

A PROOF THAT ALL SEIFERT 3-MANIFOLD GROUPS AND ALL VIRTUAL SURFACE GROUPS ARE CONJUGACY SEPARABLE

ARMANDO MARTINO

ABSTRACT. We prove that the fundamental group of any Seifert 3-manifold is conjugacy separable. That is, conjugates may be distinguished in finite quotients or, equivalently, conjugacy classes are closed in the pro-finite topology.

1. INTRODUCTION

There has been considerable interest in the separability properties of groups, both from group theorists and topologists.

A subset X of a group G is called separable if for each $g \notin X$ there exists a map $\pi : G \rightarrow Q$, to a finite group Q such that $g\pi \notin X\pi$.

It is well known that *residual finiteness*, where $X = \{1\}$, for a finitely presented group implies a solution to the word problem. Similarly, *subgroup separability*, where X is a finitely generated subgroup, implies a solution to the membership problem. And *conjugacy separability*, where X is a conjugacy class implies a solution to the conjugacy problem. Additionally, these properties are related to the problem of lifting immersed subspaces of topological spaces to embedded subspaces of finite covers.

Recently, [1] have proved that certain Seifert 3-manifold groups are conjugacy separable and in this paper we prove that *all* Seifert 3-manifold groups are conjugacy separable. They are already known to be residually finite, [5] and subgroup separable, [11]. Our approach is different from that in [1], in that we base our argument on the algebraic structure of a Seifert 3-manifold group as an extension, rather than trying to decompose it into amalgamated free products. One advantage of this approach is that it allows an essentially unified treatment of these groups, although there is a certain distinction which arises depending on whether the Seifert fibred space is ‘built’ from a free group or the fundamental group of a compact surface.

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Our proof relies on the fact that virtually surface groups are all conjugacy separable. However, while this is already known that virtually free groups are conjugacy separable, it was surprisingly unknown for virtually surface groups. It is important to realise the contrast between conjugacy separability and residual finiteness at this point, since the latter is easily shown to pass to finite extensions whereas for the former this is unclear and possibly untrue. However, it is known that Fuchsian groups are conjugacy separable, and we show how to argue from here to deduce that all virtually surface groups are conjugacy separable.

2. BACKGROUND

2.1. Pro-finite topology. Given a group, G , the *pro-finite topology* on G has as a basis all the cosets of finite index subgroups of G . Each such coset is both open and closed in the pro-finite topology.

While the separability properties of G can be described in terms of the finite quotients of G , it is sometimes more convenient to talk about the pro-finite topology instead.

For instance, G is called *residually finite* if for any $1 \neq g \in G$, there exists a finite quotient of G in which g does not map to the identity element. In other words, there is a normal subgroup N of G of finite index such that $g \notin N$. Equivalently, this means that $\{1\}$ is closed in the pro-finite topology of G . In fact, this is the same as saying that any one element subset of G is closed or, the seemingly stronger statement, that G is Hausdorff.

The group G is called *subgroup separable* if for every finitely generated subgroup, H of G and every $g \in G - H$ there exists a normal subgroup of finite index, N of G such that $g \notin HN$. Thus, there is a finite quotient of G in which the images of g and H are disjoint. As before, this is the same as saying that every finitely generated subgroup is closed.

More generally, a subset $S \subseteq G$ is called separable if for every $g \notin S$, there is a finite quotient of G in which g and S have disjoint images. Equivalently, S is separable if it is closed.

A group G is called *conjugacy separable* if conjugacy classes are separable.

If we have a subgroup H of G , then there are two possible topologies one can put on H . Namely, the subspace topology and the pro-finite topology of H itself. In general the subspace topology may be more coarse, but not if H has finite index.

Lemma 2.1. *Let H be a finite index subgroup of G . Then the subspace topology and the pro-finite topology of H are the same.*

Proof. If K is a finite index subgroup of G then $Kx \cap H$ is either empty, or is a coset of $K \cap H$, which has finite index in H . Hence the subspace topology is contained in the pro-finite topology.

Conversely, for any K_0 which is a finite index subgroup of H , K_0x is open in G , since K_0 has finite index in G . Hence the pro-finite topology is contained in the subspace topology. \square

2.2. Seifert fibred spaces. Throughout the paper we shall use the term *surface group* to mean a group isomorphic to the fundamental group of a compact surface. This includes the free abelian group of rank 2, the closed hyperbolic surface groups as well as any finitely generated free group. Since we are generally working up to finite index, we will mainly restrict our attention to the orientable case. Using standard terminology, we will call a group a *virtual surface group* if it has a surface subgroup of finite index.

A Seifert fibred space is a 3-manifold that is ‘almost’ a bundle. This property can be made more precise when one considers the fundamental group, $\pi_1(M)$ of a Seifert fibred space, M .

Theorem 2.2. *Let M be a Seifert fibred space. Then there exists a short exact sequence,*

$$1 \rightarrow C \rightarrow \pi_1(M) \rightarrow H$$

such that C is cyclic and H contains a finite index subgroup isomorphic to a surface group.

Proof. This is proved in [12], where it is shown that H is the fundamental group of a 2-orbifold. By [12], such a group is virtually a surface group. We note that it is sufficient to consider the case where the 2-orbifold is without boundary, by the remarks in [12], and since finitely generated subgroups of surface groups are surface groups (recall that we are including free groups as surface groups). In fact, it turns out that if $\pi_1(M)$ is infinite - the only real case of interest for us - then C is also infinite. But we shall not need to use this fact. \square

There is a rich literature dealing with the subject of Seifert fibred spaces ([6], [14], for instance) and their role in the theory of 3-manifolds. For our purposes, however, the Theorem above provides sufficient information on their fundamental groups in order to demonstrate conjugacy separability.

3. VIRTUAL SURFACE GROUPS

The goal of this section is to prove that any finitely generated group having a surface subgroup of finite index is conjugacy separable. Since virtually free and virtually free abelian groups are already known to be conjugacy separable, we shall in fact only be considering the case of closed

hyperbolic surface groups. In fact, since in [3] it has been proved that Fuchsian groups are conjugacy separable, we shall show that this is sufficient for one to deduce that *any* finite extension of a closed hyperbolic surface group is conjugacy separable. The connection between the general case and the Fuchsian case is provided by the Nielsen Realisation Theorem.

First, however, we start with some terminology.

Definition 3.1. *Given a group S we call an automorphism, ϕ , of S virtually inner if ϕ^n is inner for some integer n . Given a virtually inner automorphism, ϕ , let $S*_\phi$ denote the group*

$$\langle S, t : t^n = x, t^{-1}gt = g\phi, \text{ for all } g \in S \rangle,$$

where n is chosen to be the least integer such that ϕ^n is inner and x is chosen so that $g\phi^n = x^{-1}gx$ for all $g \in S$. This group has S as a subgroup of finite index.

Also, given an automorphism ϕ of S , we consider the equivalence relation, \sim_ϕ on S where $g_1 \sim_\phi g_2$ if and only if there exists an $h \in S$ such that $(h^{-1}\phi)g_1h = g_2$. We call this relation "twisted- ϕ conjugacy" with "twisted- ϕ conjugacy classes". We denote the twisted- ϕ conjugacy class of an element h by $[h]_\phi$.

The main results we shall use in the section are the following,

Theorem 3.2 (Nielsen Realisation Theorem, [8], [10]). *Let S be the fundamental group of a closed hyperbolic surface. Let H be a finite subgroup of $\text{Out}(S)$ and let G be the pre-image of H in $\text{Aut}(S)$. Then G has $\text{Inn}(S) \cong S$ as a subgroup of finite index and is a Fuchsian group.*

Theorem 3.3 ([3]). *Fuchsian groups are conjugacy separable.*

Putting the above two together immediately produces the following crucial corollary.

Corollary 3.4. *Let S be the fundamental group of a closed hyperbolic surface, and ϕ a virtually inner automorphism of S . Then $S*_\phi$ is conjugacy separable.*

Proof. Clearly, the image of ϕ in $\text{Out}(S)$ is a finite cyclic subgroup whose pre-image in $\text{Aut}(S)$ is isomorphic to $S*_\phi$. \square

The purpose of introducing our seemingly cumbersome terminology (twisted- ϕ conjugacy) is that it enables us to pass between finite extensions of a group, S , by describing the separability properties of the extension solely in terms of S .

Proposition 3.5. *Let S be a conjugacy separable group and ϕ a virtually inner automorphism of S . If $S*_\phi$ is conjugacy separable, then twisted- ϕ conjugacy classes in S are closed in the pro-finite topology of S .*

Proof. Note that S is a finite index subgroup of S^*_ϕ , so a subset of S is closed in the pro-finite topology of S if and only if it is closed in the pro-finite topology of S^*_ϕ .

Now consider the element, tg , for $g \in S$. Clearly, every conjugate of tg is a conjugate of tg by some element of S . Moreover, if $x \in S$ then,

$$x^{-1}tgx = t(x^{-1}\phi)gx.$$

Hence the conjugacy class of tg in S^*_ϕ is equal to

$$t([g]_\phi),$$

where the twisted- ϕ conjugacy class is understood to be in S . Now, since group multiplication is a homeomorphism, $[g]_\phi$ must be closed in S^*_ϕ and hence in S . \square

Proposition 3.6. *Let G be a group which has S as a normal subgroup of finite index. Suppose that all twisted- ϕ conjugacy classes of S are closed in the pro-finite topology of S , for all virtually inner automorphisms ϕ . Then G is conjugacy separable.*

Proof. Again, as S has finite index, a subset of S is closed in S if and only if it is closed in G .

Now consider a $g \in G$. Let x_1, \dots, x_r be a set of coset representatives for S in G and let $g_i = x_i^{-1}gx_i$. Let ϕ_i be the automorphism of S induced by conjugation by the element g_i . Clearly, each ϕ_i is virtually inner. We observe the following,

$$\begin{aligned} \{w^{-1}gw : w \in G\} &= \bigcup_{i=1}^r \{x_i^{-1}g_ix : x \in S\} \\ &= \bigcup_{i=1}^r g_i[1]_{\phi_i}, \end{aligned}$$

where $[1]_{\phi_i}$ is understood to be a twisted- ϕ_i conjugacy class in S . By hypothesis, each $[1]_{\phi_i}$ is closed in S and thence in G . Thus each $g_i[1]_{\phi_i}$ is closed in G and thus so is the (finite) union. Thus we have shown that every conjugacy class in G is closed, which is another way of saying that G is conjugacy separable. \square

It is now clear how to put these results together to get,

Theorem 3.7. *Let G be a group having a surface subgroup of finite index. Then G is conjugacy separable.*

Proof. Let S be a surface subgroup of finite index which we assume, without loss of generality, is normal in G . If S is free this is proved by [2]. If S is free abelian (the torus case), this is proved by [4].

So we are left with the case where S is the fundamental group of a closed hyperbolic surface. Proposition 3.5 and Corollary 3.4 imply that all twisted- ϕ conjugacy classes of S are closed, for any virtually inner ϕ . Proposition 3.6 then implies that G is conjugacy separable. \square

4. EXTENSIONS

The main goal of this section is to prove Theorem 4.3, which says that finite-by-surface groups are surface-by-finite. That is, they are virtually surface groups. We note that this is already known, see [7], exercise 4.7, but we felt that since it seems to be a key observation which allows us to construct our proof of the fact that Seifert 3-manifold groups are conjugacy separable, it would be beneficial to include a proof of it. Additionally, the proof we present here is, as far as we are aware, new and is elementary up to some well known results concerning the residual properties of free groups and nilpotent groups.

We start with the following lemma.

Lemma 4.1. *Let F be a finitely generated free group and $1 \neq g \in F$. For any integer n there exists a finite quotient, Q_n of F , such that the image of g is a central element of order n in Q_n .*

Proof. It is clearly sufficient to prove the Lemma in the case where n is a power of some prime. So we consider an arbitrary prime p and will show that there is a quotient of F in which the image of g is central and has order p^k for any natural number k .

Now consider the central series for F . Write $\gamma_0 = F$ and $\gamma_{i+1} = [F, \gamma_i]$. As F is residually nilpotent ([9]) we can find a least i such that $g \notin \gamma_i$. Hence g is non-trivial and central in the free nilpotent group F/γ_i . The group F/γ_i is a torsion free nilpotent group, and hence is residually a finite p -group for all primes p ([13] or deduce this from [9] again). Thus we can find a sequence of quotients Q_k of F such that each Q_k is a p -group in which the image of g is central and g has order p^{n_k} for some unbounded sequence n_k . (Specifically, Q_k is a finite p -group quotient of F/γ_i in which the images of g^{p^i} , for $1 \leq i \leq k$, are non-trivial.)

However, we note that if h is a central element of a finite p -group Q with order p^r , then there is a quotient of Q in which h is central and has order p^{r-1} . Namely, quotient out by all central elements of order p . Hence we are done. \square

Lemma 4.2. *Consider the short exact sequence of groups,*

$$1 \rightarrow G \rightarrow \Gamma \xrightarrow{\pi} H \rightarrow 1$$

where G is a finite subgroup of the centre of Γ and H is a surface group. Then Γ has a finite index subgroup Γ_0 such that $\Gamma_0 \cap G$ is trivial.

Proof. This is trivial if H is free so we assume that H is a one-relator group and we write $H = \langle X : r \rangle$. Now let F be the free group on the set X , and $\rho : F \rightarrow H$ the natural epimorphism. By the universal property of

free groups, there is a map $\sigma : F \rightarrow \Gamma$ and a commuting triangle,

$$\begin{array}{ccc} & F & \\ \sigma \swarrow & & \searrow \rho \\ \Gamma & \xrightarrow{\pi} & H \end{array}$$

Now $r\sigma$ is clearly in the kernel of π and hence is an element of G . Suppose the order of $r\sigma$ is n . Then, by Lemma 4.1, there exists a finite quotient, Q , of F in which the image of r is central and has order n . Let F_0 be the kernel of the map from F to Q and consider the subgroup $\Gamma_0 = F_0\sigma$. Clearly, Γ_0 is a finite index subgroup of Γ . We claim that Γ_0 is the required subgroup.

For consider an element $w \in \Gamma_0 \cap G$. Then $w = u\sigma$ for some $u \in F_0$. Moreover, as w lies in the kernel of π , u must lie in the kernel of ρ . Hence we can write u as a product of conjugates of $r^{\pm 1}$ as follows,

$$u = \prod_{i=1}^k g_i^{-1} r^{\epsilon_i} g_i$$

where each $\epsilon_i = \pm 1$ and the $g_i \in F$. Note that since r is central in F/F_0 we have that $u = r^m$ in F/F_0 , where $m = \sum_{i=1}^k \epsilon_i$. In fact, since r has order exactly n in F/F_0 and $u \in F_0$ we also have that m is a multiple of n .

However, since $r\sigma$ is central in Γ we also get that $w = u\sigma = (r\sigma)^m$. As $r\sigma$ has order n and m is a multiple of n , we deduce that $\Gamma_0 \cap G$ is trivial. \square

Theorem 4.3. *Consider a short exact sequence of groups,*

$$1 \rightarrow G \rightarrow \Gamma \rightarrow H \rightarrow 1,$$

where G is a finite group and H is a surface group. Then Γ has a subgroup of finite index which is a surface group. In particular, Γ is conjugacy separable.

Proof. Note that Γ acts on G by conjugation. As this is a finite group, the kernel of this action, Γ_1 is a finite index subgroup of Γ which centralises G . In particular, $G \cap \Gamma_1$ is a finite central subgroup of Γ_1 . Since the image of Γ_1 in H is a finite index subgroup of H and hence a surface group, we may apply Lemma 4.2 to deduce that Γ_1 and hence Γ has a subgroup of finite index which intersects G trivially. Since this subgroup is isomorphic to a subgroup of finite index in H , we have shown that Γ is virtually a surface group. Hence, by Theorem 3.7, Γ is conjugacy separable. \square

5. MAIN ARGUMENT

Given a Seifert 3-manifold, M , we will show that $\pi_1(M)$ is conjugacy separable by considering the short exact sequence given by Theorem 2.2.

$$1 \rightarrow C \rightarrow \pi_1(M) \rightarrow H \rightarrow 1,$$

where C is cyclic and H is virtually a surface group. In particular, we do this by analysing the action on the normal subgroup C . In fact, by Theorem 4.3, it is sufficient to consider the case when C is infinite cyclic, so henceforth we shall assume that $C \cong \mathbb{Z}$.

We let h denote the generator of this \mathbb{Z} and note that every conjugate of h is equal to either h or h^{-1} . In particular, the subgroup $\langle h^k \rangle$ is a normal subgroup of $\pi_1(M)$ for each integer k . Our strategy will be to show that if two elements in $\pi_1(M)$ are not conjugate, then they will also fail to be conjugate in a quotient of $\pi_1(M)$ by some $\langle h^k \rangle$. Since these quotients are all conjugacy separable by the previous section, this is clearly sufficient.

The crux of the argument that follows is the content of the following lemma.

Lemma 5.1. *Let $g \in \pi_1(M)$. Then there exist integers λ, λ_0 so that g conjugate to gh^n in $\pi_1(M)$ if and only if $n = k\lambda$ or $n = k\lambda + \lambda_0$, for some integer k .*

Proof. Let C be the pre-image in $\pi_1(M)$ of the centraliser of g in $\pi_1(X)$. That is, C is precisely the subgroup of elements which conjugate g to an element of the form gh^n for some integer n . Let C^+ be the subgroup of C of index at most 2 which centralise h .

Consider $x_1, x_2 \in C^+$ and suppose that $x_i^{-1}gx_i = gh^{n_i}$. Then it is easy to check that

$$x_2^{\mp 1}x_1^{-1}gx_1x_2^{\pm 1} = gh^{n_1 \pm n_2}.$$

In other words, the set $\{h^n : x^{-1}gx = gh^n \text{ for some } x \in C^+\}$ is a subgroup of $\langle h \rangle$. Thus it has a generator h^λ and we have shown that $gh^n = x^{-1}gx$ for some $x \in C^+$ if and only if $n = k\lambda$.

If $C = C^+$ we are done by setting $\lambda_0 = 0$. Otherwise we can find a $y \in C - C^+$ and an integer λ_0 such that $y^{-1}gy = gh^{\lambda_0}$. The conclusion of this lemma is now immediate on noting that,

$$y^{-1}gh^{k\lambda}y = gh^{-k\lambda + \lambda_0},$$

and that $C = C^+ \cup C^+y$. \square

Theorem 5.2. *All Seifert 3-manifold groups are conjugacy separable.*

Proof. By Theorem 4.3, it is sufficient to prove the theorem in the case where we have a short exact sequence of groups associated to a Seifert 3-manifold, M , of the form

$$1 \rightarrow \mathbb{Z} \rightarrow \pi_1(M) \rightarrow H \rightarrow 1,$$

where H is a virtual surface group and \mathbb{Z} is generated by an element h .

So let us consider two elements $g, g_1 \in \pi_1(M)$ which are not conjugate. In order to show that these elements are not conjugate in some finite quotient,

it is clearly sufficient to show that they are not conjugate in some conjugacy separable quotient of $\pi_1(M)$. Thus we may assume that the images of g and g_1 are conjugate in H , since H is a conjugacy separable quotient of $\pi_1(M)$, by Theorem 3.7. After taking suitable conjugates, we may further assume that g and g_1 have the same image in H .

Thus $g_1 = gh^n$ for some integer n . However, by Lemma 5.1, there exist integers λ, λ_0 so that an element of the form gh^m is conjugate to g in $\pi_1(M)$ if and only if $m = k\lambda$ or $m = k\lambda + \lambda_0$ for some integer k . In particular, since g and g_1 are not conjugate in $\pi_1(M)$, neither can they be conjugate in the quotient $\pi_1(M)/\langle h^\lambda \rangle$. However, again by Theorem 4.3, $\pi_1(M)/\langle h^\lambda \rangle$ is conjugacy separable and we are done. \square

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