CAT(0) AND CAT(-1) DIMENSIONS OF TORSION FREE HYPERBOLIC GROUPS

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Abstract. We show that a particular free-by-cyclic group has CAT(0) dimension equal to 2, but CAT(-1) dimension equal to 3. We also classify the minimal proper 2-dimensional CAT(0) actions of this group; they correspond, up to scaling, to a 1-parameter family of locally CAT(0) piecewise Euclidean metrics on a fixed presentation complex for the group.

This information is used to produce an infinite family of 2-dimensional hyperbolic groups, which do not act properly by isometries on any proper CAT(0) metric space of dimension 2. This family includes a free-by-cyclic group with free kernel of rank 6.

Contents

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1. Introduction

One of the foundational problems in the theory of hyperbolic groups is to determine the relationship between coarse and continuous notions of negative curvature. Specifically, one is interested in the relationship between coarse notions such as Gromov's δ-hyperbolicity, and the more continuous notions due to Alexandrov and Toponogov of CAT(0) and CAT(-1) metric spaces. It is known that if a group acts properly and cocompactly by isometries on a CAT(-1) metric space (or on a CAT(0) metric space which contains no isometrically embedded flat planes [9]) then the group is hyperbolic in the sense of Gromov. It is still an open problem as to whether all hyperbolic groups act properly and cocompactly by isometries on CAT(0) metric spaces.

The main results of this paper show that if one is trying to find a proper CAT(0) or CAT(-1) metric space on which a given torsion free hyperbolic group acts properly and cocompactly by isometries, then the dimension of the space may have to be be strictly greater than the geometric dimension (usual topological dimension) of the group. We find that this is so in the case of hyperbolic free-by-cyclic groups. The basic example upon which everything else is built is an $F_3 \rtimes \mathbb{Z}$ group which has CAT(0) dimension equal to 2, but has CAT(-1) dimension equal to 3. Furthermore, there is a one-parameter family of CAT(0) piecewise Euclidean 2-complexes associated to this group with the following property: every 2-dimensional proper CAT(0) space on which this group acts properly isometrically contains an isometrically embedded scaled copy of one of these 2-complexes.

Theorem 1. *The group G with presentation $\langle a, b \mid aba^2 = b^2 \rangle$ is of the form $F_3 \rtimes \mathbb{Z}$, is hyperbolic, and admits a compact locally CAT(-1) 3-dimensional $K(G,1)$.*

Furthermore, there is a one-parameter family $\{K_t\}_t$ of compact, locally CAT(0), piecewise Euclidean 2-complexes with the following properties:

1. *Each K_t is a $K(G,1)$. In particular, G has CAT(0) dimension equal to 2.*
2. *Every proper CAT(0) space of dimension 2 on which G acts properly discontinuously by isometries contains an isometrically embedded, G-equivariant copy of the universal cover of some K_t (up to a constant scaling of the metric on K_t). In particular, G does not act properly isometrically on a proper CAT(-1) space of dimension 2, and so G has CAT(-1) dimension equal to 3.*

Property (2) in Theorem 1 can be viewed as a first (weak) hyperbolic analogue to the Flat Torus Theorem. Here is the analogy. Say that a group

G acts minimally on a CAT(0) space X if X contains no proper convex G-invariant subspace. Then the Flat Torus Theorem states that any minimal proper semi-simple action of $\mathbb{Z}^n$ on a CAT(0) space is an action by translations on $\mathbb{E}^n$. Our Theorem 1 (2) states that the minimal 2-dimensional CAT(0) actions of the group G are classified by the model complexes K_t for $t \in \mathbb{R}$. The analogy is "weak" in the sense that we impose the dimension restriction. On the other hand, we do not suppose the actions are semi-simple. This mimicks the statement of the Torus Theorem of Fijuwara, Shioya and Yamagata [13] which includes the dimension restriction, but does not require semi-simplicity. In fact we shall use Proposition 4.4 of [13] in order to find a minimum for combined displacement function without recourse to any cocompactness hypothesis.

The Flat Torus Theorem has been very useful in proving that certain groups are not CAT(0) [14]. The groups typically contain a $\mathbb{Z}^2$ subgroup, together with a lot of conjugation relations, so that any putative, non-positively curved $K(\pi, 1)$ for them will contain an impossibly shaped flat torus. In the present case the K_t complexes play the role of flat 2-tori. They can be used to construct groups which will not be CAT(0) in dimension 2, because extra conjugation relations will somehow contradict the geometry of the K_t. For example, we have the following theorem.

Theorem 2. *There is an infinite family of hyperbolic groups of geometric dimension 2, which do not act properly by isometries on any proper CAT(0) metric space of dimension 2.*

This family includes infinitely many free-by-cyclic groups, one of which is $F_6 \rtimes \mathbb{Z}$.

Remark 3. This theorem offers the first conclusive proof that in tackling the question of whether all (torsion free) hyperbolic groups are CAT(0), one is obliged to look for CAT(0) structures above the geometric dimension of the group. This is the case even within the class of hyperbolic free-by-cyclic groups.

It was already known that it is easier to find CAT(0) structures for hyperbolic groups if one looks above the geometric dimension. The work of Wise [20] and of Brady-McCammond [7] all find high dimensional CAT(0) piecewise Euclidean cubical structures for various classes of hyperbolic groups (certain small-cancellation groups, and certain families of ample twisted face pairing 3-manifold groups). The results of this paper imply that it is not only easier, but that in some cases it is also necessary to look above the geometric dimension in the search for CAT(0) structures.

By the work of Bridson [10] and of Brady-Crisp [5] (see also [12, 13]) one knows that the minimal dimension of a CAT(0) structure for a CAT(0) group, may be strictly greater than its geometric dimension. However, all these papers used some version of the flat torus theorem, and made heavy use of periodic flats in 2-dimensional CAT(0) spaces. The key idea in the current paper is to find a very special hyperbolic group, and corresponding 2-complexes, which play the role of $\mathbb{Z}^2$ subgroups and flat 2-tori.

We wish to thank Luisa Paoluzzi, Robert Roussarie, Sylvain Crovisier and Christian Bonatti for (collectively) bringing to our attention the full variety of 2-dimensional structures K_t, and for several helpful observations and simplifications of the arguments of Section 3.3. We also wish to thank Lee Mosher and Leonid Potyagailo for raising the questions mentioned in Section 3.1.

2. Definitions and background

A **metric space** X is said to be **proper** if every closed ball $B_r(a)$ in X is compact. An **action** of a group G by isometries on a metric space X is said to be **proper** if for each $x \in X$ there is an open ball $B_r(x)$ about x $(r > 0)$ such that $g(B_r(x)) \cap B_r(x)$ is nonempty for only finitely many $g \in G$. This is slightly more restrictive than the usual notion of a proper action on a topological space, namely that the set $\{g \in G : g(K) \cap K \text{ nonempty }\}$ is finite for any compact $K \subset X$, but the two notions are equivalent for actions on proper metric spaces – see Remarks on page 132 of [11].

We refer to [11] for details on CAT(κ) spaces, for $\kappa \leq 0$; metric spaces of global non-positive curvature bounded above by $\kappa \in \mathbb{R}$.

Let g be an isometry of a CAT(κ) space X, $\kappa \leq 0$. The **translation length** of g is defined as $l(g) = \inf\{d(x, gx) : x \in X\}$. The isometry g is said to be **semi-simple** if it attains its translation length at some point of x. In this paper we make no assumptions on the semi-simplicity or otherwise of our group actions. This is in contrast to previous works [5, 10, 12] where semi-simplicity is assumed (because it is needed to apply the usual Flat Torus Theorem). In these cases this hypothesis can be removed by using instead the 2-dimensional Torus Theorem of [13], at the expense of supposing that action is on a proper CAT(0) space.

In this paper we shall use the notion of "geometric dimension" introduced by Bruce Kleiner in [18] for the class of metric spaces with curvature bounded above in the sense of Alexandrov [1], termed CBA spaces. These spaces include all complete CAT(κ) spaces ($\kappa \in \mathbb{R}$). Associated to any point p in a CBA space X is the space of directions $\Sigma_p X$, which is known to be a complete CAT(1) space (see [19]). The **geometric dimension** of a CBA space is "the largest number of times we can pass to spaces of directions without getting the empty set"– more precisely, the smallest function $GD : \{\text{CBA spaces}\} \to \mathbb{N} \cup \{\infty\}$ such that $GD(X) = 0$ if X is discrete, and otherwise $GD(X) \geq 1 + GD(\Sigma_p X)$ for all p in X.

In [18], Kleiner shows that the geometric dimension is a lower bound for the usual covering dimension (defined in general for topological spaces, see [17]). Moreover, he remarks (on p.412) that these two dimensions coincide for separable CBA spaces, which include proper CAT(κ) spaces.

Since it will be useful later, we recall that the space of directions $\Sigma_p X$ is defined as the space of all equivalence classes of geodesics emanating from p, where two geodesics are said to be equivalent if the Alexandrov angle between them is zero, and carries a metric induced by the Alexandrov angle.

3. The group $G = \langle a, b | aba^2 = b^2 \rangle$: geometric structures

In this section we prove all of the statements contained in Theorem 1 with the exception of part (2), which we defer until the next section. The work is broken into three subsections: in (3.1) we show that the group G is $F_3 \rtimes \mathbb{Z}$, in (3.2) we exhibit a 3-dimensional CAT(-1) structure, and in (3.3) we introduce the one-parameter family of 2-dimensional CAT(0) structures. We shall give a Morse theory argument that the group is free-by-cyclic. This will easily extend to show that certain HNN-extensions with base G and $\mathbb{Z}$ edge groups are also free-by-cyclic (see Proposition 19 of Section 5).

3.1. **The free-by-cyclic structure.** The group G has presentation $\langle\, a, b \,|\, abaa = bb \,\rangle$, and the corresponding presentation 2-complex has one vertex (labeled v), two 1-cells (labeled a and b), and a single hexagonal 2-cell (labeled by the relation).

Any map of G to $\mathbb{Z}$ takes the generators a and b to integers A and B respectively which satisfy the equation $3A + B = 2B$ or $B = 3A$. Thus we may assume that a is taken to a generator of $\mathbb{Z}$ and b to three times this generator. We can realize this homomorphism topologically by a map from the presentation 2-complex to the circle (with one 0-cell and one 1-cell). This map sends the vertex v to the base vertex of S^1 and maps a once around the circle, and b three times around the circle. Extend this map linearly over the 2-cell. This lifts to a Morse function on the universal cover.

Figure 1 shows how a typical 2-cell of the universal cover looks with respect to the Morse function. The preimage of the base vertex of S^1 is a graph in the 2-complex, and is shown as the graph Γ in Figure 1. The vertices $[b/3]$ and $[2b/3]$ denote points which are respectively 1/3 and 2/3 along the edge b, and which map to the vertex of S^1. Note that $\pi_1(\Gamma)$ is F_3. One can check that the preimage of a generic point of S^1 will be a graph, Δ, with 4 vertices, and 6 edges. As the generic point on the circle moves towards the base vertex, an edge of the preimage graph collapses to a point, [1] giving a homotopy equivalence with the graph Γ.

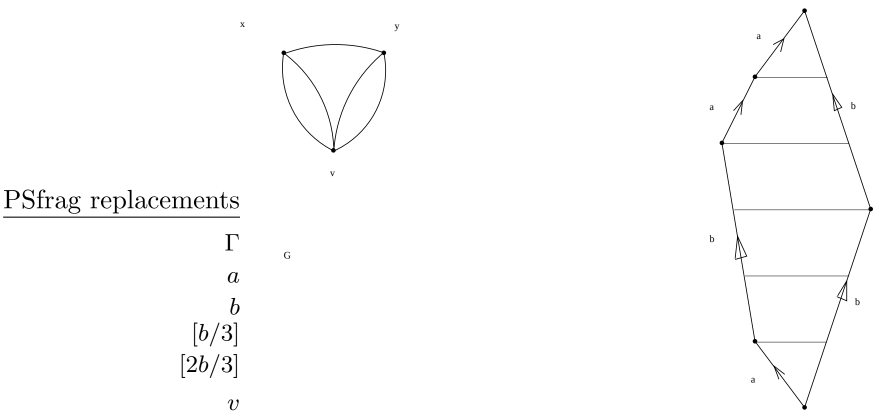


FIGURE 1. The Morse function on the 2–cell of the $F_3 \rtimes \mathbb{Z}$ group, and the level set Γ through the vertex v.

Thus the presentation 2-complex of G can be viewed as a graph of spaces whose underlying graph is the circle (with one vertex and one 1-cell), whose edge space is Δ, whose vertex space is Γ, and whose maps are the homotopy equivalences $\Delta \to \Gamma$ obtained by collapsing particular single edges of Δ. Thus G is isomorphic to the fundamental group of this graph of spaces, and so is $F_3 \rtimes \mathbb{Z}$ where the monodromy automorphism is obtained by composing the "ascending" homotopy equivalence $\Delta \to \Gamma$ with the inverse of the "descending" one.

It is a good exercise to work out this automorphism explicitly from the graph of spaces description; it is a "change of tree" automorphism, although it is not a change of maximal trees. Here we give an explicit description of the automorphism in terms of the original presentation of G.

The group G is an extension of the free group $F_3\langle x, y, z\rangle$ by $\mathbb{Z}$ where $\mathbb{Z}$ acts via the automorphism $\varphi : x \mapsto y \mapsto z \mapsto x^{-1}y$; that is G is the HNN-extension $F_3 *_\varphi$. The automorphism is just conjugation by a. Putting $x = a^{-2}bx^{-1}$ (so $b = a^2xa$) the relation $aba^2 = b^2$ is rewritten

$$a^3xa^3 = a^2xa^3xa\,,$$

[1]This collapsing edge corresponds to either the ascending or the descending link of the Morse function. See Bestvina-Brady [3] for terminology, or Brady-Miller [8] where the connection between Morse theory and free-by-free groups is made explicit.

which easily rearranges to $\varphi(x) = x\varphi^3(x)$, or rather $\varphi^3(x) = x^{-1}\varphi(x)$, where φ denotes conjugation by a. Thus G is isomorphic to the given HNN-extension.

The automorphism φ is exponential, but with a very low expansion rate: $\lambda = 2.325 =$ (the solution to $\lambda^3 = \lambda + 1$). This may give a heuristic explanation for the reluctance of G to act on a CAT(-1) 2-complex. We do not know whether the extensions $F_3 *_{\varphi^n}$ for $n > 1$ (which are subgroups of index n in G) are CAT(-1) in dimension two. This latter problem was raised by Lee Mosher and is closely related to the following question suggested by Leonid Potyagailo.

Question 1. Does every 2-dimensional word hyperbolic group contain a finite index subgroup which acts properly cocompactly by isometries on a 2-dimensional CAT(-1) proper metric space?

3.2. **The 3-dimensional CAT(-1) structure.** Let P denote a regular solid octahedron of "small" volume in hyperbolic 3-space, as illustrated in Figure 3.2. We label the vertices $1, 2, .., 6$ as indicated in the figure, and define the piecewise hyperbolic 3-complex M to be obtained from P by identifying the pair of faces labelled $(1, 4, 6)$ and $(6, 3, 5)$ to a single face A, and the pair $(1, 5, 2)$ and $(2, 4, 3)$ to a single face B (respecting the order of vertices in each case). The remaining four faces are left open.

Choose a basepoint in the interior of P and define oriented paths a and b in M passing through the faces A and B, respectively, as indicated in the figure. One easily checks that the loops a and b generate $\pi_1(M)$ subject to the single relation $abaa = bb$. That is $\pi_1(M) \cong G$. (In fact the $K(G, 1)$ complex K discussed the next subsection can be embedded in M as a "2-spine"– the complex K is a deformation retract of M, showing that M is also a $K(G, 1)$).

The complex M has a single vertex v with link $\mathrm{Lk}(v, M)$ as illustrated in Figure 3.2. This is a Moebius band composed of six spherical quadrilaterals with sidelengths all equal to $\pi/3 - \varepsilon_1$ and diagonals all of length $\pi/2 - \varepsilon_2$ where $\varepsilon_1, \varepsilon_2$ both tend to zero as the chosen volume of the octahedron P tends towards 0. A systole for $\mathrm{Lk}(v, M)$ is shown in bold in the figure. If P were chosen Euclidean, then the length of this systole would be $4.(\pi/3) + 2\mu$ where μ (the length of the segment crossing quadrilateral 2) lies strictly between $\pi/3$ and $\pi/4$ (in fact $\mu > 72°$). In the small volume hyperbolic case the systole measures $4.(\pi/3) + 2\mu - \varepsilon$ where ε also tends to 0 with the volume of P. For sufficiently small choice of volume of P this value is larger than 2π and M is a locally CAT(-1) space. This also gives a further way of seeing that M is indeed a compact $K(G, 1)$ for our group G.

We refer the reader to [6] for further details concerning determination of the systole in $\mathrm{Lk}(v, M)$ and the calculation of its length.

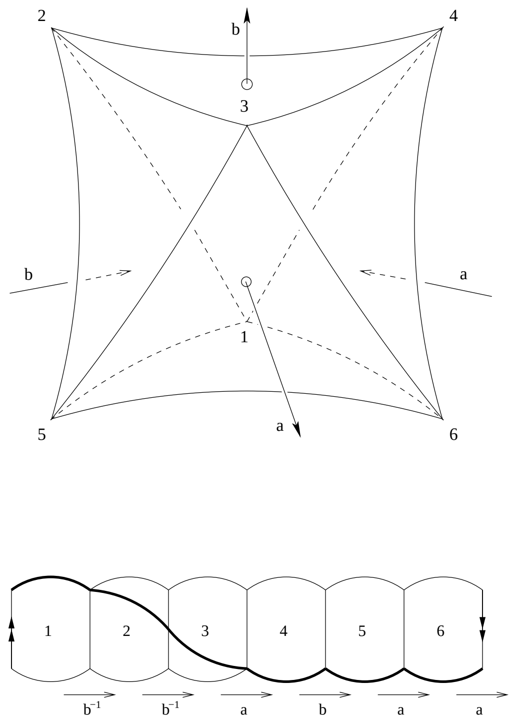
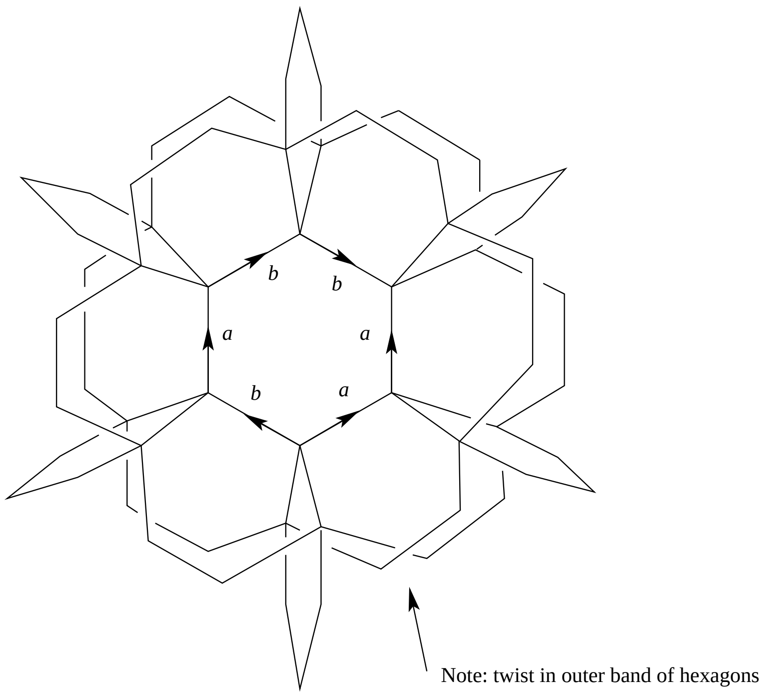


FIGURE 2. The 3-dimensional piecewise hyperbolic complex $M = P/\sim$, and its vertex link $\mathrm{Lk}(v, M)$.

3.3. **The 1-parameter family of CAT(0) structures.** Let K denote the presentation complex defined by the 1-relator presentation $G = \langle a, b \mid abaa = bb \rangle$. The associated Cayley complex $\widetilde{K}$ (the universal cover of K) has been previously studied by both Haglund [15] and Ballman and Brin [2]. It is one of the *two* completely regular simply connected polyhedra which can be built out of regular hexagonal cells in such a way that every vertex link is a complete graph on 4 vertices. The other is the Cayley complex associated to the one-relator presentation with relation $baa = abb$, which defines the Geisking 3-manifold group. A portion of the complex $\widetilde{K}$ is illustrated in Figure 3. Note that the band of hexagons immediately surrounding the central one in the figure is twisted, so that their union is a Moebius band rather than an annulus as in the case of the Geisking complex.

The complex $\widetilde{K}$ very naturally admits a CAT(0) metric in which each cell is a regular Euclidean hexagon. Consequently, the quotient presentation 2-complex K is a locally CAT(0) $K(G,1)$, and the group G therefore has CAT(0) dimension equal to 2. Moreover, since this Haglund, Ballman-Brin complex $\widetilde{K}$ does not contain any isometrically embedded flat planes (because of the twist), we have another way of concluding that the group G is hyperbolic. [2]



PSfrag replacements

b

a

FIGURE 3. The Cayley complex for $\langle a, b \,|\, aba^2 = b^2 \rangle$.

[2]In [16], M. Kapovich considered the full isometry group $\mathrm{Isom}(\widetilde{K})$ of $\widetilde{K}$ and showed, by using the existence of torsion elements, that $\mathrm{Isom}(\widetilde{K})$ does not act properly isometrically on any CAT(-1) space. This line of argument was pursued in [6] for this and various similar examples. However the techniques used here to study the torsion free subgroup G are necessarily quite different and we have not as yet succeeded in extending them in this way to other examples.

It is natural to ask whether there are any piecewise Euclidean, G-equivariant, CAT(0) structures on the complex $\widetilde{K}$ other than the regular one just described. In fact, we have the following classification of such structures.

Proposition 4. *There exists a continuous family $\{K_t : t \in \mathbb{R}\}$ of piecewise Euclidean locally CAT(0) metrics on the presentation complex K. Furthermore, any locally CAT(0) metric on K which is obtained by edge identifications on a convex euclidean hexagon is isometric, up to a linear scaling, to K_t for some t.*

Proof. We start with an edge identification on a convex euclidean hexagon H as illustrated in Figure 4, where we identify the edges with common labels. The figure H need not be a *regular* hexagon, however all three edges labelled a must have the same length, all three b-edges the same length, and when identifications are made the link condition at the vertex must be satisfied. Label the angles of H as shown in Figure 4: namely, we label the angle from a^+ to b^+ by α_0, from a^- to b^- by α_3, from a^+ to a^- by α_1, from b^+ to b^- by α_4, from a^+ to b^- by α_2, and from b^+ to a^- by α_5. The link of the vertex v in the presentation 2-complex for G is the complete graph on 4 vertices, with each edge α_i complementary to (sharing no vertices with) α_{i+3}, where indices are taken mod 6. The link condition requires that the sum of the angles contributing to each simple circuit in this graph is at least 2π. On the other hand, since the angles α_i are angles in a Euclidean hexagon, they must sum to 4π. The next lemma deduces relations among the α_i. We will need to use it again later on, with the weaker assumption that the sum of the α_i is at most 4π, so we prove it in that generality now.

Lemma 5. *Suppose that the complete graph on 4 vertices has a CAT(1) metric, where each edge length is in the range $(0, \pi]$, and where the total of all six edge lengths is at most 4π. Then the following are true.*

1. *The total of all 6 edges is exactly 4π.*
2. *The total of the edge lengths in any circuit of combinatorial length 3 is exactly 2π.*
3. *The lengths of complementary edges (no vertices in common) are equal.*

Proof. Label the edges by α_i where $i \in \{0, 1, 2, 3, 4, 5\}$, so that α_i and α_{i+3} (indices are (mod 6)) are labels of complementary edges. The CAT(1) condition requires that the sum of all edges in each complete subgraph on

3 vertices is at least 2π. This gives 4 linear inequalities:

$$\begin{aligned}
\alpha_0 + \alpha_1 + \alpha_2 &\geq 2\pi \\
\alpha_0 + \alpha_4 + \alpha_5 &\geq 2\pi \\
\alpha_3 + \alpha_1 + \alpha_5 &\geq 2\pi \\
\alpha_3 + \alpha_4 + \alpha_2 &\geq 2\pi
\end{aligned}$$

These combine with the hypothesis that $\sum_{i=0}^{5} \alpha_i \leq 4\pi$ to give 5 equalities. To see this, simply add the 4 inequalities and divide by 2 to get $\sum_{i=0}^{5} \alpha_i \geq 4\pi$. These two opposite inequalities force equality, and hence equalities in all of the above.

Finally, since the 4 inequalities become 4 equations, one can reduce them to get $\alpha_i = \alpha_{i+3}$ where indices are taken (mod 6). □

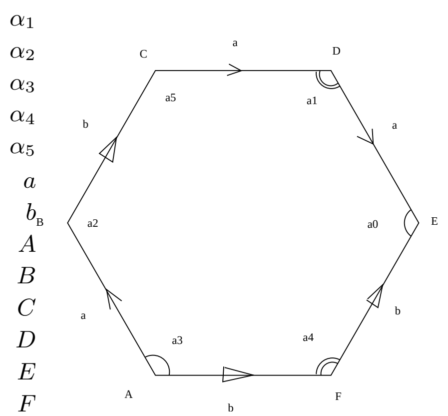


FIGURE 4. The 1-parameter family of 2-dimensional CAT(0) structures for G.

Thus we have extra information about the hexagonal 2-cell. Namely, $\alpha_0 = \alpha_3$, $\alpha_1 = \alpha_4$, $\alpha_2 = \alpha_5$ (as indicated in Figure 4) and $\alpha_0 + \alpha_1 + \alpha_2 = 2\pi$. (This is also *sufficient* to ensure that the link condition is satisfied). Note that all the vertices A, B, C, D, E must lie on a common circle. This is seen in two steps. First, $\square(ABCD)$ is a cyclic quadrilateral, since $|AB| = |CD|$ and the angle α_2 at B equals the angle α_5 at C (it is an isosceles trapesium). Secondly, $\square(ABDE)$ is a cyclic quadrilateral, since it is also an isosceles trapesium; $|AB| = |ED|$ and angle $\measuredangle BAE$ equals angle $\measuredangle DEA$. These last two angles are equal since we are given that $\alpha_0 = \alpha_3$, and the triangle $\triangle(AFE)$ is isosceles. These two cyclic quadrilaterals have three points A, B, D in common, and so all 5 points lie on a common circle.

An arbitrary locally CAT(0) piecewise Euclidean strcture on K may now be described as follows. Take a circle with center O and points A, B, C, D, E on its circumference, so that

$$\measuredangle AOB \;=\; \measuredangle COD \;=\; \measuredangle DOE \;=\; 2x$$

and that $\measuredangle BOC = 2y$ for positive numbers x, y satisfying $3x + y < \pi$. Now construct an isosceles triangle $\triangle(FAE)$ on the base AE which is similar to the triangle $\triangle(DCE)$. Choose F so that it lies outside of the pentagon $ABCDE$. We now have a hexagon $ABCDEF$ which satisfies all the conditions to be a 2-cell in a non-positively curved presentation 2-complex for G, with the possible exception that the edge length $|AF| = |FE|$ may not be equal to the edge length $|BC|$. Moreover, the construction depends only on the choice of angles x and y (subject to $3x + y < \pi$).

We suppose without loss of generality that the circle containing A, B, C, D, E has unit radius. Using the facts that $|BC| = 2\sin y$, $|CD| = 2\sin x$, $|CE| = 2\sin(2x)$, $|AE| = 2\sin(\pi - (3x + y)) = 2\sin(3x + y)$, and the fact that the triangles $\triangle(DCE)$ and $\triangle(FAE)$ are similar, we have

$$\frac{|BC|}{|CD|} = \frac{\sin y}{\sin x} \qquad \text{and} \qquad \frac{|AF|}{|CD|} = \frac{\sin(3x+y)}{\sin(2x)} = \frac{\sin(3x+y)}{2\sin x \cos x}$$

Therefore $|AF| = |BC|$ if and only if the following trigonometric identity is satisfied:

$$\sin(3x + y) \;=\; 2\cos(x)\sin(y)\,. \tag{1}$$

Expanding the left hand side gives

$$\sin(3x)\cos(y) + \cos(3x)\sin(y) \;=\; 2\cos(x)\sin(y)\,.$$

Grouping the $\sin(y)$ terms and solving for $\tan(y)$ yields

$$\tan y \;=\; \frac{\sin(3x)}{2\cos x - \cos(3x)}\,. \tag{2}$$

This expresses $\tan y$ as a smooth function of x for $0 < x < \pi/3$. Thus, for each $x \in (0, \frac{\pi}{3})$, there is a unique y-value in the interval $(0, \frac{\pi}{2})$ for which the corresponding hexagon yields a non-positively curved, 2-dimensional $K(G, 1)$. These $K(G, 1)$ spaces form a 1-parameter family K_t, for $t \in \mathbb{R}$, where we set $t = \cot(3x)$, say, for x ranging over the interval $(0, \frac{\pi}{3})$. (Note: With this convention K_0 denotes the regular hexagonal structure, $x = \frac{\pi}{6}$). This completes the proof of Proposition 4. □

We observe that the equation (2) given in the above proof may be re-expressed by using the identities $\sin(3x) = 3\sin x \cos^2 x - \sin^3 x$ and

$\cos(3x) = \cos^3 x - 3\cos x \sin^2 x$. Thus

$$\tan y \;=\; \frac{\sin x(3\cos^2 x - \sin^2 x)}{\cos x(5\sin^2 x + \cos^2 x)} \;=\; \frac{\tan x(3 - \tan^2 x)}{5\tan^2 x + 1} \tag{3}$$

Note that, for $x \in (0, \frac{\pi}{3})$, we have $\tan^2(x) \in (0, 3)$.

Lemma 6. *Let $U = \tan^2(x)$ and $V = \tan^2(y)$, and suppose throughout that $x \in (0, \frac{\pi}{3})$ and $y \in (0, \frac{\pi}{2})$. Given that the identity (3) holds, then the following identities also hold, with $U \in (0, 3)$:*

$$\frac{\sin(y)}{\sin(x)} = \sqrt{\frac{V(1+U)}{U(1+V)}} = \frac{(3-U)}{\sqrt{U^2 + 18U + 1}} > 0 \tag{4}$$

$$\frac{\cos(y)}{\cos(x)} = \sqrt{\frac{(1+U)}{(1+V)}} = \frac{(5U+1)}{\sqrt{U^2 + 18U + 1}} > 0 \tag{5}$$

Proof. We first of note that for x, y in the given ranges, all expressions in the statement of the Lemma take positive values (we consider only positive valued square roots).

The first equalities in (4) and (5) are immediate consequences of the usual trigonometric identities expressing $\sin\theta$ and $\cos\theta$ in terms of $\tan\theta$: namely,

$$\sin^2\theta = \frac{\tan^2\theta}{1 + \tan^2\theta} \qquad \text{and} \qquad \cos^2\theta = \frac{1}{1 + \tan^2\theta}\,.$$

Equation 3 also gives us the fundamental identity

$$V = \frac{U(3-U)^2}{(5U+1)^2}$$

whence

$$1 + V \;=\; \frac{(3-U)^2 U + (5U+1)^2}{(5U+1)^2} \;=\; \frac{(U+1)(U^2 + 18U + 1)}{(5U+1)^2}\,.$$

The remaining equalities in (4) and (5) now follow easily. □

Lemma 7. *Label a fundamental domain hexagon in the universal cover of K_t as in Figure 4, and suppose that the scaling on the metric for K_t is such that the circle on which $A, \ldots, E$ all lie has unit radius. Let O be the center of this circle, and let $2x$ and $2y$ be the respective measures of the angles AOB and BOC. (We have $t = \cot(3x)$).*

For $u, v \in \{a, b\}$ we define δ_{uv} to be the distance in K_t between the midpoints of the two edges in any edge path labelled uv in the 1-skeleton of K_t. Then we have

(1) $\delta_{ab} = |AC|/2 = \sin(x + y)$.

(2) $\delta_{ba} = |BD|/2 = \sin(x+y)$.
(3) $\delta_{aa} = |CE|/2 = \sin(2x)$.
(4) $\delta_{bb} = |AE|/2 = \sin(3x+y) = 2\cos x \sin y$.

Proof. The proof uses just the following observation from trigonometry. The length of the base of an isosceles triangle with two edges of length 1 subtending an angle of θ is $2\sin(\theta/2)$. The segments AC, BD, CE and AE subtend angles at the center O of the circle measuring (respectively) $2(x+y)$, $2(x+y)$, $2x$ and $2(\pi-(3x+y))$. The result follows from the fact that each path ab, ba, aa and bb occur on the boundary of the given hexagon, and that δ_{uv} is exactly half the length of the interval spanned by the endpoints of the path uv. In case (4) we apply the identity $\sin(\pi-\theta) = \sin(\theta)$ and the equation (1) derived in the proof of Proposition 4. □

Proposition 8. *Let w be a positive word in the generators a, b of the group G which contains at least one occurrence of b ($w \neq a^k$). Let*

$$L : G \to \mathbb{Z}$$

denote the abelianisation homomorphism ($L(a) = 1$, $L(b) = 3$). Then in any of the 2-dimensional CAT(0) structures K_t for G the translation length of w is strictly less than that of $a^{L(w)}$:

$$\frac{l(w)}{l(a)} < L(w)\,.$$

Moreover, we have $\frac{l(w)}{l(a)} \to L(w)$ as $t \to \infty$ ($x \to 0$).

Proof. Without loss of generality we suppose that the metric on K_t is scaled as in the statement of Lemma 7. We first observe that, in the universal cover of any of the K_t, the piecewise geodesic which connects midpoints of successive a-edges is actually a geodesic. This allows us to compute the translation length of a precisely to be $l(a) = \delta_{aa} = \sin(2x)$.

On the other hand, we get an upper bound estimate for the translation length of w obtained by measuring the length of the piecewise geodesic path drawn between successive midpoints of the edges of the hexagons in the edge-path corresponding to w. More precisely, let $w = u_1u_2\ldots u_n$ be a positive word in a and b ($u_i \in \{a,b\}$ for all i), viewed as a cyclic word. Then

$$l(w) \le \sum_{i=1}^{n} \delta_{u_iu_{i+1}}\,, \qquad \text{while} \qquad L(w) = \frac{1}{2}\sum_{i=1}^{n} L(u_iu_{i+1})\,.$$

The inequality stated in the Lemma now follows by showing that $\delta_{uv}/l(a) < \frac{1}{2}L(uv)$ in each of the cases $uv = ab, ba$, and bb. (Since, in addition we have

$\delta_{aa}/l(a) = 1 = \frac{1}{2}L(aa)$, we obtain an inequality $l(w)/l(a) \le L(w)$ which is strict if and only if $w \ne a^k$ for some k).

Case $uv = ab$ **or** ba**:** By Lemma 7 we have

$$2\delta_{ab}/l(a) = \frac{2\sin(x+y)}{\sin(2x)} = \frac{\sin x \cos y + \cos x \sin y}{\sin x \cos x} = \frac{\cos y}{\cos x} + \frac{\sin y}{\sin x}\,.$$

Applying Lemma 6 this gives

$$2\delta_{ab}/l(a) = \frac{(5U+1)+(3-U)}{\sqrt{U^2+18U+1}} = \frac{4(U+1)}{\sqrt{U^2+18U+1}} \;<\; 4 = L(ab)\,.$$

The inequality follows since $\sqrt{U^2+18U+1} > (U+1) > 1$, for $U > 0$. The case $uv = ba$ is identical.

Case $uv = bb$**:** This time, by Lemmas 7 and 6, we have

$$\delta_{bb}/l(a) = \frac{2\cos x \sin y}{\sin(2x)} = \frac{\sin y}{\sin x} = \frac{(3-U)}{\sqrt{U^2+18U+1}} \;<\; 3 = \frac{1}{2}L(bb)\,.$$

The inequality follows once again since $U > 0$.

This completes the proof that $l(w)/l(a) < L(w)$ for positive words $w \ne a^k$.

Finally, we observe that as x tends to zero the hexagon H of Figure 4 degenerates towards an interval with endpoints A and E and length $2|BC| = 3|AB| + |BC|$. Thus K_t collapses onto a real line where translation lengths are determined by the abelianisation homomorphism: $l(g)/l(a) = L(g)$ for all $g \in G$. This completes the proof of Proposition 8. □

4. Proof of Theorem 1 (2): an analogue Flat Torus Theorem

In this section we complete the proof of Theorem 1 by establishing part (2), the "analogue Flat Torus Theorem". This section forms the geometric heart of this paper.

Theorem 9. *Let $G = \langle a, b \,|\, aba^2 = b^2\rangle$. Every proper CAT(0) space of dimension 2 on which G acts properly discontinuously by isometries contains an isometrically embedded, G-equivariant copy of the universal cover of some K_t (up to a constant scaling of the metric on K_t).*

Proof. Let X denote a proper CAT(0) space of geometric dimension two on which G acts properly isometrically. We shall show that X contains a convex subspace which is G-equivariantly isomorphic to the universal cover of K_t for some $t \in \mathbb{R}$.

The proof takes on two parts. We first claim that there exists a point x in X which minimises the combined displacement function $f(x) = d(x, a(x)) +$

$d(x, b(x))$. This is a straightforward consequence of Lemma 10 below, since here G is a torsion free hyperbolic group of cohomlogical dimension 2.

The second step involves constructing a map $\widetilde{K} \to X$. Let Γ denote the 1-skeleton of $\widetilde{K}$ (the Cayley graph of G with respect to $\{a, b\}$) and let v denote a base vertex in Γ. Given any point $x \in X$ we may define a continuous map $\varphi_x : \Gamma \to X$ by sending v to x, extending G-equivariantly on the vertex set of Γ and then mapping each edge to the (unique) geodesic joining the images of its endpoints. This construction also leads to a natural choice of "lengths" for each edge in $\mathrm{Lk}(v, \widetilde{K})$. For simplicity we write $L = \mathrm{Lk}(v, \widetilde{K})$ and $\Sigma = \Sigma_x X$, the space of directions at x in X. We recall that L is simply the complete graph on 4 vertices. If p denotes a vertex of L, determined by the edge e say, then we write $\overline{p}$ for the direction in Σ determined by the geodesic segment $\varphi_x(e)$. We now assign to each edge (p, q) in L a "length" given by the distance between $\overline{p}$ and $\overline{q}$ in Σ. Note that, since it is possible that $\overline{p} = \overline{q}$, which endows the edge (p, q) with *zero* length, this choice determines a *pseudo*-metric, rather than a metric, on L. We shall write L_x to denote the graph L equipped with this pseudo-metric.

We shall prove further on (see Lemma 12 and the Remark which follows it) that if the point x is chosen so as to minimise the combined displacement function f then L_x turns out to satisfy the link condition for a CAT(1) metric graph: each circuit has length at least 2π. On the other hand, the six edge lengths in L_x appear as the angles of a "geodesic hexagon" C in X (take the image of any hexagonal circuit in Γ which bounds a 2-cell of $\widetilde{K}$). Nonpositive curvature in X implies that the sum of these angles is at most 4π (see[12], Lemma 1, for example). It now follows by Lemma 5 that the total of all six angles is *exactly* 4π and that each simple circuit in L has length exactly 2π. (The above arguments apply even when there are zero length edges in L). By the Flat Triangle Lemma [11], it now follows that the geodesic hexagon C actually bounds a genuine convex (but possibly degenerate) 2-dimensional Euclidean hexagon H isometrically embedded in X. By properness of the action of G on X this hexagon cannot degenerate on to an interval (for then the orbit of x would lie on a single line!), so has non-empty interior and nonzero angles. We now choose a G-equivariant metric on $\widetilde{K}$ by letting each 2-cell be isometric to the hexagon H, and extend the map φ_x to a map $\varphi_x : \widetilde{K} \to X$ which is locally isometric on the interior of 2-cells. Note that the link of each vertex in $\widetilde{K}$ is isometric to L_x and CAT(1). Thus, by Proposition 4, $\widetilde{K}$ equipped with this metric is (up to scaling) G-equivariantly isometric to the universal cover of one of the model complexes K_t for some $t \in \mathbb{R}$. In particular, all edge-lengths in the link L are strictly less than π.

It now only remains to show that φ_x is locally isometric at each vertex. For this we need to prove that the induced map $\varphi : L \to \Sigma$ is π-distance preserving: if $d_\Sigma(\overline{p}, \overline{q}) < \pi$ then $d_L(p,q) = d_\Sigma(\overline{p}, \overline{q})$. Since L has diameter equal to π it suffices to show that $\varphi : L \to \Sigma$ is an isometric embedding.

Let p_1, p_2, p_3 be three vertices of L, spanning a triangle Δ with edge lengths ϕ_1, ϕ_2 and ϕ_3. We have that each length ϕ_i lies in the range $(0, \pi)$, and $\phi_1 + \phi_2 + \phi_3 = 2\pi$. By Lemma 12, the image $\varphi(\Delta)$ (in Σ) of the triangle Δ is a closed geodesic of length 2π. Thus φ maps the triangle Δ isometrically into Σ (it is a locally isometric embedding of a space of diameter π into a uniquely π-geodesic space). Since any two points in L lie in a common triangle Δ it now follows that $\varphi : L \to \Sigma$ is an isometric embedding, as required. □

4.1. **Finding a minimum for the combined displacement.** In this section we prove the following result:

Lemma 10. *Let X be a proper CAT(0) space, $g_1, g_2, \ldots, g_n$ a finite collection of isometries of X which generate a group G acting properly on X, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ strictly positive real numbers. Then either*

(1) *there exists a point in X which minimises the "combined displacement function":*

$$f : X \to \mathbb{R}^+ : f(x) = \sum_{i=1}^{n} \lambda_i d(x, g_i(x))$$

or

(2) *the group G fixes a point x_∞ on the ideal boundary ∂X of X.*

If G happens to be a word hyperbolic group with finite $K(G,1)$ then case (2) above implies that either $G \cong \mathbb{Z}$ or $cd(G) \leq dim(X) - 1$.

Proof. In the first instance, if f is a proper map then it clearly achieves a minimum. Supposing otherwise, and using the fact that X is proper (and so $X \cup \partial X$ is a compact space – see, for example, Exercise II.8.15(2) of [11]), one can find a sequence of points $\{x_k\}_\mathbb{N}$ which converges to a point x_∞ in ∂X yet such that the sequence $\{f(x_k)\}$ is bounded. It follows that each g_i fixes x_∞ and so (2) holds.

Now suppose that G is a hyperbolic group with finite $K(G,1)$ (ie: G is torsion free). In particular, G is finitely generated. By considering the action of G on the horofunctions at x_∞ we deduce an exact sequence

$$H \to G \to \mathbb{Z}$$

where H is the subgroup of elements which act by leaving invariant every horosphere at x_∞.

Suppose firstly that the map $G \to \mathbb{Z}$ is nontrivial. By properness of the action of G, any two elements which map nontrivially to $\mathbb{Z}$ must share a common power. Thus all elements of $G - H$ leave fixed a common pair of points $\{p, q\}$ in ∂G. Moreover H must also fix the pair $\{p, q\}$ (since it conjugates elements of $G - H$ to elements of $G - H$). But this implies that G is virtually $\mathbb{Z}$, or rather $\mathbb{Z}$ since it is torsion free.

We may now suppose that $G \cong H$ and acts by leaving invariant all horospheres at x_∞. But then our conclusion that $\mathrm{cd}(G) \leq \dim(X) - 1$ follows directly from Proposition 4.4 of [13]. □

4.2. **Angle measurements.** In this section we wish to use an idea from elementary calculus: that the "rate of change" of a function in any direction from a local minimum is never negative. The functions that we consider are linear combinations of distance functions in a CAT(0) space. For these reasons we introduce the following:

Lemma 11. *Let X be a CAT(0) space, and $[p, q]$ a nontrivial geodesic segment ($d(p, q) > 0$), let $\gamma : [0, \epsilon] \to X$ denote a nontrivial constant speed geodesic with $\gamma(0) = p$, and let θ denote the Alexandrov angle at p between $[p, q]$ and γ. Let $f : [0, \epsilon] \to \mathbb{R}$ be the (necessarily continuous) function such that $f(t) = d(\gamma(t), q)$. Then*

$$\lim_{t\to 0; t>0} \frac{f(t) - f(0)}{t} = -\cos\theta \ .$$

We refer to the above limit as the derivative of f in the direction of γ.

Remark. The above Lemma asserts, if you like, the existence of a directional derivative in the first variable of the distance function, which is defined, for a point $(p, q) \in X \times X$, over the space of directions $\Sigma_p X$ at p in X.

Proof. Note firstly that the lemma is precisely true in the Euclidean plane $\mathbb{E}^2$. Now, given the general situation described above, choose in $\mathbb{E}^2$ points $\widehat{p}$, $\widehat{q}$ such that $d_{\mathbb{E}}(\widehat{p}, \widehat{q}) = d(p, q)$, and a geodesic $\widehat{\gamma}$ from $\widehat{p}$ such that the Alexandrov angle at $\widehat{p}$ between $\widehat{\gamma}$ and $[\widehat{p}, \widehat{q}]$ equals θ. This configuration determines the function $\widehat{f}(t) = d_{\mathbb{E}}(\widehat{\gamma}(t), \widehat{q})$ where $\widehat{f}(0) = f(0) = f_0$ say. By one version of the comparison axiom (Proposition II.1.7 (5) of [11]), we have that $\widehat{f}(t) \leq f(t)$ for all $t \in [0, \epsilon]$. Since $\widehat{f}$ is a convex function we then have $f_0 - t\cos\theta \leq \widehat{f}(t) \leq f(t)$ and hence $\frac{f(t)-f_0}{t} \geq -\cos\theta$ for all $t \in [0, \epsilon]$.

Fix $s \in (0, \epsilon]$ and let $\Delta'(p', q', r')$ denote the Euclidean comparison triangle for the triangle in X with corners p, q and $r = \gamma(s)$. That is $d_{\mathbb{E}}(p', q') = d(p, q)$, etc... Let θ_s denote the angle in $\mathbb{E}^2$ between the sides of Δ' meeting at p'. Also, define a function $f_s : [0, s] \to \mathbb{R}$ such that $f_s(t)$

is the distance from q' to a point a distance t from p' along the side of Δ' between p' and r'. By the comparison axiom, we have $f(t) \leq f_s(t)$ for every $t \in [0,s]$. Also by the comparison axiom, $\theta \leq \theta_s$ for each s. Moreover, by the interpretation of Alexandrov angle as the "strong upper angle" (see Proposition I.1.16 of [11]), we have that $\lim_{s\to 0; s>0} \theta_s = \theta$.

Now, suppose we are given a small $\varepsilon > 0$. Then there exists $s \in (0, \epsilon]$ such that $\theta \leq \theta_s < \theta + \varepsilon$. Since f_s is a differentiable function with derivative $-\cos\theta_s < -\cos(\theta+\varepsilon)$ (ε sufficiently small) we may find a sufficiently small δ such that $f(\delta) \leq f_s(\delta) < f_0 - \delta\cos(\theta+\varepsilon)$. But this implies that for all sufficiently small $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$-\cos\theta \leq \frac{f(\delta) - f_0}{\delta} < -\cos(\theta+\varepsilon) \ .$$

This establishes the Lemma. □

As in the main body of the proof of Theorem 9, we suppose that $G = \langle a, b | bab^2 = a^2 \rangle$ acts properly by isometries on a proper CAT(0) space X of geometric dimension two. We recall the notation introduced in the proof of Theorem 9. In particular, for a choice of point $x \in X$, we defined the map $\varphi_x : \Gamma \to X$, where Γ denotes the Cayley graph of G (with respect to $\{a, b\}$). As before we write $L = \mathrm{Lk}(v, \widetilde{K})$ and $\Sigma = \Sigma_x X$, the space of directions at x in X, and we write $\overline{p}$ for the point in Σ associated to a vertex p of L via the map φ_x. Recall that L is a complete graph on 4 vertices.

Since X is CAT(0) the space of directions Σ is CAT(1). We recall that CAT(1) spaces are uniquely π-geodesic: if $d(x,y) < \pi$ then there exists a unique geodesic in the space joining x to y. We also note that, by the dimension constraint on X, Σ is 1-dimensional (a CAT(1) metric $\mathbb{R}$-graph). We note that, in a 1-dimensional CBA space, a path is locally godesic if and only if it is locally embedded.

Lemma 12. *Suppose that $x \in X$ is chosen so as to minimise the combined displacement $f(x) = d(x, a(x)) + d(x, b(x))$. Let p_1, p_2, p_3 denote three distinct vertices of L and, for each $i = 1, 2, 3$, let ϕ_i denote the angle measured between $\overline{p}_i$ and $\overline{p}_{i+1}$ in Σ (indices taken mod 3), and suppose that each $\phi_i < \pi$. Then either $\phi_1 + \phi_2 + \phi_3 > 2\pi$ or the (unique) geodesic triangle in Σ spanned by the vertices $\overline{p}_1, \overline{p}_2, \overline{p}_3$ is a closed geodesic of length exactly 2π.*

Remark. Note that if $\phi_i = \pi$, for some i, then the triangle inequality implies straightaway that $\phi_1 + \phi_2 + \phi_3 \geq 2\pi$.

Proof. Suppose that $\phi_1 + \phi_2 + \phi_3 \leq 2\pi$. We show that the geodesic triangle spanned by $\overline{p}_1, \overline{p}_2, \overline{p}_3$ in Σ is a closed geodesic of length exactly 2π. Since each $\phi_i < \pi$, and since Σ is uniquely π-geodesic space, there is a unique (but possibly degenerate) geodesic triangle Δ spanned by $\overline{p}_1, \overline{p}_2, \overline{p}_3$. Either this triangle supports a simple closed circuit in Σ or it is in fact a (possibly degenerate) tripod. In the former case, since Σ is 1-dimensional and CAT(-1), the simple circuit is a closed geodesic and must have length at least 2π. Thus Δ is exactly a 2π closed geodesic, as required. In the latter case we shall obtain a contradiction.

Suppose then that the geodesic triangle Δ is a tripod. We let m denote the branch point of the tripod (or rather the median of $\overline{p}_1, \overline{p}_2, \overline{p}_3$ – the unique point which lies on all three sides of the triangle Δ) and let θ_i denote the angle between $\overline{p}_i$ and m, for each $i = 1, 2, 3$. We have $\theta_1 + \theta_2 + \theta_3 = \frac{1}{2}(\phi_1 + \phi_2 + \phi_3) \leq \pi$.

We now consider the effect of moving the point x a very small distance in the direction m. For convenience we G-equivariantly subdivide all edges in the Cayley graph Γ. Let e_1, e_2, e_3, e_4 denote the geodesic segments at x in X which are the images under φ_x of the four half-edges of Γ which emanate from the vertex v. We suppose that the labels are such that e_i determines the point $\overline{p}_i$ in Σ for each $i = 1, 2, 3$. We now allow x to move in the direction of m while fixing the other endpoints of the "half-edges" e_i. Using Lemma 11 we may compute the derivative of $\ell(e_i)$ in the direction m at x to be simply $-\cos\theta_i$ for each $i = 1, 2, 3$. On the other hand the derivative of $\ell(e_4)$ is at most 1 (in any direction). Thus it follows from Lemma 13 below that the sum of the lengths of the e_i strictly decreases under a sufficiently small perturbation of the point x. (Note that since we suppose each $\phi_i < \pi$ we cannot have equality in Lemma 13).

Performing this disturbance G-equivariantly, it is clear that the sum of the lengths of the new segments e'_i is an upper bound for $f(x')$ and hence that $f(x') < f(x)$ for a sufficiently small disturbance. But this contradicts the choice of x. □

Lemma 13. *Given real numbers $\theta_1, \theta_2, \theta_3 \geq 0$ such that $\theta_1 + \theta_2 + \theta_3 \leq \pi$, we have*

$$\cos\theta_1 + \cos\theta_2 + \cos\theta_3 \geq 1$$

with equality if and ony if $\theta_1 + \theta_2 + \theta_3 = \pi$ and $\theta_i = 0$ for some i.

Proof. The region of interest in $\mathbb{R}^3$ is a right simplex

$$R = \{(\theta_1, \theta_2, \theta_3) : \theta_i \geq 0 \text{ and } \theta_1 + \theta_2 + \theta_3 \leq \pi\}.$$

Write $g(\theta_1, \theta_2, \theta_3) = \cos\theta_1 + \cos\theta_2 + \cos\theta_3$. We first consider the problem of minimisig the function g over the 2-simplex $R_\pi = \{(\theta_1, \theta_2, \theta_3) \in R : \theta_1 + \theta_2 + \theta_3 = \pi\}$. Observe that $g = 1$ on the boundary of R_π, namely when

$\theta_1 + \theta_2 + \theta_3 = \pi$ and $\theta_i = 0$ for some i. For, if $\theta_1 = 0$ and $\theta_2 = \pi - \theta_3$, for instance, then $g(0, \theta_2, \theta_3) = 1 + \cos(\pi - \theta_3) + \cos\theta_3 = 1$.

Consider now the possibility of local minima in the interior of R_π. At such points the gradient of g is normal to R_π. That is,

$$\nabla g = -(\sin\theta_1, \sin\theta_2, \sin\theta_3) = \lambda(1,1,1) \quad \text{for some } \lambda \in \mathbb{R}\,.$$

Thus $\sin\theta_1 = \sin\theta_2 = \sin\theta_3 = -\lambda$. But then one sees that $\theta_1 = \theta_2 = \theta_3 = \frac{\pi}{3}$ (since if, for some $i \neq j$, we had $\theta_i \neq \theta_j$ but $\sin\theta_i = \sin\theta_j$ we would have $\theta_i + \theta_j = \pi$ contradicting the choice of point in the interior of R_π). Now $g(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}) = 3\cos\frac{\pi}{3} = \frac{3}{2} > 1$. Therefore g is always strictly greater that 1 on the interior of R_π.

Finally, since g is strictly decreasing along rays from the origin (through R), we deduce that $g > 1$ at all points of $R \setminus R_\pi$. □

Remark 14. In fact the barycentre of R_π is a local maximum of g over R_π, as can be seen by looking at the Hessian matrix which is $-\cos(\frac{\pi}{3})I_n$ at the barycentre. More generally, an easy induction shows that

$$g(\theta_1, .., \theta_n) = \sum_{i=1}^{n} \cos\theta_i \geq n - 2$$

whenever θ_i are non-negative and sum to at most π, with equality precisely on the 1-skeleton of the "level π simplex" R_π.

5. The hyperbolic versus CAT(0) problem: Theorem 2

Consider the following family of 2-dimensional groups:

$$G_{w,m} \;=\; \langle\, a, b, t \,|\, abaa = bb,\, ta^m t^{-1} = w \,\rangle$$

with $m \in \mathbb{Z} \setminus \{0\}$ and w a positive word in a, b which contains at least one b.

These are easily seen to be 2-dimensional, since they are HNN's of the 2-dimensional group G over infinite cyclic subgroups. In particular, they are all torsion free groups.

That this family contains an infinite collection of hyperbolic groups which do not act properly isometrically on any proper CAT(0) space of dimension 2 follows by combining Proposition 16 and Proposition 17 below. More precisely, we obtain:

Theorem 15. *Let w to be any positive word which represents a primitive element of G different from a, and let $m \in \mathbb{Z}$ such that $|m| \geq L(w)$. Then $G_{w,m}$ is a 2-dimensional hyperbolic group but does not act properly by isometries on any 2-dimensional CAT(0) space.*

For example, one could take $w = b$ and $|m| \geq 3$. (For other possible words w see Remark 18 below). Finally, in Proposition 19, we see that when one chooses $m = L(w)$, the resulting group is always a free-by-cyclic group. In the case of the previous example, the group $G_{b,3}$ is isomorphic to $F_6 \rtimes \mathbb{Z}$. These observations complete the proof of Theorem 2.

Proposition 16. *Let w denote a positive word in the letters a, b which contains at least one b. If $|m| \geq L(w)$ then $G_{w,m}$ admits no proper isometric action on a proper CAT(0) space of dimension 2.*

Proof. Suppose that $G_{w,m}$ acts properly isometrically on a proper CAT(0) space X of dimension 2. The group $G_{w,m}$ is an HNN-extension of the group $G = \langle\, a, b \,|\, abaa = bb \,\rangle$ of Theorem 1. Thus G acts properly isometrically on X (as a subgroup of $G_{w,m}$) and by Theorem 1 we have a copy of $\widetilde{K}_t$ isometrically and G-equivariantly embedded in X. It follows that the translation lengths of the elements a and w (acting on X) may be measured in $\widetilde{K}_t$, and from Proposition 4 we have that $l(w)/l(a) < L(w)$. However, the relation $ta^m t^{-1} = w$ in $G_{w,m}$ forces $l(w)/l(a) = |m| \geq L(w)$, a contradiction. □

Proposition 17. *Let w denote a positive word in the letters a, b which contains at least one b. If w represents a primitive element of the group G (and $m \neq 0$) then $G_{w,m}$ is a word hyperbolic group.*

Proof. The groups $G_{w,m}$ are HNN extensions of the hyperbolic group G via an isomorphism identifying the cyclic subgroup $\langle a^m \rangle$ with the cyclic subgroup $\langle w \rangle$, so we can apply criterion (2) of Corollary 2.3 of the Bestvina-Feighn Combination Theorem [4].

Since the centralizer of any element in a torsion free hyperbolic group is always an infinite cyclic group, it follows that any *primitive* element w of G generates its own centralizer. Therefore $\langle w \rangle$ is malnormal in G, and so one of the conditions (a), (b) of criterion (2) in Corollary 2.3 holds.

Note also that no non-trivial power of a is conjugate to a non-trivial power of w. For if this were the case then the ratio $l(w)/l(a)$ would be constant over the full range of model spaces K_t for G. However, the fact that $l(w)/l(a)$ tends towards a strict upper bound (Proposition 8) shows that this is not the case. Thus, the set $CC'(x)$ of criterion (2) of Corollary 2.3 is always finite (actually is always $\{1\}$).

By criterion (2) of Corollary 2.3 of [4] and the results of the preceding two paragraphs, we conclude that the HNN extension $G_{w,m}$ is torsion free hyperbolic whenever w is a (positive) primitive element of G different from a. □

Remark 18. The only technical obstacle to applying the above propositions is knowing when a positive word w represents a primitive element of G. In many cases, however, we can give a geometric argument using geodesics in the universal cover of the regular hexagonal structure K_0 to prove primitivity.

For suitable w, we observe that the piecewise geodesic which connects midpoints of successive edges in the bi-infinite edge-path determined by w is actually an axis for w. [3] This is the case if w is a positive word which is required not to contain either of the two positive, length 3 subwords (aba and baa) which form half of the hexagonal relator. If, moreover, w has odd wordlength then the axis is unique (w acts as a "glide reflection" along this axis). Since any root of w must leave this axis invariant, it follows that w is primitive in G if it is primitive in the free group $F_{\{a,b\}}$ (i.e: if it is not obviously a nontrivial power). Examples of such primitive elements w include b, ab^{2n} $(n \geq 1)$, ab^2ab^3 etc. We note however that $ab^3 = (a^2b)^2$ is not primitive. Many further elements may be seen to be primitive by variations on this argument. These include elements ab and a^2b, as well as positive words which contain no subword aba or baa, other than the exception ab^3 just mentioned.

The next proposition shows that the examples afforded by Theorem 15 include infinitely many free-by-cyclic groups.

Proposition 19. *If $m = L(w)$, then the group $G_{w,m}$ is free-by-cyclic.*

Proof. Consider the group

$$G_{b,3} \;=\; \langle\, a, b, t \,|\, abaa = bb, ta^3t^{-1} = b \,\rangle$$

for example.

Recall that the original group G admits an epimorphism $L : G \to \mathbb{Z}$, where $L(b) = 3L(a)$. Since the abelianization of the new relation still implies that $L(b) = 3L(a)$, we can extend the circle-valued Morse function from the presentation 2-complex of G to the presentation 2-complex for $G_{b,3}$ by mapping t once around the circle, and extending "linearly" over the new 2-cell as shown in Figure 5. Ascending and descending links are trees (segments of length two each), so the space is homeomorphic to the total space of a graph of spaces where the underlying graph is again a one vertex circle, the vertex space is as shown on the left hand side of Figure 5, and the edge space is a graph which is homotopy equivalent to this, and maps are homotopy equivalences. Thus $G_{b,3}$ is isomorphic to a semidirect product $F_6 \rtimes \mathbb{Z}$.

[3] This reasoning enabled us earlier to compute exact translation lengths for the element a.

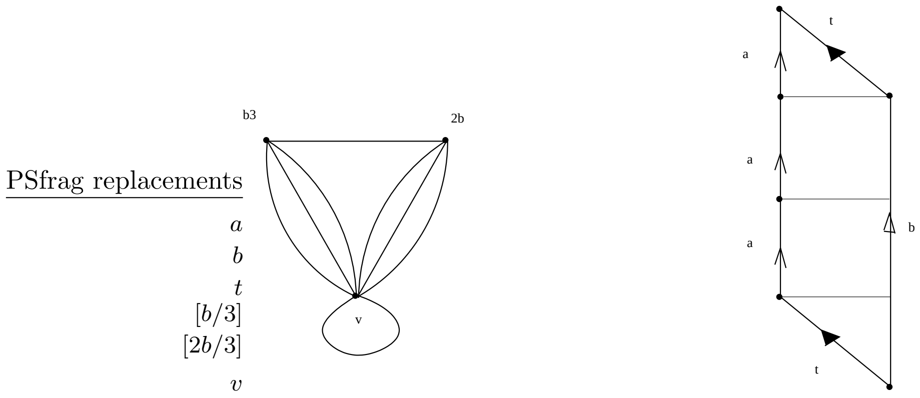


FIGURE 5. The Morse function on the extra 2-cell in the $F_6 \rtimes \mathbb{Z}$ group, and the level set through v.

In the general case of the group $G_{w,L(w)}$ the picture of the Morse function on the new 2-cell will be as in Figure 5, with a^3 replaced by $a^{L(w)}$ and with b replace by w. This has the effect of adding $L(w)$ new edges to the level set Γ passing through v. Just as in the preceding paragraph, the ascending and descending links will still be contractible (segments of length 2 each). Thus

$$G_{w,L(w)} \; = \; \langle\, a, b, t \,|\, abaa = bb, ta^{L(w)}t^{-1} = w \,\rangle$$

is free-by-cyclic with free kernel of rank $3 + L(w)$. □

Remark 20. We do not know if any of the groups $G_{w,m}$ where $|m| \geq L(w)$ are CAT(0). Some of them may indeed have 3-dimensional CAT(0) structures. However, it is hard to imagine low dimensional CAT(0) structures for $G_{w,m}$ when $|m| \gg L(w)$, or when one takes further HNN extensions over suitably chosen $\mathbb{Z}$ subgroups of these $G_{w,m}$. There is more to explore here.

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