ON THE NUMBER OF LIMIT CYCLES BIFURCATING FROM A NON-GLOBAL DEGENERATED CENTER

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Abstract. We give an upper bound for the number of zeros of an Abelian integral. This integral controls the number of limit cycles that bifurcate, by a polynomial perturbation of arbitrary degree, from the periodic orbits of the integrable system $(1+x)dH = 0$, where H is the quasi-homogeneous Hamiltonian $H(x,y) = x^{2k}/(2k) + y^2/2$. The main tool used in the proof is the Argument Principle applied to a suitable complex extension of the Abelian integral.

1. Introduction and Main Result

Consider the perturbed integrable system

$$\begin{cases} \dot{x} = -\dfrac{\partial H(x,y)}{\partial y} R(x,y) + \varepsilon P(x,y)\,, \\ \dot{y} = \dfrac{\partial H(x,y)}{\partial x} R(x,y) + \varepsilon Q(x,y)\,, \end{cases} \tag{1$_\varepsilon$}$$

and associated to it define the Abelian integral

$$I(h) = \int_{\Gamma_h} \frac{P(x,y)dy - Q(x,y)dx}{R(x,y)}\,, \tag{2}$$

taken along the real ovals of the level curves $\{H(x,y) = h\}$. It is well known that each isolated zero h^* of $I(h)$ gives rise, for ε small enough, to a limit cycle of the perturbed system $(1)_\varepsilon$, which tends to the oval $\{H(x,y) = h^*\}$ when ε goes to zero, see for instance [8].

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The first author is partially supported by the DGES grant number BFM2002-04236-C02-2 and CONACIT grant number 2001SGR00173. The second author is partially supported by the grant NSFC–10231020 and RFDP of China.

A challenging problem consists in, fixed H and R, consider P and Q polynomials of degree at most n and give upper bounds for the maximum number of isolated zeroes of $I(h)$ in terms of n. This question is related with the known as weak or infinitesimal Hilbert's sixteenth problem. Also, considered as a method for obtaining limit cycles for polynomial vector fields it is strongly related with the Hilbert's sixteenth problem.

The most studied cases are the ones for which system $(1)_0$ is Hamiltonian (*i.e.* those for which $R(x,y)\equiv 1$) and more concretely the ones having H either a cubic polynomial or a function of hyperelliptic type, $H(x,y) = y^2+V(x)$.

This paper is devoted to study the above problem when $H(x,y)$ is the quasi-homogeneous hyperelliptic function $x^{2k}/(2k)+y^2/2$, $k\geq 1$ and $R(x,y)=1+x$. Notice that in this case the phase portrait of $(1)_0$ is a non-global center, which phase portrait consists of a global center (degenerated if $k>1$) cut by the line of critical points $\{x=-1\}$. Our main result is the following theorem:

Theorem A. *Let $P(x,y)$ and $Q(x,y)$ be real polynomials of degree at most $n\in\{2\ell,2\ell+1\}$, $\ell\geq 1$. Then the maximum number of isolated zeros of the Abelian integral*

$$I(h)=\int_{\Gamma_h}\frac{P(x,y)dy-Q(x,y)dx}{1+x}\,, \tag{3}$$

where $\Gamma_h=\{(x,y)\,:\,x^{2k}/(2k)+y^2/2=h,\ 0<h<1/(2k)\}$, is $k(\ell+1)+2\ell+3-\sin^2\left(\frac{k\pi}{2}\right)\ell$.

The above result has been already proved in [4] when $k=1$. Indeed in that paper it is proved that the maximum number of isolated zeros of $I(h)$ is n and that this number can be attained. Notice that our upper bound, $2\ell+4$, is slightly bigger. This small difference can be easily corrected following carefully our proof of Theorem A when $k=1$.

As it will be seen along the proof of our theorem, the expression of $I(h)$ includes some polynomials in $h^{\frac{1}{k}}$ and the function

$$J(h)=\sqrt{1-2kh}\int_0^h\frac{t^{\frac{1-k}{2k}}p_{k-1}(t^{\frac{1}{k}})}{(1-2kt)^{3/2}}\,dt,$$

where p_{k-1} a real polynomial of degree $k-1$. In the particular case $k=1$, $J(h)=2\pi\left[\sqrt{1-2h}-1\right]$ and the problem of the study of the zeros of $I(h)$ can be reduced to the study of the zeros of a polynomial in a new variable $w=\sqrt{1-2h}$. For $k>1$ the expression of $I(h)$ is more complicated and our study relies on the study of an extension of $I(h)$ in a suitable complex domain. Theorem A will be proved in Section 3 by applying the Argument Principle to this extension of $I(h)$. As far as we know this method to estimate the number of zeros of Abelian integrals was introduced by Petrov in [5, 6, 7].

It is also worth to mention that in [2] a related question is considered by using a slightly different approach. In that paper by using the Poincaré-Lyapunov polar coordinates, see [1, 3], a lower bound for the number of zeros of $I(h)$, when $k=\ell$, is obtained. This result is studied there to get a lower bound for the number of nested limit cycles of a polynomial system of degree $n \in \{2\ell, 2\ell+1\}$. For instance when $n=2\ell$ the lower bound given in that paper is $(\ell^2+3\ell)/2$. Notice that by applying Theorem A we get that an upper bound for the zeros of $I(h)$ is $\ell^2+3\ell+3-\sin^2\left(\frac{\ell\pi}{2}\right)\ell$. The gap between these two results remains to be studied.

2. Preliminary Results

Along this paper any polynomial in $\mathbb{R}[x]$ of degree m is denoted by $p_m(x)$ although its coefficients may vary from one expression to another. If it is necessary we will also use indistinctly $q_m(x)$. Notice that in particular the constants will be denoted by $p_0(x) \equiv p_0$, $p'_m(x) = p_{m-1}(x)$, $p_m(x)p_n(x) = p_{n+m}(x)$, and so on.

Define

$$I_j(h) = \int_{\Gamma_h} x^j y dx\,, \quad j=0,1,2,\dots\,; \text{ and } \qquad J(h) = \int_{\Gamma_h} \frac{y}{1+x} dx\,. \quad (4)$$

We give several preliminary results on the above functions which, as we will see, generate the Abelian integral $I(h)$.

Lemma 1. *The functions $I_j(h)$ introduced in (4) are*

$$I_{2j+1}(h) \equiv 0 \quad \textit{and} \quad I_{2j}(h) = p_0 h^{\frac{2j+k+1}{2k}}\,, \quad j=0,1,2,\dots$$

where $p_0 \neq 0$ is a constant independent on h.

Proof. The oval of $H(x,y)=h$ is given by

$$\frac{x^{2k}}{2k} + \frac{y^2}{2} = h\,, \quad (5)$$

or, equivalently, by

$$y = y_\pm(x,h) = \pm\sqrt{2h - \frac{x^{2k}}{k}}\,. \quad (6)$$

Hence

$$I_j(h) = 2\int_{-x^*}^{x^*} x^j \sqrt{2h - \frac{x^{2k}}{k}}\, dx\,,$$

where $x^* = \sqrt[2k]{2kh}$. Thus, by the symmetry of the integrator it is clear that $I_{2m+1} = 0$ for all m. In case of even subindex we have that

$$I_{2m}(h) = 4\int_0^{x^*} x^{2m}\sqrt{2h - \frac{x^{2k}}{k}}\,dx\,.$$

By making the substitution $x = \sqrt[2k]{2kh\sin^2\theta}$, we obtain

$$I_{2m}(h) = C_{k,m}h^{\frac{2m+k+1}{2k}}\,,$$

where

$$p_0 := C_{k,m} = \frac{4\sqrt{2}(2k)^{\frac{2m+1}{2k}}}{k}\int_0^{\frac{\pi}{2}}(\sin\theta)^{\frac{2m+1-k}{k}}\cos^2\theta\,d\theta \neq 0\,,$$

as we wanted to prove. □

The proof of next result is straightforward.

Lemma 2. *Consider the functions $I_j(h)$ and $J(h)$ introduced in (4). Their derivatives can be expressed as*

$$I_j'(h) = \int_{\Gamma_h}\frac{x^j}{y}dx\,,\quad j = 0,1,2,\ldots\,;\qquad J'(h) = \int_{\Gamma_h}\frac{1}{y(1+x)}dx\,.$$

By using previous lemma and Lemma 1 we get:

Lemma 3. *Write $m = 2r$ or $m = 2r+1$. The following equalities hold:*

$$\int_{\Gamma_h} yp_m(x)\,dx = h^{\frac{k+1}{2k}}p_r(h^{\frac{1}{k}}),\qquad \textit{and}\qquad \int_{\Gamma_h}\frac{p_m(x)}{y}\,dx = h^{\frac{1-k}{2k}}p_r(h^{\frac{1}{k}}).$$

Lemma 4. *(i) The function $J(h)$ introduced in (4) is also given by the expression*

$$J(h) = \sqrt{1-2kh}\int_0^h \frac{s^{\frac{1-k}{2k}}p_{k-1}(s^{\frac{1}{k}})}{(1-2ks)^{3/2}}\,ds\,. \tag{7}$$

where $p_{k-1}(x)$ is a polynomial of degree exactly $k-1$. Furthermore, $\lim_{h\to(\frac{1}{2k})^-} J(h)$ exists, and it is finite and non-zero.

(ii) When $k = 1$ then $J(h) = 2\pi\left[\sqrt{1-2h}-1\right]$.

Proof. By Lemmas 2, 3 and equations (4) and (5) we have

$$\begin{aligned}
J(h) &= \int_{\Gamma_h} \frac{y^2}{y(1+x)}\,dx = \int_{\Gamma_h} \frac{2h-\frac{x^{2k}}{k}}{y(1+x)}\,dx = 2hJ'(h) - \frac{1}{k}\int_{\Gamma_h} \frac{x^{2k}}{y(x+1)}\,dx \\
&= 2hJ'(h) - \frac{1}{k}\int_{\Gamma_h} \frac{(x+1)p_{2k-1}(x)+1}{y(x+1)}\,dx \\
&= \left(2h-\frac{1}{k}\right)J'(h) + \int_{\Gamma_h} \frac{p_{2k-1}(x)}{y}\,dx \\
&= \left(2h-\frac{1}{k}\right)J'(h) + h^{\frac{1-k}{2k}}p_{k-1}(h^{\frac{1}{k}}).
\end{aligned}$$

Notice that from the construction of p_{2k-1}, all its coefficients are non null. From the computations used to prove Lemma 3 it is easy to see that the same property happens for the coefficients of p_{k-1}. In particular it is exactly of degree $k-1$.

By definition of $J(h)$, we have $J(0) = 0$. By integration of the above ordinary differential equation for $J(h)$ we obtain (7). From (6) we have that near $x = -1$,

$$y_{\pm}(x, \frac{1}{2k}) = \pm\sqrt{1+x}\left(\sqrt{2} + O(x+1)\right)^{1/2}.$$

Hence

$$\lim_{h\to(\frac{1}{2k})^-} J(h) = 2\int_{-1}^{1} \frac{y_+(x,\frac{1}{2k})}{1+x}\,dx < \infty,$$

and is different from zero, as we wanted to prove.

To end the proof of the lemma let us study $J(h)$ when $k = 1$. In this case notice that the integral appearing in (7) can be computed explicitly. By using that $p_0(h) = -2\pi$, the expression for $J(h)$ follows. □

Proposition 5. *Assume that $n = 2\ell$ or $n = 2\ell + 1$. Then the Abelian integral $I(h)$ given in* (2) *can be expressed in the form*

$$I(h) = \frac{1}{2h-1/k}\left(h^{\frac{k+1}{2k}}p_{k\ell+k-1}(h^{\frac{1}{k}}) + q_{\ell+1}(h)J(h)\right), \tag{8}$$

where as usual $p_{k\ell+k-1}(h)$ and $q_{\ell+1}(h)$ are real polynomials in h of degree given by their respective subindexes.

Proof. We assume $n = 2\ell + 1$, the proof for the case $n = 2\ell$ is similar. Let us start by studying the integral $\int_{\Gamma_h} \frac{Q(x,y)}{1+x}\,dx$. The polynomial $Q(x, y)$ can be decomposed as the sum of its even and odd parts with respect to the variable y, *i.e.* $Q(x, y) = Q^e(x, y) + Q^o(x, y)$, where $Q^e(x, -y) = Q^e(x, y)$

and $Q^o(x,-y) = -Q^o(x,y)$. In particular $Q^e(x,y) = \tilde{Q}(x,y^2)$ being $\tilde{Q}$ another suitable polynomial. Notice that

$$\int_{\Gamma_h} \frac{Q^e(x,y)}{1+x}\,dx = \int_{\Gamma_h} \frac{\tilde{Q}(x,y^2)}{1+x}\,dx = \int_{\Gamma_h} \frac{\tilde{Q}(x,2h-x^{2k}/(2k))}{1+x}\,dx \equiv 0.$$

Thus $\int_{\Gamma_h} \frac{Q(x,y)}{1+x}\,dx = \int_{\Gamma_h} \frac{Q^o(x,y)}{1+x}\,dx$. Write

$$\begin{aligned} Q^o(x,y) &= \\ &= p_{n-1}(x)y + p_{n-3}(x)y^3 + \cdots + p_2(x)y^{2\ell-1} + p_0(x)y^{2\ell+1} \\ &= y\left(p_{n-1}(x)+p_{n-3}(x)\left[2h-x^{2k}/(2k)\right]^2+\cdots+p_0(x)\left[2h-x^{2k}/(2k)\right]^\ell\right) \\ &= y\left(p_{2k\ell}(x) + hp_{2k(\ell-1)}(x) + h^2 p_{2k(\ell-2)}(x) + \cdots + h^\ell p_0(x)\right). \end{aligned} \tag{9}$$

Hence

$$\frac{Q^o(x,y)}{1+x} = \\ = y\left(p_{2k\ell-1}(x) + hp_{2k(\ell-1)-1}(x) + h^2 p_{2k(\ell-2)-1}(x) + \cdots + h^{\ell-1} p_{2k-1}(x)\right) + q_\ell(h)\frac{y}{1+x}.$$

By using Lemma 3 we get

$$\begin{aligned} \int_{\Gamma_h} \frac{Q^o(x,y)}{1+x}\,dx &= h^{\frac{k+1}{2k}}\Big(p_{k\ell-1}(h^{\frac{1}{k}})+hp_{k\ell-k-1}(h^{\frac{1}{k}})+h^2 p_{k\ell-2k-1}(h^{\frac{1}{k}})+\cdots \\ &\quad +h^{\ell-1}p_{k-1}(h^{\frac{1}{k}})\Big) + q_\ell(h)J(h) \\ &= h^{\frac{k+1}{2k}} p_{k\ell-1}(h^{\frac{1}{k}}) + q_\ell(h)J(h). \end{aligned}$$

Let us study the term $\int_{\Gamma_h} \frac{P(x,y)}{1+x}\,dy$ in the expression of $I(h)$. By using (5) we have that over Γ_h, $\frac{dy}{dx} = -\frac{x^{2k-1}}{y}$, hence

$$\int_{\Gamma_h} \frac{P(x,y)}{1+x}\,dy = -\int_{\Gamma_h} \frac{x^{2k-1}P(x,y)}{y(1+x)}\,dx.$$

As in the previous case we decompose $P(x,y) = P^e(x,y) + P^o(x,y)$, being P^e and P^o the even and odd parts, respectively, with respect to the variable y. Thus

$$\int_{\Gamma_h} \frac{x^{2k-1}P(x,y)}{y(1+x)}\,dx = \int_{\Gamma_h} \frac{x^{2k-1}P^e(x,y)}{y(1+x)}\,dx = \\ \int_{\Gamma_h}\frac{x^{2k-1}p_{2\ell+1}(x)}{y(1+x)}\,dx + \int_{\Gamma_h}\frac{\left(p_{2\ell-1}(x)+p_{2\ell-3}(x)y^2+\cdots+p_1(x)y^{2\ell-2}\right)x^{2k-1}y}{1+x}\,dx.$$

Notice that the numerator of the last integral above, when the points (x, y) are on Γ_h can be written as

$$\left(p_{2\ell-1}(x) + p_{2\ell-3}(x)y^2 + \cdots + p_1(x)y^{2\ell-2}\right) x^{2k-1}y =$$
$$= \left(p_{2\ell-1}(x) + p_{2\ell-3}(x)\left[2h - \frac{x^{2k}}{2k}\right] + \cdots + p_1(x)\left[2h - \frac{x^{2k}}{2k}\right]^{\ell-1} \right) x^{2k-1}y =$$
$$= \left(p_{2k\ell}(x) + hp_{2k(\ell-1)}(x) + h^2 p_{2k(\ell-2)}(x) + \cdots + h^{\ell-1}p_{2k}(x)\right) y.$$

This last expression coincides with the one of (9). Thus following the same steps that in that case we obtain that

$$\int_{\Gamma_h} \frac{x^{2k-1}P(x,y)}{y(1+x)}\, dx = \int_{\Gamma_h} \frac{x^{2k-1}p_{2\ell+1}(x)}{y(1+x)}\, dx + h^{\frac{k+1}{2k}} p_{k\ell-1}(h^{\frac{1}{k}}) + q_\ell(h)J(h). \tag{10}$$

To end the proof we study the integral appearing in the righthand side of the above expression. We have

$$\begin{aligned} \int_{\Gamma_h} \frac{x^{2k-1}p_{2\ell+1}(x)}{y(1+x)}\, dx &= \int_{\Gamma_h} \frac{p_{2(\ell+k)}(x)}{y(1+x)}\, dx = \\ &= \int_{\Gamma_h} \frac{p_{2(\ell+k)-1}(x)}{y}\, dx + \int_{\Gamma_h} \frac{p_0}{y(1+x)}\, dx = \\ &= h^{\frac{1-k}{2k}} p_{\ell+k-1}(h^{\frac{1}{k}}) + p_0 J'(h), \end{aligned}$$

where we have used Lemmas 2 and 3. Finally, from the proof of Lemma 4 we have that

$$J'(h) = \frac{1}{2h - 1/k}\left(J(h) + h^{\frac{1-k}{2k}} p_{k-1}(h^{\frac{1}{k}})\right).$$

Collecting all the results we get

$$\begin{aligned} I(h) &= h^{\frac{k+1}{2k}} p_{k\ell-1}(h^{\frac{1}{k}}) + q_\ell(h)J(h) + \frac{p_0}{2h-1/k}\left(J(h) + h^{\frac{1-k}{2k}} p_{k-1}(h^{\frac{1}{k}})\right) \\ &= \frac{1}{2h-1/k}\left(h^{\frac{k+1}{2k}} p_{k\ell+k-1}(h^{\frac{1}{k}}) + q_{\ell+1}(h)J(h)\right), \end{aligned}$$

as we wanted to prove. □

From previous proposition to study the zeros of $I(h)$ in the real interval $(0, \frac{1}{2k})$ it suffices to study a function of the form

$$M(h) := h^{\frac{k+1}{2k}} p_{k\ell+k-1}(h^{\frac{1}{k}}) + q_{\ell+1}(h)J(h). \tag{11}$$

One of the standard tools to give an upper bound of its number of zeros consists in extending the function to a suitable subset of the complex plane, and later to apply the Argument Principle to this extended function. In this case this can be done in the set

$$D = \mathbb{C} \setminus \left(\{h \in \mathbb{R}\colon h \le 0\} \cup \left\{ h \in \mathbb{R}\colon h \ge \frac{1}{2k} \right\} \right).$$

Essentially the problems to get a full extension to all $\mathbb{C}$ come from the functions $h^{\frac{1}{k}}$ and $(1-2kh)^{3/2}$.

Lemma 6. *The function $M(h)$ given in (11) and defined for $h \in (0, \frac{1}{2k})$, can be extended to D as a single-valued analytic function of h. We denote this extension also by $M(h)$.*

Proof. We first rewrite the real function $J(h)$ as

$$J(h) = \sqrt{1-2kh} \left[a_k + \int_{\frac{1}{4k}}^{h} \frac{s^{\frac{1-k}{2k}} p_{k-1}(s^{\frac{1}{k}})}{(1-2ks)^{3/2}}\, ds \right],$$

where $h \in (0, \frac{1}{2k})$ and $a_k = \int_0^{\frac{1}{4k}} \frac{s^{\frac{1-k}{2k}} p_{k-1}(s^{\frac{1}{k}})}{(1-2ks)^{3/2}}\, ds$ is a constant with parameter k. Then it can be extended to

$$J(h) = \sqrt{1-2kh} \left[a_k + \int_{\gamma_h} \frac{s^{\frac{1-k}{2k}} p_{k-1}(s^{\frac{1}{k}})}{(1-2ks)^{3/2}}\, ds \right], \tag{12}$$

where γ_h is any path contained in D joining $\frac{1}{4k}$ and h. Thus, the function $J(h)$ in form (12) is single-valued and analytic in D. On the other hand, the functions $h^{\frac{1}{k}}$ and $h^{\frac{k+1}{2k}}$ are obviously single-valued and analytic for $h \in D$. Hence, $M(h)$ can be extended to D as a single-valued analytic function of h. □

In order to use the Argument Principle to $M(h)$, we define $G = G_{R,\varepsilon} \subset D$ (a simply connected region) with $\partial G = C = C_{R,\varepsilon}$ a simple closed curve,

$$C_{R,\varepsilon} = C_R \cup C_\varepsilon^1 \cup C_\varepsilon^2 \cup L_\pm^1(R,\varepsilon) \cup L_\pm^2(R,\varepsilon),$$

where $C_R = \{z \in \mathbb{C},\ |z| = R \gg 1\}$, $C_\varepsilon^1 = \{z \in \mathbb{C},\ |z - \frac{1}{2k}| = \varepsilon \ll 1\}$, $C_\varepsilon^2 = \{z \in \mathbb{C},\ |z| = \varepsilon \ll 1\}$, $L_\pm^1(R,\varepsilon) = \{z \in \mathbb{R},\ \frac{1}{2k} + \varepsilon \le z \le R\}$, $L_\pm^2(R,\varepsilon) = \{z \in \mathbb{R},\ -R \le z \le -\varepsilon\}$. See Figure 1.

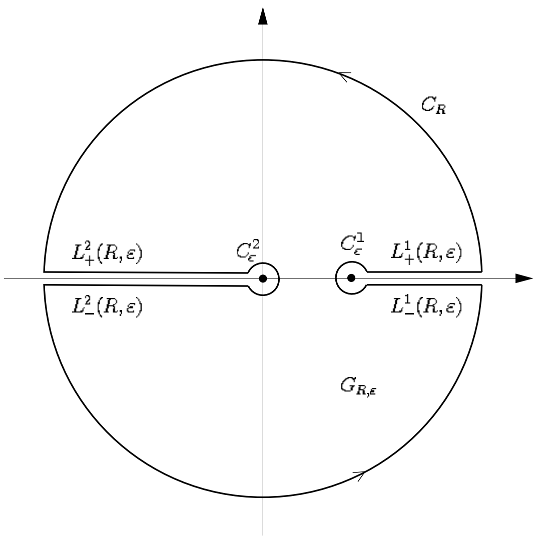

Figure 1. Domain $G = G_{R,\varepsilon} \subset D$.

Lemma 7. *Let $M(h)$ and $J(h)$ be the extensions to $D \subset \mathbb{C}$ of the functions given in Lemma 6 and (12). The following statements hold:*

- *(i)* $\mathrm{Im}(M(h))$ *has at most* $\ell + 1$ *zeros for* $h \in L^1_\pm$.
- *(ii)* $\mathrm{Im}(M(h))$ *has at most* $k\ell + k - 1 - \sin^2\left(\frac{k\pi}{2}\right)\ell$ *zeros for* $h \in L^2_\pm$.
- *(iii)* $J(h) \sim 0$ *as* $h \to 0$ *and* $J(h) \sim p_0$ *(constant) as* $h \to \frac{1}{2k}$.
- *(iv) There exist constants* B *and* $C \neq 0$ *such that* $J(h) \sim Bh^{\frac{1}{2}} + Ch^{\frac{1}{2}-\frac{1}{2k}}$ *as* $|h| \to \infty$.

Proof. (i) Write $J(h) = \sqrt{1-2kh}K(h)$ where $K(h) = \left(a_k + \int_{\gamma_h} \frac{s^{\frac{1-k}{2k}} p_{k-1}(s^{\frac{1}{k}})}{(1-2ks)^{3/2}}\, ds\right)$, being γ_h any path contained in D joining $\frac{1}{4k}$ and h. Notice that both functions $\sqrt{1-2kh}$ and $K(h)$ are holomorphic in D. To get more information on $K(h)$ at $L^1_\pm(R,\varepsilon)$ we consider in each case a concrete path joining $\frac{1}{4k}$ and h. Let us fix $h \in L^1_+(R,\varepsilon)$; the another case can be similarly studied. Take the path Γ_h formed by a curve contained in the first quadrant between $\frac{1}{4k}$ and $\frac{1}{2k} + 2\varepsilon \in \mathbb{R}$, followed by the segment $\{z \in \mathbb{R}, : \frac{1}{2k} + 2\varepsilon \le z \le h\}$. The function $K(h)$ can be computed by using Γ_h. In particular

$$K(h) - K(\frac{1}{2k} + 2\varepsilon) = \int_{\frac{1}{2k}+2\varepsilon}^{h} \frac{s^{\frac{1-k}{2k}} p_{k-1}(s^{\frac{1}{k}})}{(1-2ks)^{3/2}}\, ds = i \int_{\frac{1}{2k}+2\varepsilon}^{h} \frac{t^{\frac{1-k}{2k}} p_{k-1}(t^{\frac{1}{k}})}{(2kt-1)^{3/2}}\, dt \in i\mathbb{R}.$$

Thus

$$\begin{aligned} J(h) &= \sqrt{1-2kh}K(h) = \\ &= i\sqrt{2kh-1}\left(K(\frac{1}{2k}+2\varepsilon)+i\int_{\frac{1}{2k}+2\varepsilon}^{h}\frac{t^{\frac{1-k}{2k}}p_{k-1}(t^{\frac{1}{k}})}{(2kt-1)^{3/2}}\,dt\right). \end{aligned}$$

Hence $\mathrm{Im}(J(h)) = \mathrm{Re}(K(\frac{1}{2k}+2\varepsilon))\sqrt{2kh-1} \neq 0$ for $h \in L_+^1(R,\varepsilon)$ and $0<\varepsilon \ll 1$, since by Lemma 4 (i), when h is real $\lim_{h\to(\frac{1}{2k})^-} J(h)$ is a non-zero real number. Note that since all functions in $M(h)$, except $J(h)$, are real for $h \in L_+^1(R,\varepsilon)$, we get that $\mathrm{Im}(M(h)) = \mathrm{Im}(q_{\ell+1}(h)J(h))$ has at most $\ell+1$ zeros in $L_+^1(R,\varepsilon)$, as we wanted to see.

(ii) Notice that the second term of $M(h)$, $q_{\ell+1}(h)J(h)$, is real for $h \in L_\pm^2$. So to study the imaginary part of $M(h)$ it suffices to consider its first term. Let $m = k\ell+k-1$ and $h=-t=t\,\mathrm{e}^{\pi i}$, where t is a positive real number. Write $p_m(x)=\sum_{j=0}^m a_jx^j$, with $a_j \in \mathbb{R}$. Then

$$\begin{aligned} \mathrm{Im}(M(h)) &= \mathrm{Im}\left(h^{\frac{k+1}{2k}}p_m(h^{\frac{1}{k}})\right) \\ &= \mathrm{Im}\left(t^{\frac{k+1}{2k}}\exp\left(\tfrac{(k+1)\pi}{2k}i\right)p_m\left(t^{\frac{1}{k}}\exp(\tfrac{\pi}{k}i)\right)\right) \\ &= \mathrm{Im}\left(t^{\frac{k+1}{2k}}\textstyle\sum_{j=0}^m a_jt^{\frac{j}{k}}\exp\left(\tfrac{(2j+k+1)\pi}{2k}i\right)\right) \\ &= t^{\frac{k+1}{2k}}\left(\textstyle\sum_{j=0}^m a_j\sin\left(\tfrac{(2j+k+1)\pi}{2k}\right)t^{\frac{j}{k}}\right) := t^{\frac{k+1}{2k}}q_m(u) \end{aligned}$$

where $u=t^{\frac{1}{k}}>0$ and q_m is a real polynomial in u of degree m. Let us count the maximum number of positive real roots of q_m. By Descarte's rule this number is bounded above by one less than the number of nonzero coefficients of q_m. When k is even we have no new information and we obtain that the maximum number of positive real roots is m. On the other hand, when k is odd several coefficients of q_m vanish. In particular, when $k=1$, $q_m=0$. It is not difficult to check that when k is odd the maximum number of isolated zeros of q_m is $m-\left[\frac{m-(k-1)/2}{k}\right]$, where, as usual $[\;]$ denotes the integer part function, *i.e.* $[x]=\{\max(m)\,:\,m\le x \text{ and } m\in\mathbb{Z}\}$. When $m=k\ell+k-1$ we get that $m-\left[\frac{m-(k-1)/2}{k}\right]=k\ell+k-1-\ell$.

Hence, $\mathrm{Im}(M(h))$ has at most $m=k\ell+k-1-\sin^2\left(\frac{k\pi}{2}\right)\ell$ zeros for $h\in L_\pm^2$.

(iii) The conclusions come from the definition of $J(h)$ and Lemma 4 (i).

(iv) Let $g(s)=\frac{s^{\frac{1-k}{2k}}p_{k-1}(s^{\frac{1}{k}})}{(1-2ks)^{3/2}}$, then $J(h)=\sqrt{1-2kh}\,(a_k+\int_{\Gamma_h}g(s)ds)$, being γ_h any path contained in D joining $\frac{1}{4k}$ and h.

Recall that by Lemma (4) (i), p_{k-1} is a real polynomial of degree exactly $k-1$. Furthermore there exists a real number $A>0$, such that in D, $g(s)$ has the following analytic expansion for $|s|>A$,

$$\begin{aligned} g(s) &= s^{\frac{1-k}{2k}} p_{k-1}(s^{\frac{1}{k}}) \left(\frac{-1}{2ks} \frac{1}{1-\frac{1}{2ks}} \right)^{\frac{3}{2}} = \\ &= i s^{\frac{1}{2k}-2} p_{k-1}(s^{\frac{1}{k}}) \left(\sum_{j=0}^{\infty} \left(\frac{1}{2ks} \right)^j \right)^{\frac{3}{2}} \sim s^{-1-\frac{1}{2k}}. \end{aligned} \tag{13}$$

From (13), in $D \cap \{|s|>A\}$ we can write

$$g(s) = \sum_{j=0}^{\infty} b_j \, s^{-1-\frac{1}{2k}-\frac{j}{k}}, \tag{14}$$

with $b_0 \neq 0$. To estimate $J(h)$, we fix h_0 with $|h_0|>A$, and choose two paths γ_0 and γ_h joining $\frac{1}{4k}$ to h_0 and h_0 to h respectively, such that γ_h is contained entirely in the region $D \cap \{|s|>A\}$. Then

$$J(h) = \sqrt{1-2kh} \left(a_k + \int_{\gamma_0} g(s)ds + \int_{\gamma_h} g(s)ds \right).$$

By using the expansion (14) in the last integration, we obtain

$$J(h) = \sqrt{1-2kh} \left(a_k + \int_{\gamma_0} g(s)ds + \phi(h) - \phi(h_0) \right),$$

where

$$\phi(h) = \sum_{j=0}^{\infty} \left(-\frac{2kb_j}{2j+1} \right) h^{-\frac{1}{2k}-\frac{j}{k}} \sim -2kb_0 \, h^{-\frac{1}{2k}} \quad \text{as } |h| \to \infty.$$

Let $C_0 = a_k + \int_{\gamma_0} g(s)ds - \phi(h_0)$, then $J(h) = \sqrt{1-2kh}\,(C_0 + \phi(h))$. We get immediately that $J(h) \sim C_1 h^{\frac{1}{2}} + h^{\frac{1}{2}-\frac{1}{2k}}$ when $|h| \to \infty$ as we wanted to prove. □

3. Proof of Theorem A

We apply the Argument Principle for $M(h)$ to $G_{R,\varepsilon}$, for R and $1/\varepsilon$ positive and big enough. We will prove that the rotation number of M when h turns around the boundary of G is at most $k(\ell+1)+2\ell+3+\frac{1}{2}-\sin^2\left(\frac{k\pi}{2}\right)\ell$. From this result we get that the number of zeros of $M(h)$ in $(0,1)$ is at most $k(\ell+1)+2\ell+3-\sin^2\left(\frac{k\pi}{2}\right)\ell$, as we wanted to prove. Note that we have removed the term $1/2$ since the number of zeros has to be a natural number.

Let us compute the rotation number of M. By Lemma 7 (iii) the number of complete turns of M on $C_\varepsilon^1 \cup C_\varepsilon^2$ when ε goes to 0, tends to zero. By Lemma 7 (i) and (ii) the number of zeros of $\mathrm{Im}(M(h))$ for $h \in L_\pm^1 \cup L_\pm^2$ is at most $2\left(\ell + 1 + k\ell + k - 1 - \sin^2\left(\frac{k\pi}{2}\right)\ell\right) = 2\left(k\ell + k + \ell - \sin^2\left(\frac{k\pi}{2}\right)\ell\right)$. Since each complete turn of $M(h)$ forces at least two zeros of $\mathrm{Im}(M(h))$ we get that the number of complete turns on these four segments is at most $k\ell + k + \ell - \sin^2\left(\frac{k\pi}{2}\right)\ell + 2$ (we add less than one half turn on each bank). Finally, from Lemma 7 (iv) the number of complete turns on C_R is at most $\ell + 1 + \frac{1}{2}$. Putting all the results together we obtain that the number of turns is at most $k(\ell+1) + 2\ell + 3 + \frac{1}{2} - \sin^2\left(\frac{k\pi}{2}\right)\ell$ as we wanted to prove.

4. Acknowledgements

Chengzhi Li wants to thank to the CRM and to the Department of Mathematics of the Universitat Autònoma de Barcelona for their support and hospitality. This paper is also supported by the CRM Research Program: On Hilbert's 16th Problem.

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