EMBEDDING THEOREMS OF FUNCTION CLASSES, IV

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ABSTRACT. We study the interrelation between the strong class $S_p(\lambda)$ and the Nikol'skii class $W^r H^\omega_\beta$.

1. INTRODUCTION

Let $f(x)$ be a 2π-periodic continuous function and let

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad (1)$$

be its Fourier series. The modulus of smoothness of order β $(\beta > 0)$ of a function $f \in C$ is given by

$$\omega_\beta(f,t) = \sup_{|h|\le t} \left\| \sum_{\nu=0}^{\infty} (-1)^\nu \binom{\beta}{\nu} f(x + (\beta-\nu)h) \right\|,$$

where $\binom{\beta}{\nu} = \frac{\beta(\beta-1)\cdots(\beta-\nu+1)}{\nu!}$ for $\nu \ge 1$, $\binom{\beta}{\nu} = 1$ for $\nu = 0$ and $\|f(\cdot)\| = \max\limits_{x\in[0,2\pi]} |f(x)|$.

Denote by $S_n(x) = S_n(f,x)$ the n-th partial sum of (1). Let $E_n(f)$ be the best approximation of $f(x)$ by trigonometric polynomials of order n and let $f^{(r)}$ be the derivative of the function f of order $r > 0\,(f^{(0)} := f)$ in the sense of Weyl.

We will write $I_1 \ll I_2$, if there exists a positive constant C such that $I_1 \le C\, I_2$. If $I_1 \ll I_2$ and $I_2 \ll I_1$ hold simultaneously, then we will write $I_1 \asymp I_2$.

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A sequence $\gamma := \{\gamma_n\}$ of positive terms will be called almost increasing (almost decreasing), if there exists a constant $K := K(\gamma) \geq 1$ such that

$$K\gamma_n \geq \gamma_m \quad (\gamma_n \leq K\gamma_m)$$

holds for any $n \geq m$.

Let Ω_β be the set of nondecreasing continuous functions on $[0, 2\pi]$ such that $\omega(0) = 0$, $\omega(\delta)$ is nondecreasing and $\delta^{-\beta}\omega(\delta)$ is nonincreasing. Define the following function classes:

$$W^r H^\omega_\beta = \left\{ f \in C : \omega_\beta(f^{(r)}, \delta) = O\left[\omega\left(\delta\right)\right] \right\},$$

$$S_p(\lambda) = \left\{ f \in C : \left\| \sum_{\nu=1}^{\infty} \lambda_\nu \left| f(x) - S_\nu(x) \right|^p \right\| \, < \, \infty \right\},$$

where $\omega\left(\delta\right) \in \Omega_\beta$, $\lambda = \{\lambda_n\}_{n=1}^\infty$ is a sequence of positive numbers, $r \in [0, \infty)$, and $\beta, p \in (0, \infty)$. We also define $H^\omega_\beta := W^0 H^\omega_\beta$.

We say that the sequence $\lambda = \{\lambda_n\}_{n=1}^\infty$ satisfies the $\triangle_2^2$-condition if

$$\lambda_n \asymp \lambda_k \qquad \text{for} \qquad n \leq k \leq 2n. \tag{2}$$

We will need the following

Definition. The sequence of positive numbers $a = \{a_n\}_{n=1}^\infty$ is said to be *general monotone*, or $a \in GM$, if the relation

$$\sum_{\nu=n}^{2n-1} |a_\nu - a_{\nu+1}| \leq C a_n$$

holds for all integer n, where the constant C is independent of n.

It was proved in [11] that $a \in GM$ if and only if a satisfies

$$a_\nu \leq C a_n \qquad \text{for} \quad n \leq \nu \leq 2n \tag{3}$$

and

$$\sum_{k=n}^{N} |\triangle a_k| \leq C \left(a_n + \sum_{k=n+1}^{N} \frac{a_k}{k} \right) \qquad \text{for any} \qquad n < N. \tag{4}$$

We remark that

$$M \subsetneq QM \cup RBVS \subsetneq ORVQM \cup RBVS \subsetneq GM,$$

where M is a class of monotone sequences, QM is a class of quasi monotone sequences (see [7], [9]), $ORVQM$ is a class of O-regularly varying quasi monotone sequences (see [8]), and $RBVS$ is a class of sequences of rest bounded variation (see [6]).

We define the following two subclasses of C:

$$C^{cos} = \left\{ f \in C : f(x) = \sum_{n=1}^{\infty} a_n \cos nx, \quad \{a_n\} \in GM \right\},$$

$$C^{sin} = \left\{ g \in C : g(x) = \sum_{n=1}^{\infty} a_n \sin nx, \quad \{a_n\} \in GM \right\}.$$

In this paper we study the interrelation between $W^r H^\omega_\beta$ and $S_p(\lambda)$. Our investigation continues the findings from the book [3] and the papers [2], [5], [6] of L. Leindler.

2. Results

First we study the embedding $S_p(\lambda) \subset W^r H^\omega_\beta$. Related results can be found in [2] and [3].

Theorem 2.1. *Let $\beta, p > 0, r \geq 0$, $\omega \in \Omega_\beta$ and let $\{\lambda_n\}$ satisfy $\triangle_2^2$-condition. Suppose*

$$\lambda_n \omega^p \left(\frac{1}{n}\right) n^{1-rp} \geq C. \tag{5}$$

(i). *If $r > 0$ and ω satisfies the conditions*

$$(B) \qquad \sum_{k=n+1}^{\infty} \frac{1}{k} \omega\Big(\frac{1}{k}\Big) = O\left[\omega\Big(\frac{1}{n}\Big)\right],$$

$$(B_\beta) \qquad \sum_{k=1}^{n} k^{\beta-1} \omega\Big(\frac{1}{k}\Big) = O\left[n^\beta \omega\Big(\frac{1}{n}\Big)\right],$$

then

$$S_p(\lambda) \subset W^r H^\omega_\beta. \tag{6}$$

(ii). *If $r = 0$ and ω satisfies the condition (B_β), then*

$$S_p(\lambda) \subset H^\omega_\beta. \tag{7}$$

We note that for certain subclasses of continuous functions the conditions on ω can be relaxed.

Theorem 2.2. *Let $\beta, p > 0, r \geq 0$, $\omega \in \Omega_\beta \cap B$. Suppose $\{\lambda_n\}$ satisfy $\triangle_2^2$-condition and condition (5); then*

$$S_p(\lambda) \cap C^{cos} \subset W^r H^\omega_\beta \qquad \textit{for} \quad r + \beta = 2l - 1, \tag{8}$$

$$S_p(\lambda) \cap C^{sin} \subset W^r H^\omega_\beta \qquad \textit{for} \quad r + \beta = 2l. \tag{9}$$

Remark 2.3. *Theorems 2.1 and 2.2 provide, in particular, an answer for the following question (see* [3]*): When does the condition*

$$\left\| \sum_{\nu=1}^{\infty} \nu^{(r+\alpha)p-1} \left| f(x) - S_\nu(x) \right|^p \right\| < \infty \tag{10}$$

imply the condition

$$\omega_\beta(f^{(r)}, \delta) = O\left[\delta^\alpha\right] \ ? \tag{11}$$

In particular, the answer is: $0 < \alpha < \beta$ *(by Theorem 2.1), and* $\alpha = \beta$ *if* $f \in C^{cos}$ *and* $r + \alpha = 2l - 1$ *or if* $f \in C^{sin}$ *and* $r + \alpha = 2l$ *(by Theorem 2.2). We note that in general, if* $\alpha = \beta$*, the answer is negative. Indeed, for* $p, \alpha, \beta = 1$*, (10) implies only*

$$\omega_\beta(f^{(r)}, \delta) = O\left(\delta \log \frac{1}{\delta}\right) \tag{12}$$

and this result is the best possible (see [3]*).*

Remark 2.4. *Let* $\beta > 0, r \geq 0$*,* $\omega \in \Omega_\beta \cap B$*, and let* $\omega^*(\delta) := \delta^r \omega(\delta)$*. We have*

$$W^r H_\beta^\omega \equiv H_{r+\beta}^{\omega^*} \subset E^{\omega^*} := \left\{ f \in C : E_n(f) = O\Big[\omega^*\Big(\frac{1}{n}\Big)\Big] \right\}.$$

Moreover,

$$C^{cos} \cap W^r H_\beta^\omega \equiv C^{cos} \cap H_\alpha^{\omega^*} \equiv C^{cos} \cap E^{\omega^*}, \qquad \textit{where} \quad \alpha \geq r + \beta = 2l - 1,$$

and

$$C^{sin} \cap W^r H_\beta^\omega \equiv C^{sin} \cap H_\alpha^{\omega^*} \equiv C^{sin} \cap E^{\omega^*}, \qquad \textit{where} \quad \alpha \geq r + \beta = 2l.$$

We remark that for some strong classes one can write the embedding into $W^r H_\beta^\omega$ without the conditions (B) and (B_β) on ω.

Remark 2.5. *Let* $\beta, p > 0, r \geq 0$*,* $\omega \in \Omega_\beta$*. Suppose* $\{\lambda_n\}$ *satisfy* $\triangle_2^2$*-condition and condition (5); then*

$$\bar{S}_p(\lambda) := \left\{ f \in C : \sum_{n=1}^{\infty} \left\| \left(\sum_{\nu=2^n+1}^{2^{n+1}} \lambda_\nu \left| f(x) - S_\nu(x) \right|^p \right)^{\frac{1}{p}} \right\| < \infty \right\} \subset W^r H_\beta^\omega. \tag{13}$$

Now we study the converse embedding $W^r H_\beta^\omega \subset S_p(\lambda)$. A useful overview and a history of the question can be found in [3], [5], [6]. There the next Theorem 2.6 was proved for $r = 0$ and Theorem 2.7 was proved for $r = 0$ and $g \in \left\{ g \in C^{sin} : \{a_n\} \in RBVS \right\} \subsetneq C^{sin}$.

Theorem 2.6. *Let $\beta, p > 0, r \geq 0$, $\omega \in \Omega_\beta$ and let $\{\lambda_n\}$ satisfy $\triangle_2^2$-condition. Suppose*

$$\sum_{n=1}^{\infty} \frac{\lambda_n \omega^p\left(\frac{1}{n}\right)}{n^{rp}} < \infty; \tag{14}$$

then

$$W^r H_\beta^\omega \subset S_p(\lambda). \tag{15}$$

We remark that condition (14) implies

$$\lambda_n \omega^p\left(\frac{1}{n}\right) n^{1-rp} \leq C. \tag{16}$$

As in the case of Theorem 2.2, one can assume only this weaker condition if we consider C^{sin}.

Theorem 2.7. *Let $\beta, p > 0, r \geq 0$, $\omega \in \Omega_\beta$ and let $\{\lambda_n\}$ satisfy $\triangle_2^2$-condition and condition (16). Suppose there exists $\varepsilon \in (0,1)$ such that*

$$\{n^{1-\varepsilon}\lambda_n\} \quad \textit{is almost increasing;} \tag{17}$$

then

$$W^r H_\beta^\omega \cap C^{sin} \subset S_p(\lambda). \tag{18}$$

Remark 2.8. *In general, condition (16) is not sufficient for embedding (15). Indeed, suppose $\omega(\delta) = \delta^\alpha$ and $\alpha = \beta = 1$; then there exists a function f such that $f \in W^r H_\beta^\omega$ but $f \notin S_p(\lambda^*)$, where $\lambda_n^* = n^{rp-1}\omega^{-p}\left(\frac{1}{n}\right)$ satisfies (17) (see* [3]*).*

From Theorems 2.2 and 2.7 and Remark 2.4 we have

Corollary 2.9. *Let $\beta, p > 0, r \geq 0$, $\omega \in \Omega_\beta \cap B$, and let $\omega^*(\delta) := \delta^r \omega(\delta)$. Suppose $\alpha \geq r + \beta = 2l$; then*

$$S_p^{sin}(\lambda^*) \equiv C^{sin} \cap W^r H_\beta^\omega \equiv C^{sin} \cap H_\alpha^{\omega^*} \equiv C^{sin} \cap E^{\omega^*},$$

where

$$\begin{aligned} S_p^{sin}(\lambda^*) \quad &:= \quad S_p^{sin}\Big(\Big\{\frac{\nu^{rp-1}}{\omega^p\left(\frac{1}{\nu}\right)}\Big\}\Big) = \\ &= \quad \left\{ f \in C^{sin} : \left\| \sum_{\nu=1}^{\infty} \frac{\nu^{rp-1}}{\omega^p\left(\frac{1}{\nu}\right)} |f(x) - S_\nu(x)|^p \right\| < \infty \right\}. \end{aligned}$$

3. Proofs

We start with the following lemmas.

Lemma 1. ([3, Theorem 8.1]). *Let* $p > 0$ *and let* $\{\gamma_n\}$ *be a positive sequence such that* $\gamma_{2^n} \le C\gamma_{2^n+i}$ $(C \ge 1, n \in \mathbf{N}, 1 \le i \le 2^n)$. *Then*

$$V_n(f,p) := \Big\| \Big(\frac{1}{n} \sum_{\nu=[\frac{n}{2}]+1}^{n} |f(x) - S_\nu(x)|^p \Big)^{\frac{1}{p}} \Big\| \ll \gamma_n \tag{19}$$

implies

$$E_n(f) \ll \gamma_n. \tag{20}$$

Lemma 2. ([1]). *If* $f(x) \in C$, *then*

$$\omega_{\beta+r}\Big(f,\delta\Big) \ll \delta^r \omega_\beta \Big(f^{(r)},\delta\Big) \qquad \text{for} \quad r,\beta > 0.$$

Lemma 3. ([13]). *If* $f(x) \in C$ *has a Fourier series*

$$\sum_{k=1}^{\infty} a_k \sin kx, \qquad a_k \ge 0, \tag{21}$$

then

$$n^{-\beta} \sum_{k=1}^{n} k^\beta a_k \ll \omega_\beta\Big(f, \frac{1}{n}\Big), \quad \text{for} \quad \beta \ne 2l, l = 1, 2, \cdots$$

Lemma 4. ([4]). *Let* $a_n \ge 0$, $\lambda_n > 0$.

(a): *If* $p \ge 1$, *then*

$$\sum_{n=1}^{\infty} \lambda_n \left(\sum_{\nu=n}^{\infty} a_\nu \right)^p \ll \sum_{n=1}^{\infty} \lambda_n^{1-p} a_n^p \left(\sum_{\nu=1}^{n} \lambda_\nu \right)^p,$$

(b): *If* $0 < p < 1$ *and* $a_{\nu+j} \le K a_\nu$ *for* $1 \le j \le \nu$, *then*

$$\sum_{n=1}^{\infty} \lambda_n \left(\sum_{\nu=n}^{\infty} a_\nu \right)^p \ll \sum_{n=1}^{\infty} n^{p-1} a_n^p \left(n\lambda_n + \sum_{\nu=1}^{n-1} \lambda_\nu \right).$$

Proof of Theorem 2.1. Since $\{\lambda_n\}$ satisfies $\triangle_2^2$-condition it is clear that $f \in S_p(\lambda)$ implies $V_n(f,p) \ll (n\lambda_n)^{-\frac{1}{p}}$. By Lemma 1 and condition (5), we have

$$E_n(f) \ll (n\lambda_n)^{-\frac{1}{p}} \ll \omega\left(\frac{1}{n}\right) n^{-r}.$$

Further we use the conditions (B) and (B_β) and the following inequalities: in the case of $r > 0$

$$\omega_\beta \left(f^{(r)}, \frac{1}{n} \right) \ll \frac{1}{n^\beta} \sum_{k=1}^{n} k^{r+\beta-1} E_k(f) + \sum_{k=n}^{\infty} k^{r-1} E_k(f) \tag{22}$$

and in the case of $r = 0$

$$\omega_\beta\left(f, \frac{1}{n}\right) \ll \frac{1}{n^\beta}\sum_{k=1}^{n} k^{\beta-1}E_k(f). \tag{23}$$

Finally, we have $\omega_\beta\left(f^{(r)}, \frac{1}{n}\right) \ll \omega\left(\frac{1}{n}\right)$, which finishes the proof.

Proof of Theorem 2.2. As in the proof of Theorem 2.1 we have $E_n(f) \ll \omega\left(\frac{1}{n}\right) n^{-r}$.

Let $f \in C^{cos}$. Then because of $\sum\limits_{k=2n}^{\infty} a_k \ll E_n(f)$, we get

$$\sum_{k=n+1}^{\infty} a_k \ll \omega\left(\frac{1}{n}\right) n^{-r}. \tag{24}$$

It was proved in [12] that if $\omega \in B$ and $r + \beta = 2l - 1$, then condition (24) is equivalent to $f \in W^r H^\omega_\beta$.

If $g \in C^{sin}$, then by inequality (23), $E_n(f) \ll \omega\left(\frac{1}{n}\right) n^{-r}$ gives for $\beta < \beta_1$

$$\begin{aligned}\omega_{\beta_1+r}\left(f, \frac{1}{n}\right) &\ll n^{-r-\beta_1}\sum_{k=1}^{n} k^{\beta_1-1}\omega\left(\frac{1}{k}\right)\\ &\ll \omega\left(\frac{1}{n}\right) n^{-r+\beta-\beta_1}\sum_{k=1}^{n} k^{\beta_1-\beta-1} \ll n^{-r}\omega\left(\frac{1}{n}\right).\end{aligned}$$

Therefore, by this, Lemma 3, and inequality (3), we write

$$na_n \ll \omega\left(\frac{1}{n}\right) n^{-r}. \tag{25}$$

From [12], if $\omega \in B$ and $r + \beta = 2l$, then condition (25) is equivalent to $g \in W^r H^\omega_\beta$. This completes the proof.

Proof of Remark 2.4 follows from [12] and Lemma 2.

Proof of Remark 2.5. Let $f \in \bar{S}_p(\lambda)$. Then Lemma 1 of [2] implies

$$\sum_{n=1}^{\infty} \frac{n^{r-1}}{\omega\left(\frac{1}{n}\right)} E_n(f) < \infty.$$

Therefore because of $\omega \in \Omega_\beta$, we write

$$\frac{1}{n^\beta \omega\left(\frac{1}{n}\right)}\sum_{k=1}^{n} k^{r+\beta-1}E_k(f) + \frac{1}{\omega\left(\frac{1}{n}\right)}\sum_{k=n}^{\infty} k^{r-1}E_k(f) < \infty,$$

and, by (22) and (23), we have $f \in W^r H^\omega_\beta$. The proof is now complete.

Proof of Theorem 2.6. Let $f \in W^r H^\omega_\beta$. By the Jackson inequality and Lemma 2, we have

$$E_n(f) \ll \omega_{\beta+r}\left(f, \frac{1}{n}\right) \ll n^{-r}\omega\left(\frac{1}{n}\right).$$

Further, we use the following important result of Leindler [3, Theorem 8.2, p. 32 and (2.75), p.65]

$$\frac{1}{n}\sum_{\nu=n+1}^{2n}|S_n-f|^p \ll E_n^p(f), \qquad p>0,\ n\in\mathbf{N}.$$

Then this inequality implies

$$\sum_{\nu=1}^{\infty}\lambda_\nu\,|f(x)-S_\nu(x)|^p \ll \sum_{n=1}^{\infty}2^n\lambda_{2^n}E_{2^n}^p(f)$$

$$\ll \sum_{n=1}^{\infty}2^{n(1-rp)}\lambda_{2^n}\omega^p\left(\frac{1}{2^n}\right) \ll \sum_{n=1}^{\infty}\frac{\lambda_n\omega^p\left(\frac{1}{n}\right)}{n^{rp}} < \infty.$$

Thus, (14) implies (15). The proof is complete.

Proof of Theorem 2.7. Let $f\in W^rH_\beta^\omega\cap C^{sin}$. By Lemmas 2 and 3 and inequality (3), we get

$$na_n \ll \omega_{\beta+r}\Big(f,\frac{1}{n}\Big) \ll n^{-r}\omega\Big(\frac{1}{n}\Big).$$

This and (16) give

$$a_n^p \ll \frac{1}{n^{1+p}\lambda_n}. \tag{26}$$

Let us prove that (26) and $\{a_n\}\in GM$ imply $f\in S_p(\lambda)$.

First we note that for $x>0$ one has

$$\left|\sum_{k=n}^{\infty}a_k\sin kx\right| \ll \frac{1}{x}\left(a_n+\sum_{k=n+1}^{\infty}\frac{a_k}{k}\right). \tag{27}$$

Using Abel's transformation, (27) follows immediately from $\left|\widetilde{D}_k(x)\right| \equiv$ $\equiv \left|\sum_{n=1}^{k}\sin nx\right| = O\left(\frac{1}{x}\right)$ and inequality (4).

Let now $x>0$ and $N\in\mathbf{N}$ such that $\frac{\pi}{N+1}<x\le\frac{\pi}{N}$. Then

$$\sum_{\nu=1}^{\infty}\lambda_\nu\,|f(x)-S_\nu(x)|^p = \left(\sum_{\nu=1}^{N}+\sum_{\nu=N+1}^{\infty}\right)\lambda_\nu\,|f(x)-S_\nu(x)|^p =: I_1+I_2.$$

Using (27), we write

$$I_2 \ll N^p\sum_{\nu=N}^{\infty}\lambda_\nu a_\nu^p + N^p\sum_{\nu=N}^{\infty}\lambda_\nu\left(\sum_{k=\nu}^{\infty}\frac{a_k}{k}\right)^p =: I_{21}+I_{22}.$$

By (26), we have

$$I_{21} \ll N^p \sum_{\nu=N}^{\infty} \lambda_\nu \frac{1}{\nu^{1+p}\lambda_\nu} \ll C.$$

To estimate I_{22}, we note that if $\{\lambda_n\}$ satisfies $\triangle_2^2$-condition, then condition (17) is equivalent to the following condition

$$\sum_{k=1}^{n} \lambda_k \ll n\lambda_n. \tag{28}$$

If $p \geq 1$, then by Lemma 4(a) and condition (28)

$$I_{22} \ll N^p \sum_{k=N}^{\infty} \left(\frac{a_k}{k}\right)^p \lambda_k^{1-p} \left(\sum_{\nu=N}^{k} \lambda_\nu\right)^p \ll I_{21} \ll C.$$

If $0 < p < 1$, then we use Lemma 4(b):

$$I_{22} \ll N^p \sum_{k=N}^{\infty} k^{p-1} \left(\frac{a_k}{k}\right)^p \left(k\lambda_k + \sum_{\nu=N}^{k} \lambda_\nu\right) \ll I_{21} \ll C.$$

Now let us estimate I_1.

$$I_1 \leq \sum_{\nu=1}^{N} \lambda_\nu \left|\sum_{k=\nu+1}^{N-1} a_k \sin kx\right|^p + \sum_{\nu=1}^{N} \lambda_\nu \left|\sum_{k=N}^{\infty} a_k \sin kx\right|^p =: I_{11} + I_{12}.$$

By (27), we have

$$I_{12} \ll N^p \sum_{\nu=1}^{N} \lambda_\nu \left[a_N^p + \left(\sum_{k=N}^{\infty} \frac{a_k}{k}\right)^p\right].$$

Because of $\{n^{1-\varepsilon}\lambda_n\}$ is almost increasing, one can write

$$\left(\sum_{k=N}^{\infty} \frac{a_k}{k}\right)^p \ll \left(\sum_{k=N}^{\infty} \frac{1}{k^2(k\lambda_k)^{\frac{1}{p}}}\right)^p \ll \frac{1}{N^{1-\varepsilon}\lambda_N} \left(\sum_{k=N}^{\infty} \frac{1}{k^{2+\frac{\varepsilon}{p}}}\right)^p \ll \frac{1}{N^{1+p}\lambda_N}.$$

Hence, we get

$$I_{12} \ll N^{1+p}\lambda_N \left[a_N^p + \frac{1}{N^{1+p}\lambda_N}\right] \ll C.$$

To estimate I_{11}, we write

$$I_{11} \ll x^p \sum_{\nu=1}^{N} \lambda_\nu \left(\sum_{k=\nu}^{N-1} k a_k\right)^p.$$

Using inequality (28) and Lemma 4(a) for the case $p \geq 1$ and Lemma 4(b) for the case $0 < p < 1$, we have

$$I_{11} \ll N^{-p} \sum_{\nu=1}^{N} \nu^{2p} \lambda_\nu a_\nu^p \ll N^{-p} \sum_{\nu=1}^{N} \nu^{p-1} \ll C.$$

Thus, collecting estimates for I_1 and I_2, we obtain $f \in S_p(\lambda)$. The proof of Theorem 2.7 is complete.

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