THE PERIOD FUNCTION OF THE GENERALIZED LOTKA-VOLTERRA CENTERS

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Abstract. The present paper deals with the period function of the quadratic centers. In the literature it is used different terminology to classify these centers, but essentially there are four families: Hamiltonian, reversible Q_3^R, codimension four Q_4 and generalized Lotka-Volterra systems Q_3^{LV}. According to Chicone's conjecture [2] the reversible centers have at most two critical periods, and the centers of the three other families have a monotonic period function. With regard to the second part of this conjecture, until now it has been proved the monotonicity in the Hamiltonian and Q_4 families [11, 35]. Concerning the Q_3^{LV} family, no substantial progress has been made since the middle 80s, when several authors showed independently the monotonicity of the classical Lotka-Volterra centers [23, 28, 33]. By means of the first period constant one can easily conclude that the period function of the centers in the Q_3^{LV} family is monotonous increasing near the inner boundary of its period annulus (i.e., the center itself). Thus, according to Chicone's conjecture, it should be also monotonous increasing near the outer boundary, which in the Poincaré disc is a polycycle. In this paper we show that this is true. In addition we prove that, except for a zero measure subset of the parameter plane, there are no bifurcation of critical periods from the outer boundary. Finally we show that the period function is globally (i.e., in the whole period annulus) monotonous increasing in two other cases different from the classical one.

1. Setting of the problem and results

This paper concerns with the period function of centers. A critical point p of a planar differential system is a *center* if it has a punctured neighbourhood that consists entirely of periodic orbits surrounding p. The largest punctured neighbourhood with this property is called the *period annulus* of

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the center and, in what follows, it will be denoted by $\mathcal{P}$. The *period function* of the center assigns to each periodic orbit in $\mathcal{P}$ its period. Questions related to the behaviour of the period function have been extensively studied. Let us quote, for instance, the problems of isochronicity (see [8, 10, 18]), monotonicity (see [3, 4, 29]) or bifurcation of critical periods (see [7, 26, 32]). In this setting the most studied polynomial family is the *quadratic* one. Taking a complex coordinate $z = x + iy$ and using the terminology from [38], the list of quadratic centers at $z = 0$ is

$\dot{z} = -iz - z^2 + 2\|z\|^2 + (b + ic)\bar{z}^2,$	Hamiltonian (Q_3^H)
$\dot{z} = -iz + az^2 + 2\|z\|^2 + b\bar{z}^2,$	reversible (Q_3^R)
$\dot{z} = -iz + 4z^2 + 2\|z\|^2 + (b + ic)\bar{z}^2,\ \|b + ic\| = 2,$	codimension four (Q_4)
$\dot{z} = -iz + z^2 + (b + ic)\bar{z}^2,$	generalized Lotka-Volterra (Q_3^{LV})
$\dot{z} = -iz + \bar{z}^2,$	Hamiltonian triangle

where a, b and c stand for arbitrary real constants. There has been a substantial amount of work devoted to understand the behaviour of the period function of these centers. The quadratic *isochronous* (i.e., those centers with constant period function) were classified by Loud [17]. Coppel and Gavrilov [11] proved that the period function of any Hamiltonian quadratic center is monotonous and, more recently, Zhao [35] showed that the codimension four centers have the same property. Recall that the period function of a center is *monotonous increasing* (respectively, *decreasing*) if for any pair of periodic orbits inside $\mathcal{P}$, say γ_1 and γ_2 with $\gamma_1 \subset \operatorname{Int}(\gamma_2)$, we have that the period of γ_2 is greater (respectively, smaller) than the one of γ_1. (Here by $\operatorname{Int}(\gamma)$ we mean the bounded connected component of $\mathbb{R}^2 \setminus \{\gamma\}$.)

The period function of the quadratic reversible centers is not monotonous in general. The first example of non-monotonic reversible center is due to Chicone and Dumortier [5]. In this setting it is to be referred the results in [20], where are determined some regions in the parameter space of Q_3^R for which the corresponding period function has at least one or two critical periods. It is important to note that the period function is defined on the set of periodic orbits in $\mathcal{P}$. So usually the first step is to parametrize this set, let us say $\{\gamma_s\}_{s\in(0,1)}$, and then one can study the qualitative properties of the period function by means of the map $s \longmapsto$ period of γ_s, which is smooth on $(0, 1)$. The *critical periods* are the critical points of this function and its number, character (maximum or minimum) and distribution does not depend on the particular parametrization of the set of periodic orbits used.

Concerning the bifurcation of critical periods, there are three situations to study [20]. Indeed, compactifying $\mathbb{R}^2$ to the Poincaré disc, the boundary of $\mathcal{P}$ has two connected components, the center itself and a polycycle. We call them respectively the *inner* and *outer boundary* of the period annulus. We have thus:

(a) Bifurcations of critical periods from the inner boundary of the period annulus.
(b) Bifurcations of critical periods from the interior of the period annulus.
(c) Bifurcations of critical periods from the outer boundary of the period annulus.

Chicone and Jacobs [7] described completely the bifurcation of critical periods from the inner boundary for the whole quadratic family. However the bifurcation from the outer boundary is far from being solved (see [19, 20] for results about the reversible family). The reason for this lack of results is twofold. The first one is that, contrary to the situation in the inner boundary, the period function does not extend smoothly to the outer boundary. The second one is that, in order to prove that a parameter is not a bifurcation value, one needs an asymptotic development which is uniform with respect to the parameters. This is not easily achieved because the shape of the polycycle at the outer boundary changes as the parameters vary. The study of the bifurcations from the interior of the period annulus seems out of reach for the moment because there are not specific tools to investigate them.

Concerning the period function of the generalized Lotka-Volterra centers, apart from its behaviour near the inner boundary, almost nothing is known. The purpose of the present paper is to study the bifurcation of critical periods from the outer boundary and the monotonicity problem. By means of a rotation of axes (see [30] for instance), any generalized Lotka-Volterra system can be brought (not exhaustively) to

$$\begin{cases} \dot{x} = -y - bx^2 - cxy + by^2, \\ \dot{y} = x + Axy. \end{cases}$$

It is proved in [15] that if $A = 0$ then the period function of the above center is monotonous increasing. Hence the most interesting stratum is $A \neq 0$, which can be brought to $A = 1$ by means of a rescaling, i.e., to the system

$$\begin{cases} \dot{x} = -y - bx^2 - cxy + by^2, \\ \dot{y} = x + xy. \end{cases} \tag{1}$$

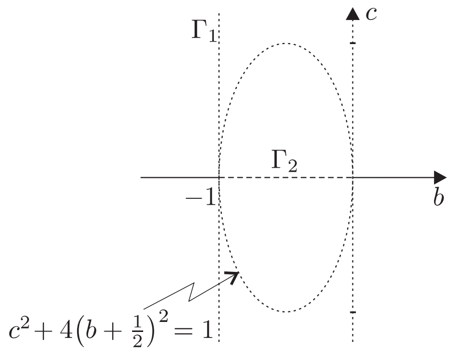

FIGURE 1. The curves Γ_1 and Γ_2 in Theorem A.

Henceforth we shall use the notation $\mu := (b, c)$ for the parameters. It is to be pointed out that the above family intersects the reversible one at $c = 0$. Furthermore the center at the origin is isochronous if and only if $\mu = (-1/2, 0)$. Note on the other hand that the first period constant [7, 14] of the center of system (1) is given by $\frac{\pi}{12}\big(c^2 + (2b+1)^2\big)$. Therefore, except for the isochronous, the period function of (1) is increasing near the inner boundary of $\mathcal{P}$ (i.e., for periodic orbits near the center). In fact Chicone and Jacobs [7] conjectured that the period function of these centers is (globally) monotonous increasing.

Define $\kappa(\mu) := c^2+4b(b+1)$. Then, see Figure 1, the conic $c^2+4(b+\frac{1}{2})^2 = 1$ is given by $\kappa(\mu) = 0$. We also consider $\Gamma_1 := \big\{\mu \in \mathbb{R}^2 : b(b+1)\kappa(\mu) = 0\big\}$ and the segment $\Gamma_2 := (-1, 0) \times \{0\}$. With this notation we can now state the main result of the present paper:

Theorem A. *Let $\{X_\mu,\ \mu \in \mathbb{R}^2\}$ be the family of vector fields in* (1) *and consider the period function of the center at the origin. Then, except for the isochronous at $\mu = (-1/2, 0)$, the period function is monotonous increasing near the outer boundary of the period annulus. In addition,*

(*a*) *There are no bifurcation of critical periods from the outer boundary of the period annulus at the parameters inside $\mathbb{R}^2 \setminus \big\{\Gamma_1 \cup \Gamma_2\big\}$.*
(*b*) *If $\mu \in \Gamma_1$ then the period function of X_μ is* (*globally*) *monotonous increasing.*

Let us now comment on the results in Theorem A in detail. Aside from reinforce Chicone's conjecture, the fact that the period function is increasing near the outer boundary of $\mathcal{P}$ implies that either the center of system (1) is isochronous or has a finite number of critical periods (see [6, 27] for related results on this question). Indeed, to show this consider an analytic parametrization of the period function, taking for instance an analytic transverse section from the inner to the outer boundary of $\mathcal{P}$. It is well known that this parametrization extends analytically to the inner boundary and, consequently, the critical periods can not accumulate there. Although it is not possible to extend it analytically to the outer boundary (not even smoothly), Theorem A shows that the critical periods do not accumulate there either. (Clearly if there exists a finite critical point at the outer boundary of $\mathcal{P}$ then the period function tends to infinity. However, even in this case, one can not assert in general that the period function is monotonous increasing near the outer boundary.)

Roughly speaking the assertion in (a) guarantees that if we move slightly $\mu \notin \Gamma_1 \cup \Gamma_2$ then the critical periods of X_μ can not emerge or disappear at the outer boundary of $\mathcal{P}_\mu$. (That this neither occurs at the inner boundary is an easy consequence, except for the isochronous at $\mu = (-1/2, 0)$, of the fact that the first period constant is positive.) Let us precise this. To this end we first parametrize the set of periodic orbits near the outer boundary of the period annulus $\mathcal{P}_\mu$ of X_μ by means of a family of transverse sections Σ_μ. More concretely, $\Sigma_\mu = \xi_\mu\big([0,\delta]\big)$, where $\xi_\mu \colon [0,\delta] \longrightarrow \mathbb{R}^2$ is a continuous family of analytic functions so that:

(1) $\xi_\mu(0)$ belongs to the outer boundary of $\mathcal{P}_\mu$,
(2) $\xi_\mu(s) \in \mathcal{P}_\mu$ and $\xi'_\mu(s)$ is transverse to $X_\mu\big(\xi_\mu(s)\big)$ for all $s \in (0,\delta)$.

Then, for each $s \in (0,\delta)$, we denote the period of the periodic orbit of X_μ passing through the point $\xi_\mu(s)$ by $P(s;\mu)$. Now the statement in (a) means by definition that for any $\mu_0 \notin \Gamma_1 \cup \Gamma_2$ there exist $\varepsilon > 0$ and a neighbourhood U of μ_0 such that $P'(s;\mu) \neq 0$ for all $s \in (0,\varepsilon)$ and $\mu \in U$. (Of course this property does not depend on the particular transverse section used.) As we mentioned before, one of the difficulties to prove (a) is that the shape of the outer boundary of $\mathcal{P}_\mu$ changes as μ varies. Let us advance that the parameters in Γ_1 correspond to bifurcations in the phase portrait of X_μ that affect $\mathcal{P}_\mu$ (see Figure 2). It is worth noting that the result in (a) is the first step to prove the existence of a uniform bound for the number of critical periods of the family $\{X_\mu,\, \mu \in \mathbb{R}^2\}$. More precisely, that there exists some $n \in \mathbb{N}$ so that if $\mu \neq (-1/2, 0)$ then the center of X_μ has at most n critical periods. According to Chicone's conjecture, $n = 0$.

Let us mention that it is also possible to derive global properties of the period function from the result in (a). To illustrate this let us first quote the work of Zhao [36, 37]. In these papers the author studies the period function of several families of quadratic centers that intersect with (1). Thus from the results in [36] and [37] respectively it follows the monotonicity for $\mu = (-3/2, 0)$ and $\mu = (-2, 0)$. Then, since at these parameters there are no bifurcation of critical periods from the boundary of $\mathcal{P}$, we can conclude that the period function is monotonous increasing not only for $\mu = (-3/2, 0)$ and $\mu = (-2, 0)$, but in a small neighbourhood of them. Let us finally comment on the results in (b). The fact that the period function is monotonous increasing for $b = 0$ is not a new result. It corresponds to the classical Lotka-Volterra system and its monotonicity was proved by Rothe [23], Schaaf [28] and Waldvogel [33] independently. However, as far as we know, the monotonicity for $b = -1$ and the conic $c^2 + 4(b + \frac{1}{2})^2 = 1$ constitutes a new result.

The paper is organized as follows. In Section 2 we first describe the different phase portraits of (1). In short, X_μ has three invariant straight lines in case that $\kappa(\mu)$ is positive and only one otherwise. We shall refer to them as the *real* and *complex* case respectively. In both cases there are situations in which the period annulus is unbounded and then its outer boundary has vertices, usually saddles, at infinity. It is therefore necessary a result about the time function associated to the passage through a saddle at infinity. In Section 2 we recall the tools developed in [19] to study this passage and introduce the related definitions. Sections 3 and 4 are devoted respectively to the complex and real case, that will cover the whole parameter plane except for $\mu \in \Gamma_1$. Finally in Section 5 we show the monotonicity for the parameters in Γ_1 and we collect all the results to prove Theorem A.

2. Tools and definitions

Recall that the family of vector fields $\{X_\mu,\, \mu \in \mathbb{R}^2\}$ under consideration is given by

$$X_\mu := \big(-y - bx^2 - cxy + by^2\big)\partial_x + x\big(1+y\big)\partial_y,$$

where $\mu = (b, c)$, and that $\kappa(\mu) := c^2 + 4b(b+1)$. Let us begin with the following property (see [30, 31]).

Remark 2.1 The multiple-valued function

$$H_\mu(x,y) = (1+y)^{2b\sqrt{\kappa}} f_+(x,y)^{\sqrt{\kappa}-c} f_-(x,y)^{\sqrt{\kappa}+c},$$

where $f_\pm(x,y) = (c \pm \sqrt{\kappa}\,)\,x - 2by + 2$, is a first integral of X_μ. □

Therefore the differential system (1) has three invariant straight lines, namely $f_\pm(x,y) = 0$ (that may not be real) and $y + 1 = 0$. In what follows we use the notation $\ell_\pm := \big\{2by - (c \pm \sqrt{\kappa}\,)\,x = 2\big\}$, which correspond to real straight lines in case that $\kappa(\mu) > 0$, i.e., μ is outside the conic $c^2 + 4(b + \frac{1}{2})^2 = 1$. Another important property of the vector fields X_μ is the following:

Remark 2.2 The transformation $(x, y, t) \longmapsto (-x, y, -t)$ brings $X_{(b,c)}$ to $X_{(b,-c)}$. □

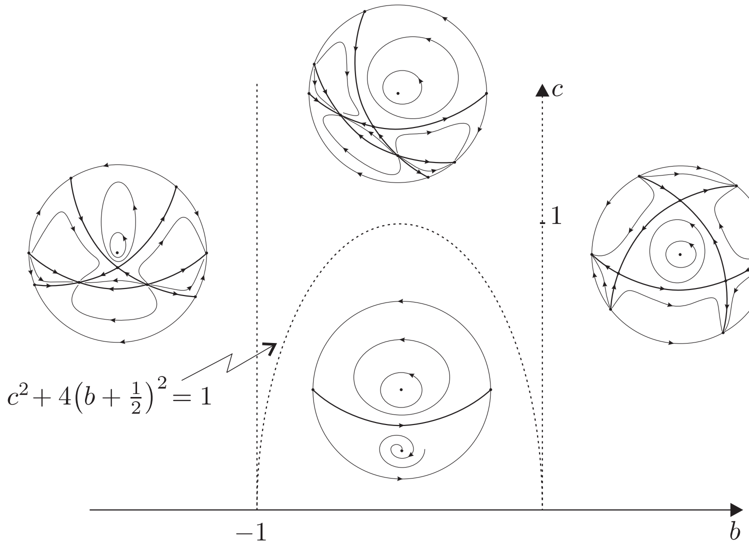

FIGURE 2. Phase portraits of the generalized Lotka-Volterra systems in the Poincaré disc

Using this property one can easily obtain the phase portrait of $X_{(b,c)}$ by means of the one corresponding to $X_{(b,-c)}$. Taking this into account, Figure 2 provides the phase portraits of (1) in the Poincaré disc. The dotted curve in the diagram corresponds to bifurcations in the phase portrait that affect the period annulus of the center at the origin. This type of bifurcation does not occur on $c = 0$. However it is to be pointed out that on the segment $(-1, 0)\times\{0\}$ the focus in the lower half-plane becomes a center. (This is so due to the symmetry explained in Remark 2.2.) Accordingly there are two centers coexisting on that segment. The bifurcation diagram of the phase portraits of all the quadratic centers can be found in [31].

Let us describe now the different type of period annuli in the family under consideration. Let us focus first in those parameters outside the conic $\kappa(\mu) = 0$. In this case there are three different situations, namely $b > 0$, $b \in (-1, 0)$ and $b < -1$. If $b > 0$ then the polycycle at the outer boundary of $\mathcal{P}_\mu$ is a triangle made up with segments of the straight lines ℓ_+, $\{y+1 = 0\}$ and ℓ_-. The critical points at the vertices of this triangle are hyperbolic saddles. For $b < -1$ and $b \in (-1, 0)$ the outer boundary of $\mathcal{P}_\mu$ is a triangle as well, but replacing one finite straight line by the line at infinity, say ℓ_∞. More concretely, if $b < -1$ then the triangle is made up with ℓ_+, ℓ_∞ and ℓ_-, whereas in case that $b \in (-1, 0)$ it is made up with ℓ_+, ℓ_∞ and $\{y+1 = 0\}$

for $c > 0$ and ℓ_-, ℓ_∞ and $\{y+1=0\}$ for $c<0$. As before the vertices are hyperbolic saddles, but now two of them are at infinity. Finally, for those parameters inside the conic $\kappa(\mu)=0$, the outer boundary of $\mathcal{P}_\mu$ is a bicycle with both vertices at infinity, hyperbolic saddles too.

Remark 2.3 If we denote the period of the periodic orbit of X_μ passing through the point $(0,y)\in\mathbb{R}^2$ with $y>0$ by $P(y;\mu)$, then from Remark 2.2 it follows that $P\big(y;(b,c)\big)=P\big(y;(b,-c)\big)$. □

It is clear at this point that to study the period function of the center at the origin we have to consider the time function associated to the passage through a hyperbolic saddle. If this saddle is at infinity then we must first perform a convenient compactification, and this yields to a meromorphic vector field. The rest of this section is devoted to introduce Proposition 2.8, which is the main result of a previous paper [19]. In short, it provides the asymptotic development of the time function associated to the passage through a hyperbolic saddle of a family of meromorphic vector fields. This result and the related definitions constitute the main ingredient in the proof of Theorem A.

Let W be an open set of $\mathbb{R}^m$ and let $\{\widetilde{Y}_\mu,\, \mu\in W\}$ be an analytic family of vector fields defined on some open set V of $\mathbb{R}^2$. Assume that each vector field $\widetilde{Y}_\mu$ has a hyperbolic saddle p_μ as the unique critical point inside V and let $\mathcal{S}_\mu$ and $\mathcal{T}_\mu$ be its stable and unstable manifolds. We consider an analytic family of *meromorphic* vector fields Y_μ proportional to $\widetilde{Y}_\mu$ with a pole of order $n>0$ along $\mathcal{T}_\mu$. We can take a coordinate system (u,v,μ) on $V\times W\subset\mathbb{R}^{2+m}$ such that $p_\mu=(0,0,\mu)$, $\mathcal{S}_\mu=\{(u,v,\mu)\colon u=0\}$ and $\mathcal{T}_\mu=\{(u,v,\mu)\colon v=0\}$. In these coordinates the family $\{Y_\mu,\,\mu\in W\}$ writes as

$$Y_\mu(u,v)=\frac{1}{v^n}\big(uP(u,v;\mu)\partial_u+vQ(u,v;\mu)\partial_v\big), \tag{2}$$

where P and Q are analytic functions such that $P(u,0;\mu)>0$ and $Q(0,v;\mu)<0$ for any $(0,v,\mu)\in\mathcal{S}_\mu$ and $(u,0,\mu)\in\mathcal{T}_\mu$. Moreover, by hypothesis, we have that

$$\lambda(\mu):=-\frac{Q(0,0;\mu)}{P(0,0;\mu)}>0.$$

The family $\{Y_\mu,\,\mu\in W\}$ can be thought of as a single vector field Y defined on $V\times W\subset\mathbb{R}^{2+m}$ whose trajectories lie on the submanifolds $\{\mu=\text{const}\}$. Let $\sigma: I\times W\longrightarrow\Sigma_\sigma$ and $\tau: I\times W\longrightarrow\Sigma_\tau$ be two analytic transverse sections to Y defined by

$$\sigma(s;\mu)=\big(\sigma_1(s;\mu),\sigma_2(s;\mu);\mu\big)\ \text{ and }\ \tau(s;\mu)=\big(\tau_1(s;\mu),\tau_2(s;\mu);\mu\big)$$

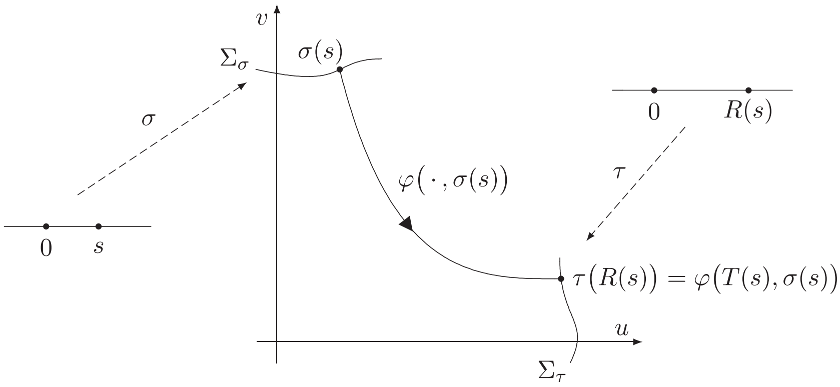

FIGURE 3. Definition of T in Proposition 2.8.

such that $\sigma(0;\mu) \in \mathcal{S}_\mu$ and $\tau(0;\mu) \in \mathcal{T}_\mu$. (Here I stands for a small interval of $\mathbb{R}$ containing 0.) We denote the Dulac and time mappings between the transverse sections Σ_σ and Σ_τ by R and T respectively. More precisely (see Figure 3), if $\varphi\big(t,(u_0,v_0);\mu\big)$ is the solution of Y_μ passing through (u_0,v_0) at $t=0$, for each $s>0$ we define $R(s;\mu)$ and $T(s;\mu)$ by means of the relation

$$\varphi\big(T(s;\mu),\sigma(s);\mu\big) = \tau\big(R(s;\mu)\big). \tag{3}$$

Definition 2.4 We say that $\{Y_\mu,\ \mu \in W\}$ verifies the *family linearization property* (FLP in short) if there exist an open set $U \subset \mathbb{R}^2$ containing the origin and an analytic *local* diffeomorphism $\Phi : U \times W \to V \times W$ of the form $\Phi(x,y;\mu) = \big(x + \text{h.o.t.}, y + \text{h.o.t.}, \mu\big)$ such that

$$Y_\mu = \Phi_*\left(\frac{1}{f(x,y;\mu)}\big(x\partial_x - \lambda(\mu)y\partial_y\big)\right),$$

where f is an analytic function on $U \times W$. □

Remark 2.5 It is easy to show that the family of meromorphic vector fields $\{Y_\mu,\ \mu \in W\}$ defined in (2) verifies FLP if it has a Darboux first integral

$$H_\mu(x,y) = f_1(x,y;\mu)^{\beta_1(\mu)} \cdots f_k(x,y;\mu)^{\beta_k(\mu)},$$

where $f_j \in \mathcal{C}^\omega(U \times W)$ for some open set $U \subset \mathbb{R}^2$ containing the origin and $\beta_j \in \mathcal{C}^\omega(W)$. □

Definition 2.6 Let W be an open subset of $\mathbb{R}^m$. We denote by $\mathcal{I}(W)$ the set of germs of analytic functions $h(s;\mu)$ defined on $(0,\varepsilon) \times W$ for some $\varepsilon > 0$ such that

$$\lim_{s\to 0} h(s;\mu) = 0 \ \text{ and } \ \lim_{s\to 0} s\,\frac{\partial h(s;\mu)}{\partial s} = 0$$

uniformly (on μ) on every compact subset of W. □

Definition 2.7 The function defined for $s>0$ and $\alpha\in\mathbb{R}$ by means of

$$\omega(s;\alpha)=\begin{cases}\frac{s^{\alpha-1}-1}{\alpha-1} & \text{if } \alpha\neq 1,\\ \log s & \text{if } \alpha=1,\end{cases}$$

is called the *Roussarie-Ecalle compensator.* □

In order to simplify the expressions that appear in the statement of the next result we introduce the functions

$$L(u;\mu):=\exp\left(\int_{\sigma_2(0)}^{u}\left(\frac{P(0,y)}{Q(0,y)}+\frac{1}{\lambda}\right)\frac{dy}{y}\right),$$

$$M(u;\mu):=\exp\left(\int_{0}^{u}\left(\frac{Q(x,0)}{P(x,0)}+\lambda\right)\frac{dx}{x}\right),$$

and the covering of the parameter space W given by the open subsets

$$W_1:=\left\{\mu\in W:\lambda>\frac{1}{n}\right\},\ W_2:=\left\{\mu\in W:\lambda<\frac{1}{n}\right\}\ \text{ and}$$

$$W_3:=\left\{\mu\in W:\frac{1}{n+1}<\lambda<\frac{2}{n}\right\}.$$

The following result [19] constitutes the main tool in order to prove Theorem A.

Proposition 2.8. *Let $\{Y_\mu,\ \mu\in W\}$ be the family of vector fields defined in* (2) *and assume that it verifies the FLP. Let T be the time function associated to the transverse sections Σ_σ and Σ_τ as introduced in* (3). *Denote*

$$\Delta_0(\mu)=\int_{\sigma_2(0)}^{0}\frac{v^{n-1}}{Q(0,v)}\,dv.$$

Then the time function $T(s;\mu)$ verifies the following:

(*a*) *If $\mu\in W_1$ then $T(s;\mu)=\Delta_0(\mu)+\Delta_1(\mu)s+s\mathcal{I}(W_1)$, where*

$$\Delta_1(\mu)=-\frac{\sigma_2'(0)\,\sigma_2(0)^{n-1}}{Q(0,\sigma_2(0))}+\sigma_1'(0)\sigma_2(0)^{1/\lambda}\int_0^{\sigma_2(0)}\frac{Q_u(0,v)L(v)v^{n-1/\lambda}}{Q(0,v)^2}\frac{dv}{v}.$$

(*b*) *If $\mu\in W_2$ then $T(s;\mu)=\Delta_0(\mu)+\Delta_2(\mu)s^{\lambda n}+s^{\lambda n}\mathcal{I}(W_2)$, where*

$$\Delta_2(\mu)=\sigma_1'(0)^{\lambda n}\sigma_2(0)^nL(0)^{\lambda n}\left\{\frac{\tau_1(0)^{-\lambda n}}{nQ(0,0)}+\int_0^{\tau_1(0)}\left(\frac{M(u)^n}{P(u,0)}-\frac{M(0)^n}{P(0,0)}\right)\frac{du}{u^{\lambda n+1}}\right\}.$$

(c) *If* $\mu \in W_3$ *then* $T(s;\mu) = \Delta_0(\mu) + \Delta_3(\mu)s\omega(s;\lambda n) + \Delta_4(\mu)s + s\mathcal{I}(W_3)$, *where the functions* $\Delta_3(\mu)$ *and* $\Delta_4(\mu)$ *are analytic on* W_3. *Furthermore, if* $\lambda(\mu_0) = 1/n$ *then*

$$\Delta_3(\mu_0) = -n\sigma_1'(0)\sigma_2(0)^n L(0)\frac{Q_u(0,0)}{P(0,0)^2}.$$

3. The complex case

In this section we consider the case in which the straight lines ℓ_+ and ℓ_- are not real, i.e., those parameters inside the subset $U := \big\{\mu \in \mathbb{R}^2 : \kappa(\mu) < 0\big\}$, and the main result is the following:

Theorem 3.1. *Consider the center at the origin of system* (1) *for the parameters inside U. Then, except for the parameter* $\mu = (-1/2, 0)$, *its period function is monotonous increasing near the outer boundary of the period annulus. Moreover, there are no bifurcation of critical periods from the outer boundary at the parameters inside* $U \setminus \{c = 0\}$.

As we already mentioned, the main ingredient in the proof of the above result is Proposition 2.8. To apply it we must first compactify X_μ and then check that the resulting family of vector fields verifies the FLP. We can not guarantee this property in the whole U because the first integral in Remark 2.1 degenerates at the segment $U \cap \{c = 0\}$. This is the reason why the second part of Theorem 3.1 does not deal with the parameters in that segment. Interestingly enough, if $\mu \in U \cap \{c = 0\}$ then system (1) has another center apart from the one at the origin (see the discussion after Remark 2.2).

In order to prove Theorem 3.1 it is first of all necessary to parametrize the period function. To this end we note, see Figure 2, that the straight line $y + 1 = 0$ is at the outer boundary of $\mathcal{P}_\mu$ for all $\mu \in U$. Thus, for each $\mu \in U$ and $s \in (0, 1)$, let $P(s;\mu)$ denote the period of the periodic orbit of X_μ passing through the point $(-1 + s, 0) \in \mathbb{R}^2$. Our aim is to study the period function near the outer boundary of $\mathcal{P}_\mu$, i.e, $P(s;\mu)$ with $s \approx 0$. Let us take the covering of U given by the open subsets

$$U_1 := \big\{\mu \in U : -1 < b < -1/2\big\},\ U_2 := \big\{\mu \in U : -1/2 < b < 0\big\} \text{ and}$$

$$U_3 := \big\{\mu \in U : -2/3 < b < -1/3\big\}.$$

Finally we define $\widehat{U}_i = U_i \setminus \big\{c = 0\big\}$ for $i = 1, 2, 3$. Now, with these notations, Theorem 3.1 is a corollary of the following result:

Proposition 3.2. *Denoting* $\Delta_0(\mu) = 2\pi|\kappa(\mu)|^{-1/2}$ *and* $\lambda(\mu) = \frac{-b}{b+1}$, *the following hold:*

(*a*) *If* $\mu \in \widehat{U}_1$ *then* $P(s;\mu) = \Delta_0(\mu) + \Delta_1(\mu)s + s\mathcal{I}\big(\widehat{U}_1\big)$ *where* Δ_1 *is an analytic negative function on* $\widehat{U}_1$.

(*b*) *If* $\mu \in \widehat{U}_2$ *then* $P(s;\mu) = \Delta_0(\mu) + \Delta_2(\mu)s^\lambda + s^\lambda\mathcal{I}\big(\widehat{U}_2\big)$ *where* Δ_2 *is an analytic negative function on* $\widehat{U}_2$.

(*c*) *If* $\mu \in \widehat{U}_3$ *then* $P(s;\mu) = \Delta_0(\mu) + \Delta_3(\mu)s\omega(s;\lambda) + \Delta_4(\mu)s + s\mathcal{I}\big(\widehat{U}_3\big)$, *where* $\Delta_3(\mu)$ *and* $\Delta_4(\mu)$ *are analytic functions on* $\widehat{U}_3$. *Furthermore* $\Delta_3(\mu_0) > 0$ *at any parameter* $\mu_0 \neq (-1/2, 0)$ *with* $\lambda(\mu_0) = 1$ (*i.e., such that* $b = -1/2$).

Finally, if $c = 0$ *then the above statements hold replacing* $\widehat{U}_i$ *by* $U_i \cap \big\{c = 0\big\}$ *for* $i = 1, 2, 3$.

It should be mentioned that U_1, U_2 and U_3 correspond respectively to those parameters $\mu \in U$ verifying that $\lambda(\mu) > 1$, $\lambda(\mu) < 1$ and $1/2 < \lambda(\mu) < 2$. Accordingly (*a*) and (*b*) in Proposition 3.2 cover all the parameters with $\lambda(\mu) \neq 1$. This is the reason why (*c*) is only concerned with the signum of $\Delta_3(\mu_0)$ in case that $\lambda(\mu_0) = 1$. Note in addition that $\Delta_3(-1/2, 0) = 0$, since $\widehat{\mu} = (-1/2, 0)$ corresponds to an isochronous center and consequently $P(s;\widehat{\mu}) = \Delta_0(\widehat{\mu}) = 2\pi$ for all s. On the other hand, to conclude that there are no bifurcation of critical periods (i.e., zeros of the derivative of $P(s;\mu)$ with respect to s) from the outer boundary of $\mathcal{P}_\mu$ (i.e., $s = 0$) at some parameter $\mu^\star$, it is necessary not only that the remainder tends to zero as $s \longrightarrow 0$ but that the limit is uniform (with respect to μ) near $\mu^\star$. The problem with the parameters on $c = 0$ is that we can not guarantee this uniformity in the remainder and this forces us to introduce the sets $\widehat{U}_i = U_i \setminus \big\{c = 0\big\}$. To clarify all this and since the proof of Proposition 3.2 is long and technical, for reader's convenience we prove Theorem 3.1 first.

Proof of Theorem 3.1 We claim that if $\mu^\star \in U \setminus \{c = 0\}$ then there exist $\varepsilon > 0$ and a neighbourhood $U^\star$ of $\mu^\star$ such that $P'(s;\mu) < 0$ for all $s \in (0, \varepsilon)$ and $\mu \in U^\star$. This implies that there are no bifurcation of critical periods from the outer boundary at $\mu^\star$. In turn, recall the definition of $P(s;\mu)$, it shows moreover that the period function of $X_{\mu^\star}$ is *increasing* near the outer boundary because the point $(-1 + s, 0) \in \mathbb{R}^2$ approaches to the outer boundary as s *decreases.*

To prove the claim let us assume first that $\mu^\star \in \widehat{U}_1$. In this case from (*a*) in Proposition 3.2 it follows that, for $\mu \approx \mu^\star$, $P(s;\mu) = \Delta_0(\mu) + \Delta_1(\mu)s + sf(s;\mu)$ with $f \in \mathcal{I}\big(\widehat{U}_1\big)$. The derivative of this expression with respect to s yields to $P'(s;\mu) = \Delta_1(\mu) + f(s;\mu) + sf'(s;\mu)$. Therefore, taking Definition 2.6 into account, it turns out that

$$P'(s;\mu) \longrightarrow \Delta_1(\mu^\star) \text{ as } (s,\mu) \longrightarrow (0,\mu^\star).$$

So the claim follows because $\Delta_1(\mu^\star) < 0$. Similarly, if $\mu^\star \in \widehat{U}_2$ then (b) in Proposition 3.2 shows that

$$\frac{P'(s;\mu)}{s^{\lambda(\mu)-1}} \longrightarrow \lambda(\mu^\star)\Delta_2(\mu^\star) \text{ as } (s,\mu) \longrightarrow (0,\mu^\star).$$

Hence, due to $\Delta_2(\mu^\star) < 0$, the claim is also true in this case. Finally if $\mu^\star \notin \widehat{U}_1 \cup \widehat{U}_2$ then, by definition, we have that $\mu^\star \in \widehat{U}_3$ with $\lambda(\mu^\star) = 1$. In this case (c) in Proposition 3.2 shows that, for $\mu \approx \mu^\star$,

$$P'(s;\mu) = \Delta_3(\mu)\big(\lambda\omega(s;\lambda)+1\big) + \Delta_4(\mu) + f(s;\mu) + sf'(s;\mu)$$

where $f \in \mathcal{I}\big(\widehat{U}_3\big)$. Note that $\omega\big(s;\lambda(\mu)\big) \longrightarrow -\infty$ as $(s,\mu) \longrightarrow (0,\mu^\star)$ since $\lambda(\mu^\star) = 1$. Accordingly

$$\frac{P'(s;\mu)}{\lambda\omega(s;\lambda)+1} \longrightarrow \Delta_3(\mu^\star) \text{ as } (s,\mu) \longrightarrow (0,\mu^\star)$$

and, since $\Delta_3(\mu^\star) > 0$, this shows the claim when $\mu^\star \notin \widehat{U}_1 \cup \widehat{U}_2$. This completes the proof of the claim.

At this point it only remains to prove that if $\mu^\star \in U \cap \{c = 0\}$ with $\mu^\star \neq (-1/2, 0)$ then the period function of the center at the origin of $X_{\mu^\star}$ is monotonous increasing near the outer boundary. (Note that to this end it is not necessary the uniformity in the remainder term.) There are two cases to consider, namely $U_1 \cap \{c = 0\}$ and $U_2 \cap \{c = 0\}$. In the first case $\mu^\star = (b^\star, 0)$ with $b^\star \in (-1, -1/2)$ and by the last statement in Proposition 3.2 we can assert that, for $b \approx b^\star$, $P\big(s;(b,0)\big) = \Delta_0(b,0) + \Delta_1(b,0)s + sf(s;b)$ where $f \in \mathcal{I}\big((-1,-1/2)\big)$. In particular, the derivative of this function at $b = b^\star$ shows that

$$P'\big(s;(b^\star,0)\big) \longrightarrow \Delta_1(b^\star,0) \text{ as } s \longrightarrow 0.$$

Therefore, due to $\Delta_1(b^\star, 0) < 0$, this implies that $P'\big(s;(b^\star,0)\big)$ is negative for $s \approx 0$ as desired. Finally the case $\mu^\star \in U_2 \cap \{c = 0\}$ follows exactly the same way and it is left to the reader. ■

Let us note that, since the linear part of the center at the origin of X_μ is $-y\partial_x + x\partial_y$, the period of the periodic orbits tends to 2π as we approach to the center. With our notation this means that $\lim_{s\to 1} P(s;\mu) = 2\pi$. The existence of some parameter $\mu_0 \in U$ such that $\Delta_0(\mu_0) < 2\pi$ would imply that Chicone's conjecture is false. Indeed, in that case $\lim_{s\to 0} P(s;\mu_0) = \Delta_0(\mu_0) < 2\pi$ and then, using that the period function is increasing near the two boundaries of the period annulus, one would derive the existence of at least two critical periods for X_{μ_0}. However, except for the isochronous at $\mu = (-1/2, 0)$, we have that $\Delta_0(\mu) > 2\pi$ for all $\mu \in U$. This is another argument in support that, as Chicone conjectures, the period function is (globally) monotonous increasing.

For the sake of clarity, the proof of Proposition 3.2 is carried out in several steps. Firstly, in Lemma 3.3 we shall obtain the asymptotic developments and compute the leading coefficients. Secondly, in Lemmas 3.5 and 3.6 we show some inequalities that lead to the assertion about the signum of Δ_1 and Δ_2 respectively. In order to simplify the expressions that appear in the statement of the next result we introduce the functions

$$F(u;\mu)=\frac{(b+1)^{\frac{1}{2b}}\big((2b+1)u+c\big)}{\big((b+1)u^2+cu-b\big)^{\frac{1}{2b}+2}}\exp\!\left(\frac{c}{b|\kappa|^{1/2}}\arctan\!\left(\frac{|\kappa|^{1/2}}{c+2(b+1)u}\right)\right)$$

and

$$G(u;\mu)=$$

$$\left(1+\frac{cu-bu^2}{b+1}\right)^{-\frac{2b+1}{2(b+1)}}\exp\!\left(\frac{-c|\kappa|^{-1/2}}{b+1}\left(\arctan\!\left(\frac{2bu-c}{|\kappa|^{1/2}}\right)+\arctan\!\left(\frac{c}{|\kappa|^{1/2}}\right)\right)\right)-1.$$

Now with these definitions we prove:

Lemma 3.3. *The asymptotic developments in Proposition 3.2 hold. Moreover, for $c\geqslant 0$,*

$$\Delta_1(\mu)=\int_0^{\frac{c}{2(b+1)}}\left(F(u)-F(-u)\exp\!\left(\frac{-c\pi}{b|\kappa|^{1/2}}\right)\right)\frac{du}{u^{1/\lambda}}+$$

$$+\int_{\frac{c}{2(b+1)}}^{+\infty}\big(F(u)-F(-u)\big)\frac{du}{u^{1/\lambda}}\,,$$

$$\Delta_2(\mu)=\frac{\lambda^{\frac{1}{2(b+1)}}}{b+1}\exp\!\left(\frac{-c|\kappa|^{-1/2}}{b+1}\arctan\!\left(\frac{|\kappa|^{1/2}}{c}\right)\right)\cdot$$

$$\cdot\int_0^{+\infty}\left(G(u)+G(-u)\exp\!\left(\frac{c\pi|\kappa|^{-1/2}}{b+1}\right)\right)\frac{du}{u^{\lambda+1}}$$

and

$$\Delta_3\big(-1/2,c\big)=4c\exp\!\left(\frac{4c}{\sqrt{1-c^2}}\arctan\!\left(\frac{c-1}{\sqrt{1-c^2}}\right)\right)\left(\exp\!\left(\frac{2\pi c}{\sqrt{1-c^2}}\right)-1\right).$$

Finally, $P(s;\mu)$ is an even function with respect to c, i.e., $P\big(s;(b,c)\big)=P\big(s;(b,-c)\big)$.

In the proof of the above result we shall use the following equality (see [1]):

Lemma 3.4. *For any* $\xi_1, \xi_2 \in \mathbb{R}$ *with* $\xi_1\xi_2 \neq 1$ *the following equalities hold:*

$$\arctan\xi_1 + \arctan\xi_2 - \arctan\left(\frac{\xi_1+\xi_2}{1-\xi_1\xi_2}\right) =$$

$$= \begin{cases} \pi & \text{if } \xi_1\xi_2 > 1 \text{ and } \xi_1+\xi_2 > 0, \\ 0 & \text{if } \xi_1\xi_2 < 1, \\ -\pi & \text{if } \xi_1\xi_2 > 1 \text{ and } \xi_1+\xi_2 < 0. \end{cases}$$

Proof of Lemma 3.3 Consider the transverse sections Σ_1 and Σ_2 given respectively by $s \longmapsto (-1+s, 0)$ and $s \longmapsto (-1/s, 0)$ for $s \gtrsim 0$. Define $T(s;\mu)$ as the time that spends the solution of X_μ with initial condition at $(-1+s,0) \in \Sigma_1$ to reach Σ_2. Then taking Remark 2.2 into account it follows that

$$P\big(s;(b,c)\big) = T\big(s;(b,c)\big) + T\big(s;(b,-c)\big), \tag{4}$$

and this shows that $P(s;\mu)$ is an even function with respect to c.

In order to study $T(s;\mu)$ we first compactify $\mathbb{R}^2$. To this end we choose an arbitrary straight line $y+\alpha x+\beta = 0$ not intersecting $\mathcal{P}_\mu \cap \{x \geqslant 0\}$ (i.e., such that $\alpha > 0$ and $\beta > 1$) and perform the projective coordinate transformation

$$(u,v) = \phi(x,y) := \left(\frac{y+1}{y+\alpha x+\beta}, \frac{1}{y+\alpha x+\beta}\right).$$

Taking $\beta = \alpha+1$ to obtain shorter expressions, a computation shows that it brings X_μ to

$$\widetilde{X}_\mu = \frac{1}{v}\big(u\,P(u,v)\,\partial_u + v\,Q(u,v)\,\partial_v\big),$$

where

$$P(u,v) = \frac{b+1}{\alpha} + \frac{c\alpha - 2(b+1)}{\alpha}\,u - (2b+c+1)\,v + \frac{b+1-b\alpha^2-c\alpha}{\alpha}\,u^2 + \big((2b+1-c)(\alpha+1)+2c\big)uv + \alpha(c-1)\,v^2$$

and

$$Q(u,v) = \frac{b}{\alpha} + \frac{c\alpha-2b-1}{\alpha}\,u - (2b+c)\,v + \frac{b+1-b\alpha^2-c\alpha}{\alpha}\,u^2 + \big((2b+1-c)(\alpha+1)+2c\big)uv + \alpha(c-1)\,v^2.$$

Thus the coordinate transformation brings $\{X_\mu,\, \mu \in U\}$ to a family of meromorphic vector fields as in (2) with a pole of order $n = 1$. Note moreover that $T(s;\mu)$ is precisely the time function associated to the passage between the transverse sections $\Sigma_\sigma := \phi(\Sigma_1)$ and $\Sigma_\tau := \phi(\Sigma_2)$. One can verify that Σ_σ and Σ_τ are given respectively by

$$\sigma(s) = \left(\frac{s}{s+\alpha}, \frac{1}{s+\alpha}\right) \text{ and } \tau(s) = \left(\frac{s+1}{1+(\alpha+1)s}, \frac{1}{1+(\alpha+1)s}\right).$$

We will apply Proposition 2.8 to study $T(s;\mu)$ near $s = 0$. However it is first necessary to check that the FLP is fulfilled. To this end we first note that, from Remark 2.1, the single-valued *real* function

$$H_\mu(x,y) = \\ = (y+1)\Big(\big(cx+2(1-by)\big)^2 + |\kappa|\,x^2\Big)^{\frac{1}{2b}} \exp\Big(\tfrac{c}{b|\kappa|^{1/2}} \arctan\Big(\tfrac{|\kappa|^{1/2}x}{2(by-1)-cx}\Big)\Big)$$

is a first integral of X_μ for all $\mu \in U$. The exponential factor above is analytic everywhere except at the straight line $2(by-1) - cx = 0$. Thus $\widetilde{H}_\mu(u,v) := H_\mu\big(\phi^{-1}(u,v)\big)$ is a Darboux first integral of $\widetilde{X}_\mu$ near the saddle at $(u,v) = (0,0)$ in case that $c \neq 0$. (If $c = 0$ then $2(by-1) - cx = 0$ cuts the saddle at infinity.) Therefore from Remark 2.5 it follows that $\big\{\widetilde{X}_\mu,\, \mu \in U \setminus \{c = 0\}\big\}$ verifies the FLP. On the other hand, since the exponential factor in $\widetilde{H}_\mu(u,v)$ disappears for $c = 0$, by Remark 2.5 again we can also assert that the 1-parameter family $\big\{\widetilde{X}_\mu,\, \mu \in U \cap \{c = 0\}\big\}$ fulfils the FLP.

As a result of the above discussion, in order to apply Proposition 2.8 we must split $\big\{X_\mu,\, \mu \in U\big\}$ into two subfamilies, the ones given by $U \setminus \{c = 0\}$ and $U \cap \{c = 0\}$. This affects the properties that we can guarantee in the remainder term of the asymptotic development of $T(s;\mu)$ at $s = 0$. However, as we will see, it has no implications in the computation of the coefficients, that will be denoted by $\widetilde{\Delta}_i(\mu)$. Thus, for the sake of shortness, when we study the case $c \neq 0$ we shall also compute the coefficients for $c = 0$. Define

$$\lambda(\mu) := -\frac{Q(0,0)}{P(0,0)} = \frac{-b}{b+1},$$

and note that then the subsets U_1, U_2 and U_3 correspond to those $\mu \in U$ such that $\lambda(\mu) > 1$, $\lambda(\mu) < 1$ and $1/2 < \lambda(\mu) < 2$ respectively.

Let us prove first the assertions concerning the case $c \neq 0$. To this end define $\widehat{U}_i = U_i \setminus \{c = 0\}$. Hence, taking (4) into account and applying Proposition 2.8, we can assert that if $\mu \in \widehat{U}_1$ then

$$\begin{aligned} P(s;\mu) &= T\big(s;(b,c)\big) + T\big(s;(b,-c)\big) \\ &= \underbrace{\widetilde{\Delta}_0(b,c) + \widetilde{\Delta}_0(b,-c)}_{\Delta_0(\mu)} + \underbrace{\big(\widetilde{\Delta}_1(b,c) + \widetilde{\Delta}_1(b,-c)\big)}_{\Delta_1(\mu)}\, s + s\mathcal{I}\big(\widehat{U}_1\big). \end{aligned}$$

It is to be pointed out that here we apply Proposition 2.8 twice, to the vector fields $\widetilde{X}_{(b,c)}$ and $\widetilde{X}_{(b,-c)}$, and that we use that $\lambda(\mu)$ does not depend on c. Exactly the same way, if $\mu \in \widehat{U}_2$ then

$$P(s;\mu) = \underbrace{\widetilde{\Delta}_0(b,c) + \widetilde{\Delta}_0(b,-c)}_{\Delta_0(\mu)} + \underbrace{\big(\widetilde{\Delta}_2(b,c) + \widetilde{\Delta}_2(b,-c)\big)}_{\Delta_2(\mu)} s^{\lambda} + s^{\lambda}\mathcal{I}\big(\widehat{U}_2\big).$$

Finally, if $\mu \in \widehat{U}_3$ then

$$\begin{aligned} P(s;\mu) &= \underbrace{\widetilde{\Delta}_0(b,c) + \widetilde{\Delta}_0(b,-c)}_{\Delta_0(\mu)} + \underbrace{\big(\widetilde{\Delta}_3(b,c) + \widetilde{\Delta}_3(b,-c)\big)}_{\Delta_3(\mu)} s\omega(s;\lambda) + \\ &+ \underbrace{\big(\widetilde{\Delta}_4(b,c) + \widetilde{\Delta}_4(b,-c)\big)}_{\Delta_4(\mu)} s + s\mathcal{I}\big(\widehat{U}_3\big). \end{aligned}$$

So far we have proved the assertions with regard to the asymptotic development of the period function. It remains to compute the coefficients. The first one is given by

$$\begin{aligned} \widetilde{\Delta}_0(\mu) &= \int_{1/\alpha}^{0} \frac{\alpha\, dv}{b - \alpha(2b+c)\, v + \alpha^2(c-1)\, v^2} \\ &= \frac{2}{|\kappa|^{1/2}} \left(\arctan\left(\frac{2(b+1)-c}{|\kappa|^{1/2}} \right) - \arctan\left(\frac{2b+c}{|\kappa|^{1/2}} \right) \right) \\ &= \frac{2}{|\kappa|^{1/2}} \left(\arctan\left(\frac{|\kappa|^{1/2}}{c} \right) + \chi(c)\, \pi \right), \quad \text{where } \chi(c) = \begin{cases} 0 & \text{if } c \geqslant 0, \\ 1 & \text{if } c < 0. \end{cases} \end{aligned}$$

Let us note that the second equality above follows from direct integration and the third one by applying Lemma 3.4. Therefore

$$\Delta_0(b,c) = \widetilde{\Delta}_0(b,c) + \widetilde{\Delta}_0(b,-c) = 2\pi|\kappa(\mu)|^{-1/2},$$

where recall that by definition $\kappa(\mu) = c^2 + 4b(b+1)$.

The "higher order" coefficients are more complicated to compute. The key point to simplify them will be that $T(s;\mu)$ does not depend on α. Indeed, this implies that, although $\widetilde{X}_\mu$ and the transverse sections depend on α, the coefficients $\widetilde{\Delta}_i(\mu)$ do not (cf. the computation of $\widetilde{\Delta}_0(\mu)$ above). Let us study first $\widetilde{\Delta}_1(\mu)$. By applying (a) in Proposition 2.8, some easy computations (which are not included here for the sake of brevity) show that

$$\widetilde{\Delta}_1(\mu) = \frac{-1}{\alpha(b+1)} + (b+1)^{\frac{1}{2b}} \exp\left(\frac{c}{b|\kappa|^{1/2}} \arctan\left(\frac{2(b+1)-c}{|\kappa|^{1/2}}\right)\right)$$
$$\int_0^1 \frac{\big((\alpha+1)(2b+c+1)-2\alpha c\big)u+\alpha c-2b-1}{\alpha\big((1-c)u^2+(2b+c)u-b\big)^{\frac{1}{2b}+2}} \cdot$$
$$\cdot \exp\left(\frac{c}{b|\kappa|^{1/2}} \arctan\left(\frac{2(c-1)u-2b-c}{|\kappa|^{1/2}}\right)\right) \frac{du}{u^{1/\lambda}}.$$

Now, since in fact $\widetilde{\Delta}_1(\mu)$ does not depend on α, we can make $\alpha \longrightarrow +\infty$ to compute it. This yields to

$$\widetilde{\Delta}_1(\mu) = (b+1)^{\frac{1}{2b}} \exp\left(\frac{c}{b|\kappa|^{1/2}} \arctan\left(\frac{2(b+1)-c}{|\kappa|^{1/2}}\right)\right)$$
$$\int_0^1 \frac{(2b-c+1)u+c}{\big((1-c)u^2+(2b+c)u-b\big)^{\frac{1}{2b}+2}} \cdot$$
$$\cdot \exp\left(\frac{c}{b|\kappa|^{1/2}} \arctan\left(\frac{2(c-1)u-2b-c}{|\kappa|^{1/2}}\right)\right) \frac{du}{u^{1/\lambda}}.$$

Our next goal is to simplify the above expression by means of the equalities in Lemma 3.4. To this end note first that if

$$\xi_1 = \frac{2(b+1)-c}{|\kappa|^{1/2}} \text{ and } \xi_2 = \frac{2(c-1)u-2b-c}{|\kappa|^{1/2}},$$

then $\xi_1+\xi_2 = 2|\kappa|^{-1/2}(c-1)(u-1) > 0$ for $\mu \in U$ and $u \in (0,1)$. On the other hand

$$1-\xi_1\xi_2 = 2(1-c)\,\frac{\big(2(b+1)-c\big)u+c}{|\kappa|^{1/2}}.$$

If $c \geqslant 0$ then one can easily show that $1-\xi_1\xi_2 > 0$ on the region under consideration. If $c < 0$ then, setting $u^\star := \frac{c}{c-2(b+1)}$, it follows that $1-\xi_1\xi_2$ is negative for $u \in (0,u^\star)$ and positive for $u \in (u^\star,1)$. Therefore, since

$$\frac{\xi_1+\xi_2}{1-\xi_1\xi_2} = \frac{|\kappa|^{1/2}(1-u)}{c(1-u)+2(b+1)u},$$

in case that $c \geqslant 0$ we obtain

$$
\begin{aligned}
\widetilde{\Delta}_1(\mu) &= (b+1)^{\frac{1}{2b}} \int_0^1 \frac{(2b-c+1)u+c}{\big((1-c)u^2+(2b+c)u-b\big)^{\frac{1}{2b}+2}} \cdot \\
&\quad \cdot \exp\left(\frac{c}{b|\kappa|^{1/2}} \arctan\left(\frac{|\kappa|^{1/2}}{c+2(b+1)\frac{u}{1-u}}\right)\right) \frac{du}{u^{1/\lambda}} \\
&= (b+1)^{\frac{1}{2b}} \int_0^{+\infty} \frac{(2b+1)v+c}{\big((b+1)v^2+cv-b\big)^{\frac{1}{2b}+2}} \cdot \\
&\quad \cdot \exp\left(\frac{c}{b|\kappa|^{1/2}} \arctan\left(\frac{|\kappa|^{1/2}}{c+2(b+1)v}\right)\right) \frac{dv}{v^{1/\lambda}} \\
&= \int_0^{+\infty} F(v;\mu) \frac{dv}{v^{1/\lambda}}.
\end{aligned}
$$

The first equality above follows from applying Lemma 3.4, and to obtain the second one we perform the change $v = \frac{u}{1-u}$. In case that $c < 0$, exactly the same way but taking $\frac{u^\star}{1-u^\star} = \frac{-c}{2(b+1)}$ into account, we get

$$
\widetilde{\Delta}_1(\mu) = \exp\left(\frac{c\pi}{b|\kappa|^{1/2}}\right) \int_0^{\frac{-c}{2(b+1)}} F(v;\mu)\frac{dv}{v^{1/\lambda}} + \int_{\frac{-c}{2(b+1)}}^{+\infty} F(v;\mu)\frac{dv}{v^{1/\lambda}}.
$$

Consider finally some $\mu=(b,c)$ with $c\geqslant 0$. Then, using that $F\big(u;(b,-c)\big)=-F\big(-u;(b,c)\big)$, we obtain

$$
\begin{aligned}
\Delta_1(\mu) &= \widetilde{\Delta}_1(b,c)+\widetilde{\Delta}_1(b,-c)\\
&= \int_0^{+\infty} F\big(v;(b,c)\big)\frac{dv}{v^{1/\lambda}}+\exp\!\left(\frac{-c\pi}{b|\kappa|^{1/2}}\right)\int_0^{\frac{c}{2(b+1)}} F\big(v;(b,-c)\big)\frac{dv}{v^{1/\lambda}}\\
&\quad+\int_{\frac{c}{2(b+1)}}^{+\infty} F\big(v;(b,-c)\big)\frac{dv}{v^{1/\lambda}}\\
&= \int_0^{+\infty} F(v;\mu)\frac{dv}{v^{1/\lambda}}-\exp\!\left(\frac{-c\pi}{b|\kappa|^{1/2}}\right)\int_0^{\frac{c}{2(b+1)}} F(-v;\mu)\frac{dv}{v^{1/\lambda}}-\\
&\quad\int_{\frac{c}{2(b+1)}}^{+\infty} F(-v;\mu)\frac{dv}{v^{1/\lambda}}\\
&= \int_0^{\frac{c}{2(b+1)}}\left(F(v)-F(-v)\exp\!\left(\frac{-c\pi}{b|\kappa|^{1/2}}\right)\right)\frac{dv}{v^{1/\lambda}}+\\
&\quad\int_{\frac{c}{2(b+1)}}^{+\infty}\big(F(v)-F(-v)\big)\frac{dv}{v^{1/\lambda}}.
\end{aligned}
$$

This shows the validity of the expression of Δ_1 in the statement.

Let us turn now to the computation of $\widetilde{\Delta}_2(\mu)$ by means of (b) in Proposition 2.8. To this end we note first that some easy simplifications yield to

$$
\begin{aligned}
&\sigma_1'(0)^{\lambda}\sigma_2(0)L(0)^{\lambda}=\\
&=\frac{\lambda^{\frac{1}{2(b+1)}}}{\alpha^{\lambda+1}}\exp\!\left(\frac{c|\kappa|^{-1/2}}{b+1}\left(\arctan\!\left(\frac{c-2(b+1)}{|\kappa|^{1/2}}\right)+\arctan\!\left(\frac{2b+c}{|\kappa|^{1/2}}\right)\right)\right)\\
&=\frac{\lambda^{\frac{1}{2(b+1)}}}{\alpha^{\lambda+1}}\exp\!\left(\frac{-c|\kappa|^{-1/2}}{b+1}\left(\arctan\!\left(\frac{|\kappa|^{1/2}}{c}\right)+\chi(c)\,\pi\right)\right),\\
&\qquad\text{where } \chi(c)=\begin{cases}0 & \text{if } c\geqslant 0,\\ 1 & \text{if } c<0.\end{cases}
\end{aligned}
$$

(Here the second equality follows from applying Lemma 3.4.) On the other hand, after the change of variable given by $v=\frac{\alpha u}{1-u}$ in the integral, the second factor in $\widetilde{\Delta}_2(\mu)$ becomes

$$\frac{\tau_1(0)^{-\lambda}}{Q(0,0)}+\int_0^{\tau_1(0)}\left(\frac{M(u)}{P(u,0)}-\frac{M(0)}{P(0,0)}\right)\frac{du}{u^{\lambda+1}}=$$
$$=\frac{\alpha}{b}+$$
$$+\frac{\alpha^{\lambda+1}}{b+1}k\int_0^{+\infty}\!\!\left(\!\left(1+\frac{cv-bv^2}{b+1}\right)^{-\frac{2b+1}{2(b+1)}}\!\exp\!\left(\frac{-c|\kappa|^{-1/2}}{b+1}\,\mathcal{S}_\alpha(v;\mu)\right)\!-\!\left(1+\frac{v}{\alpha}\right)^{\lambda-1}\right)\frac{dv}{v^{\lambda+1}},$$

where

$$\mathcal{S}_\alpha(v;\mu):=\arctan\!\left(\frac{(2bv-c)\alpha+cv+2(b+1)}{|\kappa|^{1/2}(v+\alpha)}\right)+\arctan\!\left(\frac{\alpha c-2(b+1)}{\alpha|\kappa|^{1/2}}\right).$$

Recall at this point that $\widetilde{\Delta}_2(\mu)$ does not depend on α. Thus, by making $\alpha\longrightarrow+\infty$ in the product of the two factors above, we conclude that

$$\widetilde{\Delta}_2(\mu)=$$
$$=\frac{\lambda^{\frac{1}{2(b+1)}}}{b+1}\exp\!\left(\frac{-c|\kappa|^{-1/2}}{b+1}\left(\arctan\!\left(\frac{|\kappa|^{1/2}}{c}\right)+\chi(c)\,\pi\right)\right)\int_0^{+\infty}\frac{G(v;\mu)}{u^{\lambda+1}}\,dv.$$

(Here we apply the Dominate Convergence Theorem to commute the limit with the integration.) Consider finally some $\mu=(b,c)$ with $c\geqslant 0$. Then, since one can verify that $G\big(u;(b,-c)\big)=G\big(-u;(b,c)\big)$, we obtain

$$\begin{aligned}\Delta_2(\mu)&=\widetilde{\Delta}_2(b,c)+\widetilde{\Delta}_2(b,-c)\\&=\frac{\lambda^{\frac{1}{2(b+1)}}}{b+1}\exp\!\left(\frac{-c|\kappa|^{-1/2}}{b+1}\arctan\!\left(\frac{|\kappa|^{1/2}}{c}\right)\right)\\&\qquad\int_0^{+\infty}\left(G(v)+G(-v)\exp\!\left(\frac{c\pi|\kappa|^{-1/2}}{b+1}\right)\right)\frac{dv}{v^{\lambda+1}}.\end{aligned}$$

This proves the validity of the expression of Δ_2 in the statement.

It remains to compute $\Delta_3(\mu)$ in case that $\lambda(\mu)=1$, i.e., for those $\mu=(b,c)$ such that $b=-1/2$. To this end we use that $\Delta_3(\mu)=\widetilde{\Delta}_3(b,c)+\widetilde{\Delta}_3(b,-c)$ and apply (c) in Proposition 2.8. Fix some $\mu_0=(-1/2,c_0)$. Then some computations show that

$$\widetilde{\Delta}_3(\mu_0)=-4c_0\exp\!\left(\frac{4c_0}{\sqrt{1-c_0^2}}\arctan\!\left(\frac{c_0-1}{\sqrt{1-c_0^2}}\right)\right).$$

Therefore

$$\begin{aligned}\Delta_3(\mu_0) &= \widetilde{\Delta}_3(-1/2, c_0) + \widetilde{\Delta}_3(-1/2, -c_0)\\ &= 4c_0\left[\exp\left(\frac{4c_0}{\sqrt{1-c_0^2}}\arctan\left(\frac{c_0+1}{\sqrt{1-c_0^2}}\right)\right)-\exp\left(\frac{4c_0}{\sqrt{1-c_0^2}}\arctan\left(\frac{c_0-1}{\sqrt{1-c_0^2}}\right)\right)\right]\\ &= 4c_0\exp\left(\frac{4c_0}{\sqrt{1-c_0^2}}\arctan\left(\frac{c_0-1}{\sqrt{1-c_0^2}}\right)\right)\left(\exp\left(\frac{2\pi c_0}{\sqrt{1-c_0^2}}\right)-1\right).\end{aligned}$$

In the last equality above we use that $\arctan(\varrho)+\arctan(1/\varrho) = \pi/2$ for $\varrho > 0$ taking $\varrho = \frac{c_0+1}{\sqrt{1-c_0^2}}$. This concludes the proof of the assertions concerning the case $c \neq 0$.

Finally, in order to study the case $c = 0$ we apply Proposition 2.8 to the family $\{\widetilde{X}_\mu,\ \mu \in (-1, 0)\times\{0\}\}$. Clearly, since λ does not depend on c, the development is exactly the same as before but replacing $\widehat{U}_i$ by $U_i \cap \{c = 0\}$ in the remainder term. Recall in addition that when we computed the coefficients for $c \neq 0$ we also contemplated the case $c = 0$. Thus the coefficients in this case follow from the substitution $c = 0$ in that ones. This concludes the proof of the result. ∎

Lemma 3.5. *Fix some $\mu = (b, c)$ in U_1 with $c \geqslant 0$ and define $u^\star = \frac{c}{2(b+1)}$. Then the following hold:*

(a) $F(u;\mu) - F(-u;\mu)\exp\left(\frac{-c\pi}{b|\kappa(\mu)|^{1/2}}\right) < 0$ for all $u \in (0, u^\star)$.
(b) $F(u;\mu) - F(-u;\mu) < 0$ for all $u \in (u^\star, +\infty)$.

Proof. Note that if $\mu \in U_1$ then $b \in (-1, -1/2)$ and that, by assumption, $c \geqslant 0$. This easily implies that

$$\frac{(2b+1)u+c}{-(2b+1)u+c} < 1 \text{ and } \frac{(b+1)u^2-cu-b}{(b+1)u^2+cu-b} < 1 \text{ for all } u > 0. \tag{5}$$

In the second inequality above we also use that $c^2 + 4b(b+1) = \kappa(\mu) < 0$ for all $\mu \in U$. Since $F(-u;\mu) > 0$ for all $u > 0$, the assertion in (a) is equivalent to prove that

$$\frac{F(u;\mu)}{F(-u;\mu)}\exp\left(\frac{c\pi}{b|\kappa|^{1/2}}\right) < 1 \text{ for all } u \in (0, u^\star). \tag{6}$$

In fact this holds for all $u > 0$. Indeed, the function on the left of the above inequality writes as

$$\frac{(2b+1)u+c}{-(2b+1)u+c}\left(\frac{(b+1)u^2-cu-b}{(b+1)u^2+cu-b}\right)^{\frac{1}{2b}+2}\exp\left(\frac{c}{b|\kappa|^{1/2}}\left(\mathcal{A}(u;\mu)+\pi\right)\right),$$

where

$$\mathcal{A}(u;\mu) := \arctan\left(\frac{|\kappa|^{1/2}}{2(b+1)u+c}\right) + \arctan\left(\frac{|\kappa|^{1/2}}{2(b+1)u-c}\right).$$

Thus, on account of the inequalities in (5) and due to $\frac{1}{2b}+2>0$, in order to show (6) it suffices to verify that $A(u;\mu)+\pi>0$. However this is clear because $\arctan x > -\pi/2$ for all x.

Let us prove next the assertion in (b). In this case, using again that $F(-u;\mu)>0$ for all $u>0$, it is equivalent to show that

$$\frac{F(u;\mu)}{F(-u;\mu)} < 1 \text{ for all } u \in (u^\star, +\infty). \tag{7}$$

Now the function on the left writes as

$$\frac{(2b+1)u+c}{-(2b+1)u+c}\left(\frac{(b+1)u^2-cu-b}{(b+1)u^2+cu-b}\right)^{\frac{1}{2b}+2} \exp\left(\frac{c}{b|\kappa|^{1/2}}\mathcal{A}(u;\mu)\right),$$

and therefore, taking account of (5) again, it suffices to show that $\mathcal{A}(u;\mu) > 0$ for all $u>u^\star$. However this is once again clear because $2(b+1)u+c>0$ for all $u>0$ and $2(b+1)u-c>0$ for all $u>u^\star$. Hence the result is proved. ∎

Lemma 3.6. *Fix some $\mu \in U_2$ with $c \geqslant 0$. Then* $G(u;\mu)+G(-u;\mu)\exp\left(\frac{c\pi|\kappa(\mu)|^{-1/2}}{b+1}\right) < 0$ *for all $u>0$.*

Proof. Define $g(u;\mu) := G(u;\mu)+G(-u;\mu)\exp\left(\frac{c\pi|\kappa(\mu)|^{-1/2}}{b+1}\right)$. We claim that $g'(u;\mu)<0$ for all $u>0$. Due to $g(0;\mu)=0$, it is clear that the result will follow once we prove this. To this end we first note that

$$G'(u;\mu) = f(u;\mu)\big(G(u;\mu)+1\big) \text{ with } f(u;\mu) := \frac{b}{b+1}\frac{(2b+1)u-c}{-bu^2+cu+b+1}.$$

Accordingly

$$\begin{aligned} g'(u;\mu) &= \\ &= G'(u;\mu) - G'(-u;\mu)\exp\left(\frac{c\pi|\kappa(\mu)|^{-1/2}}{b+1}\right) \\ &= f(u;\mu)\big(G(u;\mu)+1\big) - f(-u;\mu)\big(G(-u;\mu)+1\big)\exp\left(\frac{c\pi|\kappa(\mu)|^{-1/2}}{b+1}\right). \end{aligned}$$

Recall at this point that $c \geqslant 0$ and that, since $\mu \in U_2$, $b \in (-1/2,0)$. Taking this into account and using that $c^2+4b(b+1)=\kappa(\mu)<0$ for all $\mu \in U$, it easily follows that $f(-u;\mu)>0$ for all $u>0$. Consequently, since it is obvious that $G(u;\mu)+1>0$ for all $u\in\mathbb{R}$, the claim is equivalent to prove that

$$\frac{f(u;\mu)}{f(-u;\mu)}\,\frac{G(u;\mu)+1}{G(-u;\mu)+1}\exp\left(\frac{-c\pi|\kappa(\mu)|^{-1/2}}{b+1}\right)<1 \text{ for all } u>0.$$

Straightforward manipulations show that the function on the left of the above inequality writes as

(8)
$$\frac{c-(2b+1)u}{c+(2b+1)u}\left(\frac{b+1-cu-bu^2}{b+1+cu-bu^2}\right)^{\frac{4b+3}{2(b+1)}}\exp\left(\frac{-c|\kappa(\mu)|^{-1/2}}{b+1}\big(\mathcal{B}(u;\mu)+\pi\big)\right),$$

where

$$\mathcal{B}(u;\mu):=\arctan\left(\frac{2bu-c}{|\kappa|^{1/2}}\right)+\arctan\left(\frac{2bu+c}{|\kappa|^{1/2}}\right).$$

Now, using that $\mu\in U_2$ and $c\geqslant 0$ by assumption, one can verify that

$$\frac{c-(2b+1)u}{c+(2b+1)u}<1 \text{ and } \frac{b+1-cu-bu^2}{b+1+cu-bu^2}<1 \text{ for all } u>0.$$

These inequalities, due to $\frac{4b+3}{2(b+1)}>0$ for $b\in(-1/2,0)$, imply that the two first factors in (8) are smaller than one. However this is also the case for the last one. Indeed, $\mathcal{B}(u;\mu)+\pi>0$ because $\arctan x>-\pi/2$ for all x. This concludes the proof of the result. ■

Proof of Proposition 3.2 In view of Lemma 3.3 it only remains to prove the assertions with regard to the signum of the coefficients. It suffices to study them on $c\geqslant 0$ because we claim that $\Delta_i(b,-c)=\Delta_i(b,c)$ for $i=1,2,3$. Indeed, note that $P(s;\mu)$ and $\Delta_0(\mu)$ are even functions with respect to c by Lemma 3.3 and, on the other hand, $\lambda(\mu)$ depends only on b. This shows the claim because, for instance,

$$\Delta_2(\mu)=\lim_{s\longrightarrow 0}\frac{P(s;\mu)-\Delta_0(\mu)}{s^{\lambda(\mu)}}.$$

The expressions of Δ_1, Δ_2 and Δ_3 for $c\geqslant 0$ are given in Lemma 3.3. Taking them into account, Lemmas 3.5 and 3.6 show respectively that $\Delta_1(\mu)<0$ for all $\mu\in U_1\cap\{c\geqslant 0\}$ and that that $\Delta_2(\mu)<0$ for all $\mu\in U_2\cap\{c\geqslant 0\}$. The fact that $\Delta_3(-1/2,c)>0$ for $c\in(0,1)$ is clear from its expression. This concludes the proof of the result. ■

4. The real case

This section is devoted to study the period function of the center at the origin in case that the straight lines $\ell_\pm$ are real. More concretely, those parameters inside $V := \{\mu \in \mathbb{R}^2 : \kappa(\mu) > 0 \text{ and } b(b+1) \neq 0\}$. We shall prove the following:

Theorem 4.1. *Consider the center at the origin of system* (1) *for the parameters inside V. Then its period function is monotonous increasing near the outer boundary of the period annulus. Moreover there are no bifurcation of critical periods from the outer boundary.*

To prove this result we must first take a parametrization of the period function. With this aim in view note that, on account of Remark 2.3, it suffices to study $W := V_1 \cup V_2^+ \cup V_3$, where

$$V_1 := \{\mu \in V : b < -1\},\ V_2^+ := \{\mu \in V : b \in (-1,0) \text{ and } c > 0\} \text{ and}$$
$$V_3 := \{\mu \in V : b > 0\}.$$

Taking this subset of V enables us to parametrize the period function in a simple way. Recall, see Figure 2, that the polycycle at the outer boundary of $\mathcal{P}_\mu$ is a triangle for all $\mu \in V$. If we consider only $\mu \in W$ then the triangle has always one side on the straight line ℓ_+. Furthermore, the point $\ell_+ \cap \{y = 0\} = (q_\mu, 0)$, with $q_\mu := \frac{-2}{c+\kappa^{1/2}}$, is at the outer boundary of $\mathcal{P}_\mu$ for all $\mu \in W$. We take advantage of the fact that the segment that joins the origin with $(q_\mu, 0)$ is always inside $\mathcal{P}_\mu$ to parametrize the period function. Thus, for any $s \in (0,1)$ and $\mu \in W$, we denote by $P(s;\mu)$ the period of the periodic orbit of X_μ passing through the point $\big((1-s)\,q_\mu, 0\big) \in \mathbb{R}^2$. Notice that $s \approx 0$ corresponds to periodic orbits near the outer boundary of $\mathcal{P}_\mu$. Theorem 4.1 is an easy corollary of the following result, which provides the asymptotic expansion of $P(s;\mu)$ near $s = 0$.

Proposition 4.2. *For any $\mu \in W$ it holds $P(s;\mu) = \Delta_0(\mu)\ln s + \Delta_1(\mu) + \mathcal{I}(W)$, where Δ_i are analytic functions on V. Moreover $\Delta_0(\mu) < 0$ for all $\mu \in W$.*

The main part of this section is devoted to prove Proposition 4.2. For the reader's convenience we prefer to postpone its proof and show Theorem 4.1 first.

Proof of Theorem 4.1 By Remark 2.3 it is enough to consider those parameters inside W. Hence let us fix some $\mu^\star \in W$ and note that then, by applying Proposition 4.2,

$$sP'(s;\mu) \longrightarrow \Delta_0(\mu^\star) \text{ as } (s,\mu) \longrightarrow (0,\mu^\star).$$

(Here we took Definition 2.6 into account.) Since Δ_0 is negative, this implies that there exist a neighbourhood $U^\star$ of $\mu^\star$ in W and $\varepsilon > 0$ such that $P'(s;\mu) < 0$ for all $s \in (0,\varepsilon)$ and $\mu \in U^\star$. Accordingly there are no bifurcation of critical periods from the outer boundary at the parameter $\mu^\star$.

Finally the fact that $P'(s;\mu^\star) < 0$ for $s \in (0,\varepsilon)$ shows that the period function of $X_{\mu^\star}$ is *increasing* near the outer boundary. Indeed, by definition, $P(s;\mu)$ is the period of the periodic orbit passing through the point $\big((1-s)\,q_\mu, 0\big)$, which approaches to the outer boundary as *s decreases.* ■

The key point to prove Proposition 4.2 is that the period annulus has always a hyperbolic *finite* saddle at its outer boundary, because this fact forces that the principal term in the development is logarithmic. We will show first that the passage through a finite saddle contributes with this type of monomial to the time function. To state this precisely some notation is needed.

Consider a family of analytic vector fields $\{X_\mu, \mu \in V\}$ defined on some open set U of $\mathbb{R}^2$. Assume that each X_μ has a hyperbolic saddle p_μ as unique critical point inside U. If $\lambda_2(\mu) < 0 < \lambda_1(\mu)$ are the eigenvalues of the linear part of X_μ at p_μ, we denote by $r(\mu) := -\frac{\lambda_2(\mu)}{\lambda_1(\mu)}$ its *ratio of hyperbolicity.* In addition, let $\mathcal{S}_\mu$ and $\mathcal{T}_\mu$ be respectively the stable and unstable manifolds passing through p_μ. We consider Σ_1^μ and Σ_2^μ, two analytic transverse sections on $\mathcal{S}_\mu$ and $\mathcal{T}_\mu$ respectively. Let $s \longmapsto \sigma(s;\mu)$ and $s \longmapsto \tau(s;\mu)$ be parametrizations of Σ_1^μ and Σ_2^μ respectively with $\sigma(0;\mu) = \Sigma_1^\mu \cap \mathcal{S}_\mu$ and $\tau(0;\mu) = \Sigma_1^\mu \cap \mathcal{T}_\mu$. Denote the Dulac and time mappings associated to the passage from Σ_1^μ to Σ_2^μ by R and T respectively (see Figure 3).

Lemma 4.3. *With the above definitions the following hold:*

(*a*) $R(s;\mu) = s^{r(\mu)}\big(\rho(\mu) + \mathcal{I}(V)\big)$, *where ρ is an analytic positive function on V.*

(*b*) $T(s;\mu) = \frac{-1}{\lambda_1(\mu)} \ln s + \Delta(\mu) + \mathcal{I}(V)$, *where Δ is analytic on V.*

The assertion about the Dulac map in Lemma 4.3 is well known (see [12, 22] for instance). The one concerning the time function can be found in [16] taking $\mathcal{C}^k$ *normalized* transverse sections Σ_1^N and Σ_2^N near p_μ instead of the arbitrary ones Σ_1^μ and Σ_2^μ. More concretely, Σ_1^N and Σ_2^N are parametrized by means of the $\mathcal{C}^k$ diffeomorphism Φ that brings the family to normal form. To prove (b) with arbitrary sections one introduces Σ_1^N and Σ_2^N as auxiliary sections as shown in Figure 4. Then it suffices to show that the regular passages from Σ_1^μ to Σ_1^N and from Σ_2^N to Σ_2^μ contribute with higher order monomials to the development. For the sake of shortness we prefer not to give the proof of this. In fact we shall apply Lemma 4.3 to saddles verifying the FLP (recall Definition 2.4) and, under this additional assumption, (*b*) is a particular case of the results in [21].

In the proof of Proposition 4.2 we shall also use the following result:

Lemma 4.4. *Let $a(\mu)$ and $k(\mu)$ be positive analytic functions on V. Consider in addition some function $g(s;\mu)$ in $\mathcal{I}(V)$ and set $\varphi := s^k\big(a+g\big)$. In this case,*

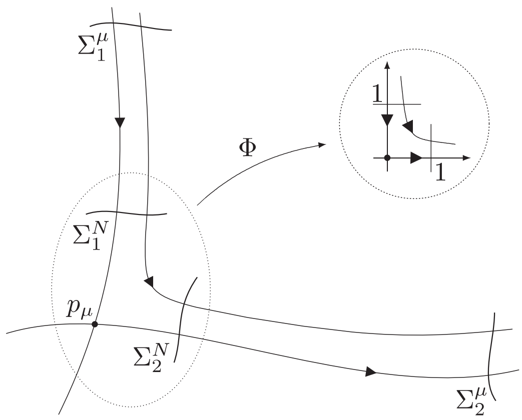

FIGURE 4. Remark on Lemma 4.3, where $\Sigma_1^N = \Phi^{-1}\big((0,\varepsilon)\times\{1\}\big)$ and $\Sigma_2^N = \Phi^{-1}\big(\{1\}\times(0,\varepsilon)\big)$.

(*a*) $s\omega(s;k) \in \mathcal{I}(V)$.
(*b*) *If* $f \in \mathcal{I}(V)$ *then* $f \circ \varphi \in \mathcal{I}(V)$.
(*c*) $\ln \varphi = k \ln s + \ln a + \mathcal{I}(V)$.

Proof. The assertion in (a) is shown in [25]. Concerning the one in (b), it is clear that $\big(f \circ \varphi\big)(s)$ tends to zero uniformly on μ as $s \longrightarrow 0$. On the other hand, straightforward computations yield to

$$s\big(f \circ \varphi\big)'(s) = s^k(a+g)f'\big(s^k(a+g)\big)\left(k + \frac{sg'}{a+g}\right),$$

which also tends to zero uniformly on μ as $s \longrightarrow 0$ since $f, g \in \mathcal{I}$. This shows the validity of (b). Finally, $\ln\varphi = k\ln s + \ln(a+g)$ and, on the other hand, $\ln(a+g) = \ln a + \mathcal{I}$ since $s \longmapsto \ln(a+s) - \ln a$ is analytic at $s=0$ and $g \in \mathcal{I}$. This proves (c) and concludes the proof of the result. ■

Proof of Proposition 4.2 Assume first that $\mu \in V_1$. In this case (see Figure 2 and recall the discussion before Remark 2.3) the triangle at the outer boundary of $\mathcal{P}_\mu$ is made up of segments of the straight lines ℓ_+ and ℓ_-, given by $2by - \big(c \pm \kappa^{1/2}\big)x = 2$, and the line at infinity ℓ_∞. One can easily verify that the vertex at $\ell_+ \cap \ell_-$, which corresponds to the point $(0, 1/b)$, is a hyperbolic saddle with eigenvalues $\frac{-c\pm\kappa^{1/2}}{2b}$. The two other vertices, $p_\pm^\infty := \ell_\pm \cap \ell_\infty$, are hyperbolic saddles too. Indeed, if we consider the chart of $\mathbb{RP}^2$ given by $(u,v)=\phi(x,y):=\left(\frac{x}{y+1},\frac{1}{y+1}\right)$, then X_μ becomes

$$\widetilde{X}_\mu(u,v)=\frac{1}{v}\left(\big(b-cu-(1+2b)v-(b+1)u^2+cuv+(b+1)v^2\big)\partial_u-uv\partial_v\right).$$

In these coordinates $p_\pm^\infty=\left(\frac{-c\pm\kappa^{1/2}}{2(b+1)},0\right)$ and their corresponding eigenvalues are $\frac{c\mp\kappa^{1/2}}{2(b+1)}$ and $\mp\kappa^{1/2}$. Let us denote by Σ_1 the transverse section at ℓ_+ given by $s\longmapsto\big((1-s)q_\mu,0\big)$ for $s\approx 0$. Note that $P(s;\mu)$ is defined precisely with respect to Σ_1. We consider in addition two auxiliary transverse sections on the other sides of the triangle. Let us denote the one at ℓ_- by Σ_2 and the one at ℓ_∞ by Σ_3. Now, let T_1 and T_2 be the time functions for X_μ associated respectively to the passage from Σ_1 to Σ_2 and from Σ_2 to Σ_3. Denote also the time function for $-X_\mu$ associated to the passage from Σ_1 to Σ_3 by T_3. Then, denoting the Dulac map from Σ_1 to Σ_2 by R_1, we can split up the period function as

$$P(s;\mu)=T_1(s;\mu)+T_2\big(R_1(s;\mu);\mu\big)+T_3(s;\mu). \tag{9}$$

Notice at this point that we can apply Proposition 2.8 (with $n=1$) to obtain the asymptotic expansion of the time functions associated to the passages through p_+^∞ and p_-^∞. Indeed, from Remark 2.1 it follows that $\widetilde{H}_\mu(u,v)=H_\mu\big(\phi^{-1}(u,v)\big)$ is a Darboux first integral of $\widetilde{X}_\mu$ and hence, according to Remark 2.5, the FLP is fulfilled. Thus, since the "second order" monomials s, s^λ and $s\omega(s;\lambda)$ belong to $\mathcal{I}\big((0,+\infty)\big)$, we can assert that

$$T_2(s;\mu)=\Delta_0^2(\mu)+f_2(s;\mu)\ \text{ and }\ T_3(s;\mu)=\Delta_0^3(\mu)+f_3(s;\mu)\ \text{ with } f_i\in\mathcal{I}(V_1).$$

(Here we took (a) in Lemma 4.4 into account.) Concerning the passage through the finite saddle at $(0,1/b)$, from Lemma 4.3 it follows that

$$R_1(s;\mu)=s^{r(\mu)}\big(\rho(\mu)+\mathcal{I}(V_1)\big)\ \text{ and }\ T_1(s;\mu)=\tfrac{-1}{\lambda_1(\mu)}\ln s+\Delta_0^1(\mu)+\mathcal{I}(V_1),$$

where $r(\mu)=\frac{-c+\kappa^{1/2}}{c+\kappa^{1/2}}$ and $\lambda_1(\mu)=-\frac{c+\kappa^{1/2}}{2b}$. Therefore, since $f_2\circ R_1\in\mathcal{I}(V_1)$ by (b) in Lemma 4.4, the combination of the three terms in (9) yields to

$$P(s;\mu)=\Delta_0(\mu)\ln s+\Delta_1(\mu)+\mathcal{I}(V_1), \tag{10}$$

with $\Delta_0(\mu)=\frac{2b}{c+\kappa^{1/2}}$ and $\Delta_1(\mu)=\Delta_0^1(\mu)+\Delta_0^2(\mu)+\Delta_0^3(\mu)$. Since $\Delta_0(\mu)<0$ for all $\mu\in V_1$, this proves the result in this case.

If $\mu\in V_2^+$ then the triangle at the outer boundary consists of segments of the straight lines $\{y=-1\}$, ℓ_+ and ℓ_∞. One can easily verify that the two vertices at infinity, namely $\ell_+\cap\ell_\infty$ and $\{y=-1\}\cap\ell_\infty$, are hyperbolic saddles. The third one, $\{y=-1\}\cap\ell_+$, is a hyperbolic saddle too (finite in this case) with eigenvalues $\lambda_1(\mu)=\kappa^{1/2}$ and $\lambda_2(\mu)=\frac{c-\kappa^{1/2}}{2b}$. We have thus the same configuration as in the previous case and, exactly the same way, one can show that (10) holds with $\Delta_0(\mu) = -\kappa^{-1/2}$.

Assume finally that $\mu \in V_3$. Then $\mathcal{P}_\mu$ is bounded and the triangle at its outer boundary is made up of segments of the straight lines ℓ_+, ℓ_- and $\{y = -1\}$. The two vertices at $\ell_\pm \cap \{y = -1\}$ are hyperbolic saddles with eigenvalues $\frac{c \mp \kappa^{1/2}}{2b}$ and $\pm\kappa^{1/2}$. The vertex at $\ell_+ \cap \ell_- = (0, 1/b)$ is a hyperbolic saddle too with $\lambda_1(\mu) = \frac{-c+\kappa^{1/2}}{2b}$ and $\lambda_2(\mu) = \frac{-c-\kappa^{1/2}}{2b}$. Recall that the period function $P(s;\mu)$ is parametrized with respect to the transverse section Σ_1 at ℓ_+, which is given by $s \longmapsto \big((1-s)\,q_\mu, 0\big)$. As before, we introduce two auxiliary transverse sections at $\{y = -1\}$ and ℓ_-, say Σ_2 and Σ_3 respectively. Let T_1 and T_2 be the time functions for X_μ associated to the passage from Σ_1 to Σ_2 and from Σ_2 to Σ_3 respectively. Denote also the time function for $-X_\mu$ associated to the passage from Σ_1 to Σ_3 by T_3. Thus, if we denote the Dulac map from Σ_1 to Σ_2 by R_1, then we can split up the period function as in (9). According to (*b*) in Lemma 4.3, for $i = 1, 2, 3$ we have that

$$T_i(s;\mu) = \Delta_0^i(\mu)\ln s + \Delta_1^i(\mu) + \mathcal{I}(V_3),$$

where

$$\Delta_0^1(\mu) = \tfrac{-1}{\kappa^{1/2}} \text{ and } \Delta_0^2(\mu) = \Delta_0^3(\mu) = \tfrac{-2b}{c+\kappa^{1/2}}.$$

(Note here that $\Delta_0^3(\mu) = \frac{1}{\lambda_2(\mu)}$ since we use $-X_\mu$ instead of X_μ.) Thus, since $R_1(s;\mu) = s^{r(\mu)}\big(\rho(\mu) + \mathcal{I}(V_3)\big)$ with $r(\mu) = -\frac{c-\kappa^{1/2}}{2b\kappa^{1/2}}$ by (*a*) in Lemma 4.3, it follows that

$$T_2\big(R_1(s;\mu);\mu\big) = \Delta_0^2(\mu)\,r(\mu)\ln s + \Delta_0^2(\mu)\ln\rho(\mu) + \Delta_1^2(\mu) + \mathcal{I}(V_3).$$

(To obtain this equality we use (*b*) and (*c*) in Lemma 4.4.) Accordingly, the combination of the three terms in (9) shows that $P(s;\mu) = \Delta_0(\mu)\ln s + \Delta_1(\mu) + \mathcal{I}(V_3)$ with

$$\Delta_0(\mu) = \Delta_0^1(\mu) + \Delta_0^2(\mu)\,r(\mu) + \Delta_0^3(\mu) = -\tfrac{2(b+1)}{c+\kappa^{1/2}}$$

and $\Delta_1(\mu) = \Delta_1^1(\mu) + \Delta_0^2(\mu)\ln\rho(\mu) + \Delta_1^2(\mu) + \Delta_1^3(\mu)$. Consequently, since $\Delta_0(\mu) < 0$ for all $\mu \in V_3$, this completes the proof of the result. ■

5. Monotonicity in $\mathbf{\Gamma}_1$ and proof of the main result

The last ingredient in the proof of Theorem A is the following result:

Theorem 5.1. *The period function of the center at the origin of system* (1) *with* $\mu \in \Gamma_1$ *is* (*globally*) *monotonous increasing.*

Proof. Recall that $\Gamma_1 = \big\{\mu \in \mathbb{R}^2 : b(b+1)\kappa(\mu) = 0\big\}$. The monotonicity on $b = 0$ follows from previous results. Indeed, if $c \neq 0$ then it corresponds to the classical Lotka-Volterra center and the monotonicity of its period function has been proved by several authors [23, 28, 33]. The monotonicity for the remaining case $\mu = (0,0)$ is proved in [3].

So let us consider the parameters on the straight line $b+1=0$ or the conic $\kappa(\mu)=0$ (see Figure 1). The key point, except for the parameters on $c=0$, is the same for both cases. Namely, that we have found a coordinate transformation $(u,v)=\phi(x,y)$ that brings system (1) with $(b+1)\kappa(\mu)=0$ to a Hamiltonian system of the form

$$\begin{cases} \dot{u} = -G'(v), \\ \dot{v} = F'(u). \end{cases} \tag{11}$$

The period function of this type of Hamiltonian center is studied in [9] and the authors provide a sufficient condition in order that it is monotonous increasing. Let us recall it briefly. To this end assume that $F(s)$ and $G(s)$ have a non-degenerate minimum at $s=0$ and that $F(0)=G(0)=0$. This guarantees that system (11) has a center at the origin and the sufficient condition mentioned before is that

$$\frac{F(s)}{F'(s)^2} \text{ and } \frac{G(s)}{G'(s)^2} \text{ are both convex functions.}$$

This constitutes a useful result for our purposes. Indeed, if $b=-1$ and $c\neq 0$ then one can check that the change of coordinates given by

$$(u,v)=\left(\ln\left(1+\frac{cx}{y+1}\right),\frac{y}{y+1}\right)$$

brings (1) to system (11) with $F(u)=\frac{1}{c}\left(e^u-u-1\right)$ and $G(v)=-c\big(\ln(1-v)+v\big)$. Since it is easy to show that both functions verify the condition above, the monotonicity follows. (To study the case $c<0$ we must first reverse time.) On the other hand, in case that $\kappa(\mu)=0$ and $c\neq 0$, the coordinate transformation

$$(u,v)=\left(\ln\left(\frac{2+2y}{2+cx-2by}\right),\frac{cx-2by}{2+cx-2by}\right)$$

brings (1) to system (11) with $F(u)=\frac{2b}{c}\left(e^u-u-1\right)$ and $G(v)=\frac{2(b+1)}{c}\big(\ln(1-v)+v\big)$. These functions are the same as before up to a constant factor and so the period function is also increasing in this case. (The reader interested on the idea underneath these two changes of coordinates is referred to Remark 5.2 below.)

It only remains to show the monotonicity for $\mu=(-1,0)$. By applying Lemma 5 in [15] we obtain a change of variables that brings system (1) with $\mu=(-1,0)$ to a potential system. Indeed, it is given by

$$(u,v)=\left(\frac{-y}{y+1},\frac{x}{y+1}\right)$$

and it brings the original system to (11) with $F(u) = u - \ln(1+u)$ and $G(v) = \frac{1}{2}v^2$. Then, by means of Chicone's criteria for potential systems (see [3]) one can easily conclude that in this case the period function is monotonous increasing too. This completes the proof of the result. ■

Remark 5.2 In the three cases considered in Theorem 5.1, namely $b = 0$, $b = -1$ and $\kappa(\mu) = 0$, the outer boundary of $\mathcal{P}_\mu$ is a triangle with two vertices at infinity. In the three cases as well, there are the same type of critical points at the vertices: one hyperbolic saddle and two saddle-nodes. However for $b = -1$ and $\kappa(\mu) = 0$ the finite vertex is one of the saddle-nodes, whereas for the classical Lotka-Volterra system it is the hyperbolic saddle. We obtained the coordinate transformations for $b = -1$ and $\kappa(\mu) = 0$ as follows. First we considered a projective change of variables that sends the straight line joining the two saddle-nodes to the line at infinity. We required moreover that it brings the hyperbolic saddle at infinity to the origin. Our goal with this was to obtain the same distribution of vertices as in the case $b = 0$. For instance, for $b = -1$ this is achieved by means of $(u,v) = \left(\frac{y+cx+1}{y+1}, \frac{1}{y+1}\right)$. Fortunately, in both cases the transformation brought the original system to one of the form

$$\begin{cases} \dot{u} = p_1(u)\, p_2(v), \\ \dot{v} = q_1(u)\, q_2(v), \end{cases}$$

and the authors in [9] explain a method to transform this type of system to a Hamiltonian as in (11). For the centers of the latter there are several monotonicity criterions (see [13, 24, 34]). □

Proof of Theorem A Recall that Theorems 3.1 and 4.1 deal with the parameters on U and V respectively. Note in addition that if $\mu \notin U \cup V$ then $b(b+1)\kappa(\mu) = 0$, which is the case treated in Theorem 5.1. The fact that, except for $\mu = (-1/2, 0)$, the period function of X_μ is monotonous increasing near the outer boundary follows from the application of Theorems 3.1, 4.1 and 5.1 in the corresponding cases. The assertion in (a) follows by applying Theorems 3.1 and 4.1. Finally Theorem 5.1 shows the assertion in (b). ■

As we already mentioned, system (1) with $\mu \in (-1,0) \times \{0\}$ has another center located at $(0, 1/b)$ apart from the one at the origin. For the sake of completeness let us not note that, by an affine transformation and a constant rescaling of time, this center can be brought to the origin and system (1) writes then as

$$\begin{cases} \dot{x} = -y + xy, \\ \dot{y} = x + Dx^2 + Fy^2, \end{cases}$$

with $(D,F) = (-b-1, -b)$. The period function of the center at the origin of the above systems, the so called dehomogenized Loud's systems, is studied in [19, 20]. In our case it follows that, near the outer boundary of its period annulus, it is monotonous increasing for $b \in (-1/2, 0)$ and decreasing for $b \in (-1, -1/2)$. The center at the origin is isochronous for $(D, F) = (-1/2, 1/2)$ and hence system (1) with $\mu = (-1/2, 0)$ has two isochronous centers coexisting.

References

[1] M. Abramowitz and I.A. Stegun, "Handbook of mathematical functions with formulas, graphs, and mathematical tables", Dover Publications, New York, 1992.

[2] C. Chicone, review in MathSciNet, ref. 94h:58072.

[3] C. Chicone, *The monotonicity of the period function for planar Hamiltonian vector fields,* J. Differential Equations **69** (1987), 310–321.

[4] C. Chicone, *Geometric methods of two-point nonlinear boundary value problem,* J. Differential Equations **72** (1988), 360–407.

[5] C. Chicone and F. Dumortier, *A quadratic system with a nonmonotonic period function,* Proc. Amer. Math. Soc. **102** (1988), 706–710.

[6] C. Chicone and F. Dumortier, *Finiteness for critical periods of planar analytic vector fields,* Nonlinear Anal. **20** (1993), 315–335.

[7] C. Chicone and M. Jacobs, *Bifurcation of critical periods for plane vector fields,* Trans. Amer. Math. Soc. **312** (1989), 433–486.

[8] C.J. Christopher and C.J. Devlin, *Isochronous centres in planar polynomial systems,* SIAM Jour. Math. Anal. **28** (1997), 162–177.

[9] A. Cima, A. Gasull and F. Mañosas, *Period function for a class of Hamiltonian systems,* J. Differential Equations **168** (2000), 180–199.

[10] A. Cima, F. Mañosas and J. Villadelprat, *Isochronicity for several classes of Hamiltonian systems,* J. Differential Equations **157** (1999), 373–413.

[11] W.A. Coppel and L. Gavrilov, *The period function of a Hamiltonian quadratic system,* Differential Integral Equations **6** (1993), 1357–1365.

[12] F. Dumortier, R. Roussarie and C. Rousseau, *Elementary graphics of cyclicity* 1 *and* 2, Nonlinearity **7** (1994), 1001–1043.

[13] E. Freire, A. Gasull and A. Guillamon, *First derivative of the period function with applications,* J. Differential Equations **204** (2004), 139-162.

[14] A. Gasull, A. Guillamon and V. Mañosa, *An explicit expression of the first Liapunov and period constants with applications,* J. Math. Anal. Appl. **211** (1997), 190–212.

[15] A. Gasull, A. Guillamon and J. Villadelprat, *The period function for second-order quadratic ODEs is monotone,* Qual. Theory Dyn. Syst. **5** (2004), 201–224.

[16] A. Gasull, V. Mañosa and J. Villadelprat, *On the period of the limit cycles appearing in one-parameter bifurcations,* preprint, to appear in J. Differential Equations (2004).

[17] W.S. Loud, *Behaviour of the period of solutions of certain plane autonomous systems near centers,* Contrib. Differential Equations **3** (1964), 21–36.

[18] P. Mardešić, L. Moser-Jauslin and C. Rousseau, *Darboux linearization and isochronous centers with a rational first integral,* J. Differential Equations **134** (1997), 216–268.

[19] P. Mardešić, D. Marín and J. Villadelprat, *On the time function of the Dulac map for families of meromorphic vector fields,* Nonlinearity **16** (2003), 855–881.

[20] P. Mardešić, D. Marín and J. Villadelprat, *The period function of reversible quadratic centers,* preprint (2004), Departament de Matemàtiques UAB.

[21] D. Marín and J. Villadelprat, *On the time function of the Dulac map,* in preparation (2005).

[22] A. Mourtada, *Cyclicité finie des polycycles hyperboliques de champs de vecteurs du plan: mise sous forme normale,* in "Bifurcations of planar vector fields" (Luminy, 1989), 272–314, Lecture Notes in Math., 1455, Springer, Berlin, 1990.

[23] F. Rothe, *The periods of the Volterra-Lokta system,* J. Reine Angew. Math. **355** (1985), 129–138.

[24] F. Rothe, *Remarks on periods of planar Hamiltonian systems,* SIAM J. Math. Anal. **24** (1993), 129–154.

[25] R. Roussarie, *On the number of limit cycles which appear by perturbation of separatrix loop of planar vector fields,* Bol. Soc. Brasil. Mat. **17** (1986), 67–101.

[26] C. Rousseau and B. Toni, *Local bifurcations of critical periods in the reduced Kukles system,* Can. J. Math. **49** (1997), 338–358.

[27] M. Saavedra, *Asymptotic expansion of the period function,* J. Differential Equations **193** (2003), 359–373.

[28] R. Schaaf, *Global behaviour of solution branches for some Neumann problems depending on one or several parameters,* J. Reine Angew. Math. **346** (1984), 1–31.

[29] R. Schaaf, *A class of Hamiltonian systems with increasing periods,* J. Reine Angew. Math. **363** (1985), 96–109.

[30] D. Schlomiuk, *Algebraic and geometric aspects of the theory of polynomial vector fields,* in "Bifurcations and periodic orbits of vector fields" (Montreal, PQ, 1992), 429–467, Kluwer Acad. Publ., Dordrecht, 1993.

[31] D. Schlomiuk, *Algebraic particular integrals, integrability and the problem of the center,* Trans. Amer. Math. Soc. **338** (1993), 799–841.

[32] J. Smoller and A. Wasserman, *Global bifurcation of steady-state solutions,* J. Differential Equations **39** (1981), 269–290.

[33] J. Waldvogel, *The period in the Lotka-Volterra system is monotonic,* J. Math. Anal. Appl. **114** (1986), 178–184.

[34] A.A. Zevin and M.A. Pinsky, *Monotonicity criteria for an energy-period function in planar Hamiltonian systems,* Nonlinearity **14** (2001), 1425-1432.

[35] Y. Zhao, *The monotonicity of period function for codimension four quadratic system* Q_4, J. Differential Equations **185** (2002), 370–387.

[36] Y. Zhao, *The period function for quadratic integrable systems with cubic orbits,* J. Math. Anal. Appl. **301** (2005), 295-312.

[37] Y. Zhao, *On the monotonicity of the period function of a quadratic system,* Discrete Contin. Dyn. Syst. **13** (2005), 795–810.

[38] H. Żołądek, *Quadratic systems with center and their perturbations,* J. Differential Equations **109** (1994), 223–273.

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