dimension m with a non-degenerate bilinear form $(\, , \,)$. For simplicity we assume that m is even. We let $H = \mathrm{O}(V)$ be the orthogonal group of V, and we let $G = \mathrm{Sp}(n)$ be the symplectic group acting on a symplectic space of dimension $2n$ over $\mathbb{Q}$. The embedding of $\mathrm{U}(n)$ into $G(\mathbb{R})$ given by $\mathbf{k} = A + iB \mapsto k = \left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right)$ gives rise to a maximal compact subgroup $K_\infty \subset G(\mathbb{R})$. At the finite places, we pick the open compact subgroup $K_p = \mathrm{Sp}(n, \mathbb{Z}_p)$. Then $K = K_\infty \times \prod_p K_p$ is the corresponding maximal compact subgroup of $G(\mathbb{A})$, the symplectic group over the ring of adeles of $\mathbb{Q}$. We let $\omega = \omega_n$ be the Schrödinger model of the Weil representation of $G_\mathbb{A}$ acting on $\mathcal{S}(V^n(\mathbb{A}))$, the space of Schwartz-Bruhat functions on $V^n(\mathbb{A})$, associated to the standard additive character of $\mathbb{A}/\mathbb{Q}$ (which on $\mathbb{R}$ is given by $t \mapsto e(t) = e^{2\pi i t}$). Note that since m is even we do not have to deal with metaplectic coverings. We form the theta series associated to $\varphi \in \mathcal{S}(V^n(\mathbb{A}))$ by

$$\theta(g, h, \varphi) = \sum_{\mathbf{x} \in V^n(\mathbb{Q})} (\omega(g)\varphi)(h^{-1}\mathbf{x}), \tag{2.1}$$

with $g \in G(\mathbb{A})$ and $h \in H(\mathbb{A})$. We assume $\varphi = \varphi_\infty \otimes \varphi_f$ with $\varphi_\infty \in \mathcal{S}(V^n(\mathbb{R}))$ and $\varphi_f \in \mathcal{S}(V^n(\mathbb{A}_f))$.

We now briefly review the Siegel-Weil formula, see e.g. [16]. We put

$$I(g, \varphi) = \int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \theta(g, h, \varphi)dh, \tag{2.2}$$

where dh is the invariant measure on $H(\mathbb{Q})\backslash H(\mathbb{A})$ normalized to have total volume 1. By Weil's convergence criterion [29], $I(g, \varphi)$ is absolutely convergent if either V is anisotropic or if

$$m - r > n + 1. \tag{2.3}$$

Here r is the Witt index of V, i.e., the dimension of a maximal isotropic subspace of V over $\mathbb{Q}$.

We set $n(b) = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ for b a symmetric $n \times n$ matrix and $m(a) = \left(\begin{smallmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{smallmatrix}\right)$ for $a \in \mathrm{GL}(n)$. Then the Siegel parabolic is given by $P(\mathbb{A}) = N(\mathbb{A})M(\mathbb{A})$ with $N = \{n(b);\ b \in \mathrm{Mat}_n,\ b = {}^t b\}$ and $M = \{m(a);\ a \in \mathrm{GL}(n)\}$. Then using the Iwasawa decomposition $G(\mathbb{A}) = P(\mathbb{A})K$ we define

$$\Phi(g, s) = (\omega(g)\varphi)(0) \cdot \det |a(g)|_\mathbb{A}^{s - s_0}, \tag{2.4}$$

where

$$s_0 = \frac{m}{2} - \frac{n+1}{2}. \tag{2.5}$$

Thus Φ defines a section in a certain induced parabolic induction space (see [16] (I.3.6)). Note that Φ is determined by its values on K. Since Φ comes from $\varphi \in \mathcal{S}(V^n(\mathbb{A}))$, we also see that Φ is a standard section, i.e., its restriction to K does not depend on s, and we write $\Phi(k) = \Phi(k, s)$ for $k \in K$. Furthermore, Φ factors as $\Phi = \Phi_\infty \otimes \Phi_f$.

We then define the Eisenstein series associated to Φ by

$$E(g,s,\Phi) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \Phi(\gamma g, s), \tag{2.6}$$

which for $\mathrm{Re}(s) > \rho_n := (n+1)/2$ converges absolutely and has a meromorphic continuation to the whole complex plane. The extension of Weil's work [29] by Kudla and Rallis in the convergent range is:

Theorem 2.1. *([20], [21].) Assume Weil's convergence criterion holds.*

(i) *Then $E(g,s,\Phi)$ is holomorphic at $s = s_0$.*

(ii) *We have*

$$I(g,\varphi) = c_0 E(g, s_0, \Phi),$$

where $c_0 = 1$ if $m > n+1$ and $c_0 = 2$ if $m \leq n+1$.

We translate the adelic Eisenstein series into more classical language, see [16] section IV.2. We let $K_f(N) \subset \prod_p K_p$ be a subgroup of finite index of level N, i.e.,

$$\Gamma := G(\mathbb{Q}) \cap (G(\mathbb{R}) K_f(N))$$

contains the principal congruence subgroup $\Gamma(N) \subset \mathrm{Sp}(n,\mathbb{Z})$. We assume that Φ_f is $K_f(N)$-invariant. Furthermore, if φ_f corresponds to the characteristic function of a coset of an even lattice L of level N in V, then we have

$$\Phi_f(\gamma) = \prod_{p|N} \Phi_p(\gamma)$$

for $\gamma \in \Gamma$. Via $G(\mathbb{A}) = G(\mathbb{Q})G(\mathbb{R})K_f(N)$ we see that the Eisenstein series $E(g,s,\Phi)$ is determined by its restriction to $G(\mathbb{R})$. We assume that the restriction of $\Phi(g,s)$ to K_∞ is given by

$$\Phi^{\kappa}_{\infty}(k,s) := \det(\mathbf{k})^{\kappa}. \tag{2.7}$$

We denote the unique section at the Archimedean prime with this property by Φ^{κ}_{∞}. Let $g_\tau = n(u)m(a)$ with ${}^t a a = v$ be an element moving the base point $i1_n$ of the Siegel upper half plane $\mathbb{H}_n$ to $\tau = u + iv$. Then we obtain a classical Eisenstein series of weight κ (and level N):

$$\begin{aligned} E(g_\tau, s, \Phi) &= \sum_{\gamma \in (P(\mathbb{Q})\cap\Gamma)\backslash\Gamma} \Phi^{\kappa}_{\infty}(\gamma g_\tau)\Phi_f(\gamma) \\ &= \det(v)^{\kappa/2} \sum_{\gamma \in (P(\mathbb{Q})\cap\Gamma)\backslash\Gamma} \left(\frac{\det(v)}{|\det(c\tau+d)|^2}\right)^{(s+\rho_n-\kappa)/2} \det(c\tau+d)^{-\kappa}\, \Phi_f(\gamma), \end{aligned}$$

with $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. In particular, if $N = 1$ then

$$E(g_\tau, s, \Phi) = \det(v)^{\kappa/2} E^{(n)}_{\kappa}\big(\tau, (s+\rho_n-\kappa)/2\big), \tag{2.8}$$

where

$$E_{\kappa}^{(n)}(\tau,s)=\sum_{\gamma\in\Gamma_\infty\backslash\operatorname{Sp}(n,\mathbb{Z})}\big(\det\Im(\gamma\tau)\big)^s\det(c\tau+d)^{-\kappa} \tag{2.9}$$

is the classical Siegel Eisenstein series for $\operatorname{Sp}(n,\mathbb{Z})$ of weight κ.

For later use, we introduce an embedding ι_0 of $\operatorname{Sp}(n)\times\operatorname{Sp}(n)$ into $\operatorname{Sp}(2n)$ by

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\times\begin{pmatrix} a' & b'\\ c' & d'\end{pmatrix}\mapsto\begin{pmatrix} a & & b & \\ & a' & & b'\\ c & & d & \\ & c' & & d'\end{pmatrix}. \tag{2.10}$$

3. Special Schwartz forms

We change the setting in this section and consider the real place only. We assume that V is now a real quadratic space of signature (p,q) of dimension m. Since it does not make any extra work we do allow m odd in this section. We set $H=\operatorname{O}(V)$. We pick an oriented orthogonal basis $\{v_i\}$ of V such that $(v_\alpha,v_\alpha)=1$ for $\alpha=1,\dots,p$ and $(v_\mu,v_\mu)=-1$ for $\mu=p+1,\dots,m$, and we denote the corresponding coordinate functions by x_α and x_μ. We let z_0 be the q-dimensional subspace $\operatorname{span}\{v_{p+1},\dots,v_m\}$ with the induced orientation. We let K^H be the maximal compact subgroup of H stabilizing z_0. We realize the symmetric space $D=H/K^H$ associated to V as the Grassmannian of oriented negative q-planes in V. Thus D has two components

$$D=D_+\amalg D_-,$$

where

$$D_+=\{z\subset V;\quad \dim z=q,\ (\ ,\ )|_z<0,\ z\text{ has the same orientation as }z_0\}.$$

Thus $D_+\simeq H_0/K^{H_0}$ with $H_0=\operatorname{SO}_0(V)$, the connected component of the orthogonal group, and K^{H_0} the maximal compact subgroup of H_0 stabilizing z_0. We associate to $z\in D$ the standard majorant $(\ ,\ )_z$ given by

$$(x,x)_z=(x_{z^\perp},x_{z^\perp})-(x_z,x_z),$$

where $x=x_z+x_{z^\perp}\in V$ is given by the orthogonal decomposition $V=z\oplus z^\perp$. We write $(\ ,\ )_0=(\ ,\ )_{z_0}$.

Let $\mathfrak{h}$ be the Lie algebra of H and let $\mathfrak{h}=\mathfrak{p}\oplus\mathfrak{k}$ with $\mathfrak{k}^H=\operatorname{Lie}(K^H)$ be the associated Cartan decomposition. Then $\mathfrak{p}\simeq\mathfrak{h}/\mathfrak{k}^H$ is isomorphic to the tangent space at the base point z_0 of D. With respect to the above basis of V we have

$$\mathfrak{p}\simeq\left\{\begin{pmatrix} 0 & X\\ {}^tX & 0\end{pmatrix};\ X\in\operatorname{Mat}_{p,q}(\mathbb{R})\right\}. \tag{3.1}$$

We let $X_{\alpha\mu}$ ($1 \le \alpha \le p$, $p+1 \le \mu \le p+q$) denote the element of $\mathfrak{p}$ which interchanges v_α and v_μ and annihilates all the other basis elements of V. We write $\omega_{\alpha\mu}$ for the element of the dual basis corresponding to $X_{\alpha\mu}$.

We let $\omega = \omega_n$ be the Weil representation of the metaplectic cover $\mathrm{Mp}(n,\mathbb{R})$ of $\mathrm{Sp}(n,\mathbb{R})$ acting on the Schwartz functions $\mathcal{S}(V^n)$. We let $K = \widetilde{\mathrm{U}}(n)$ be the maximal compact subgroup of $\mathrm{Mp}(n,\mathbb{R})$ given by the inverse image of the standard maximal compact subgroup $\mathrm{U}(n)$ in $\mathrm{Sp}(n,\mathbb{R})$. Recall that K admits a character $\det^{1/2}$ whose square descends to the determinant character of $\mathrm{U}(n)$. We also write ω for the associated Lie algebra action on the space of K-finite vectors in $\mathcal{S}(V^n)$. It is given by the so-called polynomial Fock space $S(V^n) \subset \mathcal{S}(V^n)$. It consists of those Schwartz functions on V^n of the form $p(\mathbf{x})\varphi_0(\mathbf{x})$, where $p(\mathbf{x})$ is a polynomial function on V^n. Here $\varphi_0(\mathbf{x})$ is the standard Gaussian on V^n. More precisely, for $\mathbf{x} = (x_1, \dots, x_n) \in V^n$ and $z \in D$, we let

$$\varphi_0(\mathbf{x}, z) = \exp\left(-\pi \sum_{i=1}^{n} (x_i, x_i)_z\right),$$

and set $\varphi_0(\mathbf{x}) = \varphi_0(\mathbf{x}, z_0)$. We view

$$\varphi_0 \in [\mathcal{S}(V^n) \otimes C^\infty(D)]^H \simeq [\mathcal{S}(V^n) \otimes \bigwedge\nolimits^0(\mathfrak{p}^*)]^{K^H}, \tag{3.2}$$

where the isomorphism is given by evaluation at the base point z_0 of D. In the following we will identify corresponding objects under this isomorphism.

Kudla and Millson (see [18]) constructed (in much greater generality) Schwartz forms φ_{KM} on V taking values in $\mathcal{A}^q(D)$, the differential q-forms on D. More precisely,

$$\varphi_{KM} \in [\mathcal{S}(V) \otimes \mathcal{A}^q(D)]^H \simeq [\mathcal{S}(V) \otimes \bigwedge\nolimits^q(\mathfrak{p}^*)]^{K^H},$$

where the isomorphism is again given by evaluation at the base point of D. The Schwartz form φ_{KM} is given by

$$\varphi_{KM} = \frac{1}{2^{q/2}} \prod_{\mu=p+1}^{p+q} \left[\sum_{\alpha=1}^{p} \left(x_\alpha - \frac{1}{2\pi}\frac{\partial}{\partial x_\alpha}\right) \otimes A_{\alpha\mu}\right] \varphi_0.$$

Here $A_{\alpha\mu}$ denotes the left multiplication by $\omega_{\alpha\mu}$. More generally, we consider the Schwartz forms

$$\varphi_{q,\ell} \in [\mathcal{S}(V) \otimes \bigwedge\nolimits^q(\mathfrak{p}^*) \otimes \mathrm{Sym}^\ell(V)]^{K^H}$$

with values in the ℓ-th symmetric powers of V introduced by Funke and Millson [12]. Viewing the Kudla-Millson form φ_{KM} as the form $\varphi_{q,0} \in$

$[\mathcal{S}(V) \otimes \bigwedge^q(\mathfrak{p}^*) \otimes \mathrm{Sym}^0(V)]^{K_H}$, the forms $\varphi_{q,\ell}$ are given by

$$\begin{aligned}\varphi_{q,\ell} &= \left[\frac{1}{2}\sum_{\alpha=1}^{p}\left(x_\alpha - \frac{1}{2\pi}\frac{\partial}{\partial x_\alpha}\right)\otimes 1 \otimes A_{v_\alpha}\right]^{\ell}\varphi_{KM} \\ &= \frac{1}{2^\ell}\sum_{\alpha_1,\dots,\alpha_\ell=1}^{p}\left[\prod_{i=1}^{\ell}\left(x_{\alpha_i} - \frac{1}{2\pi}\frac{\partial}{\partial x_{\alpha_i}}\right)\otimes 1 \otimes \prod_{i=1}^{\ell} A_{v_{\alpha_i}}\right]\varphi_{KM}.\end{aligned}$$

Here A_v denotes the multiplication with the vector v in the symmetric algebra of V. Note that $\mathrm{Sym}^\ell(V)$ is not an irreducible representation of H, and we denote by $\varphi_{q,[\ell]}$ the projection of $\varphi_{q,\ell}$ onto $\mathcal{H}^\ell(V)$, the harmonic ℓ-tensors in V. It consists of those symmetric ℓ-tensors which are annihilated by the signature (p,q)-Laplacian $\Delta = \sum_{\alpha=1}^{p}\frac{\partial^2}{\partial v_\alpha^2} - \sum_{\mu=p+1}^{m}\frac{\partial^2}{\partial v_\mu^2}$. Here we view v_α and v_μ as independent variables. It can be also characterized as the space of symmetric ℓ-tensors in V which are orthogonal with respect to the induced inner product on $\mathrm{Sym}^\ell(V)$ to vectors of the form r^2w. Here $w \in \mathrm{Sym}^{\ell-2}(V)$ and r^2 denotes the multiplication with $\sum_{\alpha=1}^{p} v_\alpha^2 - \sum_{\mu=p+1}^{m} v_\mu^2$. Recall that we have $\mathrm{Sym}^\ell(V) = \mathcal{H}^\ell(V) \oplus r^2\,\mathrm{Sym}^{\ell-2}(V)$ as representations of H.

The Schwartz form $\varphi_{q,\ell}$ (and also $\varphi_{q,[\ell]}$) is an eigenfunction of weight $m/2+\ell$ under the action of $k \in K$, see [18, 12], i.e.,

$$\omega(k)\varphi_{q,\ell} = \det(\mathbf{k})^{m/2+\ell}\varphi_{q,\ell}. \tag{3.3}$$

Here $\mathbf{k}$ is the element in $\widetilde{\mathrm{U}}(1)$ corresponding to $k \in \widetilde{\mathrm{SO}}(2) \subset \mathrm{Mp}(1,\mathbb{R})$. Moreover, $\varphi_{q,\ell}(x)$ is a closed differential form on D.

We normalize the inner product on $\mathrm{Sym}^\ell(V)$ inductively by setting

$$(w_1\cdots w_\ell, w_1'\cdots w_\ell') = \frac{1}{\ell}\sum_{j=1}^{\ell}(w_1,w_j')(w_2\cdots w_\ell, w_1'\cdots\widehat{w_j'}\cdots w_\ell').$$

With this normalization we easily see that for the restriction of $(\ ,\ )$ to the positive definite subspace $\mathrm{span}\{v_\alpha;\ 1 \le \alpha \le p\}$ of V we have

$$\sum_{\substack{\alpha_1,\dots,\alpha_\ell=1\\ \beta_1,\dots,\beta_\ell=1}}^{p}\left(\prod_{i=1}^{\ell} v_{\alpha_i}, \prod_{i=1}^{\ell} v_{\beta_i}\right) = p^\ell.$$

We let $\widetilde{\mathrm{Sym}}^\ell(V)$ be the local system on D associated to $\mathrm{Sym}^\ell(V)$. Then for the wedge product, we have $\wedge : \mathcal{A}^r(D, \widetilde{\mathrm{Sym}}^\ell(V)) \times \mathcal{A}^s(D, \widetilde{\mathrm{Sym}}^\ell(V)) \to \mathcal{A}^{r+s}(D)$ by taking the inner product on the fibers $\mathrm{Sym}^\ell(V)$. We are ultimately more interested in the form $\varphi_{q,[\ell]}$, but calculations with $\varphi_{q,\ell}$ are more convenient. In this context the following lemma will be important later.

Lemma 3.1. *Let* $\eta \in \mathcal{A}^{(p-1)q}(D, \widetilde{\mathrm{Sym}}^{\ell-2}(V))$. *Then*

$$\varphi_{q,\ell} \wedge r^2\eta = -\frac{1}{2\pi}(\omega(R)\varphi_{q,\ell-2}) \wedge \eta.$$

Here $R = \frac{1}{2}\left(\begin{smallmatrix} 1 & i \\ i & -1 \end{smallmatrix}\right) \in \mathfrak{sl}(2,\mathbb{C})$ *is the standard* $\mathrm{SL}(2)$*-raising operator.*

Proof. By the adjointness of $\frac{1}{\ell(\ell-1)}\Delta$ and r^2 with respect to the inner product in $\mathrm{Sym}^\bullet(V)$, we have $\varphi_{q,\ell} \wedge r^2\eta = \frac{1}{\ell(\ell-1)}(\Delta\varphi_{q,\ell}) \wedge \eta$. Note that Δ operates on the coefficient part of $\varphi_{q,\ell}$. Then switching to the Fock model of the Weil representation, see the proof of Lemma 3.5, and using (3.11) one easily sees $\Delta\varphi_{q,\ell} = -\frac{\ell(\ell-1)}{2\pi}\omega(R)\varphi_{q,\ell-2}$. We leave the details to the reader. □

We let $*$ denote the Hodge $*$-operator on D. Then $\varphi_{q,\ell}(x_1) \wedge *\varphi_{q,\ell}(x_2)$ with $\mathbf{x} = (x_1, x_2) \in V^2$, being a top degree differential form, gives rise to a scalar-valued Schwartz function $\phi_{q,\ell}$ on V^2 defined by

$$\phi_{q,\ell}(\mathbf{x}, z)\mu = \varphi_{q,\ell}(x_1, z) \wedge *\varphi_{q,\ell}(x_2, z). \tag{3.4}$$

Here μ is the volume form on D induced by the Riemannian metric coming from the Killing form on $\mathfrak{g}$. For convenience we scale the metric such that the restriction of μ to the base point z_0 is given by

$$\mu = \omega_{1,p+1} \wedge \cdots \wedge \omega_{1,p+q} \wedge \omega_{2,p+1} \wedge \cdots \wedge \omega_{p,p+q}. \tag{3.5}$$

Note that

$$\phi_{q,\ell} \in [\mathcal{S}(V^2) \otimes C^\infty(D)]^{H_0} \simeq [\mathcal{S}(V^2) \otimes \textstyle\bigwedge^0(\mathfrak{p}^*)]^{K^{H_0}}.$$

Lemma 3.2. *We have*

$$\begin{aligned}\phi_{q,\ell}(\mathbf{x}) &= \frac{p^\ell}{2^{q+2\ell}} \sum_{\alpha_1,\dots,\alpha_{q+\ell}=1}^{p} \prod_{i=1}^{q+\ell} \left(x_{\alpha_i 1} - \frac{1}{2\pi}\frac{\partial}{\partial x_{\alpha_i 1}}\right)\left(x_{\alpha_i 2} - \frac{1}{2\pi}\frac{\partial}{\partial x_{\alpha_i 2}}\right)\varphi_0(\mathbf{x}) \\ &= \frac{p^\ell}{2^{q+2\ell}} \left(\sum_{\alpha=1}^{p} \left(x_{\alpha 1} - \frac{1}{2\pi}\frac{\partial}{\partial x_{\alpha 1}}\right)\left(x_{\alpha 2} - \frac{1}{2\pi}\frac{\partial}{\partial x_{\alpha 2}}\right)\right)^{q+\ell} \varphi_0(\mathbf{x}).\end{aligned}$$

Example 3.3. For signature $(p, 2)$, we have

$$\phi_{q,0}(\mathbf{x}) = \sum_{\alpha=1}^{p}(x_{\alpha 1}^2 - \frac{1}{4\pi})(x_{\alpha 2}^2 - \frac{1}{4\pi})\varphi_0(\mathbf{x}) + 4\sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{p} x_{\alpha 1}x_{\beta 1}x_{\alpha 2}x_{\beta 2}\varphi_0(\mathbf{x}).$$

Note that (3.3) immediately implies:

Lemma 3.4. *For $k_1, k_2 \in \widetilde{\mathrm{SO}}(2) \subset \mathrm{Mp}(1,\mathbb{R})$, we have*

$$\omega(\iota_0(k_1,k_2))\phi_{q,\ell} = \det(\mathbf{k_1k_2})^{m/2+\ell}\phi_{q,\ell}.$$

The action of the full maximal compact $K \subset \mathrm{Mp}(2,\mathbb{R})$ on $\phi_{q,\ell}$ via the Weil representation is more complicated, as we now explain. We let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_+ \oplus \mathfrak{p}_- \tag{3.6}$$

be a Harish-Chandra decomposition of $\mathfrak{g} = \mathfrak{sp}(2,\mathbb{C})$, where $\mathfrak{k} = \mathrm{Lie}(K)_{\mathbb{C}}$,

$$\mathfrak{p}_+ = \left\{ p_+(X) = \frac{1}{2}\begin{pmatrix} X & iX \\ iX & -X \end{pmatrix};\ X \in \mathrm{Mat}_2(\mathbb{C}), {}^tX = X \right\}, \tag{3.7}$$

and $\mathfrak{p}_- = \overline{\mathfrak{p}_+}$. Note that $\mathfrak{p}_+$ is the holomorphic tangent space of $\mathbb{H}_2$ at the base point $i1_2$ and is spanned by the raising operators

$$R_1 = R_{11} = p_+\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \qquad R_2 = R_{22} = p_+\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right), \tag{3.8}$$

$$R_{12} = \frac{1}{2}p_+\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right). \tag{3.9}$$

Note that $R_1 = \iota_0(R,0)$ and $R_2 = \iota_0(0,R)$ are the images of the SL_2-raising operator R in $\mathfrak{sp}(2,\mathbb{C})$ under the two standard embeddings of $\mathfrak{sl}(2)$ into $\mathfrak{sp}(2)$.

Recall that the adjoint action of K on $\mathfrak{p}_+$ is isomorphic to the standard action of K on $\mathrm{Sym}^2(\mathbb{C}^2)$. Explicitly, the intertwiner is given by $R_{rs} \mapsto e_r e_s$, where e_1, e_2 denotes the standard basis of $\mathbb{C}^2$. We obtain an isomorphism of K-modules

$$\mathrm{Sym}^\bullet \mathrm{Sym}^2 \mathbb{C}^2 = \bigoplus_{j=0}^{\infty} \mathrm{Sym}^j \mathrm{Sym}^2 \mathbb{C}^2 \simeq U(\mathfrak{p}_+) \tag{3.10}$$

of the symmetric algebra on $\mathrm{Sym}^2 \mathbb{C}^2$ with the universal enveloping algebra of $\mathfrak{p}_+$.

Lemma 3.5. *We have*

$$\phi_{q,\ell} = \frac{p^\ell(-1)^{q+\ell}}{2^\ell \pi^{q+\ell}} \omega(R_{12})^{q+\ell}\varphi_0.$$

Proof. We indicate a quick proof using the Fock model of the Weil representation. For more details for what follows, see the appendix of [12]. There is an intertwining map $\iota : S(V^n) \to \mathcal{P}(\mathbb{C}^{n(p+q)})$ from the polynomial Fock space to the infinitesimal Fock model of the Weil representation acting on the space of complex polynomials $\mathcal{P}(\mathbb{C}^{n(p+q)})$ in $n(p+q)$ variables such that $\iota(\varphi_0) = 1$. We denote the variables in $\mathcal{P}(\mathbb{C}^{n(p+q)})$ by $z_{\alpha i}$ $(1 \leq \alpha \leq p)$ and $z_{\mu i}$ $(p+1 \leq \mu \leq p+q)$ with $i = 1,\dots,n$. Moreover, the intertwining map ι satisfies

$$\iota\left(x_{\alpha i} - \frac{1}{2\pi}\frac{\partial}{\partial x_{\alpha i}}\right)\iota^{-1} = \frac{1}{2\pi i} z_{\alpha i}.$$

Hence in the Fock model, we have

$$\phi_{q,\ell} = \frac{p^\ell}{2^{q+2\ell}} \left(\frac{1}{2\pi i}\right)^{2(q+\ell)} \left[\sum_{\alpha=1}^{p} z_{\alpha 1} z_{\alpha 2}\right]^{q+\ell}.$$

On the other hand, for the action of the raising operators, we find

$$\omega(R_{rs}) = \frac{1}{8\pi} \sum_{\alpha=1}^{p} z_{\alpha r} z_{\alpha s} - 2\pi \sum_{\mu=p+1}^{m} \frac{\partial^2}{\partial z_{\mu r} \partial z_{\mu s}}. \tag{3.11}$$

In the Fock model, we therefore have $\omega(R_{12})^{q+\ell}\varphi_0 = \left[\frac{1}{8\pi}\sum_{\alpha=1}^{p} z_{\alpha 1} z_{\alpha 2}\right]^{q+\ell}$, and the lemma follows. □

We obtain:

Proposition 3.6. *For $k \in K \simeq \widetilde{\mathrm{U}}(2)$, we have*

$$\omega(k)\phi_{q,\ell} = \frac{p^\ell(-1)^{q+\ell}}{2^\ell \pi^{q+\ell}} \det(\mathbf{k})^{(p-q)/2} \left(\mathrm{Ad}(k)R_{12}\right)^{q+\ell} \varphi_0.$$

Proof. This follows immediately from Lemma 3.5 and the fact that the Gaussian φ_0 has weight $(p-q)/2$. □

Remark 3.7. The Kudla-Millson forms φ_{KM} cannot be expressed in terms of elements in $\mathfrak{p}_+$.

Proposition 3.6 reduces the K-action on $\phi_{q,\ell}$ to the representation theory of the group $\mathrm{U}(2)(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C})$ on $\mathrm{Sym}^\bullet \mathrm{Sym}^2 \mathbb{C}^2$, which is given as follows.

Lemma 3.8. *The $\mathrm{GL}_2(\mathbb{C})$-representation $\mathrm{Sym}^j \mathrm{Sym}^2 \mathbb{C}^2$ decomposes as*

$$\mathrm{Sym}^j \mathrm{Sym}^2 \mathbb{C}^2 \simeq \bigoplus_{i=0}^{[j/2]} \mathrm{Sym}^{2j-4i} \mathbb{C}^2 \otimes \det{}^{2i}$$

into its irreducible constituents. The summand for $i = [j/2]$ is given by

$$\mathrm{Sym}^{2j-4[j/2]} \mathbb{C}^2 \otimes \det{}^{2[j/2]} = \begin{cases} \det{}^j & \text{if } j \text{ is even,} \\ \mathrm{Sym}^2 \mathbb{C}^2 \otimes \det{}^{j-1} & \text{if } j \text{ is odd,} \end{cases} \tag{3.12}$$

and is generated by the vector

$$\alpha_j = \sum_{i=0}^{[j/2]} \binom{[j/2]}{i} (-1)^i (e_1^2)^i (e_2^2)^i (e_1 e_2)^{j-2i} = \tag{3.13}$$

$$= \begin{cases} \left[(e_1e_2)^2 - e_1^2 e_2^2\right]^{j/2} & \text{if } j \text{ is even,} \\ (e_1 e_2)\left[(e_1e_2)^2 - e_1^2 e_2^2\right]^{[j/2]} & \text{if } j \text{ is odd.} \end{cases}$$

Proof. For the first statement, see e.g. [11], p.81/82. For (3.13), note that in

$$\operatorname{Sym}^2\operatorname{Sym}^2\mathbb{C}^2 = \operatorname{Sym}^4\mathbb{C}\oplus\det{}^2,$$

the vector

$$\alpha_2 = (e_1e_2)^2 - e_1^2e_2^2$$

generates the one-dimensional sub-representation. Then, for j even, α_j is given by the image of $(\alpha_2)^{j/2}\in\operatorname{Sym}^{j/2}\operatorname{Sym}^2\operatorname{Sym}^2\mathbb{C}^2$ under the projection onto $\operatorname{Sym}^j\operatorname{Sym}^2\mathbb{C}^2$. The argument for j odd is analogous. □

By slight abuse of notation, we also write α_j for the corresponding element in $\mathrm{U}(\mathfrak{p}_+)$ and define another Schwartz function $\xi=\xi_{q,\ell}\in\mathcal{S}(V^2)$ by

$$\xi=\xi_{q,\ell}=\frac{p^\ell(-1)^{q+\ell}}{2^\ell\pi^{q+\ell}}\omega(\alpha_j)\varphi_0. \tag{3.14}$$

Proposition 3.9. *For the Schwartz function $\phi_{q,\ell}$, there exists a $\psi\in\mathcal{S}(V^2)$ such that*

$$\phi_{q,\ell}=\xi_{q,\ell}+\omega(R_1)\omega(R_2)\psi. \tag{3.15}$$

Proof. We have

$$(e_1e_2)^{q+\ell}-\alpha_{q+\ell}=e_1^2e_2^2\sum_{i=1}^{[(q+\ell)/2]}\binom{[(q+\ell)/2]}{i}(-1)^i(e_1^2)^{i-1}(e_2^2)^{i-1}(e_1e_2)^{q+\ell-2i}.$$

Using the intertwiner with $\mathrm{U}(\mathfrak{p}_+)$, we recall that e_i^2 corresponds to R_i. Thus ψ is given by

$$\psi=\frac{p^\ell(-1)^{q+\ell}}{2^\ell\pi^{q+\ell}}\sum_{i=1}^{[(q+\ell)/2]}\binom{[(q+\ell)/2]}{i}(-1)^i\omega\left(R_1^{i-1}R_2^{i-1}R_{12}^{q+\ell-2i}\right)\varphi_0.$$

□

One easily sees using (3.11):

Lemma 3.10. *The Schwartz function ξ vanishes identically if and only if $p=1$ and $q+\ell>1$.*

Example 3.11. For $q=2$, $p>1$, and $\ell=0$, we have

$$\phi_{2,0}\cdot\Omega^p = C\varphi_{KM}\wedge\varphi_{KM}\wedge\Omega^{p-2}+C'\omega(R_1)\omega(R_2)\varphi_0\cdot\Omega^p$$

for some nonzero constants C and C'. Here Ω denotes the Kähler form on the Hermitian domain D. But we will not need this.

In view of Lemma 3.8 and Proposition 3.6, we see for $q+\ell$ even that

$$\omega(k)\xi = \det(\mathbf{k})^{m/2+\ell}\xi \tag{3.16}$$

for $k \in K$. We let $\Xi(g,s)$ be the section in the induced representation corresponding to the Schwartz function ξ via (2.4).

Proposition 3.12. *Let $q+\ell$ be even. Then Ξ is the standard section* (2.7) *at the infinite place of weight $m/2+\ell$. More precisely,*

$$\Xi(s) = C(s)\Phi_\infty^{m/2+\ell}(s) \tag{3.17}$$

for a certain (explicit) polynomial $C(s)$. Moreover,

$$C(s_0) \neq 0$$

with $s_0 = (m-3)/2$ as in (2.5) *for $p>1$, while $C(s)\equiv 0$ for $p=1$.*

Proof. The identity (3.17) follows from (3.16) and the uniqueness of $\Phi_\infty^{m/2+\ell}$. The precise statement follows from considerations in [22]. The element $\alpha_{q+\ell}$ is trivially a highest weight vector of weight $\mu=(q+\ell,q+\ell)$ of $\mathrm{GL}_2(\mathbb{C})$. Therefore we can take $\alpha_{q+\ell}$ equal to the element $u_\mu^0 \in \mathrm{U}(\mathfrak{p}_+)$ (or $u_\mu \in \mathrm{U}(\mathfrak{g})$) in the notation of [22], p.31/32. Then by Corollary 1.4 of [22], we have $\Xi(s) = u_\mu \Phi_\infty^{(p-q)/2}(s) = cP_\mu^{(p-q)/2}(s)\Phi^{m/2+\ell}(s)$, for a certain polynomial $P_\mu^{(p-q)/2}$ and a nonzero constant c. One easily sees $P_\mu^{(p-q)/2}(s_0)\neq 0$ for $p>1$. See also [22], p. 38. For $p=1$, Ξ vanishes identically, since already $\xi=0$ by Lemma 3.10. □

Remark 3.13. For $q+\ell$ odd, we see in the same way

$$\Xi(s) = C(s)R_{12}\Phi_\infty^{m/2+\ell-1}(s)$$

for a certain polynomial $C(s)$. Note that $\alpha_{q+\ell}$ is *not* a highest weight vector for $\mathrm{Sym}^2\,\mathbb{C}^2\otimes \det^{\,q+\ell-1}$ (which has weight $(q+\ell+1,q+\ell-1)$).

4. The L^2-norm of the theta lift

We now return to the global situation and retain the notation of Section 2. Let V be a non-degenerate quadratic space over $\mathbb{Q}$ of signature (p,q) and even dimension $m=p+q$. We let $L\subset V$ be an even lattice and write $L^\#$ for the dual lattice. For each prime p, we let $L_p = L\otimes\mathbb{Z}_p$ and let K_p^H be the subgroup of $\mathrm{O}(L_p)$ given by the kernel of $\mathrm{O}(L_p)\to\mathrm{O}(L_p^\#/L_p)$. Then $K_f^H=\prod_p K_p^H$ is an open compact subgroup of $H(\mathbb{A}_f)$. We write $H(\mathbb{R})_0 = \mathrm{SO}_0(V(\mathbb{R}))$, and we let K_∞^H be a maximal compact subgroup of $H(\mathbb{R})$. Then $D=H(\mathbb{R})/K_\infty^H$ is the symmetric domain of oriented negative

q-planes considered in the previous section. By strong approximation we write

$$H(\mathbb{A}) = \coprod_j H(\mathbb{Q})H(\mathbb{R})_0 h_j K_f^H \tag{4.1}$$

with $h_j \in H(\mathbb{A}_f)$. Then we put

$$X = X_{K_f^H} = H(\mathbb{Q})\backslash(D \times H(\mathbb{A}_f))/K_f^H \tag{4.2}$$

such that

$$X \simeq \coprod_j X_j \tag{4.3}$$

with $X_j = \Gamma_j\backslash D_+$, where $\Gamma_j = H(\mathbb{Q}) \cap (H(\mathbb{R})_0 h_j K_f^H h_j^{-1})$. We let $\varphi_f \in \mathcal{S}(V(\mathbb{A}_f))^{K_f^H}$ be a K_f^H-invariant Schwartz function on the finite adeles. Then φ_f corresponds to a linear combination of characteristic functions on the discriminant group $L^\#/L$. Since $\varphi_{q,\ell}$ is an eigenfunction of weight

$$\kappa = m/2 + \ell$$

under the action of $\mathrm{U}(1)$, we can form the classical theta function on $\mathbb{H}$, the upper half space, by setting

$$\begin{aligned}\theta(\tau, z, \varphi_{q,\ell} \otimes \varphi_f) &= v^{-\kappa/2} \sum_{x \in V(\mathbb{Q})} \varphi_f(x)\omega_\infty(g_\tau)\varphi_{q,\ell}(x, z)\\ &= v^{-\ell/2} \sum_{x \in V(\mathbb{Q})} \varphi_f(x)\varphi_{q,\ell}(\sqrt{v}x, z)e^{\pi i(x,x)u}.\end{aligned}$$

Here $\tau = u + iv \in \mathbb{H}$, and $g_\tau = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \sqrt{v} & 0 \\ 0 & \sqrt{v}^{-1} \end{smallmatrix}\right) \in G(\mathbb{R}) \subset G(\mathbb{A})$ is the standard element moving the base point $i \in \mathbb{H}$ to τ. Then $\theta(\tau, z, \varphi_{q,\ell} \otimes \varphi_f)$ transforms like a non-holomorphic modular form of weight κ for the principal congruence subgroup $\Gamma(N)$ of $\mathrm{SL}_2(\mathbb{Z})$ taking values in the differential q-forms on X. Here N is the level of L, i.e., the smallest positive integer such that $\frac{1}{2}N(x, x) \in \mathbb{Z}$ for all $x \in L^\#$. In particular, if L is unimodular, $\theta(\tau, z, \varphi_{q,\ell} \otimes \varphi_f)$ is a form for the full modular group $\mathrm{SL}_2(\mathbb{Z})$.

We write $S_\kappa(\Gamma(N))$ for the space of cusp forms of weight κ for $\Gamma(N)$. We normalize the Petersson scalar product be putting

$$(f, g) = \frac{1}{[\Gamma(1) : \Gamma(N)]} \int_{\Gamma(N)\backslash\mathbb{H}} f(\tau)\overline{g(\tau)}v^\kappa \, d\mu(\tau) \tag{4.4}$$

for $f, g \in S_\kappa(\Gamma(N))$. Here $d\mu(\tau) = \frac{du\,dv}{v^2}$ is the invariant measure on $\mathbb{H}$. For $f \in S_\kappa(\Gamma(N))$, we consider the theta lift

$$\Lambda(f) = \big(f, \theta(\tau, \varphi_{q,\ell} \otimes \varphi_f)\big) = \int_{\Gamma(N)\backslash\mathbb{H}} f(\tau)\overline{\theta(\tau, \varphi_{q,\ell} \otimes \varphi_f)}v^\kappa \, d\mu(\tau). \tag{4.5}$$

It defines a linear map

$$\Lambda : S_\kappa(\Gamma(N)) \longrightarrow \mathcal{Z}^q(X, \widetilde{\mathrm{Sym}}^\ell(V)) \tag{4.6}$$

into the $\widetilde{\mathrm{Sym}}^\ell(V)$-valued closed differential q-forms on X.

In order to show the injectivity of Λ, we study its L^2-norm given by

$$\|\Lambda(f)\|_2^2 = \int_X \Lambda(f) \wedge * \overline{\Lambda(f)}. \tag{4.7}$$

We will use the *doubling method* to compute $\|\Lambda(f)\|_2^2$, see [5, 13, 25, 26].

Proposition 4.1. *Assume that $m > 3 + r$ so that Weil's convergence criterion (2.3) in genus 2 holds. Then $\Lambda(f)$ is square integrable, and*

$$\|\Lambda(f)\|_2^2 = \left(f(\tau_1) \otimes \overline{f(\tau_2)}, \tilde{I}(\tau_1, -\bar{\tau}_2, \phi_{q,\ell} \otimes \phi_f) \right), \tag{4.8}$$

where $(\ ,\ )$ denotes the Petersson scalar product on $\Gamma(N) \times \Gamma(N)$ and

$$\tilde{I}(\tau_1, \tau_2, \phi_{q,\ell} \otimes \phi_f) = \int_X \theta(\tau_1, \tau_2, z, \phi_{q,\ell} \otimes \phi_f)\mu \tag{4.9}$$

is the integral over the locally symmetric space of the theta series
(4.10)

$$\theta(\tau_1, \tau_2, z, \phi_{q,\ell} \otimes \phi_f) = (v_1 v_2)^{-\kappa/2} \sum_{\mathbf{x} \in V^2(\mathbb{Q})} \phi_f(\mathbf{x})(\omega_\infty(\iota_0(g_{\tau_1}, g_{\tau_2}))\phi_{q,\ell}(\mathbf{x}, z),$$

(which by (3.3) defines a modular form of weight κ on $\Gamma(N) \times \Gamma(N)$). Here $\phi_f = \varphi_f \otimes \varphi_f \in \mathcal{S}(V^2(\mathbb{A}_f))$.

Proof. The formula (4.8) implies the square integrability since the right hand side of (4.8) is absolutely convergent by Weil's convergence criterion (2.3). We have

$$\|\Lambda(f)\|_2^2 = \int_X \left(\int_{\Gamma(N)\backslash\mathbb{H}} f(\tau_1)\overline{\theta(\tau_1, \varphi_{q,\ell} \otimes \varphi_f)} v_1^\kappa d\mu(\tau_1) \right) \wedge \overline{\left(\int_{\Gamma(N)\backslash\mathbb{H}} f(\tau_2)\overline{\theta(\tau_2, *\varphi_{q,\ell} \otimes \varphi_f)} v_2^\kappa d\mu(\tau_2) \right)}.$$

Interchanging the integration, we obtain

$$\iint f(\tau_1)\overline{f(\tau_2)} \overline{\left(\int_X \theta(\tau_1, \varphi_{q,\ell} \otimes \varphi_f) \wedge \overline{\theta(\tau_2, *\varphi_{q,\ell} \otimes \varphi_f)} \right)} (v_1 v_2)^\kappa d\mu(\tau_1) d\mu(\tau_2).$$

Since $\varphi_{q,\ell}$ is real valued, we easily see by the explicit formulas of the Weil representation that

$$\overline{\theta(\tau_2, *\varphi_{q,\ell} \otimes \varphi_f)} = \theta(-\bar{\tau}_2, *\varphi_{q,\ell} \otimes \varphi_f)$$

and therefore

$$\theta(\tau_1, \varphi_{q,\ell} \otimes \varphi_f) \wedge \overline{\theta(\tau_2, *\varphi_{q,\ell} \otimes \varphi_f)} = \theta(\tau_1, -\bar{\tau}_2, z, \phi_{q,\ell} \otimes \phi_f)\mu$$

by (3.4). This implies the assertion. □

Remark 4.2. For signature $(p, 2)$, the lift $\Lambda(f)$ is actually always square integrable, see [6, 8]. We expect this to be true for other signatures as well even if Weil's convergence criterion does not hold. In that case, one would need to regularize the theta integral $\tilde{I}$ as in [23].

Note that the Schwartz function ξ introduced by (3.14) is K_∞^H-invariant. We can therefore consider $\xi \in [\mathcal{S}(V^2) \otimes C^\infty(D)]^{H(\mathbb{R})}$ by setting

$$\xi(\mathbf{x}, z) = \xi(h_\infty^{-1}\mathbf{x})$$

with $h_\infty \in H(\mathbb{R})$ such that $h_\infty z_0 = z$. In particular, $\xi(\mathbf{x}, z_0) = \xi(\mathbf{x})$.

Proposition 4.3. *Define $\theta(\tau_1, \tau_2, z, \xi \otimes \phi_f)$ and $\tilde{I}(\tau_1, \tau_2, \xi \otimes \phi_f)$ in the same way as for $\phi_{q,\ell}$ in (4.10), (4.9). Then*

$$\|\Lambda(f)\|_2^2 = \Big(f(\tau_1) \otimes \overline{f(\tau_2)}, \tilde{I}(\tau_1, -\bar{\tau}_2, \xi \otimes \phi_f)\Big).$$

Proof. By Proposition 3.9 and Proposition 4.1, we see (omitting ϕ_f from the notation)

$$\begin{aligned}\|\Lambda(f)\|_2^2 &= \Big(f(\tau_1) \otimes \overline{f(\tau_2)}, \tilde{I}(\tau_1, -\bar{\tau}_2, \phi_{q,\ell})\Big) \\ &= \Big(f(\tau_1) \otimes \overline{f(\tau_2)}, \tilde{I}(\tau_1, -\bar{\tau}_2, \xi)\Big) + \Big(f(\tau_1) \otimes \overline{f(\tau_2)}, \tilde{I}(\tau_1, -\bar{\tau}_2, \omega(R_1)\omega(R_2)\psi)\Big) \\ &= \Big(f(\tau_1) \otimes \overline{f(\tau_2)}, \tilde{I}(\tau_1, -\bar{\tau}_2, \xi)\Big) + \Big(f(\tau_1) \otimes \overline{f(\tau_2)}, R_1 R_2 \tilde{I}(\tau_1, -\bar{\tau}_2, \psi)\Big).\end{aligned}$$

By the adjointness of the Maass lowering and raising operators with respect to the Petersson scalar product, the latter summand vanishes. □

Corollary 4.4. *Let $p = 1$ and $q + \ell > 1$. Then Λ vanishes identically.*

Proof. This is obvious from Proposition 4.3 and $\xi = 0$ (Lemma 3.10). □

Remark 4.5. We could have defined the lift Λ of f by using the Schwartz form $\varphi_{q,[\ell]}$ instead of the form $\varphi_{q,\ell}$. Using Lemma 3.1 we see by the argument of the proof of Proposition 4.3 that the L^2-norms $\|\Lambda(f)\|$ coincide.

We want to relate the integral $\tilde{I}(\tau_1, \tau_2, \xi \otimes \phi_f)$ to the pullback of a genus 2 Eisenstein series via the Siegel-Weil formula. We first need to relate the integral over the locally symmetric space X to an integral over $H(\mathbb{Q})\backslash H(\mathbb{A})$.

We do this following [17], pp. 332. First we define the theta series associated to ξ more generally for $g \in G(\mathbb{A})$ and $h = (h_\infty h_f) \in H(\mathbb{A})$ by

$$\theta(g, h, \xi \otimes \phi_f) = \sum_{\mathbf{x} \in V^2(\mathbb{Q})} \omega(g)\xi(h_\infty^{-1}\mathbf{x}, z_0)\phi_f(h_f^{-1}\mathbf{x}),$$

where z_0 is the base point of D. Note that

$$\theta(\tau_1, \tau_2, z, \xi \otimes \phi_f) = (v_1 v_2)^{-\kappa/2}\theta(\iota_0(g_{\tau_1}, g_{\tau_2}), h_\infty, \xi \otimes \phi_f)$$

with $h_\infty \in H(\mathbb{R})$ such that $z = h_\infty z_0$.

We normalize the Haar measure on $H(\mathbb{A})$ such that $H(\mathbb{Q})\backslash H(\mathbb{A})$ has volume 1. Moreover, we normalize the Haar measure dh_∞ on $H(\mathbb{R})$ such that

$$\int_D f(z)\,\mu = \int_{H(\mathbb{R})} f(h_\infty z_0)\,dh_\infty$$

for compactly supported functions f on D. Here μ denotes the measure on D induced by the invariant Riemann metric normalized as in (3.5). This gives rise to a factorization $dh = dh_\infty \times dh_f$.

Remark 4.6. One has, see also [17] Remark 4.18, that

$$\mathrm{vol}(K_f^H) = \frac{1}{\mathrm{vol}(X, \mu)},$$

where $\mathrm{vol}(X, \mu)$ denotes the volume of X with respect to the volume form μ on D.

Proposition 4.7. *We have*

$$\frac{1}{\mathrm{vol}(X, \mu)}\tilde{I}(\tau_1, \tau_2, \xi \otimes \phi_f) = (v_1 v_2)^{-\kappa/2} \int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \theta(\iota_0(g_{\tau_1}, g_{\tau_2}), h, \xi \otimes \phi_f) dh,$$

were dh is the invariant measure on $H(\mathbb{A})$ such that $H(\mathbb{Q})\backslash H(\mathbb{A})$ has volume 1.

Proof. We use the above normalizations of the Haar measures on $H(\mathbb{R})$ and $H(\mathbb{A}_f)$. By means of (4.1) we obtain

$$\begin{aligned}
&\int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \theta(\iota_0(g_{\tau_1}, g_{\tau_2}), h, \xi \otimes \phi_f) dh \\
&\qquad = \sum_j \int_{H(\mathbb{Q})\backslash H(\mathbb{Q})H(\mathbb{R})_0 h_j K_f^H h_j^{-1}} \theta(\iota_0(g_{\tau_1}, g_{\tau_2}), h h_j, \xi \otimes \phi_f) dh \\
&\qquad = \mathrm{vol}(K_f^H) \sum_j \int_{\Gamma_j\backslash H(\mathbb{R})_0} \theta(\iota_0(g_{\tau_1}, g_{\tau_2}), h_\infty h_j, \xi \otimes \phi_f) dh_\infty \\
&\qquad = (v_1 v_2)^{\kappa/2}\, \mathrm{vol}(K_f^H) \tilde{I}(\tau_1, \tau_2, \xi \otimes \phi_f).
\end{aligned}$$

The assertion now follows using Remark 4.6. □

Proposition 4.8. *Let $\Xi(s) \otimes \Phi_f(s)$ be the section associated to $\xi \otimes \phi_f$ via* (2.4) *and let $s_0 = (m-3)/2$. Then*

$$\frac{1}{\mathrm{vol}(X,\mu)}\|\Lambda(f)\|_2^2 = (v_1v_2)^{-\kappa/2}\Big(f(\tau_1)\otimes\overline{f(\tau_2)}, E(\iota_0(g_{\tau_1}, g_{-\bar\tau_2}), s_0, \Xi\otimes\Phi_f)\Big).$$

Proof. Using Proposition 4.7 and the Siegel-Weil formula, Theorem 2.1, we find

$$\frac{1}{\mathrm{vol}(X,\mu)}\tilde{I}(\tau_1,\tau_2,\xi\otimes\phi_f) = (v_1v_2)^{-\kappa/2}E(\iota_0(g_{\tau_1}, g_{\tau_2}), s_0, \Xi\otimes\Phi_f).$$

Now the assertion follows from Proposition 4.3. □

Corollary 4.9. *Assume that $q+\ell$ is even and $p>1$. Let $\Phi_\infty^\kappa(s)$ be the standard section defined by* (2.7)*, and let $\Phi_f(s)$ be the section associated to ϕ_f via* (2.4)*. Then*

$$\frac{1}{\mathrm{vol}(X,\mu)}\|\Lambda(f)\|_2^2 =$$

$$= C(s_0)(v_1v_2)^{-\kappa/2}\left(f(\tau_1)\otimes\overline{f(\tau_2)}, E(\iota_0(g_{\tau_1}, g_{-\bar\tau_2}), s_0, \Phi_\infty^\kappa\otimes\Phi_f)\right),$$

where $C(s_0)$ is the nonzero constant in Proposition 3.12.

Proof. We have $\Xi(g,s) = C(s)\Phi_\infty^\kappa(g,s)$ by Proposition 3.12. Hence the Corollary immediately follows from Proposition 4.8. □

Suppose that f is an eigenform of level N and let S denote the set of primes dividing N together with ∞. Then the doubling method [25, 26, 5, 13] expresses a convolution integral as on the right hand side above as a product of the standard L-function $L^S(s_0+\frac{1}{2}, f)$ with the Euler factors corresponding to $p\in S$ omitted times a product of "bad" local factors corresponding to the primes in S. If $m>4$ then $s_0+\frac{1}{2}$ lies in the region of convergence of the Euler product of $L^S(s,f)$. Hence the L-value does not vanish. Therefore the lift $\Lambda(f)$ vanishes precisely if at least one of the "bad" local factors vanishes. By the analysis of the present paper we determine the local factor at infinity.

We now specialize to the case when the lattice L is even and *unimodular*. Then φ_f corresponds to the characteristic function of L and $\Phi_f(s)=1$. The level of L is $N=1$, so that ∞ is the only "bad" place. By the above analysis we obtain a very explicit formula for $\|\Lambda(f)\|_2^2$ as we shall now explain.

In this case $\theta(\tau, z, \varphi_{q,\ell})$ is a modular form of weight $\kappa = m/2+\ell$ for $\mathrm{SL}_2(\mathbb{Z})$ and vanishes unless $q+\ell$ is even, which we assume from now on as well. Then κ is even, because $8 \mid p-q$. By Corollary 4.9 and (2.8) we have

$$\frac{1}{\mathrm{vol}(X,\mu)}\|\Lambda(f)\|_2^2 = C(s_0)\left(f(\tau_1)\otimes\overline{f(\tau_2)}, E_\kappa^{(2)}(\tau_1, -\bar\tau_2, -\ell/2)\right), \tag{4.11}$$

where $E_\kappa^{(2)}(\tau_1, \tau_2, s)$ is the pullback of the classical genus 2 Siegel Eisenstein series $E_\kappa^{(2)}(\tau, s)$ (see (2.9)) to the diagonal.

We recall the definition of the standard L-function of a Hecke eigenform $f \in S_\kappa(\Gamma(1))$. We use the normalization of [1], [5], [24]. We denote the Fourier coefficients of f by $c(n)$ and assume that f is normalized, i.e., $c(1) = 1$. Let p be a prime. The Satake parameters $\alpha_{0,p}, \alpha_{1,p}$ of f at p are defined by the factorization of the Hecke polynomial

$$(1 - c(p)X + p^{\kappa-1}X^2) = (1 - \alpha_{0,p}X)(1 - \alpha_{0,p}\alpha_{1,p}X). \tag{4.12}$$

Hence

$$\alpha_{0,p}^2\alpha_{1,p} = p^{\kappa-1}, \qquad \alpha_{0,p}(1 + \alpha_{1,p}) = c(p).$$

According to Deligne's theorem, formerly the Ramanujan-Petersson conjecture, we have $|\alpha_{1,p}| = 1$. The standard L-function of f is defined by the Euler product

$$D_f(s) = \prod_p \left[(1 - p^{-s})(1 - \alpha_{1,p}^{-1}p^{-s})(1 - \alpha_{1,p}p^{-s})\right]^{-1}. \tag{4.13}$$

It converges for $\Re(s) > 1$. The corresponding completed L-function

$$\Psi_f(s) = \pi^{-\frac{3s}{2}}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+\kappa-1}{2}\right)\Gamma\left(\frac{s+\kappa}{2}\right)D_f(s) \tag{4.14}$$

has a meromorphic continuation to $\mathbb{C}$ and satisfies the functional equation

$$\Psi_f(s) = \Psi_f(1-s) \tag{4.15}$$

(see e.g. [5], [27]). It is well known (see [27], Introduction, [30]) that $D_f(s)$ can be interpreted as the Rankin L-series

$$D_f(s) = \zeta(2s)\sum_{n=1}^\infty c(n^2)n^{-s-\kappa+1} = \frac{\zeta(2s)}{\zeta(s)}\sum_{n=1}^\infty c(n)^2 n^{-s-\kappa+1}.$$

Theorem 4.10. *Assume that $m > 3 + r$ so that Weil's convergence criterion* (2.3) *in genus* 2 *holds. Furthermore, assume that $q + \ell$ is even and that L is even unimodular. Let $f \in S_\kappa(\Gamma(1))$ be a Hecke eigenform, and write $\|f\|_2^2 = (f, f)$ for its Petersson norm normalized as in* (4.4). *We have*

$$\frac{1}{\mathrm{vol}(X,\mu)} \cdot \frac{\|\Lambda(f)\|_2^2}{\|f\|_2^2} = C(s_0)\mu(1, \kappa, -\ell/2)\frac{D_f(m/2-1)}{\zeta(m/2)\zeta(m-2)},$$

where

$$\mu(1, \kappa, -\ell/2) = 2^{3-m/2}(-1)^{\kappa/2}\pi\frac{\Gamma(m/2 + \ell/2 - 1)}{\Gamma(m/2 + \ell/2)}.$$

Proof. The statement follows from (4.11) by means of [5], identities (14) and (22). □

Remark 4.11. By the same argument it is easily seen that $(\Lambda(f), \Lambda(g)) = 0$ for two different normalized Hecke eigenforms f and g.

Corollary 4.12. *Assume that $m > \max(4, 3+r)$, $p > 1$, $q+\ell$ even, and that L is even unimodular. Then the theta lift $\Lambda : S_\kappa(\Gamma(1)) \to \mathcal{Z}^q(X, \widetilde{\mathrm{Sym}}^\ell(V))$ is injective.*

Proof. This follows from Theorem 4.10, Proposition 3.12, and the convergence of the Euler-product for $D_f(m/2-1)$ in this case. □

References

[1] A. N. Andrianov, *The multiplicative arithmetic of Siegel modular forms*, Russian Math. Surveys **34** (1979), 75–148.

[2] R. E. Borcherds, *Automorphic forms on* $\mathrm{O}_{s+2,2}(\mathbb{R})$ *and infinite products*, Invent. Math. **120** (1995), 161–213.

[3] R. E. Borcherds, *Automorphic forms with singularities on Grassmannians*, Inv. Math. **132** (1998), 491–562.

[4] R. Borcherds, *The Gross-Kohnen-Zagier theorem in higher dimensions*, Duke Math. J. **97** (1999), 219–233. Correction in: Duke Math J. **105** No. 1 p.183–184.

[5] S. Böcherer, *Über die Funktionalgleichung automorpher L-Funktionen zur Siegelschen Modulgruppe*, J. Reine Angew. Math. 362 (1985), 146–168.

[6] J. Bruinier, *Borcherds products on* $\mathrm{O}(2, l)$ *and Chern classes of Heegner divisors*, Springer Lecture Notes in Mathematics **1780**, Springer-Verlag (2002).

[7] *J. H. Bruinier and E. Freitag*, Local Borcherds products, Annales de l'Institut Fourier **51.1** (2001), 1–26.

[8] J. Bruinier and J. Funke, *On two geometric theta lifts*, Duke Math J. **125** (2004), 45-90.

[9] W. Duke, *Hyperbolic distribution problems and half-integral weight Maass forms*, Invent. Math. **92** (1988), 73-90.

[10] E. Freitag, *Stabile Modulformen*, Math. Ann. **230** (1977), 197–211.

[11] W. Fulton and J. Harris, *Representation Theory, A First Course*, Graduate Texts in Mathematics **129**, Springer, 1991.

[12] J. Funke and J. Millson, *Cycles with local coefficients for orthogonal groups and vector-valued Siegel modular forms*, to appear in American J. Math. (2006).

[13] P. Garrett, *Pullbacks of Eisenstein series; Applications.* In: Automorphic forms of several variables, Taniguchi Symposium, Katata, 1983, Birhäuser (1984).

[14] *J. Harvey and G. Moore*, Algebras, BPS states, and strings, Nuclear Phys. B **463** (1996), no. 2-3, 315–368.

[15] J.-S. Li, *Nonvanishing theorems for the cohomology of certain arithmetic quotients*, J. Reine Angew. Math. **428** (1992), 177–217.

[16] S. Kudla, *Some extensions of the Siegel-Weil formula*, unpublished manuscript (1992). Available at `www.math.umd.edu/~ssk` .

[17] S. Kudla, *Integrals of Borcherds forms*, Compositio Math. **137** (2003), 293-349.

[18] S. Kudla and J. Millson, *The Theta Correspondence and Harmonic Forms I*, Math. Ann. **274** (1986), 353-378.

[19] S. Kudla and J. Millson, *Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables*, IHES Pub. **71** (1990), 121-172.

[20] S. Kudla and S. Rallis, *On the Weil-Siegel formula*, J. Reine Angew. Math. **387** (1988), 1-68.

[21] S. Kudla and S. Rallis, *On the Weil-Siegel formula II*, J. Reine Angew. Math. **391** (1988), 65-84.

[22] S. Kudla and S. Rallis, *Degenerate principal series and invariant distributions*, Israel J. Math. **69** (1990), 25-45.

[23] S. Kudla and S. Rallis, *A regularized Weil-Siegel formula: the first term identity*, Annals of Math. **140** (1994), 1-80.

[24] R. P. Langlands, *Problems in the theory of automorphic forms*, Lecture Notes in Math. **170** (1970), 18–86, Springer-Verlag.

[25] I. Piatetski-Shapiro, S. Rallis, *L*-functions for classical groups. Lecture Notes in Mathematics **1254**, Springer-Verlag, Berlin (1987).

[26] S. Rallis, *Injectivity properties of liftings associated to Weil representations*, Compositio Math. **52** (1984), 139–169.

[27] G. Shimura, *On the holomorphicity of certain Dirichlet series*, Proc. London Math. Soc. **31** (1975), 79–98.

[28] J.-L. Waldspurger, *Engendrement par des séries têta de certains espaces de formes modulaires*, Invent. Math. **50** (1979), 135-168.

[29] A. Weil, *Sur la formule de Siegel dans la théorie des groupes classiques*, Acta Math. **113** (1965) 1-87.

[30] D. Zagier, *Modular forms whose Fourier coefficients involve zeta-functions of quadratic fields*. In: Modular Functions of One Variable VI, Lecture Notes in Math. **627**, Springer-Verlag (1977), 105-169.

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