EMBEDDING THEOREMS OF FUNCTION CLASSES, II

SERGEY TIKHONOV

Abstract. In this paper the embedding results in the questions of strong approximation on Fourier series are considered. We prove several theorems on the interrelation between class $W^r H^\omega_\beta$ and class $H(\lambda, p, r, \omega)$ which was defined by L. Leindler. Previous related results from Leindler's book [2] and the paper [5] are particular cases of our results.

1. Introduction

Let $f(x)$ be a 2π-periodic continuous function and let

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad (1)$$

be its Fourier series. The modulus of smoothness of order β $(\beta > 0)$ of function $f \in C$ is given by

$$\omega_\beta(f, t) = \sup_{|h| \le t} \left\| \sum_{\nu=0}^{\infty} (-1)^\nu \binom{\beta}{\nu} f(x + (\beta - \nu)h) \right\|,$$

where $\binom{\beta}{\nu} = \frac{\beta(\beta-1)\cdots(\beta-\nu+1)}{\nu!}$ for $\nu \ge 1$, $\binom{\beta}{\nu} = 1$ for $\nu = 0$ and $\|f(\cdot)\| = \max\limits_{x \in [0,2\pi]} |f(x)|$.

Denote by $S_n(x) = S_n(f, x)$ the n-th partial sum of (1). Let $E_n(f)$ be the best approximation of $f(x)$ by trigonometric polynomials of order n and let $f^{(r)}$ be the derivative of function f of order $r > 0$ ($f^{(0)} := f$) in the sense of Weyl (see [12]).

2000 *Mathematics Subject Classification.* 42A24, 42A25.

Key words and phrases. Strong approximation, Fourier series, Embedding theorems.

The research was funded by a EU Marie Curie fellowship (Contract MIF1-CT-2004-509465). The paper was partially written while the author was staying at the Centre de Recerca Matemàtica (CRM) in Barcelona, Spain. He is grateful to the staff of CRM for their hospitality.

We shall write $I_1 \ll I_2$, if there exists a positive constant C such that $I_1 \leq C\, I_2$. If $I_1 \ll I_2$ and $I_2 \ll I_1$ hold simultaneously, then we shall write $I_1 \asymp I_2$.

A sequence $\gamma := \{\gamma_n\}$ of positive terms will be called almost increasing (almost decreasing), if there exists a constant $K := K(\gamma) \geq 1$ such that

$$K\gamma_n \geq \gamma_m \quad (\gamma_n \leq K\gamma_m)$$

holds for any $n \geq m$. This concept was introduced by S.N. Bernstein.

For any sequence $\lambda = \{\lambda_n\}_{n=1}^{\infty}$ of positive numbers we set $\Lambda_n = \sum\limits_{\nu=1}^{n} \lambda_\nu$ and for any positive p we define the following strong mean:

$$h_n(f,\lambda,p) := \left\| \left(\frac{1}{\Lambda_n} \sum_{\nu=1}^{n} \lambda_\nu \left| f(x) - S_\nu(x) \right|^p \right)^{\frac{1}{p}} \right\|.$$

Let Ω be the set of nondecreasing continuous functions on $[0, 2\pi]$ such that $\omega(0) = 0$ and $\omega(\delta) \to 0$ as $\delta \to 0$.

Further, we define the following function classes:

$$H(\lambda,p,r,\omega) = \left\{ f \in C : h_n(f,\lambda,p) = O\left[n^{-r}\omega\left(\frac{1}{n}\right)\right]\right\},$$

$$W^r H^\omega_\beta = \left\{ f \in C : \omega_\beta(f^{(r)},\delta) = O\left[\omega\left(\delta\right)\right]\right\},$$

$$E^\omega_r = \left\{ f \in C : E_n(f) = O\left[n^{-r}\omega\left(\frac{1}{n}\right)\right]\right\}.$$

One of the first results concerning the interrelation between classes $H(\lambda,p,r,\omega)$ with $\lambda_n \equiv 1$ and $W^r H^\omega_\beta$ was obtained by G. Alexits and D. Králik [1]:

$$\text{Lip}\ \alpha \ \equiv\ W^0 H_1^{\delta^\alpha} \subset H(\lambda,1,0,\delta^\alpha), \qquad \text{if} \quad 0 < \alpha < 1.$$

In his book L. Leindler (see [2, Chapter III], see also [3]) proved embedding theorems for classes

$$H(\lambda,p,r,\omega), \quad W^r H^\omega_\beta \quad \text{and} \quad E^\omega_r,$$

where

$$\lambda_n = n^{\alpha-1}, \alpha > 0, \quad r \in \mathbf{N} \cup \{0\} \quad \text{and} \quad \beta = 1.$$

In present article we continue investigating this topic and prove the following theorems.

Theorem 1.1. *Let $\beta, p > 0, r \geq 0$, $\omega \in \Omega$ and λ_n be a positive sequence such that*

$$\Lambda_{2n} \ll \Lambda_n \ll n\lambda_n. \tag{2}$$

(i). *Then* $H(\lambda,p,r,\omega)\subset E_r^\omega$.

(ii). *If, additionally,* ω *satisfies*

$$(B)\qquad \sum_{k=n+1}^{\infty}\frac{1}{k}\omega\Big(\frac{1}{k}\Big)=O\left[\omega\Big(\frac{1}{n}\Big)\right],$$

$$(B_\beta)\qquad \sum_{k=1}^{n}k^{\beta-1}\omega\Big(\frac{1}{k}\Big)=O\left[n^\beta\omega\Big(\frac{1}{n}\Big)\right],$$

then $\quad H(\lambda,p,r,\omega)\subset E_r^\omega\equiv W^rH_\beta^\omega$.

Theorem 1.2. *Let* $\beta,p>0,r\geq 0$, $\omega\in\Omega$ *and* λ_n *be a positive sequence which satisfies condition (2). If* ω *satisfies the following condition: there exists* $\varepsilon>0$ *such that*

$$\left\{\lambda_n\omega^p\left(\frac{1}{n}\right)n^{1-rp-\varepsilon}\right\}\quad \textit{is almost increasing sequence.}\tag{3}$$

Then $W^rH_\beta^\omega\subset H(\lambda,p,r,\omega)$.

It was observed by L. Leindler in [5] that for a certain subclass of continuous functions the condition (3) can be relaxed.

We need the following definitions. Let $\mathbf{c}:=\{c_n\}$ be the positive sequence. The sequence $\mathbf{c}$ is called quasimonotonic ($\mathbf{c}\in QM$) if there exists $\rho\geq 0$ such that $n^{-\rho}c_n\downarrow 0$.

A sequence $\mathbf{c}:=\{c_n\}$ of positive numbers tending to zero is called of rest bounded variation ($\mathbf{c}\in R_0^+BVS$), if it possesses the property

$$\sum_{n=m}^{\infty}|c_n-c_{n+1}|\leq K(\mathbf{c})\,c_m$$

for all natural numbers m, where $K(\mathbf{c})$ is a constant depending only on $\mathbf{c}$.

Answering to S.A. Telyakovsky's question, L.Leindler [6] proved that the class R_0^+BVS was not comparable to the class QM.
We define the following two subclasses of C:

$$C_1=\left\{f\in C: f(x)=\sum_{n=1}^{\infty}b_n\sin nx,\quad \{b_n\}\in QM\right\},$$
$$C_2=\left\{f\in C: f(x)=\sum_{n=1}^{\infty}b_n\sin nx,\quad \{b_n\}\in R_0^+BVS\right\}.$$

Now we can formulate the theorem for classes C_1 and C_2, which gives more general conditions of the embedding $W^rH_\beta^\omega\subset H(\lambda,p,r,\omega)$ than Theorem 2.

Theorem 1.3. *Let $\beta, p > 0, r \geq 0$, $\omega \in \Omega$ and λ_n be a positive sequence, which satisfies condition (2). If ω satisfies the following condition:*

$$\left\{\lambda_n \omega^p \left(\frac{1}{n}\right) n^{1-rp}\right\} \quad \text{is almost increasing sequence.} \tag{4}$$

Then

$$W^r H^\omega_\beta \cap C_j \subset H(\lambda, p, r, \omega), \quad \text{where} \quad j = 1 \quad \text{or} \quad j = 2. \tag{5}$$

Remarks.
1. Theorem 1.3 in the case $\lambda_n = n^{\alpha-1}, \alpha > 0$, $r = 2k$, $k \in \mathbf{N} \cup \{0\}$, $\beta = 1$ and $f \in C_2$ was proved in [5].
2. In the book [2, p. 161-162] (see also [3]) for his version of Theorem 1.1 (ii) Leindler used the following conditions: $\omega \in B$ and there exists a natural number μ such that the inequalities

$$2^\mu \omega\left(\frac{1}{2^{n+\mu}}\right) > 2\omega\left(\frac{1}{2^n}\right) \tag{6}$$

hold for all n.
And for his version of Theorem 1.2: there exists a natural number μ such that the inequalities

$$2^{\mu\left(\frac{\alpha}{p}-r\right)} \omega\left(\frac{1}{2^{n+\mu}}\right) > 2^{\frac{1}{p}} \omega\left(\frac{1}{2^n}\right) \tag{7}$$

hold for all n.
It follows from [10] that (6) is equivalent to $\omega \in B_1$ and (7) is equivalent to (3) with $\lambda_n = n^{\alpha-1}, \alpha > 0$.

2. Auxiliary results

Lemma 2.1. ([9]). *Let $\beta > 0$ and $f(x) \in C$.*

(a): $E_n(f) \ll \omega_\beta\left(f, \frac{1}{n}\right) \ll n^{-\beta} \sum\limits_{k=1}^{n} k^{\beta-1} E_k(f)$;

(b): $\omega_{\alpha+\beta}(f,\delta)_p \leq C(\alpha)\omega_\beta(f,\delta)_p \quad$ *for* $\quad \alpha \geq 0$.

Lemma 2.2. ([4], [7]). *Let $a_n \geq 0$, $\lambda_n > 0$.*

(a): *If $p \geq 1$, then* $\quad \sum\limits_{n=1}^{\infty} \lambda_n \left(\sum\limits_{\nu=n}^{\infty} a_\nu\right)^p \ll \sum\limits_{n=1}^{\infty} \lambda_n^{1-p} a_n^p \left(\sum\limits_{\nu=1}^{n} \lambda_\nu\right)^p$,

(b): *If $0 < p < 1$ and $a_n \downarrow$, then*
$\sum\limits_{n=1}^{\infty} \lambda_n \left(\sum\limits_{\nu=n}^{\infty} a_\nu\right)^p \ll \sum\limits_{n=1}^{\infty} n^{p-1} a_n^p \left(n\lambda_n + \sum\limits_{\nu=1}^{n-1} \lambda_\nu\right)$.

Lemma 2.3. *Let $p>0, r\geq 0$, $\omega\in\Omega$ and λ_n be a positive sequence, which satisfies condition (2). Let $f(x)=\sum\limits_{n=1}^{\infty} b_n \sin nx \in C_j$ $(j=1$ or $j=2)$ and*

$$b_n \ll n^{-r-1}\omega\left(\frac{1}{n}\right). \tag{8}$$

If ω satisfies the condition (4), then $f(x)\in H(\lambda,p,r,\omega)$.

Proof of Lemma 2.3.

Let $x\in(0,\pi)$ and m be an integer such that $\frac{\pi}{m+1}<x\leq\frac{\pi}{m}$. Applying Abel's transformation, we obtain for $k\leq m$

$$\begin{aligned}
|f(x)-S_k(x)| &\leq \left|\sum_{\xi=k+1}^{m} b_\xi \sin\xi x\right| + \left|\sum_{\xi=m+1}^{\infty} b_\xi \sin\xi x\right| \\
&\ll \left|x\sum_{\xi=k+1}^{m}\xi b_\xi\right| + \sum_{\xi=m+1}^{\infty}|b_\xi-b_{\xi+1}|\left|\widetilde{D}_\xi(x)\right| \\
&\ll \frac{1}{m}\left|\sum_{\xi=k+1}^{m}\xi b_\xi\right| + m\sum_{\xi=m+1}^{\infty}|b_\xi-b_{\xi+1}|, \qquad (9)
\end{aligned}$$

where $\widetilde{D}_k(x)$ are the conjugate Dirichlet kernels, i.e. $\widetilde{D}_k(x):=\sum\limits_{n=1}^{k}\sin nx$, $k\in\mathbf{N}$, and we used $\left|\widetilde{D}_k(x)\right|=O\left(\frac{1}{x}\right)$. Therefore, from (9), if $k\leq m$ and $\{b_n\}\in R_0^+BVS$, then

$$|f(x)-S_k(x)| \ll \frac{1}{m}\left|\sum_{\xi=k}^{m}\xi b_\xi\right| + mb_m, \tag{10}$$

and if $k\leq m$ and $\{b_n\}\in QM$, then

$$|f(x)-S_k(x)| \ll \frac{1}{m}\left|\sum_{\xi=k}^{m}\xi b_\xi\right| + mb_m + m\sum_{\xi=m}^{\infty}\frac{b_\xi}{\xi}. \tag{11}$$

If $k\geq m$, then we have

$$|f(x)-S_k(x)| \ll m\sum_{\xi=k}^{\infty}|b_\xi-b_{\xi+1}|,$$

and if $\{b_n\}\in R_0^+BVS$, we write

$$|f(x)-S_k(x)| \ll mb_k, \tag{12}$$

and if $\{b_n\} \in QM$, we write

$$|f(x) - S_k(x)| \ll mb_k + m\sum_{\xi=k}^{\infty} \frac{b_\xi}{\xi}. \tag{13}$$

Further, we shall conduct the proof for $f \in C_1$ and use inequalities (11) and (13).

Let $n > m$. Then

$$\begin{aligned}\sum_{k=1}^{n} \lambda_k \, |f(x)–S_k(x)|^p &= \sum_{k=1}^{m} \lambda_k \, |f(x)–S_k(x)|^p + \sum_{k=m+1}^{n} \lambda_k \, |f(x)–S_k(x)|^p \\ &=: I_1 + I_2.\end{aligned}$$

Using (11), we have

$$\begin{aligned} I_1 &\ll \frac{1}{m^p} \sum_{k=1}^{m} \lambda_k \left(\sum_{\xi=k}^{m} \xi b_\xi \right)^p + m^p b_m^p \sum_{k=1}^{m} \lambda_k + \sum_{k=1}^{m} \lambda_k \left(m \sum_{\xi=m}^{\infty} \frac{b_\xi}{\xi} \right)^p \\ &=: I_{11} + I_{12} + I_{13}. \end{aligned}$$

Using (13), we have

$$I_2 \ll \sum_{k=m}^{n} \lambda_k \, (mb_k)^p + \sum_{k=m}^{n} \lambda_k \left(m \sum_{\xi=k}^{\infty} \frac{b_\xi}{\xi} \right)^p =: I_{21} + I_{22}.$$

Now we shall estimate I_{11}. Let $p \geq 1$. Then by Lemma 2.2(a) and conditions (2), (4) and (8), we have

$$\begin{aligned} I_{11} &\ll \frac{1}{m^p} \sum_{\xi=1}^{m} \xi^p b_\xi^p \lambda_\xi^{1-p} \left(\sum_{k=1}^{\xi} \lambda_k \right)^p \\ &\ll \frac{1}{m^p} \sum_{\xi=1}^{m} \xi^{p-1} \lambda_\xi \xi^{1-rp} \omega^p \left(\frac{1}{\xi} \right) \\ &\ll m^{1-rp-p} \lambda_m \omega^p \left(\frac{1}{m} \right) \sum_{\xi=1}^{m} \xi^{p-1} \\ &\ll m^{1-rp} \lambda_m \omega^p \left(\frac{1}{m} \right). \end{aligned} \tag{14}$$

If $0 < p < 1$, then we shall use inequality (8), Lemma 2.2(b), and inequalities (2), (4):

$$\begin{aligned} I_{11} &\ll \frac{1}{m^p}\sum_{k=1}^{m}\lambda_k\left(\sum_{\xi=k}^{m}\xi^{-r}\omega\left(\frac{1}{\xi}\right)\right)^p \\ &\ll \frac{1}{m^p}\sum_{\xi=1}^{m}\xi^{p-1-rp}\omega^p\left(\frac{1}{\xi}\right)\left(\xi\lambda_\xi+\sum_{k=1}^{\xi}\lambda_k\right) \\ &\ll m^{1-rp-p}\lambda_m\omega^p\left(\frac{1}{m}\right)\sum_{k=1}^{\xi}k^{p-1} \\ &\ll m^{1-rp}\lambda_m\omega^p\left(\frac{1}{m}\right). \end{aligned} \tag{15}$$

The estimate of I_{12} evidently follows from inequalities (2) and (4):

$$I_{12} \ll m^{1-rp}\lambda_m\omega^p\left(\frac{1}{m}\right). \tag{16}$$

Using (2) and monotonicity of ω, we write

$$\begin{aligned} I_{13} &\ll \sum_{k=1}^{m}\lambda_k\left(m^{1-r}\omega\left(\frac{1}{m}\right)\sum_{\xi=m}^{\infty}\frac{1}{\xi^2}\right)^p \\ &\ll m^{1-rp}\lambda_m\omega^p\left(\frac{1}{m}\right). \end{aligned} \tag{17}$$

From (8) and (4) we estimate I_2.

$$\begin{aligned} I_{21} &\ll m^p\sum_{k=m}^{n}k^{-p-rp}\lambda_k\omega^p\left(\frac{1}{k}\right) \\ &\ll m^p n^{1-rp}\lambda_n\omega^p\left(\frac{1}{n}\right)\sum_{k=m}^{n}k^{-p-1} \\ &\ll n^{1-rp}\lambda_n\omega^p\left(\frac{1}{n}\right). \end{aligned} \tag{18}$$

Using (8) and the monotonicity of ω, we write

$$I_{22} \ll m^p\sum_{k=m}^{n}k^{-p-rp}\lambda_k\omega^p\left(\frac{1}{k}\right).$$

The last expression we have already estimated in (18). Thus, collecting estimates (14) – (17) and the estimates for I_{21} and I_{22}, we have

$$\begin{aligned} I_1 &\ll m^{1-rp}\lambda_m\omega^p\left(\frac{1}{m}\right), \\ I_2 &\ll n^{1-rp}\lambda_n\omega^p\left(\frac{1}{n}\right). \end{aligned}$$

and, by (4),

$$I_1+I_2 \ll n^{1-rp}\lambda_n\omega^p\left(\frac{1}{n}\right).$$

If $n\le m$, then for the estimates of $\sum\limits_{k=1}^{n}\lambda_k\left|f(x)-S_k(x)\right|^p$ we shall use the same reasoning as for estimates of I_1 with only one difference that instead of m there should be n.

Thus, we have for any n

$$\sum_{k=1}^{n}\lambda_k\left|f(x)-S_k(x)\right|^p \ll n^{1-rp}\lambda_n\omega^p\left(\frac{1}{n}\right). \tag{19}$$

It is easy to see that the following conditions are equivalent:

(a) $\Lambda_{2n}\ll\Lambda_n\ll n\lambda_n$,
(b) $\Lambda_n\ll n\lambda_n$ and $\lambda_n\asymp\lambda_k$ for $n\le k\le 2n$,
(c) $\Lambda_{2n}\asymp\Lambda_n\asymp n\lambda_n$.

Only one nontrivial part is $(a)\Rightarrow(b)$. Let us prove it. From (4) and the monotonicity of ω one has for $n\le k\le 2n$

$$\begin{aligned} \lambda_k &\gg n^{1-rp}\lambda_n\omega^p\left(\frac{1}{n}\right)k^{rp-1}\omega^{-p}\left(\frac{1}{k}\right) \\ &\gg n^{1-rp}\lambda_n k^{rp-1} \\ &\gg \lambda_n. \end{aligned} \tag{20}$$

Let (a) be true. Using (20), we can write

$$\begin{aligned} \lambda_n &\gg \frac{1}{n}\Lambda_n \\ &\gg \frac{1}{n}\Lambda_{4n} \\ &\gg \frac{1}{n}\sum_{\nu=k}^{2k}\lambda_\nu \\ &\gg \lambda_k. \end{aligned}$$

Hence, (b) holds.

From the assumption of our Lemma, condition (a) holds and, therefore, (c) holds. Then, by (19), we have $f(x) \in H(\lambda, p, r, \omega)$.

For $f \in C_2$ we shall use estimates (10) and (12) and the comparison between couples of estimates (10), (12) and (11), (13) implies $f(x) \in H(\lambda, p, r, \omega)$. The proof of Lemma 2.3 is complete.

Lemma 2.4. ([11]). *If $g(x) \in C$ and $g(x)$ has a Fourier series $\sum_{n=1}^{\infty} b_n \sin nx$, $b_n \geq 0$, then*

$$n^{-\beta} \sum_{k=1}^{n} k^{\beta} b_k \leq C\omega_\beta \left(g, \frac{1}{n}\right), \quad \textit{for} \quad \beta \neq 2l, l = 1, 2, \cdots$$

Lemma 2.5. ([11]). *If $f(x) \in C$ and $f(x)$ has a Fourier series $\sum_{n=1}^{\infty} a_n \cos nx$, $a_n \geq 0$, then*

$$n^{-\beta} \sum_{k=1}^{n} k^{\beta} a_k \leq C\omega_\beta \left(f, \frac{1}{n}\right), \quad \textit{for} \quad \beta \neq 2l-1, l = 1, 2, \cdots$$

3. Proofs of Theorems

We shall follow the method of proof of L. Leindler.

Proof of Theorem 1.1.

Part (i) immediately follows from the inequality (see [2, Theorem 8.2, p. 147])

$$E_n(f) = O\left(h_n(f, \lambda, p)\right).$$

Let us prove part (ii). We need the following inequality (see [8])

$$\omega_\beta\left(f^{(r)}, \delta\right) \ll \int_0^\delta t^{-r-1} \omega_{r+\beta}\left(f, t\right) dt, \quad r \geq 0. \tag{21}$$

Then from (21), Lemma 2.1 (a), part (i) and conditions (B) and (B_β) on ω we have

$$\begin{aligned}
\omega_\beta\left(f^{(r)},\frac{1}{n}\right) &\ll \sum_{k=n}^{\infty} k^{r-1}\omega_{r+\beta}\left(f,\frac{1}{k}\right)\\
&\ll \sum_{k=n}^{\infty} k^{-\beta-1}\sum_{\nu=1}^{k}\nu^{r+\beta-1}E_k\left(f\right)\\
&\ll \frac{1}{n^\beta}\sum_{k=1}^{n}k^{r+\beta-1}\frac{\omega\left(\frac{1}{k}\right)}{k^r}+\sum_{k=n}^{\infty}k^{r-1}\frac{\omega\left(\frac{1}{k}\right)}{k^r}\\
&\ll \omega\left(\frac{1}{n}\right). \qquad (22)
\end{aligned}$$

Thus, $f\in W^rH^\omega_\beta$ and the proof of Theorem 1.1 is complete.
Proof of Theorem 1.2.
To prove this theorem we need the following estimates

$$E_n(f)\ll\frac{\omega_\beta\left(f^{(r)},\frac{1}{n}\right)}{n^r}\qquad r\geq 0,\ n\in\mathbf{N} \qquad (23)$$

and

$$\frac{1}{n}\sum_{\nu=n+1}^{2n}|S_n-f|^p\ll E_n^p(f)\qquad p>0,\ n\in\mathbf{N}. \qquad (24)$$

Inequality (23) follows from [9, p. 397-398] and Lemma 2.1(a), and (24) follows from [2, Theorem 8.2, p. 32 and (2.75), p.65].

By assumption on the sequence $\{\lambda_n\}$ it is clear that $\lambda_k\asymp\lambda_n$ for $n\leq k<2n$ (see the proof of equivalence of conditions (a), (b) and (c) in Lemma 2.3) and from (23) and (24) one can have

$$h_n(f,\lambda,p)\ll\left(\frac{1}{\Lambda_n}\sum_{\nu=1}^{n}\lambda_\nu\frac{\omega_\beta^p\left(f^{(r)},\frac{1}{\nu}\right)}{\nu^{rp}}\right)^{\frac{1}{p}}$$

By (3) and $n\lambda_n\ll\Lambda_n$, last expression gives $W^rH^\omega_\beta\subset H(\lambda,p,r,\omega)$. The proof of Theorem 1.2 is complete.
Proof of Theorem 1.3.
Let $f\in W^rH^\omega_\beta\cap C_j$ and $j=1$ or $j=2$. Then

$$f^{(r)}(x)\sim\sum_{n=1}^{\infty}n^rb_n sin\left(nx+\frac{\pi r}{2}\right).$$

Define $f_{\pm}^{(r)}(x) := \frac{f^{(r)}(x) \pm f^{(r)}(-x)}{2}$.
Let $r \neq 2m-1$, $m \in \mathbf{N}$. Then for $\beta \neq 2k$, $k \in \mathbf{N}$ we have from Lemma 2.4

$$\begin{aligned} \omega\left(\frac{1}{n}\right) &\gg \omega_\beta\left(f^{(r)}, \frac{1}{n}\right) \\ &\gg \omega_\beta\left(f_{-}^{(r)}, \frac{1}{n}\right) \\ &\gg n^{-\beta}\sum_{k=1}^{n} k^{r+\beta} b_k \\ &\gg n^{-(\beta+1)}\sum_{k=1}^{n} k^{r+\beta+1} b_k. \qquad (25) \end{aligned}$$

If $\beta = 2k$, $k \in \mathbf{N}$, we shall use Lemmas 2.1(b) and 2.4.

$$\begin{aligned} \omega\left(\frac{1}{n}\right) &\gg \omega_\beta\left(f_{-}^{(r)}, \frac{1}{n}\right) \\ &\gg \omega_{\beta+1}\left(f_{-}^{(r)}, \frac{1}{n}\right) \\ &\gg n^{-(\beta+1)}\sum_{k=1}^{n} k^{r+\beta+1} b_k. \qquad (26) \end{aligned}$$

Now let $r = 2m-1$. Then $f^{(r)} = f_{+}^{(r)}$ and the same estimates as (25) and (26) (we use Lemma 2.5 for $\beta \neq 2l-1$, $l \in \mathbf{N}$ and Lemmas 2.1(b) and 2.5 for $\beta = 2l-1$) give for any β

$$\omega\left(\frac{1}{n}\right) \gg n^{-(\beta+1)}\sum_{k=1}^{n} k^{r+\beta+1} b_k. \qquad (27)$$

Thus, from (25), (26) and (27), if $\{b_n\} \in QM$, then

$$\begin{aligned} \omega\left(\frac{1}{n}\right) &\gg n^{-(\beta+1)}\frac{b_n}{n^\rho}\sum_{k=1}^{n} k^{r+\beta+\rho+1} \\ &\gg b_n n^{r+1}, \qquad (28) \end{aligned}$$

and if $\{b_n\} \in R_0^+BVS$, then

$$\begin{aligned} \omega\left(\frac{1}{n}\right) &\gg n^{-(\beta+1)} b_n \sum_{k=1}^{n} k^{r+\beta+1} \\ &\gg b_n n^{r+1}. \qquad (29) \end{aligned}$$

Here we used the fact that if $\{b_n\} \in R_0^+BVS$, then $\{b_n\}$ is almost decreasing. Finally, by Lemma 2.3, estimates (28), (29) imply (5). The proof of Theorem 1.3 is complete.

References

[1] G. Alexits and D. Králik, *Über den Annäherungsgrad der Approximation im starken Sinne von stetigen Funktionen* // Magyar Tud. Akad. Mat. Kutató Int. Közl. **8**, 1963, 317-327.

[2] L. Leindler, *Strong approximation by Fourier series,* Budapest. 1985.

[3] L. Leindler, *New sequels of a problem of Alexits* // Acta Math. Acad. Sci. Hung. **35**, 1980, 117-121.

[4] L. Leindler, *Some inequalities of Hardy-Littlewood type* // Analysis Math., **20**, No 2, 1994, 95-106.

[5] L. Leindler, *Embedding results pertaining to strong approximation of Fourier series. III* // Analysis Math., **23**, No 4, 1997, 273-281.

[6] L. Leindler, *A new class of numerical sequences and its applications to sine and cosine series* // Analysis Math., **28**, No 4, 2002, 279-286.

[7] M.K.Potapov *A certain imbedding theorem* // Mathematica (Cluj), **14(37)**, 1972, 123-146.

[8] B.V. Simonov, S. Yu. Tikhonov, *Embedding theorems of function classes*, in press.

[9] R. Taberski, *Differences, moduli and derivatives of fractional orders* // Commentat. Math. V. 19, 1976-1977, 389–400.

[10] S. Tikhonov, *Generalized Lipschitz classes and Fourier series* // Matem. Zametki, **75**, 6, 2004, 947–951, translation in Math. Notes, **75**, 5-6, 2004, 885-889.

[11] S. Tikhonov, *On two theorem of Lorentz* // Izv. Ross. Akad. Nauk, Ser. Mat. **69**, 1, 2005, 165–178, translation from Russian Acad. Sci. Izv. Math, 69 (1), 2005, 163-175.

[12] A. Zygmund, *Trigonometric series.* Vol. I and II, Cambridge University Press. 1977.

Centre de Recerca Matemàtica (CRM)
Bellaterra (Barcelona) E-08193,
Spain *E-mail*: `stikhonov@crm.es`