

LARGE DEVIATION FOR BSDE WITH SUBDIFFERENTIAL OPERATOR

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ABSTRACT. In this paper we prove that the solution of a backward stochastic differential equation, which involves a subdifferential operator and associated to a family of reflecting diffusion processes, converges to the solution of a deterministic backward equation and satisfies a large deviation principle.

1. INTRODUCTION

Let $X^{s,x,\varepsilon}$ be the diffusion process that is the unique solution of the stochastic differential equation

$$(1) \quad X_t^{s,x,\varepsilon} = x + \int_s^t b(X_r^{s,x,\varepsilon})dr + \sqrt{\varepsilon} \int_s^t \sigma(X_r^{s,x,\varepsilon})dB_r, \quad 0 \leq s \leq t \leq T,$$

where $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a uniformly Lipschitz continuous function, all the elements of the diffusion matrix σ are bounded, uniformly Lipschitz continuous functions, and B is a standard Brownian motion in \mathbb{R}^d . The existence and uniqueness of the strong solution $X^{s,x,\varepsilon}$ of (1) is standard (see, for example, see Dembo and Zeitouni [4]). It is known, thanks to the works of Freidlin and Wentzell [7], that $X^{s,x,\varepsilon}$ converges in probability, as ε goes to 0, to the solution $\chi^{s,x}$ of the following deterministic equation

$$(2) \quad \chi_t^{s,x} = x + \int_s^t b(\chi_r^{s,x})dr, \quad 0 \leq s \leq t \leq T,$$

and satisfies a large deviation principle. This result has been generalized recently by Rainero [15] to the case of backward stochastic differential equation (BSDE for short), related to the family of diffusion processes $\{X^{s,x,\varepsilon}\}$,

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of the form

$$(3) \quad Y_t^{s,x,\varepsilon} = h(X_T^{s,x,\varepsilon}) + \int_t^T f(r, X_r^{s,x,\varepsilon}, Y_r^{s,x,\varepsilon}, Z_r^{s,x,\varepsilon}) dr - \int_t^T Z_r^{s,x,\varepsilon} dB_r, \\ 0 \leq s \leq t \leq T,$$

where h and f are a given functions and satisfy some appropriate assumptions. The author has proved that the solution $(Y^{s,\varepsilon}, Z^{s,\varepsilon})$ of equation (3) converges, as ε goes to 0, to $(Y^{s,x}, 0)$ solution of the following backward deterministic equation

$$Y_t^{s,x} = h(\chi_T^{s,x}) + \int_t^T f(r, \chi_r^{s,x}, Y_r^{s,x}, 0) dr, \quad 0 \leq s \leq t \leq T,$$

and satisfies a large deviation principle.

Backward stochastic differential equations of type (3) have been first introduced by Pardoux and Peng [12]. A solution for such equation is a couple of adapted processes (Y, Z) with values in $\mathbb{R}^k \times \mathbb{R}^{k \times d}$ which mainly satisfies equation (3). The aim of Pardoux and Peng was to give a probabilistic interpretation of a solution of second order quasi-linear partial differential equation. Since then, those equations have been intensively investigated due to their connections with financial mathematics, optimal control and stochastic game, non-linear PDEs and homogenization (see, for example, [5, 6, 8, 9, 14, 12, 2, 3, 10, 1] and the references therein).

In this paper, we are interested to the system of forward-backward stochastic differential equations

$$(4) \quad \left\{ \begin{array}{l} X_t^{s,x,\varepsilon} = \\ = x + \int_s^t b(X_r^{s,x,\varepsilon}) dr + \sqrt{\varepsilon} \int_s^t \sigma(X_r^{s,x,\varepsilon}) dB_r + \rho_t^{s,x,\varepsilon} - \rho_s^{s,x,\varepsilon}, 0 \leq s \leq t \leq T, \\ \rho_t^{s,x,\varepsilon} = \int_0^t \nabla \psi(X_r^{s,x,\varepsilon}) d|\rho^{s,x,\varepsilon}|_r, |\rho^{s,x,\varepsilon}|_t = \int_0^t \mathbf{1}_{\{X_r^{s,x,\varepsilon} \in \partial\Theta\}} d|\rho^{s,x,\varepsilon}|_r, \end{array} \right.$$

$$(5) \quad \left\{ \begin{array}{l} Y_t^{s,x,\varepsilon} = \\ = h(X_T^{s,x,\varepsilon}) + \int_t^T f(r, X_r^{s,x,\varepsilon}, Y_r^{s,x,\varepsilon}, Z_r^{s,x,\varepsilon}) dr - \int_t^T Z_r^{s,x,\varepsilon} dB_r - \int_t^T U_r^{s,x,\varepsilon} dr \\ (Y_t^{s,x,\varepsilon}, U_t^{s,x,\varepsilon}) \in \partial\Pi, \text{ and } \mathbb{E} \int_0^T \Pi(Y_r^{s,x,\varepsilon}) dr < +\infty, \end{array} \right.$$

where ψ is $\mathcal{C}_b^2(\mathbb{R}^d)$ function and ρ is a bounded variation process such that $\rho_0 = 0$, Π is a proper lower semicontinuous convex function and $\partial\Pi$ is a subdifferential operator. Equations of type (5) have been introduced by Pardoux and Rascanu [13]. A solution of such equations is a triple of process (Y, Z, U) with values in $\mathbb{R}^k \times \mathbb{R}^{k \times d} \times \mathbb{R}^k$ and satisfies equation (5). Our aim

is to prove that the solution $(X^{s,x,\varepsilon}, \rho^{s,x,\varepsilon}, Y^{s,x,\varepsilon}, Z^{s,x,\varepsilon}, U^{s,x,\varepsilon})$ of system (4)-(5) converges, as ε goes to 0, to the solution $(\chi^{s,x}, \rho^{s,x}, Y^{s,x}, Z^{s,x}, U^{s,x})$ of the following system of forward-backward deterministic equation

$$\left\{ \begin{array}{l} \chi_t^{s,x} = x + \int_s^t b(\chi_r^{s,x}) dr + \rho_t^{s,x} - \rho_s^{s,x} \\ \rho_t^{s,x} = \int_0^t \nabla \psi(\chi_r^{s,x}) d|\rho^{s,x}|_r, \quad |\rho^{s,x}|_t = \int_0^t \mathbf{1}_{\{\chi_r^{s,x} \in \partial\Theta\}} d|\rho^{s,x}|_r \\ Y_t^{s,x} = h(\chi_T^{s,x}) + \int_t^T f(r, \chi_r^{s,x}, Y_r^{s,x}, 0) dr - \int_t^T U_r^{s,x} dr \\ (Y_t^{s,x}, U_t^{s,x}) \in \partial\Pi, \text{ and } \mathbb{E} \int_0^T \Pi(Y_r^{s,x}) dr < +\infty, \end{array} \right.$$

and that $Y^{s,x,\varepsilon}$ satisfies a large deviation principle. Our paper is, in fact, a generalization of the two works cited before.

2. ASSUMPTIONS AND PROBLEM FORMULATION

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \leq 1})$ be a stochastic basis such that \mathcal{F}_0 contains all P -null sets of \mathcal{F} , $\mathcal{F}_{t+} = \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon} = \mathcal{F}_t$, $\forall t \leq 1$, and suppose that the filtration is generated by a d -dimensional Brownian motion $(B_t)_{t \leq 1}$.

On the other hand, let

- Θ be an open connected bounded subset of \mathbb{R}^d , which is such that for a function $\psi \in \mathcal{C}_b^2(\mathbb{R}^d)$, $\Theta = \{\psi > 0\}$, $\partial\Theta = \{\psi = 0\}$, and $|\nabla\psi(x)| = 1$, $x \in \partial\Theta$. Note that at any boundary point $x \in \partial\Theta$, $\nabla\psi(x)$ is a unit normal vector to the boundary, pointing towards the interior of Θ . The above assumptions imply that there exists a constant $\delta > 0$ such that for all $x \in \partial\Theta$, $x' \in \bar{\Theta}$

$$(6) \quad 2\langle x' - x, \nabla\psi(x) \rangle + \delta|x - x'|^2 \geq 0.$$

- $b : \bar{\Theta} \rightarrow \mathbb{R}^d$, $\sigma : \bar{\Theta} \rightarrow \mathbb{R}^{d \times d}$ be functions such that :

(A1) There exists a constant $C > 0$ such that

$$|b(x)| + |\sigma(x)| \leq C, \forall x \in \bar{\Theta}$$

$$|b(x) - b(x')| + |\sigma(x) - \sigma(x')| \leq C|x - x'|, \forall x, x' \in \bar{\Theta}.$$

(A2) The matrix $a = \sigma\sigma^*$ is uniformly elliptic, that is, there exists a constant $\gamma > 0$ such that

$$a(x) \geq \gamma|x|^2, \forall x \in \bar{\Theta}.$$

- $h \in \mathcal{C}(\bar{\Theta}; \mathbb{R}^k)$, $f \in \mathcal{C}([0, 1] \times \bar{\Theta} \times \mathbb{R}^k \times \mathbb{R}^{k \times d}; \mathbb{R}^k)$ be functions satisfying the following assumptions :

(A3) There exist constants $\alpha \in \mathbb{R}$, $K > 0$, $c > 0$, $\mu > 0$ such that

$$\begin{aligned}
& (i) \forall t, \forall x, \forall y, \forall (z, z'), \\
& \quad |f(t, x, y, z) - f(t, x', y, z')| \leq \mu(|z - z'| + |x - x'|) \\
& (ii) \forall t, \forall x, \forall z, \forall (y, y'), \\
& \quad \langle y - y', f(t, x, y, z) - f(t, x, y', z) \rangle \leq \alpha |y - y'|^2 \\
& (iii) \forall x, \forall x', |h(x) - h(x')| \leq c|x - x'|, \\
& (vi) \forall t, \forall x, \forall y, \forall z, |f(t, x, y, z)| \leq K(1 + |y| + |z|) \\
& (v) \forall x, |h(x)| \leq K(1 + |x|).
\end{aligned}$$

• $\Pi : \mathbb{R}^k \rightarrow]-\infty, +\infty]$, be a proper lower semicontinuous convex function such that

(A4) There exists a constant $C > 0$ such that

$$\Pi(h(x)) \leq C(1 + |x|), \quad \forall x \in \bar{\Theta},$$

$$\Pi(y) \geq \Pi(0) = 0, \quad \forall y \in \mathbb{R}^k.$$

We need also the following notations :

- $\mathcal{C}[0, T]$ denotes the space of continuous functions $\Phi : [0, T] \rightarrow \mathbb{R}^d$ such that $f(0) \in \bar{\Theta}$.
- $\bar{\mathcal{C}}[0, T]$ denotes the space of continuous functions $\Psi : [0, T] \rightarrow \bar{\Theta}$.
- $\mathcal{V}[0, T]$ denotes the space of functions $\rho : [0, T] \rightarrow \mathbb{R}^d$ with bounded variation and $\rho(0) = 0$.

For $\rho \in \mathcal{V}[0, T]$, $|\rho|_t$ denotes the total variation of ρ in the interval $[0, t]$.

Consider the system of forward-backward stochastic differential equations

$$(7) \quad \left\{ \begin{aligned} X_t^{s,x,\varepsilon} &= \\ &= x + \int_s^t b(X_r^{s,x,\varepsilon}) dr + \sqrt{\varepsilon} \int_s^t \sigma(X_r^{s,x,\varepsilon}) dB_r + \rho_t^{s,x,\varepsilon} - \rho_s^{s,x,\varepsilon}, 0 \leq s \leq t \leq T, \\ \rho_t^{s,x,\varepsilon} &= \int_0^t \nabla \psi(X_r^{s,x,\varepsilon}) d|\rho^{s,x,\varepsilon}|_r, |\rho^{s,x,\varepsilon}|_t = \int_0^t \mathbf{1}_{\{X_r^{s,x,\varepsilon} \in \partial\Theta\}} d|\rho^{s,x,\varepsilon}|_r \end{aligned} \right.$$

$$(8) \quad \left\{ \begin{aligned} Y_t^{s,x,\varepsilon} &= \\ &= h(X_T^{s,x,\varepsilon}) + \int_t^T f(r, X_r^{s,x,\varepsilon}, Y_r^{s,x,\varepsilon}, Z_r^{s,x,\varepsilon}) dr - \int_t^T Z_r^{s,x,\varepsilon} dB_r - \int_t^T U_r^{s,x,\varepsilon} dr \\ &(Y_t^{s,x,\varepsilon}, U_t^{s,x,\varepsilon}) \in \partial\Pi, \text{ and } \mathbb{E} \int_0^T \Pi(Y_r^{s,x,\varepsilon}) dr < +\infty, \end{aligned} \right.$$

where

$$\partial\Pi(u) = \{u^* \in \mathbb{R}^k : \langle u^*, v - u \rangle + \Pi(u) \leq \Pi(v), \forall v \in \mathbb{R}^k\}$$

Note that the subdifferential operator $\partial\Pi : \mathbb{R}^k \rightarrow 2^{\mathbb{R}^k}$ is a maximal monotone operator, that is

$$\langle u' - v', u - v \rangle \geq 0. \quad \forall (u, u'), (v, v') \in \partial\Pi.$$

The existence and uniqueness of the strong solution $X^{s,x,\varepsilon}$, under assumption **(A1)**, for equation (7) is standard (see, for example, Lions and Sznitman [11] or Saisho [16]). It follows also from the result of Pardoux and Rascanu [13] that, under assumptions **(A3)** and **(A4)**, there exists a unique triple $(Y^{s,x,\varepsilon}, Z^{s,x,\varepsilon}, U^{s,x,\varepsilon})$ for equation (8).

The objective of this work is to prove that the solution of forward-backward stochastic differential equation (7)-(8) converges and satisfies a large deviation principle.

For the sake of simplicity, we put, in general, $s = 0$. Of course, the results hold true for all $s \in [0, T]$. We denote then by $X^{x,\varepsilon} := X^{0,x,\varepsilon}$, $Y^{0,x,\varepsilon} := Y^{x,\varepsilon}, \dots$

3. LARGE DEVIATION PRINCIPLE AND CONVERGENCE OF THE SOLUTION OF THE FORWARD EQUATION

Before giving a large deviation principle for the reflecting diffusion process $X^{s,x,\varepsilon}$, we recall the following

Definition 3.1. *The family of processes $(X_t, 0 \leq t \leq T)$ which depends on a parameter ε is said to satisfy a large deviation principle with a rate function $S(\Psi)$ if the following condition hold for every Borel set $A \subseteq \mathcal{C}[0, T]$*

1. $\limsup_{\varepsilon \rightarrow 0} \varepsilon \ln(P(X^\varepsilon \in A)) \leq \inf_{\Psi \in \bar{A}} S(\Psi)$
2. $\liminf_{\varepsilon \rightarrow 0} \varepsilon \ln(P(X^\varepsilon \in A)) \geq - \inf_{\Psi \in \overset{\circ}{A}} S(\Psi),$

where \bar{A} is the closure of A and $\overset{\circ}{A}$ the interior of A .

Let $\Phi \in \mathcal{C}[0, T]$, $\Psi \in \bar{\mathcal{C}}[0, T]$, $\rho \in \mathcal{V}[0, T]$ such that

$$(9) \quad \begin{cases} \Psi(t) = \Phi(t) + \rho(t), \\ \rho_t = \int_0^t \nabla \psi(\Psi_r) d|\rho|_r, \quad |\rho|_t = \int_0^t \mathbf{1}_{\{\Psi(r) \in \partial\Theta\}} d|\rho|_r \end{cases}$$

For Φ and Ψ defined as above, let $\Psi = F(\Phi)$. It is known from Lions and Sznitman [11] or Saisho [16] that F is continuous. We have the following theorem

Theorem 3.1. *The process $X^{x,\varepsilon}$ given by equation (7) satisfies a large deviation principle with rate function $S(\Psi)$ defined by*

$$S(\Psi) = \frac{1}{2} \inf_{\Phi \in F^{-1}(\Psi)} \int_0^T (\dot{\Phi}(s) - b(\Psi(s)))^* a^{-1}(\Psi(s)) (\dot{\Phi}(s) - b(\Psi(s))) ds,$$

with the fact that $S(\Psi) = \infty$ if $F^{-1}(\psi) = \emptyset$ or Φ is not absolutely continuous.

Proof. The result follows by using the contraction principle (see Dembo and Zeitouni [4]) and a large deviation principle for diffusion processes (see Stroock [18] or [4], see also Sheu [17] for other assumptions on Θ).

Remark 3.1. *The function $S(\Psi)$ has the following properties*

1. $S(\Psi)$ is lower semi-continuous in Ψ .
2. If $S(\Psi) < \infty$, then there exists $\Phi \in \mathcal{C}[0, T]$ such that $F(\Phi) = \Psi$ and

$$S(\Psi) = \frac{1}{2} \int_0^T (\dot{\Phi}(s) - b(\Psi(s)))^* a^{-1}(\Psi(s)) (\dot{\Phi}(s) - b(\Psi(s))) ds.$$

Let $(\chi^{s,x}, \rho^{s,x})$ be the solution of the following deterministic equation

$$\begin{cases} \chi_t^{s,x} = x + \int_s^t b(\chi_r^{s,x}) dr + \rho_t^{s,x} - \rho_s^{s,x} \\ \rho_t^{s,x} = \int_0^t \nabla \psi(\chi_r^{s,x}) d|\rho^{s,x}|_r, \quad |\rho^{s,x}|_t = \int_0^t \mathbf{1}_{\{\chi_r^{s,x} \in \partial\Theta\}} d|\rho^{s,x}|_r. \end{cases}$$

We get the following

Lemma 3.1. *For all $\varepsilon \in]0, 1]$, there exists a constant $C > 0$, independent of x and ε , such that*

$$(10) \quad \mathbb{E} \sup_{0 \leq t \leq T} |X_t^{x,\varepsilon} - \chi_t^x|^2 \leq C\varepsilon.$$

Proof. Applying Itô's formula to

$$e^{-\delta(\psi(X_t^{x,\varepsilon}) + \psi(\chi_t^x))} |X_t^{x,\varepsilon} - \chi_t^x|^2,$$

where δ is given by the inequality (6), we get a.s for all $t \in [0, T]$

$$\begin{aligned}
(11) \quad & e^{-\delta(\psi(X_t^{x,\varepsilon})+\psi(\chi_t^x))} |X_t^{x,\varepsilon} - \chi_t^x|^2 \\
&= 2 \int_0^t e^{-\delta(\psi(X_r^{x,\varepsilon})+\psi(\chi_r^x))} [\langle X_r^{x,\varepsilon} - \chi_r^x, \sqrt{\varepsilon} \sigma(X_r^{x,\varepsilon}) dB_r \rangle + \\
&+ \langle X_r^{x,\varepsilon} - \chi_r^x, b(X_r^{x,\varepsilon}) - b(\chi_r^x) \rangle dr] \\
&+ 2 \int_0^t e^{-\delta(\psi(X_r^{x,\varepsilon})+\psi(\chi_r^x))} [\langle X_r^{x,\varepsilon} - \chi_r^x, \nabla \psi(X_r^{x,\varepsilon}) \rangle d|\rho^{x,\varepsilon}|_r - \\
&- \langle X_r^{x,\varepsilon} - \chi_r^x, \nabla \psi(\chi_r^x) \rangle d|\rho^x|_r] \\
&+ \varepsilon \int_0^t e^{-\delta(\psi(X_r^{x,\varepsilon})+\psi(\chi_r^x))} |\sigma(X_r^{x,\varepsilon})|^2 dr \\
&- \delta \int_0^t e^{-\delta(\psi(X_r^{x,\varepsilon})+\psi(\chi_r^x))} |X_r^{x,\varepsilon} - \chi_r^x|^2 \left((\nabla \psi(X_r^{x,\varepsilon}))^* \sqrt{\varepsilon} \sigma(X_r^{x,\varepsilon}) + \right. \\
&+ \frac{\varepsilon}{2} \text{tr}(D^2 \psi(X_r^{x,\varepsilon}) \sigma \sigma^*(X_r^{x,\varepsilon})) + \langle \nabla \psi(X_r^{x,\varepsilon}), b(X_r^{x,\varepsilon}) \rangle + \\
&+ \langle \nabla \psi(\chi_r^x), b(\chi_r^x) \rangle) dr - \delta \int_0^t e^{-\delta(\psi(X_r^{x,\varepsilon})+\psi(\chi_r^x))} |X_r^{x,\varepsilon} - \\
&- \chi_r^x|^2 \left(|\nabla \psi(X_r^{x,\varepsilon})|^2 d|\rho^{x,\varepsilon}|_r + |\nabla \psi(X_r^x)|^2 d|\rho^x|_r \right) \\
&+ \frac{\varepsilon \delta^2}{2} \int_0^t e^{-\delta(\psi(X_r^{x,\varepsilon})+\psi(\chi_r^x))} |X_r^{x,\varepsilon} - \chi_r^x|^2 |(\sigma(X_r^{x,\varepsilon}))^* \nabla \psi(X_r^{x,\varepsilon})|^2 dr \\
&- 2\varepsilon \delta \int_0^t e^{-\delta(\psi(X_r^{x,\varepsilon})+\psi(\chi_r^x))} \langle X_r^{x,\varepsilon} - \chi_r^x, \sigma(X_r^{x,\varepsilon}) \rangle (\sigma(X_r^{x,\varepsilon})^* \nabla \psi(X_r^{x,\varepsilon})) dr.
\end{aligned}$$

Since $|\nabla \phi| = 1$ for all $x \in \partial\Theta$, by inequality (6) we have

$$\begin{aligned}
(12) \quad & 2 \int_0^t e^{-\delta(\psi(X_r^{x,\varepsilon})+\psi(\chi_r^x))} \langle X_r^{x,\varepsilon} - \chi_r^x, \nabla \psi(X_r^{x,\varepsilon}) \rangle d|\rho^{x,\varepsilon}|_r \\
&- \delta \int_0^t e^{-\delta(\psi(X_r^{x,\varepsilon})+\psi(\chi_r^x))} |X_r^{x,\varepsilon} - \chi_r^x|^2 |\nabla \psi(X_r^{x,\varepsilon})|^2 d|\rho^{x,\varepsilon}|_r \leq 0,
\end{aligned}$$

and

$$\begin{aligned}
(13) \quad & -2 \int_0^t e^{-\delta(\psi(X_r^{x,\varepsilon})+\psi(\chi_r^x))} \langle X_r^{x,\varepsilon} - \chi_r^x, \nabla \psi(X_r^x) \rangle d|\rho^x|_r \\
&- \delta \int_0^t e^{-\delta(\psi(X_r^{x,\varepsilon})+\psi(\chi_r^x))} |X_r^{x,\varepsilon} - \chi_r^x|^2 |\nabla \psi(X_r^x)|^2 d|\rho^x|_r \leq 0.
\end{aligned}$$

The result is then a consequence of the boundeness of b , σ , ψ , $\nabla \psi$, $D^2 \psi$, inequalities (12)–(13) and Burkholder-Davis-Gundy inequality.

Remark 3.2. *As a consequence of Lemma 3.1, the solution of the reflecting diffusion process $X^{x,\varepsilon}$ converges to the deterministic path χ^x in L^2 .*

4. CONVERGENCE AND LARGE DEVIATION PRINCIPLE FOR THE SOLUTION OF THE BACKWARD EQUATION

Let $(\chi^{(s,x)}, \rho^{s,x}, Y^{(s,x)}, 0, U^{(s,x)})$ be the solution of the following deterministic equations

$$(14) \quad \begin{cases} \chi_t^{s,x} = x + \int_s^t b(\chi_r^{s,x}) dr + \rho_t^{s,x} - \rho_s^{s,x} \\ \rho_t^{s,x} = \int_0^t \nabla \psi(\chi_r^{s,x}) d|\rho^{s,x}|_r, \quad |\rho^{s,x}|_t = \int_0^t \mathbf{1}_{\{\chi_r^{s,x} \in \partial\Theta\}} d|\rho^{s,x}|_r \\ Y_t^{s,x} = h(\chi_T^{s,x}) + \int_t^T f(r, \chi_r^{s,x}, Y_r^{s,x}, 0) dr - \int_t^T U_r^{s,x} dr \\ (Y_t^{s,x}, U_t^{s,x}) \in \partial\Pi, \text{ and } \mathbb{E} \int_0^T \Pi(Y_r^{s,x}) dr < +\infty. \end{cases}$$

We have the following theorem

Theorem 4.1. $\forall \varepsilon \in]0, 1]$, there exists a constant $C > 0$, independent of s , x and ε , such that

$$(15) \quad \begin{aligned} & \mathbb{E} \left[\sup_{s \leq t \leq T} |Y_t^{s,x,\varepsilon} - Y_t^{s,x}|^2 + \int_s^T |Z_r^{s,x,\varepsilon}|^2 dr \right] \\ & \leq C [\mathbb{E} (X_T^{s,x,\varepsilon} - \chi_T^{s,x})^2] + \mathbb{E} \int_s^T |X_r^{s,x,\varepsilon} - \chi_r^{s,x}|^2 dr. \end{aligned}$$

Proof. Applying Itô's formula to $|Y_t^{s,x,\varepsilon} - Y_t^{s,x}|^2$, we get

$$\begin{aligned} & \mathbb{E} |Y_t^{s,x,\varepsilon} - Y_t^{s,x}|^2 + \mathbb{E} \int_s^T |Z_r^{s,x,\varepsilon}|^2 dr + 2\mathbb{E} \int_s^T \langle Y_r^{s,x,\varepsilon} - Y_r^{s,x}, U_r^{s,x,\varepsilon} - U_r^{s,x} \rangle dr \\ & \leq \mathbb{E} (h(X_T^{s,x,\varepsilon}) - h(\chi_T^{s,x}))^2 \\ & + 2\mathbb{E} \int_s^T \langle Y_r^{s,x,\varepsilon} - Y_r^{s,x}, f(r, X_r^{s,x,\varepsilon}, Y_r^{s,x,\varepsilon}, Z_r^{s,x,\varepsilon}) - f(r, \chi_r^{s,x}, Y_r^{s,x}, 0) \rangle dr. \end{aligned}$$

But $\langle Y_r^{s,x,\varepsilon} - Y_r^{s,x}, U_r^{s,x,\varepsilon} - U_r^{s,x} \rangle \geq 0$, $dP \times dr$ a.e and f satisfies conditions **A3(i) - (ii)**, then

$$\begin{aligned} & \mathbb{E} |Y_t^{s,x,\varepsilon} - Y_t^{s,x}|^2 + \mathbb{E} \int_s^T |Z_r^{s,x,\varepsilon}|^2 dr \\ & \leq \mathbb{E} (h(X_T^{s,x,\varepsilon}) - h(\chi_T^{s,x}))^2 + 2\alpha \mathbb{E} \int_s^T |Y_r^{s,x,\varepsilon} - Y_r^{s,x}|^2 dr \\ & + 2\mu \mathbb{E} \int_s^T |Y_r^{s,x,\varepsilon} - Y_r^{s,x}| |X_r^{s,x,\varepsilon} - \chi_r^{s,x}| dr + 2\mu \mathbb{E} \int_s^T |Y_r^{s,x,\varepsilon} - Y_r^{s,x}| |Z_r^{s,x,\varepsilon}| dr \end{aligned}$$

Henceforth

$$\begin{aligned} & \mathbb{E}[|Y_t^{s,x,\varepsilon} - Y_t^{s,x}|^2 + \int_s^T |Z_r^{s,x,\varepsilon}|^2 dr] \\ & \leq C[\mathbb{E}(X_T^{s,x,\varepsilon} - \chi_T^{s,x})^2] + \mathbb{E} \int_s^T |X_r^{s,x,\varepsilon} - \chi_r^{s,x}|^2 dr, \end{aligned}$$

where C is a positive constant. The result then follows from Burkholder-Davis-Gundy inequality.

Remark 4.1. *As a consequence of Theorem 4.1 and Lemma 3.1, we get*

$$(16) \quad \mathbb{E}[\sup_{s \leq t \leq T} |Y_t^{s,x,\varepsilon} - Y_t^{s,x}|^2 + \int_s^T |Z_r^{s,x,\varepsilon}|^2 dr] \leq C\varepsilon,$$

where C is a positive constant and then the solution of the BSDE (8) converges to the deterministic solution of the equation (14).

We now consider the BSDE in the case $k = 1$. We want to prove that the process $Y^{s,x,\varepsilon}$ satisfies a large deviation principle. For that reason, we recall the link between Variational Inequality (VI, for short) and BSDE. For all $\varepsilon \geq 0$, we consider the following VI

$$(17) \quad \left\{ \begin{array}{l} \frac{\partial u^\varepsilon}{\partial t}(t, x) + \mathcal{L}^{x,\varepsilon} u^\varepsilon(t, x) + \\ + f(t, x, u^\varepsilon(t, x), ((\nabla u^\varepsilon)^* \sqrt{\varepsilon} \sigma)(t, x)) \in \partial \Pi(u^\varepsilon(t, x)), \\ \quad t \in]0, T[, x \in \Theta \\ \frac{\partial u^\varepsilon}{\partial n}(t, x) \in \partial \Pi(u^\varepsilon(t, x)), x \in \partial \Theta \\ u^\varepsilon(T, x) = h(x), x \in \bar{\Theta}, \end{array} \right.$$

where $\mathcal{L}^{x,\varepsilon}$ is the second order partial differential operator

$$\mathcal{L}^{x,\varepsilon} := \frac{\varepsilon}{2} \sum_{i,j=1}^d (\sigma \sigma^*)_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i \frac{\partial}{\partial x_i},$$

and at point $x \in \partial \Theta$

$$\frac{\partial}{\partial n} := \sum_{i=1}^d \frac{\partial \psi}{\partial x_i}(x) \frac{\partial}{\partial x_i},$$

then we have, for each $(t, x) \in [0, T] \times \bar{\Theta}$,

$$(18) \quad u^\varepsilon(t, x) = Y_t^{t,x,\varepsilon},$$

both in the sense that any classical solution of the VI (17) is equal to $Y_t^{t,x,\varepsilon}$, and $Y_t^{t,x,\varepsilon}$ is, in the case where all coefficients are continuous, a viscosity solution of the VI (17) (see Pardoux and Rascanu [13]). Moreover, we have also that

$$Y_t^{s,x,\varepsilon} = u^\varepsilon(t, X_t^{s,x,\varepsilon}).$$

Let $s \in [0, T]$ and $\varepsilon \geq 0$, we define the following applications :

$$F^\varepsilon(\Psi) := [t \rightarrow u^\varepsilon(t, \Psi_t)], t \in [s, T], \Psi \in \bar{\mathcal{C}}[s, T] \text{ satisfying equation (9).}$$

Hence $Y_t^{s,x,\varepsilon} = F^\varepsilon(X^{s,x,\varepsilon})(t)$, for all $t \in [0, T]$, and $Y^{s,x,\varepsilon} = F^\varepsilon(X^{s,x,\varepsilon})$. For $\varepsilon = 0$, u and F stand for u^0 and F^0 . We have the following theorem

Theorem 4.2. *$Y^{x,\varepsilon}$ satisfies a large deviation principle with a rate function*

$$S'(\Psi') = \inf\{S(\Psi) | \Psi'_t = F(\Psi)(t) = u(t, \Psi_t), \forall t \in [0, T]\}.$$

Proof. In order to apply the contraction principle, we need to prove that F^ε , $\varepsilon \geq 0$ are continuous and $\{F^\varepsilon\}$ converges uniformly to F on every compact of $\bar{\mathcal{C}}[0, T]$. Since u^ε is continuous, it is not hard to prove that F^ε is also continuous. The uniform convergence of $\{F^\varepsilon\}$ is a consequence of Remark 4.1.

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