MIXED HODGE STRUCTURES AND VECTOR BUNDLES ON THE PROJECTIVE PLANE I

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Abstract. We describe an equivalence of categories between the category of mixed Hodge structures and a category of vector bundles on the toric complex projective plane which verify some semistability condition. We then apply this correspondence to define an invariant which generalises the notion of $\mathbf{R}$-split mixed Hodge structure and compute extensions in the category of mixed Hodge structures in terms of extensions of the corresponding vector bundles. We also give a relative version of this correspondence and apply it to define stratifications of the bases of the variations of mixed Hodge structure.

Introduction

The purpose of this note is to give a geometric equivalent of the notion of mixed Hodge structure. To this end, following Simpson, we adopt the Rees' philosophy of associating a graded ring to a ring filtered by a chain of ideals. A mixed Hodge structure is roughly speaking the data of a vector space endowed with three ordered filtrations which are in a certain position called opposed. We define a functor, named the Rees functor, which converts each pair of filtrations into a graded module, and, next, we look at the associated coherent sheaf on the affine plane. It turns out that these sheaves are locally free and equivariant for the standard action by translation of $\mathbf{C}^* \times \mathbf{C}^*$. We can glue these local descriptions to form an equivariant locally free sheaf on the projective plane $\mathbf{P}^2_{\mathbf{C}}$. Starting with such a sheaf associated to a 3-filtered vector space, the action of the torus allows us to recover the filtrations on the same vector space. We thus get an equivalence of categories between filtered vector spaces and equivariant vector bundles. As in the classic Rees' construction, one deforms a filtered object into its corresponding graded object. Indeed, the fibre over the points of the dense open orbit of the torus action of a bundle associated to three filtration is the underlying vector

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space, whereas the fibres over the origins of the affine charts are naturally isomorphic to the bigraded objects associated to each pair of filtrations.

The idea of associating a vector bundle on the complex projective plane $\mathbf{P}^2_{\mathbf{C}}$ to the three filtrations which form a mixed Hodge structure has a a double origin: Simpson's construction [21],[22] of mixed twistor structures on the complex projective line $\mathbf{P}^1_{\mathbf{C}}$ associated to mixed Hodge structures, which involves the Hodge filtration and its conjugate, and Sabbah's construction [19] of Frobenius manifolds starting from families of vector bundles over $\mathbf{P}^1_{\mathbf{C}}$ in which such families are constructed using the weight and the Hodge filtrations of a variation of mixed Hodge structure. There are two motivations for handling the three filtration simultaneously. On the one hand, the relative position of the Hodge filtration and its conjugate plays an important role in the study of degenerations of mixed Hodge structure. On the other hand, the weight filtration contains all the extension data of a mixed Hodge structure.

Although they were firstly stated without reference to toric geometry in [16], the formalism derived by Perling in [17] after Klyachko [12],[13] for describing equivariant sheaves over toric varieties in terms of filtered vector spaces seems to be appropriate for describing the equivalences between categories of vector bundles and the categories involved in Hodge theory we are interested in.

The paper is organised in four parts. In section 1 we first define the Rees functor and its inverse for 2-filtered vector spaces and next introduce Perling's formalism [17], which, given a fan Δ, allows us to describe the category of $\Delta(1)$-families of complete filtrations in terms of equivariant locally free sheaves on the corresponding toric variety X_Δ. It turns out that the locally free sheaves on $\mathbf{P}^2_k$ which are associated to 3-filtered k-vector space whose ordered filtrations are opposed verify a semistability condition. They are $\mathbf{P}^1_0$-semistable, namely their restrictions to the divisor corresponding to the first of the three filtrations are of degree 0 and μ-semistable, or, equivalently, a direct sum of line bundles of the same slope 0. Notice that this notion of semistability is stronger than the μ-semistability. The principal consequence of this geometric characterisation is the fact that the category of vector spaces endowed with three opposed filtrations is abelian. Indeed, when one defines the cokernel of a morphism in the category of vector bundles of degree 0 verifying this semistability condition to be the reflexivization of the cokernel in the category of coherent sheaves, which corresponds to the cokernel in the category of 3-filtered vector spaces by the equivalence, one gets an abelian category [15].

The section 2 consists of an application of the previous section to Hodge theory. The filtered vector spaces involved here arise from Hodge theory.

We recover in a geometric way the fact that the category of mixed Hodge structures is abelian, which was proved by Deligne in [4] using linear algebra. After defining the $\mathbf{R}$-split level $\alpha(H)$ of a mixed Hodge structure H to be the second Chern class of its associated Rees bundle on the projective plane $\mathbf{P}^2_{\mathbf{C}}$, we give some properties this invariant verifies. In particular, it generalises the notion of $\mathbf{R}$-split mixed Hodge structure which is important when one studies degenerations of Hodge structure [3]. A mixed Hodge structure is $\mathbf{R}$-split if and only if $\alpha(H) = 0$. We then give some computations of the $\mathbf{R}$-split level of the mixed Hodge structures on the cohomology of possibly non complete and singular curves of genus $0, 1$. Finally we study extensions in the abelian category of mixed Hodge structures in terms of extensions of the corresponding equivariant sheaves.

A relative version of the Rees construction is proposed in section 3. Starting from a vector bundle on an algebraic variety S endowed with three filtrations by subbundles, one builds a sheaf on $\mathbf{P}^2_k \times S$ called the relative Rees sheaf. This sheaf is reflexive, S-flat and equivariant for the action of the torus in the direction of the projective plane. However, its fibres over points of S do not in general directly provide the Rees bundles corresponding to the expected 3-filtered vector spaces. Once again we have to consider the reflexivization of these coherent sheaves to recover the right object.

In section 4, we apply the relative Rees construction to vector bundles endowed with three filtrations provided by Hodge theory. After recalling the definition of a variation of mixed Hodge structure on a variety S and constructing the corresponding classifying space M [3], one would like to define a universal relative Rees sheaf on it. Since the conjugate of the Hodge filtration do not vary algebraically nor holomorphically, in order to perform the construction, we have to consider an open subvariety M^{opp} of the product $M \times M$ which parametrizes pairs of opposed filtrations. On this variety we have a canonical 3-filtered vector bundle formed by the weight filtration, which is constant since it is flat for the canonical connection, and two filtrations which come from the universal Hodge filtration on the first and second factor of $M \times M$. Once we have this 3-filtered vector bundle we can build the universal relative Rees sheaf $\xi_{M^{opp}}$ on $\mathbf{P}^2_{\mathbf{C}} \times M^{opp}$. A variation of mixed Hodge structure on S gives rise to a morphism from its universal covering to the classifying space $\varphi : \tilde{S} \to M$. One recovers the Rees bundle associated to the mixed Hodge structure over $s \in \tilde{S}$ by looking at the fibre over s of the pullback on $\mathbf{P}^2_{\mathbf{C}} \times \tilde{S}$ of the universal relative Rees sheaf $(id_{\mathbf{P}^2_{\mathbf{C}}} \times \varphi \times \overline{\varphi})^* \xi_{M^{opp}}$. This sheaf is a real algebraic, or real analytic, coherent sheaf of $\mathcal{C}^{\infty}_{\tilde{S}} \mathcal{O}_{\mathbf{P}^2}$-module whose fibres are coherent sheaves of $\mathcal{O}_{\mathbf{P}^2}$-modules by GAGA, and are equivariant, of degree 0 and $\mathbf{P}^1_0$-semistable. Finally, we show that the $\mathbf{R}$-split level is upper semi-continuous and hence defines a stratification of the base of the variation by real algebraic or real analytic closed subsets.

In this correspondence between variations of mixed Hodge structure and families of $\mathbf{P}^1_0$-semistable equivariant sheaves, two notions are missing to be able to understand degenerations of mixed Hodge structure in terms of the compactification of the related moduli space of semistable Rees coherent sheaves: an equivalent in the category of relative Rees sheaves of the connection underlying a variation of mixed Hodge structure and a translation of the polarizations.

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1. Rees construction and toric vector bundles

Let k be an algebraically closed field of characteristic zero. By an algebraic variety over k we understand a separated scheme of finite type over $\mathrm{Spec}\, k$.

1.1. Rees construction. The technical tool we will use to geometrize multi-filtrate vector spaces is the Rees construction. This construction allows us to associate a coherent sheaf on the n-dimensional affine space $\mathbf{A}^n_k$ to a vector space endowed with n filtrations.

Let V be a finite dimensional k-vector space endowed with n decreasing filtrations $F_1^\bullet, F_2^\bullet, ..., F_n^\bullet$. The object $(V, F_1^\bullet, F_2^\bullet, ..., F_n^\bullet)$ is called a *n-filtered* vector space. All the filtrations we will consider in this section are *complete*; a filtration $F^\bullet$ of a vector space V is said to be complete if there exists two integers m and n, $m \leq n$, such that $F^m = V$ and $F^n = \{0\}$.

A morphism between two n-filtered vector spaces

$$f : (V, F_1^\bullet, F_2^\bullet, ..., F_n^\bullet) \to (V', G_1^\bullet, G_2^\bullet, ..., G_n^\bullet)$$

is a morphism between the underlying vector spaces that is *compatible with the filtrations*, or *filtered*, that is, for any integers i and p, $f(F_i^p) \subset G_i^p$. We will denote by $\mathcal{C}_{n\, filtr}$ the category whose objects are n-filtered vector spaces and morphisms are filtered morphisms.

Consider the $k[u_1, u_2, ..., u_n]$-module $R(V, F_1^\bullet, F_2^\bullet, ..., F_n^\bullet)$ generated by the elements of the form

$$(\prod_{i\in[1,n]} u_i^{p_i}) \otimes v, \text{ where } v \in F_1^{p_1} \cap F_2^{p_2} \cap ... \cap F_n^{p_n}.$$

The module $R(V, F_1^\bullet, F_2^\bullet, ..., F_n^\bullet)$ is called the *Rees module* associated with the n-filtered vector space $(V, F_1^\bullet, F_2^\bullet, ..., F_n^\bullet)$. Since V is finite dimensional and the filtrations are complete, the Rees module is a finitely generated torsion free $k[u_1, u_2, ..., u_n]$-module.

We generalise the definition and the construction of Rees coherent sheaves given in [22] for one filtration to n-tuple of filtrations.

Definition 1.1. The *Rees coherent sheaf* $\xi(V, F_1^\bullet, F_2^\bullet, ..., F_n^\bullet)$ associated to a n-filtered vector space V is the coherent sheaf on the affine space $\mathbf{A}_k^n = \operatorname{Spec} k[u_1, u_2, ..., u_n]$ associated to the Rees $k[u_1, u_2, ..., u_n]$-module $R(V, F_1^\bullet, F_2^\bullet, ..., F_n^\bullet)$,

$$\xi(V, F_1^\bullet, F_2^\bullet, ..., F_n^\bullet) = \tilde{R}(V, F_1^\bullet, F_2^\bullet, ..., F_n^\bullet).$$

Let G be an algebraic group which acts on an algebraic variety X and let $\sigma : G \times X \to X$ denote the action and p_2 the projection on the second factor. A sheaf of $\mathcal{O}_X$-module $\mathcal{E}$ is said to be *equivariant* if there exists an isomorphism $\Phi : \sigma^*\mathcal{E} \cong p_2^*\mathcal{E}$ satisfying the cocycle condition $(\pi \otimes id_X)^*\Phi = p_{12}^* \circ (id_G \otimes \sigma)^*\Phi$, where π is the group multiplication and $p_{12} : G \times G \times X \to G \times X$ the projection on the two latest factors.

Consider the standard action on the affine space $\mathbf{A}_k^n$ of the torus $(\mathbf{G}_m)^n = \operatorname{Spec} B$ where $B = k[t_1^{\pm 1}, t_2^{\pm 1}, ..., t_n^{\pm 1}]$. The morphism $\sigma : (\mathbf{G}_m)^n \times \mathbf{A}_k^n \to \mathbf{A}_k^n$ corresponds to a structure of B-comodule on $k[u_1, u_2, ..., u_n]$ given by the morphism

$$\begin{aligned} \sigma^\sharp : k[u_1, u_2, ..., u_n] &\to B \otimes_k k[u_1, u_2, ..., u_n] \\ u_i &\mapsto t_i \otimes u_i. \end{aligned}$$

Then, this morphism induces a structure of B-comodule on the Rees module defined by the morphism

$$\begin{aligned} \sigma_R^\sharp : R(V, F_1^\bullet, F_2^\bullet, ..., F_n^\bullet) &\to B \otimes_k R(V, F_1^\bullet, F_2^\bullet, ..., F_n^\bullet) \\ (\textstyle\prod_{i \in [1,n]} u_i^{p_i}) \otimes v &\mapsto (\textstyle\prod_{i \in [1,n]} t_i^{p_i}) \otimes u_i^{p_i} \otimes v. \end{aligned}$$

This morphism endows the corresponding Rees coherent sheaf with a $(\mathbf{G}_m)^n$-equivariant structure.

Thus, the Rees construction yields a functor ϕ_R from the category of n-filtered finite dimensional vector spaces $\mathcal{C}_{n\,filtr}$ to the category of equivariant coherent sheaves on $\mathbf{A}_k^n$ for the standard action of the torus $(\mathbf{G}_m)^n$.

As we will see in the following sections the functor Φ_R has an inverse. We will explicit it here in a particular case we will be interested in for the applications. Let us focus on the case of coherent sheaves on $\mathbf{A}_k^2$ associated to vector spaces endowed with two filtrations. In that case, the associated coherent sheaves are locally free. Starting with a locally free sheaf $\mathcal{E}$ on $\mathbf{A}_k^2$ which is equivariant for the action of $(\mathbf{G}_m)^2$, then, we get two filtrations on the fibre $V = \mathcal{E}_{(1,1)}$ over $(1,1)$ in the following manner: all vector bundles on the plane being trivial, $\mathcal{E}$ corresponds to a free $k[u_1, u_2]$-module $R = \mathcal{E}(\mathbf{A}_k^2)$. The action of the group gives a trivialisation over $(\mathbf{G}_m)^2 \subset \mathbf{A}_k^2$ which yields an isomorphism of $k[u_1^{\pm 1}, u_2^{\pm 1}]$-modules

$$k[u_1^{\pm 1}, u_2^{\pm 1}] \otimes_{k[u_1,u_2]} R \cong k[u_1^{\pm 1}, u_2^{\pm 1}] \otimes_k V.$$

By taking the quotient by the ideal $(u_1 - 1, u_2 - 1)$ we show that V can be canonically identified with $R/(u_1 - 1, u_2 - 1)R$, which is the fibre $\mathcal{E}_{(1,1)}$. This gives

$$R \subset k[u_1^{\pm 1}, u_2^{\pm 1}] \otimes_k V.$$

We can define two complete decreasing filtrations by taking $F_1^p \cap F_2^q$ to be the subspace of vectors v such that $u_1^{-p} u_2^{-q} \otimes v \in R$. This construction defines a functor Φ_I from the category of equivariant vector bundles on the projective plane to $\mathcal{C}_{2filtr}$.

This construction is inverse to the Rees construction. Starting with a 2-filtered vector space $(V, F^\bullet, G^\bullet)$ and applying it to the associated Rees bundle $\xi(V, F^\bullet, G^\bullet)$, we recover the same filtrations on the same vector space. It can be seen by taking a basis adapted to both filtrations corresponding to a splitting of the two filtrations $V = \oplus_{p,q} V^{p,q}$ such that $F^p = \oplus_{p' \geq p, q'} V^{p',q'}$ and $G^q = \oplus_{p', q' \geq q} V^{p',q'}$. Hence the module in the above discussion is the $k[u_1, u_2]$-module $B = \oplus_{p,q} u_1^{-p} u_2^{-q} V^{p,q}$ and, applying the functor Φ_I, we recover the filtrations on the same vector space V.

Proposition 1.2. *([16], [22]) The Rees functor Φ_R and the inverse functor Φ_I establish an equivalence of categories between the category $\mathcal{C}_{2filtr}$ and the category of equivariant vector bundles on $\mathbf{A}_k^2$.*

Let $(V, F^\bullet) \in \mathcal{C}_{1filtr}$. We denote by $Gr_{F^\bullet} V = \oplus_p F^p / F^{p+1}$ the associated graded object whose p^{th} piece is $Gr_{F^\bullet}^p V = F^p / F^{p+1}$. If $(V, F^\bullet, G^\bullet) \in \mathcal{C}_{2filtr}$, then $G^\bullet$ induces a decreasing complete filtration on each graded piece of $Gr_{F^\bullet} V$ and gives rise to a bigraded object $Gr_{G^\bullet} Gr_{F^\bullet} V$. We refer to [4] for a background on graded objects. Note that, according to Zassenhaus' lemma, the objects $Gr_{F^\bullet}^p Gr_{G^\bullet}^q V$ and $Gr_{G^\bullet}^q Gr_{F^\bullet}^p V$ are canonically isomorphic.

We denote by $\mathcal{E}(s)$ the fibre of the coherent sheaf $\mathcal{E}$ over s. By taking the quotient of the corresponding Rees modules we get:

Lemma 1.3. *All the following isomorphisms are canonical.*

(i) *Let $(V, F^\bullet) \in \mathcal{C}_{1filtr}$. Then:*
   (a) $\xi(V, F^\bullet)(s) \cong V$ *if* $s \in \mathbf{A}_k^1 \setminus \{0\}$.
   (b) $\xi(V, F^\bullet)(0) \cong Gr_{F^\bullet} V$.

(ii) *Let $(V, F^\bullet, G^\bullet) \in \mathcal{C}_{2filtr}$. Then:*

(a) $\xi(V, F^\bullet, G^\bullet)|_{\mathbf{A}^1_k \times \{s\}} = f_s^* \xi(V, F^\bullet, G^\bullet) \cong \xi(V, F^\bullet)$ *as* $\mathbf{G}_m$*-equivariant locally free sheaves on the affine line, where* $s \neq 0$ *and* $f_s : \mathbf{A}^1_k \times \{s\} \to \mathbf{A}^2_k$ *is the inclusion morphism.*

(b) $\xi(V, F^\bullet, G^\bullet)((s,t)) \cong$
$$\cong \begin{cases} V \text{ if } (s,t) \in \mathbf{A}^2_k \setminus (\mathbf{A}^1 \times \{0\} \cup \{0\} \times \mathbf{A}^1), \\ Gr_{F^\bullet} V \text{ if } s = 0 \text{ and } t \neq 0, \\ Gr_{F^\bullet} Gr_{G^\bullet} V \cong Gr_{G^\bullet} Gr_{F^\bullet} V \text{ if } (s,t) = (0,0). \end{cases}$$

Rees sheaves could be considered as total spaces of deformations of filtered vector spaces into the associated graded vector spaces.

1.2. **Rees bundles and Toric bundle.** Let X be a toric variety, that is, a normal variety which contains an algebraic torus T as a dense open subset such that the torus multiplication extends to an action of the algebraic group T on X. We refer to [9] for a background on toric varieties. The variety X is defined by a fan Δ contained in the real vector space $N_{\mathbf{R}} = N \otimes_{\mathbf{Z}} \mathbf{R}$ associated with a lattice $N \cong \mathbf{Z}^n$ and will be denoted by X_Δ.

Let M be the lattice dual to N and $\langle , \rangle : M \times N \to \mathbf{Z}$ the canonical pairing. The elements of the abelian group M are denoted by m, m' if written additively and by $\chi(m), \chi(m')$ if written multiplicatively. M is the natural group of characters of the torus $\mathbf{T} = \mathrm{Hom}_{gr}(M, k^*)$.

A cone σ of the fan Δ is a convex rational polyhedral cone contained in $N_{\mathbf{R}}$. The cones will be denoted by Greek small letters σ, ρ, τ and etc., the order relation among cones is denoted by $<$. Let us recall the following standard notations (see [9], [17]):

(i) $\sigma(i) = \{\tau < \sigma | \dim \tau = i\}$, the elements of $\Delta(1)$are called the *rays*,
(ii) $n(\rho)$ the primitive lattice element spanning the ray $\rho \in \Delta(1)$,
(iii) the conedual to σ is defined by $\check{\sigma} = \{m \in M_{\mathbf{R}} | \langle m, n\rangle \geq 0 \text{ for all } n \in \sigma\}$,
(iv) $\sigma^\perp = \{m \in M_{\mathbf{R}} | \langle m, n\rangle = 0$ for all $n \in \sigma\}$ is the orthogonal cone to σ,
(v) $\sigma_M = \check{\sigma} \cap M$ is the subsemigroup of M associated with σ.

The category of σ-families. Let σ be a cone. We introduce the notion of σ-family following [17]. Let $\mathcal{E}$ be a quasicoherent sheaf on a toric variety X_Δ. The torus action on the $\mathbf{T}$-invariant affine open sets $U_\sigma = \mathrm{Spec}\, k[\sigma_M]$, $\sigma \in \Delta$, gives an isotypical decomposition of the $\mathcal{O}_{U_\sigma}$-modules of sections of $\mathcal{E}$ into $\mathbf{T}$-eigenspaces
$$\Gamma(U_\sigma, \mathcal{E}) = \bigoplus_{m \in M} \Gamma(U_\sigma, \mathcal{E})_m.$$

The module structure over $k[\sigma_M]$ induces, for each $m, m' \in M$, a map $\Gamma(U_\sigma, \mathcal{E})_m \to \Gamma(U_\sigma, \mathcal{E})_{m'}$ defined by $e \mapsto \chi(m' - m).e$ provided $m' - m \in$

σ_M. This induces a natural preorder on M associated with σ_M by setting $m \leq_\sigma m'$ if $m' - m \in \sigma_M$.

Since we want to work with decreasing filtrations, which is more convenient for the applications to Hodge theory, the notations and conventions we adopt here differ from the one used in [17]. We denote by E_σ the $k[\sigma_M]$-module $\Gamma(U_\sigma, \mathcal{E})$ and by F_σ^m the direct summand $\Gamma(U_\sigma, \mathcal{E})_{-m}$. Consider the preceding isotypical decomposition $E_\sigma = \bigoplus_{m \in M} F_\sigma^m$. Note that the torus $\mathbf{T}$ acts by multiplication by $\chi(-m)$ on F_σ^m and that there is a morphism $F_\sigma^m \to F_\sigma^{m'}$ given by the multiplication by $\chi(m-m')$ provided $m - m' \in \sigma_M$. To recover Perling's convention it suffices to consider the increasing filtration $F_\bullet$ associated to each decreasing filtration $F^\bullet$ by letting, for each integer p, $F_p = F^{-p}$. The family of vector spaces F_σ^m and characters $\chi(m)$ form a direct family of vector spaces associated with the preorder imposed by σ_M. Following [17], such a data is called a σ-family:

Definition 1.4. Let $\{F_\sigma^m\}_{m \in M}$ be a family of k-vector spaces. Suppose that for every m and m' such that $m' \leq_\sigma m$ we have a vector space homomorphism $\chi_\sigma^{m,m'} : F_\sigma^m \to F_\sigma^{m'}$ such that for all m, $\chi_\sigma^{m,m} = id$, and, for every triple (m, m', m'') such that $m'' \leq_\sigma m' \leq_\sigma m$ we have $\chi_\sigma^{m,m''} = \chi_\sigma^{m',m''} \circ \chi_\sigma^{m,m'}$. Such a data is a *$\sigma$-family.*

A σ-family is said to be *finite* if all the vector spaces F_σ^m are finite-dimensional, if for each ascending chain of characters ... $\leq_\sigma m_{i+1} \leq_\sigma m_i \leq_\sigma m_{-1} \leq_\sigma$ ... there exists an integer i_0 such that $F_\sigma^{m_i} = 0$ for each $i < i_0$ and finally if there is only a finite number of vector spaces F_σ^m such that the map $\oplus_{m \leq_\sigma m'} F_\sigma^{m'} \to F_\sigma^m$ defined by the summation of the $\chi_\sigma^{m',m}$ is not surjective.

Definition 1.5. Suppose given two σ-families $\{F_\sigma^m\}_{m \in M}$ and $\{G_\sigma^m\}_{m \in M}$ with respective vector space homomorphisms $\chi_\sigma^{m,m'}$ and $\gamma_\sigma^{m,m'}$. A *morphism of σ-family* ϕ_σ from the first to the second σ-family is a set of vector space homomorphisms $\{\phi_\sigma^m : F_\sigma^m \to G_\sigma^m\}_{m \in M}$ such that for all m, m' verifying $m' \leq_\sigma m$, $\phi_\sigma^{m'} \circ \chi_\sigma^{m,m'} = \gamma_\sigma^{m,m'} \circ \phi_\sigma^m$.

Consider a M-graded module E_σ with associated decomposition $E_\sigma = \oplus_{m \in M} F_\sigma^m$. We define $\chi_\sigma^{m,m'} : F_\sigma^m \to F_\sigma^{m'}$ to be the morphism induced by the multiplication by $\chi(m - m')$ in the structure of M-graded module. Then the family of vector spaces F_σ^m with the morphisms $\chi_{m,m'}$ yields a σ-family. Each graded morphism provides a morphism of σ-families by its decomposition into homogenous components.

Reciprocally, consider a σ-family and let $E_\sigma = \oplus_{m \in M} F_\sigma^m$. Define an M-graded structure by letting, for each $m \in \sigma_M$ and each $v \in F_\sigma^{m'}$, $\chi(m).v = \chi_\sigma^{m',m'+m}(v)$. Morphisms of σ-families give rise to graded morphisms.

We can state a first correspondence between families of vector spaces and equivariant sheaves:

Proposition 1.6. [17] *Let U_σ be a* $\mathbf{T}$*-invariant affine open scheme of* X_Δ*. The following categories are equivalent:*

(i) *equivariant quasicoherent sheaves over* U_σ*,*
(ii) M*-graded* $k[\sigma_M]$*-modules with morphisms of degree* 0*, and*
(iii) σ*-families.*

Moreover, finite σ*-families correspond to equivariant coherent sheaves.*

The category of Δ-families. Let Δ be a fan. Suppose given a σ-family for each $\sigma \in \Delta$. We obtain a system of quasicoherent sheaves $\mathcal{E}_\sigma$ over each U_σ. If certain conditions of compatibility between the σ-families are fulfilled, these sheaves glue together to form a quasicoherent sheaf $\mathcal{E}$ on X_Δ. These conditions are that for each pair $\tau \leq \sigma$, with associated inclusion $i_\sigma^\tau : U_\tau \hookrightarrow U_\sigma$, the τ-family associated with $\mathcal{E}_\tau$ and the τ-family associated with the pullback $i_\sigma^{\tau *}\mathcal{E}_\sigma$ are isomorphic and that for each triple $\rho \leq \tau \leq \sigma$ we have an equality $\eta_{\rho\sigma} = \eta_{\rho\tau} \circ i_\tau^{\rho *}\eta_\tau^\sigma$. Such families are called Δ-*families*. They are *finite* if all the underlying σ-families are finite.

The Δ-families we will consider are of the following form:

Definition 1.7. A $\Delta(1)$-*family of complete filtrations* is the data of a vector space V and, for each $\rho \in \Delta(1)$, a complete decreasing filtration $F_\rho^\bullet$, the vector space homomorphisms $\chi_\rho^{m,m'}$ being given by the inclusion $F_\rho^{m'} \hookrightarrow F_\rho^m$ for $m' \geq_\sigma m$. . Such a family will be denoted by $(V, \{F_\rho^\bullet\}_{\rho\in\Delta(1)})$.

A morphism between $\Delta(1)$-family of complete filtrations from $(V, \{F_\rho^\bullet\}_{\rho\in\Delta(1)})$ to $(V', \{G_\rho^\bullet\}_{\rho\in\Delta(1)})$ is a morphism between the underlying vector spaces that respects the filtrations.

This definition corresponds to the notion of vector space with full filtrations associated with each ray of $\Delta(1)$ in [17].

Note that a morphism between $\Delta(1)$-families of complete filtrations is simply a morphism of Δ-families between two $\Delta(1)$-families of complete filtrations.

Consider an equivariant reflexive sheaf $\mathcal{E}$ on X_Δ. Since it is normal, it is completely determined by its restriction to an open set whose complementary is at least 2-codimensional. So, if we let $\Delta' = \Delta(0) \cup \Delta(1)$, we have

$$\Gamma(X_\Delta, \mathcal{E}) = \Gamma(X_{\Delta'}, \mathcal{E}).$$

This implies $\Gamma(U_\sigma, \mathcal{E}) = \cap_{\rho\in\sigma(1)}\Gamma(U_\rho, \mathcal{E})$ on each affine toric variety U_σ, and hence, $F_\sigma^m = \cap_{\rho\in\sigma(1)} F_\rho^m$, where $F_\rho^m = F_\rho^{m'}$ if $m - m' \in \rho_M^\perp$. For each

$\rho \in \sigma(1)$ the stabiliser of the minimal orbit of U_ρ, whose group of characters in $M/\rho_M^\perp$ can be canonically identified with $\mathbf{Z}$ using the primitive lattice element $n(\rho)$, determines a complete filtration of $\Gamma(U_{\sigma(1)}, \mathcal{E})$. So, we get a $\Delta(1)$-family of complete filtrations of $\Gamma(X_{\Delta'}, \mathcal{E})$,

$$(\Gamma(X_{\Delta'}, \mathcal{E}), \{F_\rho^\bullet\}_{\rho\in\Delta(1)}).$$

The first condition on which has been established the correspondence between sets of filtrations and equivariant sheaves is Klyachko's splitting criterion in [12]. It describes equivariant locally free sheaves. In our context it amounts to the following: a $\Delta(1)$-family of complete filtrations verifies *Klyachko's condition* if for any $\sigma \in \Delta$ there exists a $\mathbf{T}$-eigenspace decomposition

$$V = \bigoplus_{m\in M} F_\sigma^m$$

such that for each $\rho \in \sigma(1)$,

$$F_\rho^i = \sum_{m, \langle m, n_\rho\rangle \geq i} F_\sigma^m.$$

We can state the correspondence between Δ-families and equivariant sheaves on X_Δ. Below, (iv) is the result of the original work of Klyachko [12], whereas the other statements come from Perling [17].

Theorem 1.8. [12],[17] *Let Δ be a fan.*

(i) *The category of Δ-families is equivalent to the category of quasicoherent equivariant sheaves over X_Δ.*
(ii) *A quasicoherent equivariant sheaf is coherent if and only if its associated Δ-family is finite.*
(iii) *The category of $\Delta(1)$-families of complete filtrations is equivalent to the category of equivariant reflexive sheaves on X_Δ.*
(iv) *The category of $\Delta(1)$-families of complete filtrations verifying Klyachko's compatibility condition is equivalent to the category of equivariant locally free sheaves on X_Δ.*

Proof. To prove (iii), it suffices to recall that the notion of $\Delta(1)$-families of complete filtrations corresponds to the notion of vector space with full filtrations associated with each ray of $\Delta(1)$ in [17]. Then, all follows from [17], Theorems 5.9, 5.19 and 5.22. □

Since the singularity set of a reflexive sheaf is at least 3-codimensional we get:

Corollary 1.9. *Let X_Δ be a toric surface associated with a fan Δ. The category of $\Delta(1)$-families of complete filtrations is equivalent to the category of equivariant locally free sheaves on X_Δ.*

Remark 1.10. According to the corollary, on a toric surface the notions of $\Delta(1)$-families of complete filtrations and of $\Delta(1)$-families of complete filtrations verifying Klyachko's condition coincide. It could be seen directly by remarking that on a toric surface X_Δ only two filtrations are involved on each invariant affine space U_σ, $\sigma \in \Delta(0)$. Since two filtrations can always be simultaneously split Klyachko's condition is automatically verified.

Δ-families and the Rees construction. Here we compare the Rees construction defined above to the correspondence stated in [17]. Let σ be a cone and consider a $\sigma(1)$-family of complete filtrations $(V, \{F_\rho^\bullet\}_{\rho\in\sigma(1)})$. We denote by E_σ the associated M-graded module. The M-graded module $k[\sigma_M]$ decomposes into $k[\sigma_M] = \oplus_{m\in M} k[\sigma_M]_m$. Consider the Rees module

$$R(V, \{F_\rho^\bullet\}_{\rho\in\sigma(1)}) = \textstyle\sum_{m\in M} k[\sigma_M]_{-m} \otimes F_\sigma^m \subset k[M] \otimes V.$$

We can define a structure of M-graded module on $R(V, \{F_\rho^\bullet\}_{\rho\in\sigma(1)})$ by letting, for each $v \in F_\sigma^m$,

$$\chi(-m).(\chi(-m') \otimes v) = \chi(-m-m')(\chi^{m',m'+m}(v)).$$

This gives directly:

Proposition 1.11. *With the above notation,* $E_\sigma \cong R(V, \{F_\rho^\bullet\}_{\rho\in\sigma(1)})$ *as* M*-graded modules.*

From now onwards we will exclusively consider $\Delta(1)$-families of complete filtrations which correspond to equivariant reflexive sheaves on a toric model of the projective plane.

1.3. **Δ-families on the projective plane and Rees bundles.** In this section we apply the correspondence stated in the previous section to equivariant coherent sheaves on the projective plane.

Consider the projective plane $\mathbf{P}_k^2 = \mathrm{Proj}\, k[u_0, u_1, u_2]$ given by the fan Δ in $N_{\mathbf{R}} = N \otimes_{\mathbf{Z}} \mathbf{R}$, where $N = \mathbf{Z}^2$, generated by the rays ρ_0, ρ_1 and ρ_2 defined by $n(\rho_1) = e_0 = (1, 0)$, $n(\rho_2) = e_1 = (0, 1)$ and $n(\rho_0) = e_2 = (-1, -1)$. σ_i is the convex cone defined by ρ_j and ρ_l, where $\{i, j, l\} = \{0, 1, 2\}$. Denote by $\mathbf{T}$ the torus acting on it. To each $\rho_i \in \Delta(1)$, $i \in \{0, 1, 2\}$, is associated a closed subvariety $\mathbf{P}_i^1 = V(\rho_i) = \mathrm{Proj}\, k[u_j, u_l], j < l, \{i, j, l\} = \{0, 1, 2\}$. For $i < j$, $P_{ij} = \mathbf{P}_i^1 \cap \mathbf{P}_j^1$ is a fixed point of the action.

Let us consider a finite dimensional k-vector space endowed with three complete and decreasing filtrations $(V, F_{\rho_0}^\bullet, F_{\rho_1}^\bullet, F_{\rho_2}^\bullet) \in \mathcal{C}_{3filtr}$. Such an object of $\mathcal{C}_{3filtr}$ yields a $\Delta(1)$-family of complete filtrations and reciprocally. The $\mathbf{T}$-equivariant coherent sheaf on $\mathbf{P}_k^2$ associated with this family is a reflexive sheaf on the projective plane and thus a locally free sheaf.

Reciprocally, to each $\mathbf{T}$-equivariant locally free sheaf on the projective plane is associated a $\Delta(1)$-family of complete filtrations, that is an object in $\mathcal{C}_{3filtr}$, $(V, F_{\rho_0}^\bullet, F_{\rho_1}^\bullet, F_{\rho_2}^\bullet)$. For each $i \in \{0, 1, 2\}$ consider the restriction $\mathcal{E}_i$

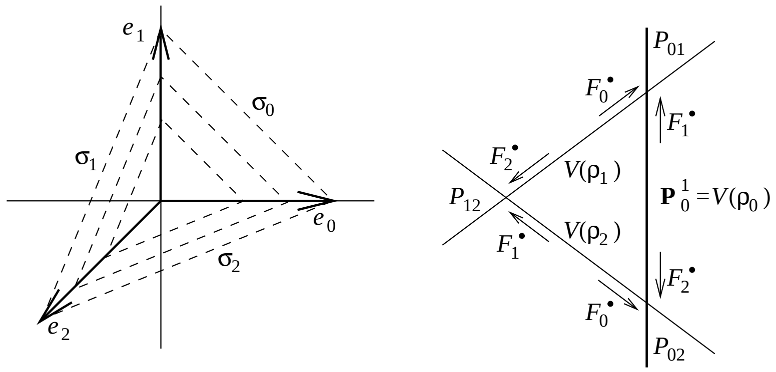

FIGURE 1

to the affine plane $\mathbf{A}_i^2 = U_{\sigma_i} = \operatorname{Spec} k[\sigma_{iM}]$. By Proposition 1.11 we have, for each integers j, l such that $j < l$ and $\{i, j, l\} = \{0, 1, 2\}$,

$$\mathcal{E}_i \cong \tilde{R}(F^\bullet_{\rho_j}, F^\bullet_{\rho_l}) = \xi(F^\bullet_{\rho_j}, F^\bullet_{\rho_l})$$

as $(\mathbf{G}_m)^2$-equivariant locally free sheaves. The equivariant vector bundle $\mathcal{E}$ is thus obtained by gluing the three Rees bundles on the affine sets associated to the three pairs of filtrations. This leads to the definition below and shows that the correspondences between filtered vector spaces and equivariant sheaves on toric varieties in [12], [17] agree with the one stated in [21], [16] in a Rees construction context.

Definition 1.12. The $\mathbf{T}$-equivariant vector bundle associated to the $\Delta(1)$-family
$(V, F^\bullet_{\rho_0}, F^\bullet_{\rho_1}, F^\bullet_{\rho_2}) \in \mathcal{C}_{3filtr}$ is denoted by $\xi(V, F^\bullet_{\rho_0}, F^\bullet_{\rho_1}, F^\bullet_{\rho_2})$ and called the *Rees bundle associated to* $(V, F^\bullet_{\rho_0}, F^\bullet_{\rho_1}, F^\bullet_{\rho_2})$.

According to theorem 1.8, we thus get:

Proposition 1.13. *Let Δ be the above fan which defines the projective space $\mathbf{P}^2_k$.*

The Rees construction and its inverse establish an equivalence of categories between the category of finite dimensional k-vector spaces endowed with three complete decreasing filtrations, namely the category of $\Delta(1)$-families of complete filtrations, $\mathcal{C}_{3filtr}$, and the category of $\mathbf{T}$-equivariant vector bundle on the toric variety $\mathbf{P}^2_k$

$$\mathcal{C}_{3filtr} \rightleftarrows Bun(\mathbf{P}^2_k/\mathbf{T})$$

To shorten the notations, from now onward $F^\bullet_{\rho_i}$, $i \in \{0, 1, 2\}$, will be denoted by $F^\bullet_i$.

Remark 1.14. By Lemma 1.3, the natural fibre of the equivariant vector bundle
$\xi(V, F_0^\bullet, F_1^\bullet, F_2^\bullet)$ over P_{ij} is $Gr_{F_i^\bullet} Gr_{F_j^\bullet} V \cong Gr_{F_j^\bullet} Gr_{F_i^\bullet} V$, whereas the natural fibre over each point of the dense orbit of the action is V. The bundle $\xi(V, F_0^\bullet, F_1^\bullet, F_2^\bullet)$ could be considered as a way to deform V into the different splittings associated to each pair of filtrations, wich are not compatible with each other in general.

Remark 1.15. Using Lemma 1.3, one can see directly that the Rees bundles on the affines planes can be glued as $\mathbf{T}$-equivariant vector bundles. The restriction to $\mathbf{A}_i^2 \cap \mathbf{A}_j^2 \cong \mathbf{G}_m \times \mathbf{A}^1$, $i \neq j$, of both Rees sheaves $\xi(V, F_j^\bullet, F_l^\bullet)$ and $\xi(V, F_i^\bullet, F_l^\bullet)$, where $\{i, j, l\} = \{0, 1, 2\}$, are indeed isomorphic as equivariant vector bundle to $f^*\xi(V, F_l^\bullet)$ where f is the projection $\mathbf{G}_m \times \mathbf{A}^1 \to \mathbf{A}^1$ onto the second factor (see the proof of Lemma 3.7 for the complete argument).

1.4. **Opposed filtrations and semistability.** The filtrations involved in Hodge theory are in specific relative positions. This leads to the following definition:

Definition 1.16. [4] Two filtrations $F_1^\bullet, F_2^\bullet$ on a vector space V are n-*opposed* if

$$Gr^p_{F_1^\bullet} Gr^q_{F_2^\bullet} V = 0 \text{ unless } p + q = n.$$

Three *ordered* filtrations $(F_0^\bullet, F_1^\bullet, F_2^\bullet)$ on V are *opposed* if

$$Gr^p_{F_1^\bullet} Gr^q_{F_2^\bullet} Gr^n_{F_0^\bullet} V = 0 \text{ unless } p + q + n = 0.$$

Recall that the objects $Gr^p_{F_1^\bullet} Gr^q_{F_1^\bullet} V$ and $Gr^q_{F_2^\bullet} Gr^p_{F_1^\bullet} V$ are canonically isomorphic. We emphasize the fact that a *triple of opposed filtrations is ordered*. Indeed, in $Gr^p_{F_1^\bullet} Gr^q_{F_2^\bullet} Gr^n_{F_0^\bullet} V$, $F_1^\bullet$ and $F_2^\bullet$ play a symmetric role but neither $F_0^\bullet$ and $F_1^\bullet$ nor $F_0^\bullet$ and $F_2^\bullet$ do.

In fact, three ordered filtrations $(F_0^\bullet, F_1^\bullet, F_2^\bullet)$ on V are *opposed* if and only if, for each integer r, $F_1^\bullet$ and $F_2^\bullet$ induce $-r$-opposed filtrations on $Gr^r_{F_0^\bullet} V$.

Let $\mathcal{E}$ be a coherent sheaf on a smooth projective variety. The slope of $\mathcal{E}$, if $\mathrm{rk}(\mathcal{E}) > 0$, is the ratio

$$\mu(\mathcal{E}) = \deg(\mathcal{E})/\mathrm{rk}(\mathcal{E})$$

and is defined to be $\mu(\mathcal{E}) = 0$ otherwise. A coherent sheaf $\mathcal{E}$ is μ-*semistable* if for every coherent subsheaf $\mathcal{F} \subset \mathcal{E}$ we have

$$\mu(\mathcal{F}) \leq \mu(\mathcal{E}).$$

We now introduce the notion of $\mathbf{P}_0^1$-semistability which is the geometric equivalent for the Rees bundles to the property to be opposed for the corresponding triples of filtrations. Let $j : \mathbf{P}_0^1 \hookrightarrow \mathbf{P}_k^2$ be the inclusion morphism.

Definition 1.17. A locally free sheaf $\mathcal{E}$ on $\mathbf{P}^2_k$ is $\mathbf{P}^1_0$-*semistable* if $j^*\mathcal{E}$, its restriction to the line $\mathbf{P}^1_0$, is μ-semistable as a locally free sheaf on the projective line.

According to a theorem of Grothendieck, every locally free sheaves on the projective line split into a sum of line bundles. A locally free sheaf on $\mathbf{P}^2_k$ is therefore $\mathbf{P}^1_0$-semistable if and only if its restriction to $\mathbf{P}^1_0$ is the direct sum of line bundles of the same slope.

Since j^* induces a monomorphism from $H^2(\mathbf{P}^2_k, \mathbf{Z})$ to $H^2(\mathbf{P}^1_0, \mathbf{Z})$ and the degree is functorial, the $\mathbf{P}^1_0$-semistability is a stronger notion than the μ-semistability: let $\mathcal{E}$ be a coherent sheaf on $\mathbf{P}^2_k$, we have

$$\mathcal{E} \text{ is } \mathbf{P}^1_0\text{-semistable} \Rightarrow \mathcal{E} \text{ is } \mu\text{-semistable.}$$

The calculation of the second Chern class we give below will be useful in the next section. Let $\omega \in H^2(\mathbf{P}^2_k, \mathbf{Z})$ be the cohomology class of a hyperplane.

Proposition 1.18. *Let $\xi(V, F_0^\bullet, F_1^\bullet, F_2^\bullet)$ be the Rees vector bundles on $\mathbf{P}^2_k$ associated with a trifiltered vector space whose filtrations are opposed, $(V, F_0^\bullet, F_1^\bullet, F_2^\bullet) \in \mathcal{C}_{3filtr,opp}$. Then,*

(i) *$\xi(V, F_0^\bullet, F_1^\bullet, F_2^\bullet)$ is $\mathbf{P}^1_0$-semistable,*
(ii) *$c_1(\xi(V, F_0^\bullet, F_1^\bullet, F_2^\bullet)) = 0$, and,*
(iii) *$c_2(\xi(V, F_0^\bullet, F_1^\bullet, F_2^\bullet)) = \frac{1}{2}\sum_{p,q}(h^{p,q} - s^{p,q})(p+q)^2\omega^2$, where $h^{p,q} = dim_k Gr^q_{F_2^\bullet} Gr^p_{F_1^\bullet} V$ and $s^{p,q} = dim_k Gr^q_{F_2^\bullet} Gr^p_{F_1^\bullet} Gr^{-p-q}_{F_0^\bullet} V$.*

Proof. (i) The restriction $\xi(V, F_0^\bullet, F_1^\bullet, F_2^\bullet)|_{\mathbf{P}^1_0}$ of the vector bundle to the divisor splits into a sum of line bundles. By Lemma 1.3, on each affine open set $\mathbf{A}^2_i$, $i \in \{1,2\}$ taking the restriction of the Rees sheaf $\xi(V, F_0^\bullet, F_i^\bullet)$ to the affine line $u_0 = 0$ amounts to make directly the Rees construction on the line with the filtered vector space $\oplus_r(Gr^r_{F_0^\bullet} V, F^\bullet_{i,r})$ (here $F^\bullet_{i,r}$ is the induced filtration on the graded piece $Gr^r_{F_0^\bullet} V$). We thus get $\xi(V, F_0^\bullet, F_1^\bullet, F_2^\bullet)|_{\mathbf{P}^1_0} \cong \oplus_r \xi(Gr^r_{F_0^\bullet} V, L^\bullet_r, F_1^\bullet, F_2^\bullet)|_{\mathbf{P}^1_0}$, where $L^\bullet_r$ is defined by $L^r_r = Gr^r_{F_0^\bullet} V$ and $L^{r+1}_r = \{0\}$. Fix r and take $v \in Gr^r_{F_0^\bullet} V$. Let p be the largest integer p such that $v \in F^p_{1,r}$. The three filtrations being opposed, $F^\bullet_{1,r}$ and $F^\bullet_{2,r}$ are $-r$-opposed on $Gr^r_{F_0^\bullet} V$, the largest integer q such that $v \in F^q_{2,r}$ is therefore $q = -r - p$. The vector v gives a section of $\xi(Gr^r_{F_0^\bullet} V, L^\bullet_r, F_1^\bullet, F_2^\bullet)$ of the form $(\frac{u_0}{u_1})^{-r}(\frac{u_2}{u_1})^{-p}.v$ over $\mathbf{A}^2_1$ and $(\frac{u_0}{u_2})^{-r}(\frac{u_1}{u_2})^{-q}.v$ over $\mathbf{A}^2_2$. Considering the vector bundle as a holomorphic vector bundle, one can take the limit of the glueing map of the two local sections as $u_0 \to 0$. We get 1. This shows that the restriction is a direct sum of trivial line bundles

$$\xi(V, F_0^\bullet, F_1^\bullet, F_2^\bullet)|_{\mathbf{P}^1_0} \cong \mathcal{O}^{\dim_k V}_{\mathbf{P}^1_0}$$

and proves (i) and (ii).

To prove (iii), we proceed in several steps in order to reduce the computation of the Chern classes to those of line bundles. Let $\pi : \widetilde{\mathbf{P}}^2_k \to \mathbf{P}^2_k$ be the blowing-up of the projective plane at P_{12} and E the exceptional curve. $\widetilde{\mathbf{P}}^2_k$ is the toric variety associated to the fan Δ_E obtained by adding a ray ρ_E to Δ in σ_0, ray whose generator is $n(\rho_E) = (1,1)$ in N. Denote by σ_0' and σ_0'' the two new 2-dimensional cones obtained. The divisor associated to ρ_E is the projective line E. Consider now the $\Delta_E(1)$-family of complete filtrations $(V, F_0^\bullet, F_1^\bullet, L^\bullet, F_2^\bullet)$, $L^\bullet$ being the filtration defined by $L^0 = V$ and $L^1 = \{0\}$. Let $\xi_E(V, F_0^\bullet, F_1^\bullet, L^\bullet, F_2^\bullet)$ be the corresponding locally free sheaf on the toric surface $\widetilde{\mathbf{P}}^2_k$.

We compare $\pi^*\xi(V, F_0^\bullet, F_1^\bullet, F_2^\bullet)$ to $\xi_E(V, F_0^\bullet, F_1^\bullet, L^\bullet, F_2^\bullet)$. By construction they coincide on $\widetilde{\mathbf{P}}^2_k \setminus E$. Since the morphism π is a morphism of toric varieties, according to Theorem 1.8, the equivariant locally free sheaf $\pi^*\xi(V, F_0^\bullet, F_1^\bullet, F_2^\bullet)$ corresponds to a $\Delta_E(1)$-family, which is of the form $(V, F_0^\bullet, F_1^\bullet, G^\bullet, F_2^\bullet)$ because the sheaves correspond on the complementary of E. When one explicits the Rees sheaves on $U_{\sigma_0'}$ and $U_{\sigma_0''}$ corresponding to the pullback of the restriction of $\xi(V, F_0^\bullet, F_1^\bullet, F_2^\bullet)$ to the chart $\mathbf{A}^2_0$ in $\mathbf{P}^2_k$, one gets $G^\bullet = F_1^\bullet \star F_2^\bullet$, where the convolution is defined by

$$G^r = \textstyle\sum_{p+q\geq r} F_1^p \cap F_2^q.$$

Suppose now that $F_1^\bullet$ and $F_2^\bullet$ are positive, that is that $F_1^0 = F_2^0 = V$; this can always be realised by shifting the indices of both $F_1^\bullet$ and $F_2^\bullet$ to get $F_1^0 = F_2^0 = V$ and, next, by shifting the indices of $F_0^\bullet$ in order to keep the filtrations opposed. $G^\bullet$ is now positive and there is therefore a filtered morphism from $(V, L^\bullet)$ to $(V, G^\bullet)$. This morphism induces an injective morphism of equivariant locally free sheaves

(1.1)

$$0 \longrightarrow \xi_E(V, F_0^\bullet, F_1^\bullet, L^\bullet, F_2^\bullet) \longrightarrow \pi^*\xi(V, F_0^\bullet, F_1^\bullet, F_2^\bullet) \longrightarrow \mathcal{T}_E \longrightarrow 0.$$

Let $V' = F_0^p$ for some p be a subvector space of V and $V'' = V/V'$ be the quotient. Keeping the notations for the filtrations induced on the subspace and on the quotient we have, on $\widetilde{\mathbf{P}}^2_k$,

$$\begin{gathered} 0 \to \xi_E(V', F_0^\bullet, F_1^\bullet, L^\bullet, F_2^\bullet) \to \xi_E(V, F_0^\bullet, F_1^\bullet, L^\bullet, F_2^\bullet) \to \\ \xi_E(V'', F_0^\bullet, F_1^\bullet, L^\bullet, F_2^\bullet) \to 0. \end{gathered}$$

Indeed, the sequence of Rees bundles associated to each pair of filtrations is exact on each invariant open sets of $\widetilde{\mathbf{P}}^2_k$, $\pi^{-1}(U_{\sigma_i})$, $i \in \{1,2,3\}$ and U_{σ_E}, and we can glue the local descriptions because only two filtrations are involved on each triple intersection. This yields $\mathrm{ch}(\xi_E(V, F_0^\bullet, F_1^\bullet, L^\bullet, F_2^\bullet)) = \sum_r \mathrm{ch}(\xi_E(Gr^r_{F_0^\bullet} V, F_0^\bullet, F_1^\bullet, L^\bullet, F_2^\bullet))$. Now, only two filtrations are involved in each bundle $\xi_E(Gr^r_{F_0^\bullet}V, F_0^\bullet, F_1^\bullet, L^\bullet, F_2^\bullet)$, we can therefore split it into a direct sum of line bundles. This gives

$\mathrm{ch}(\xi_E(V, F_0^\bullet, F_1^\bullet, L^\bullet, F_2^\bullet)) =$
$$\sum_{r,p,q} \dim_k(Gr^q_{F_2^\bullet}Gr^p_{F_1^\bullet}Gr^r_{F_0^\bullet}V)\,\mathrm{ch}(\xi_E(k, L[r]^\bullet, L[p]^\bullet, L^\bullet, L[q]^\bullet)),$$
where $L[i]^\bullet$ is the filtration $L^\bullet$ shifted by i, defined by $L[i]^k = L^{k-i}$. Denote by $\widetilde{D}_i$ and $\widetilde{D}_E$ the divisors on $\widetilde{\mathbf{P}}^2_k$ corresponding respectively to ρ_i, $i \in \{1,2,3\}$, and ρ_E. The Poincaré dual of a divisor D is denoted by η_D. Let $\widetilde{\omega} \in H^2(\widetilde{\mathbf{P}}^2_k, \mathbf{Z})$ be the cohomology class of a hyperplane. We have $\mathrm{c}_1(\xi_E(k, L[r]^\bullet, L[p]^\bullet, L^\bullet, L[q]^\bullet)) = r\eta_{\widetilde{D}_0} + p\eta_{\widetilde{D}_1} + q\eta_{\widetilde{D}_2}$ and so $\mathrm{ch}(\xi_E(k, L[r]^\bullet, L[p]^\bullet, L^\bullet, L[q]^\bullet) = 1+(r+p+q)\widetilde{\omega}+\frac{1}{2}(r^2+2rp+2rq)\widetilde{w}^2$, which gives, the filtrations being opposed, $\mathrm{ch}(\xi_E(V, F_0^\bullet, F_1^\bullet, L^\bullet, F_2^\bullet)) = -\frac{1}{2}\sum_{p,q} h^{p,q}(p+q)^2\widetilde{w}^2$.

Let us now compute the Chern character of $\mathcal{T}_E$. This coherent sheaf is supported on E and do not depend on $F_0^\bullet$. Thus, we can rewrite the exact sequence (1.1) using $F_1^\bullet$ (or $F_2^\bullet$) instead of $F_0^\bullet$

$$0 \to \xi_E(V, F_1^\bullet, F_1^\bullet, L^\bullet, F_2^\bullet) \to \pi^*\xi(V, F_1^\bullet, F_1^\bullet, F_2^\bullet) \to \mathcal{T}_E \to 0.$$

Both vector bundles $\xi_E(V, F_1^\bullet, F_1^\bullet, L^\bullet, F_2^\bullet)$ and $\pi^*\xi(V, F_1^\bullet, F_1^\bullet, F_2^\bullet)$ split into a sum of line bundles. The Chern character of the first is given by the above formula

$\mathrm{ch}(\xi_E(V, F_1^\bullet, F_1^\bullet, L^\bullet, F_2^\bullet)) =$
$$\dim_k V + \sum_{p,q} s^{p,q}\left((2p+q)\widetilde{\omega} + \tfrac{1}{2}(3p^2+2pq)\widetilde{w}^2\right).$$
For the second,

$$\begin{aligned} &\mathrm{ch}(\pi^*\xi(V, F_1^\bullet, F_1^\bullet, F_2^\bullet)) = \\ &= \pi^*\mathrm{ch}(\oplus_{p,q}\xi(V, L[p]^\bullet, L[p]^\bullet, L[q]^\bullet)^{s^{p,q}}) \\ &= \dim_k V + \sum_{p,q} s^{p,q}\left((2p+q)\widetilde{\omega} + \frac{1}{2}(4p^2+q^2+4pq)\widetilde{w}^2\right). \end{aligned}$$

This gives $\mathrm{ch}(\mathcal{T}_E) = \frac{1}{2}\sum_{p,q} s^{p,q}(p+q)^2\widetilde{w}^2$, which allows us to conclude since the Chern character is additive. □

Remark 1.19. The three filtration do not play the same role in the formula giving the Chern classes. The reason for this asymmetry is the one explained above: the condition for three filtrations to be opposed is not symmetric.

The following theorem gives the geometric property which characterises the equivariant vector bundles corresponding to opposed filtrations among the Rees bundles on the projective plane associated to triple of filtrations.

Theorem 1.20. *The Rees construction establishes an equivalence of categories between the category of finite dimensional* 3*-filtered* k*-vector spaces* *whose ordered filtrations are opposed and the category of* $\mathbf{T}$*-equivariant* $\mathbf{P}^1_0$*-semistable vector bundles of degree* 0 *on the projective plane:*

$$\mathcal{C}_{3filtr,opp} \rightleftarrows Bun_{\mathbf{P}^1_0-semistable,\mu=0}(\mathbf{P}^2_k/\mathbf{T}) \ .$$

Proof. According to the preceding proposition a Rees bundles associated to a vector space endowed with opposed filtrations is $\mathbf{P}^1_0$-semistable and has zero first Chern class.

Reciprocally, suppose given a $\mathbf{P}^1_0$-semistable degree 0 $\mathbf{T}$-equivariant vector bundle $\mathcal{E}$ on $\mathbf{P}^2_k$. Let $(V, F^\bullet_0, F^\bullet_1, F^\bullet_2)$ be the associated element in $\mathcal{C}_{3filtr}$. Suppose there exists a triple $(r_0, p_0, q_0) \in \mathbf{Z}^3$ such that $r_0 + p_0 + q_0 > 0$ and $Gr^{p_0}_{F^\bullet_1} Gr^{q_0}_{F^\bullet_2} Gr^{r_0}_{F^\bullet_0} V \neq \{0\}$. Then, there exists a one-dimensional subvector space $V' \subset V$ whose projection on $Gr^{p_0}_{F^\bullet_1} Gr^{q_0}_{F^\bullet_2} Gr^{r_0}_{F^\bullet_0} V$ is not zero. Let $(V', L^\bullet[r_0], L^\bullet[p_0], L^\bullet[q_0])$ be the trifiltered vector space whose filtrations are defined by $L^p[i] = V'$ if $p \leq i$ and $L^p[i] = 0$ otherwise. The monomorphism of Rees $k[u_0, u_1, u_2]$-modules induces an injective map of locally free sheaves

$$0 \longrightarrow \xi(V', L^\bullet(r_0), L^\bullet(p_0), L^\bullet(q_0)) \longrightarrow \mathcal{E} \ .$$

By the formula given in the proof of Proposition 1.18

$$c_1(\xi(V', L^\bullet(r_0), L^\bullet(p_0), L^\bullet(q_0))) = r_0 + p_0 + q_0 > 0,$$

which contradicts the μ-semistability and hence the $\mathbf{P}^1_0$-semistability of $\mathcal{E}$. Moreover,

$$\mathrm{c}_1(\mathcal{E}) = \textstyle\sum_{r,p,q} \dim_k (Gr^q_{F^\bullet_2} Gr^p_{F^\bullet_1} Gr^r_{F^\bullet_0} V)(r + p + q)\omega.$$

The preceding fact proves that there is no positive contribution to the first Chern class of $\mathcal{E}$ in this formula.

Every triple (r_0, p_0, q_0) such that $r_0+p_0+q_0 < 0$ and $Gr^{p_0}_{F^\bullet_1} Gr^{q_0}_{F^\bullet_2} Gr^{r_0}_{F^\bullet_0} V \neq 0$ will now give a negative contribution to the first Chern class which is zero; such a space does therefore not exist. This proves the theorem. □

1.5. **Semistable reflexive sheaves.** Let $\mathcal{R}efl_\mu(X)$ be the category whose objects are equivariant μ-semistable reflexive sheaves on an algebraic variety X. Let $\mathcal{F}$ be a coherent sheaf on X and consider the canonical morphism to its double dual $\nu : \mathcal{F} \to \mathcal{F}^{**}$. Recall from [10] that $\mathcal{F}^{**}$ is reflexive. It is called the *reflexive sheaf associated* to $\mathcal{F}$ and is canonically isomorphic to it when $\mathcal{F}$ is reflexive. The kernel, cokernel, image and coimage of a morphism in the category of reflexive sheaves are defined to be reflexive sheaves respectively associated with the kernel, cokernel, image and coimage

in the category of coherent sheaves. In fact, the kernel of a morphism $f : \mathcal{E} \to \mathcal{F}$ in $\mathcal{R}efl_\mu(X)$ is still reflexive. We get
(1.2)

$$\begin{tikzcd} 0 \arrow[r] & Ker(f) \cong Ker(f)^{**} \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{F} \arrow[r] \arrow[dr] & Coker(f) \arrow[r] \arrow[d, "\nu"] & 0 \\ & & & & Coker(f)^{**} \arrow[r] & 0, \end{tikzcd}$$

where the horizontal sequence is exact in the category of coherent sheaves and the other is exact in $\mathcal{R}efl_\mu(X)$. Suppose now X is endowed with the action of an algebraic group G and denote by $\mathcal{R}efl_\mu(X/G)$ the subcategory of $\mathcal{R}efl_\mu(X)$ whose objects and morphisms are G-equivariant.

This category is clearly additive. To prove that it is exact we use the fact that μ-semistability allows us to exhibit an isomorphism between the image and the coimage of a morphism in the complement of a subvariety of codimension at least 2. This provides an isomorphism since reflexive sheaves are normal.

Theorem 1.21. [15] *$\mathcal{R}efl_\mu(X/G)$ is an abelian category.*

Remark 1.22. Note that the abelian category $\mathcal{R}efl_\mu(X)$ *is not a sub-abelian category* of the abelian category of coherent sheaves, the cokernels in both categories do not agree.

Since reflexive sheaves on curves and surfaces are locally free, considering the action of the trivial group, we recover the classical result (see [18] for example):

Corollary 1.23. *The category of μ-semistable sheaves on a curve or a surface is abelian.*

Recall that an affine toric variety U_σ is nonsingular if and only if σ is nonsingular, which means that σ is generated by part of a basis for the lattice. A fan is said to be nonsingular if all of its cones are nonsingular.

Corollary 1.24. *Let Δ be a nonsingular fan. The category of $\Delta(1)$-families of complete filtrations is abelian.*

Let us turn back to consider the projective plane $\mathbf{P}^2_k$ defined above. One can show that when one imposes a stronger semistabily condition the category of semistable vector bundles of degree 0 on $\mathbf{P}^2_k$ is still abelian (For a proof of this fact see [15], Theorem 3.1).

Proposition 1.25. *The category of $\mathbf{T}$-equivariant $\mathbf{P}^1_0$-semistable vector bundles of degree 0 on the projective plane $Bun_{\mathbf{P}^1_0-semistable,\mu=0}(\mathbf{P}^2_k/\mathbf{T})$ is abelian.*

Thus, by Theorem 1.20, we recover in a geometric way the following result (see [4]):

Corollary 1.26. *The category $\mathcal{C}_{3filtr,opp}$ of finite dimensional trifiltered k-vector spaces whose ordered filtrations are opposed is abelian.*

1.6. **Real structures.** In this section we take $k = \mathbf{C}$. Consider the anti-holomorphic involution of $\mathbf{P}^2_{\mathbf{C}}$, $\tau : (u_0, u_1, u_2) \mapsto (\overline{u}_0, \overline{u}_2, \overline{u}_1)$. Let $\mathcal{E}$ be an $\mathcal{O}_{\mathbf{P}^2_{\mathbf{C}}}$-module. We define the sheaf $\tau^*(\mathcal{E})$ by letting, for each Zariski open set U,

$$\tau^*\mathcal{E}(U) = \mathcal{E}(\tau(U)).$$

It is canonically endowed with an $\mathcal{O}_{\mathbf{P}^2_{\mathbf{C}}}$-module structure by setting, for each $e \in \tau^*\mathcal{E}(U)$ and each $f \in \mathcal{O}_{\mathbf{P}^2_{\mathbf{C}}}(U)$,

$$f.e = \overline{\tau^*(f)}e.$$

Note that $\mathbf{P}^1_0$ is globally invariant by τ.

A *τ-equivariant* coherent sheaf on the complex projective plane is the data of a coherent sheaf $\mathcal{E}$ and of a morphism of $\mathcal{O}_{\mathbf{P}^2_{\mathbf{C}}}$-modules $f : \mathcal{E} \to \tau^*\mathcal{E}$ such that $\tau^*(f) \circ f = id_{\mathcal{E}}$.

A *$\mathbf{T}^\tau$-equivariant* coherent sheaf is a coherent sheaf that is both $\mathbf{T}$ and τ-equivariant. $\mathbf{T}^\tau$-equivariant sheaves are naturally associated with finite dimensional trifiltered complex vector spaces with underlying real structure whose both latest filtrations are conjugated one another, that is, for each integer p, $\overline{F^p_2} = F^p_1$. We denote by $\mathcal{C}_{3filtr,opp,\mathbf{R}}$ the category whose objects are of this form and whose morphisms are morphisms of real vector spaces which induce filtered morphisms when passing to the complex structure. There is a forgetful functor from $\mathcal{C}_{3filtr,opp,\mathbf{R}}$ to $\mathcal{C}_{3filtr,opp}$ which consists in forgetting the real structure.

Theorem 1.27. *The Rees construction establishes an equivalence of categories:*

$$\mathcal{C}_{3filtr,opp,\mathbf{R}} \rightleftarrows Bun_{\mathbf{P}^1_0-semistable,\mu=0}(\mathbf{P}^2_{\mathbf{C}}/\mathbf{T}^\tau)\ .$$

Moreover, the category $C_{3filtr,opp,\mathbf{R}}$ is abelian.

Proof. One immediately verifies that τ-equivariant objects correspond bijectively to triples of filtrations whose filtrations are conjugate. The fact that the category is abelian is a direct consequence of the fact that $Bun_{\mathbf{P}^1_0-\mathrm{semistable},\mu=0}(\mathbf{P}^2_{\mathbf{C}}/\mathbf{T})$ is and that the functor τ^* is exact since τ is an homeomorphism. $\square$

2. Application to Hodge theory

In this section we apply the correspondence stated before to filtered vector spaces arising from Hodge theory. We take $k = \mathbf{C}$. Let us recall some definitions and results about Hodge structures. Notice that we *do not* consider mixed Hodge structures defined on $\mathbf{Z}$ nor $\mathbf{Q}$.

Definition 2.1. A *pure* $\mathbf{R}$*-Hodge structure of weight* r is a triple $(H_{\mathbf{R}}, F^\bullet, \overline{F}^\bullet)$ consisting of a finite dimensional $\mathbf{R}$-vector space $H_{\mathbf{R}}$ and two decreasing filtrations of $H_{\mathbf{C}} = H_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$, the *Hodge filtration* $F^\bullet$ and the conjugate filtration with respect to the underlying real structure $\overline{F}^\bullet$, such that $F^\bullet$ and $\overline{F}^\bullet$ are r-opposed.

Definition 2.2. A $\mathbf{R}$*-mixed Hodge structure* is a quadruple $(H_{\mathbf{R}}, W_\bullet, F^\bullet, \overline{F}^\bullet)$ which consists of a finite dimensional $\mathbf{R}$-vector space $H_{\mathbf{R}}$, an increasing filtration of this real vector space $W_\bullet$, called the *weight filtration*, two filtrations $F^\bullet$ and $\overline{F}^\bullet$ of $H_{\mathbf{C}} = H_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$, conjugate to each other with respect to the underlying real structure, such that $F^\bullet$ and $\overline{F}^\bullet$ induce a pure Hodge structure of weight r on each quotient $Gr_r^W = W_r/W_{r-1}$, or, equivalently, such that the three ordered filtrations $(W_\bullet, F^\bullet, \overline{F}^\bullet)$ of $H_{\mathbf{C}}$ are opposed; here the weight filtration is viewed as a filtration of $H_{\mathbf{C}}$.

We will denote by $\mathbf{R}$-MHS the category whose objects are $\mathbf{R}$-mixed Hodge structures and morphisms are morphisms between vector spaces compatible with the filtrations.

In the same way we can define the category of complex mixed Hodge structures, denoted by $\mathbf{C}$-MHS, by only requiring the objects to consist of a finite dimensional complex vector space endowed with three ordered opposed filtrations, the first being increasing and the other decreasing. An element $H \in \mathbf{C}$-MHS corresponds to a quadruple $(H_{\mathbf{C}}, W_\bullet, F^\bullet, \hat{F}^\bullet)$.

When a definition or a result concerns both categories, we do not specify any of them.

Definition 2.3. The *length* of a mixed Hodge structure is the length of the largest interval $[a,b]$ such that $Gr_r^W \neq 0$ for each $r \in \{a,b\}$.

The *level* of a mixed Hodge structure is the length of the largest interval $[a,b]$ such that $Gr_p^F \neq 0$ for each $r \in \{a,b\}$.

A *Tate Hodge structure of weight* l, denoted by $T\langle l\rangle$, is the unique Hodge structure of rank 1 and of pure type (l,l).

Real mixed Hodge structures of length 0 are pure real Hodge structures and mixed Hodge structures of level 0 are extensions of Tate's Hodge structures.

A bigrading of a real mixed Hodge structure $H = (H_{\mathbf{R}}, W_\bullet, F^\bullet, \overline{F}^\bullet)$ is a direct sum decomposition $H_{\mathbf{C}} = \oplus_{p,q} V^{p,q}$ of the underlying vector space which verifies $W_r = \oplus_{j+l \leq r} V^{j,l}$ and $F^p = \oplus_{j \geq p, l} V^{j,l}$.

Following Deligne, one obtains an analogue of the Hodge decomposition for mixed Hodge structures:

Lemma 2.4. [4] *Let $H = (H_{\mathbf{R}}, W_\bullet, F^\bullet, \overline{F}^\bullet)$ be a mixed Hodge structure. Then, there exists a unique bigrading of H, denoted by $\{I^{p,q}\}_{p,q}$, such that*

$$I^{p,q} = \overline{I}^{q,p} \ mod \ \oplus_{j<p,l<q} I^{j,l}.$$

The following notion is important when one considers degeneration of mixed Hodge structures (see [3]).

Definition 2.5. A $\mathbf{R}$-mixed Hodge structure $H = (H_{\mathbf{R}}, W_\bullet, F^\bullet, \overline{F}^\bullet)$ is said to be $\mathbf{R}$-*split* if for each p, q the $I^{p,q}$ vector spaces verify $I^{q,p} = \overline{I}^{p,q}$.

In this case the $I^{p,q}$ spaces give a decomposition of $H_{\mathbf{C}}$ which is compatible with the three filtrations. By Lemma 2.4 every mixed Hodge structure whose length is lower than 2 is $\mathbf{R}$-split. In particular, every pure Hodge structure is $\mathbf{R}$-split.

2.1. **Vector bundles associated with mixed Hodge structures.** We associate to each mixed Hodge structure $H \in \mathbf{R}$-MHS (resp. $\mathbf{C}$-MHS) an object in $\mathcal{C}_{3filtr,opp,\mathbf{R}}$ (resp. $\mathcal{C}_{3filtr,opp}$). For this purpose, we take the *decreasing filtration associated with the weight filtration $W_\bullet$, denoted by $W^\bullet$*, by setting, for each $r \in \mathbf{Z}$, $W^r = W_r$.

Then, we associate to each object $H \in \mathcal{C}_{3filtr,opp,\mathbf{R}}$ (resp. $\mathcal{C}_{3filtr,opp}$) the corresponding Rees bundle on the toric complex projective plane as described in section (1.3), or, equivalently, the equivariant vector bundle corresponding to the $\Delta(1)$-family of complete filtrations that H represents.

Definition 2.6. Let H be a mixed Hodge structure in $\mathbf{R}$-MHS (resp. in $\mathbf{C}$-MHS). The *Rees bundle associated to H* is the Rees bundle associated with the trifiltered vector space $(H_{\mathbf{C}}, W^\bullet, F^\bullet, \overline{F}^\bullet)$ (resp. $(H_{\mathbf{C}}, W^\bullet, F^\bullet, \hat{F}^\bullet)$):

$$\xi_{\mathbf{P}^2_{\mathbf{C}}}(H) = \xi_{\mathbf{P}^2_{\mathbf{C}}}(H_{\mathbf{C}}, W^\bullet, F^\bullet, \overline{F}^\bullet) \ \text{(resp. } \xi_{\mathbf{P}^2_{\mathbf{C}}}(H_{\mathbf{C}}, W^\bullet, F^\bullet, \hat{F}^\bullet)).$$

Remark that to give a splitting of a mixed Hodge structure is equivalent to show that the $\Delta(1)$-family of complete filtrations verifies Klyachko's condition. As a direct outcome of Theorem 1.20 and Theorem 1.27 we get:

Theorem 2.7. *The category of complex mixed Hodge structures $\mathbf{C}$-MHS is equivalent to the category of $\mathbf{T}$-equivariant $\mathbf{P}^1_0$-semistable vector bundles of degree 0 on the projective plane*

$$\mathbf{C}\text{-MHS} \rightleftarrows Bun_{\mathbf{P}^1_0-semistable,\mu=0}(\mathbf{P}^2_{\mathbf{C}}/\mathbf{T}) \ .$$

The category of real mixed Hodge structures $\mathbf{R}$-MHS is equivalent to the category of $\mathbf{T}^\tau$-equivariant $\mathbf{P}^1_0$-semistable vector bundles of degree 0 on the projective plane

$$\mathbf{R}\text{-MHS} \rightleftarrows Bun_{\mathbf{P}^1_0-semistable,\mu=0}(\mathbf{P}^2_{\mathbf{C}}/\mathbf{T}^\tau) \ .$$

By Corollary 1.26, we thus recover Deligne's result in a geometric way:

Corollary 2.8. [4] *The category of real and complex mixed Hodge structures are abelian.*

Consider an exact sequence of mixed Hodge structures in $\mathbf{R}$-MHS

$$0 \longrightarrow A \xrightarrow{i} H \xrightarrow{\pi} B \longrightarrow 0 \ ,$$

and the associated exact sequence in $Bun_{\mathbf{P}^1_0-\text{semistable},\mu=0}(\mathbf{P}^2_{\mathbf{C}}/\mathbf{T}^{\tau})$

$$0 \longrightarrow \xi_{\mathbf{P}^2_{\mathbf{C}}}(A) \xrightarrow{i} \xi_{\mathbf{P}^2_{\mathbf{C}}}(H) \xrightarrow{\pi} \xi_{\mathbf{P}^2_{\mathbf{C}}}(B) \longrightarrow 0 \ .$$

By construction, $\xi_{\mathbf{P}^2_{\mathbf{C}}}(B)$ is the reflexive sheaf associated with the cokernel of i in the category of coherent sheaves, $Coker(i)$, whose singularity set is included in P_{12}. Let $\mathcal{T}_{P_{12}}$ be the cokernel of the canonical injective morphism $Coker(i) \to Coker(i)^{**} \cong \xi_{\mathbf{P}^2_{\mathbf{C}}}(B)$. In the category of coherent sheaves we thus have the exact sequence

$$0 \longrightarrow \xi_{\mathbf{P}^2_{\mathbf{C}}}(A) \xrightarrow{i} \xi_{\mathbf{P}^2_{\mathbf{C}}}(H) \xrightarrow{\tilde{\pi}} \xi_{\mathbf{P}^2_{\mathbf{C}}}(B) \longrightarrow \mathcal{T}_{P_{12}} \longrightarrow 0. \tag{2.1}$$

2.2. **New Hodge numbers and R-split level.** The fact that the Rees bundle associated to a cokernel in the category of mixed Hodge structures is not a cokernel in the category of coherent sheaves is due to the behaviour of some integers similar to the Hodge numbers, classically denoted by $h^{p,q}$ and given by $h^{p,q}_H = \dim_{\mathbf{C}} Gr^p_F Gr^q_{\overline{F}} Gr^{-p-q}_W H_{\mathbf{C}} = \dim_{\mathbf{C}} Gr^p_F Gr^{-p-q}_W H_{\mathbf{C}} = \dim_{\mathbf{C}} Gr^q_{\overline{F}} Gr^{-p-q}_W H_{\mathbf{C}}$. These integers, which we denote by $s^{p,q}$, measure in some sense the relative position of the filtrations but are not additive contrary to the Hodge numbers.

Definition 2.9. Let H be a mixed Hodge structure. We let

$$s^{p,q}_H = \dim_{\mathbf{C}} Gr^p_F Gr^q_{\overline{F}} H_{\mathbf{C}}.$$

We make the identification $H^4(\mathbf{P}^2_{\mathbf{C}}, \mathbf{Z}) = \mathbf{Z}$.

Definition 2.10. Let $H = (H_{\mathbf{R}}, W_{\bullet}, F^{\bullet}, \overline{F}^{\bullet})$ be a mixed Hodge structure. We define the $\mathbf{R}$-split level of H to be the integer

$$\alpha(H) = \mathrm{c}_2(\xi_{\mathbf{P}^2_{\mathbf{C}}}(H)).$$

Remark 2.11. The definition makes sense for $\mathbf{C}$-MHS too. The term *split level* would be, however, more appropriate for the objects of $\mathbf{C}$-MHS.

Since the Rees bundles associated with mixed Hodge structures are of degree 0, one easily verifies that for each mixed Hodge structures H, H' and each $k \in \mathbf{Z}$:

(i) $\alpha(H \oplus H') = \alpha(H) + \alpha(H')$.

(ii) $\alpha(H^*) = \alpha(H)$, where $H^* = \mathrm{Hom}_{\mathbf{R}-MHS}(H, T\langle 0\rangle)$.
(iii) $\alpha(H \otimes H') = \dim_{\mathbf{C}} H'_{\mathbf{C}}\, \alpha(H) + \dim_{\mathbf{C}} H_{\mathbf{C}}\, \alpha(H')$.
(iv) $\alpha(H \otimes T\langle k\rangle) = \alpha(H)$.

Next we give an explicit formula for the **R**-split level. It is a direct consequence of Proposition 1.18.

Proposition 2.12. *The* **R***-split level is expressed by*

$$\alpha(H) = \frac{1}{2} \sum_{p,q} (p+q)^2 (h_H^{p,q} - s_H^{p,q}). \tag{2.2}$$

We can define extensions in the abelian categories **R**-MHS and **C**-MHS (see [1] for example).

Theorem 2.13. *The* **R***-split level is sub-additive that is, for A and B two mixed Hodge structures in* **R***-MHS (resp.* **C***-MHS) and $H \in Ext^1_{\mathbf{R}-MHS}(A, B)$ (resp. $Ext^1_{\mathbf{C}-MHS}(A, B)$),*

$$\alpha(H) \geq \alpha(A) + \alpha(B).$$

Proof. The exact sequence given by the extension leads to the exact sequence (2.1) in the category of coherent sheaves. We thus have $c_2(\xi_{\mathbf{P}^2_{\mathbf{C}}}(H)) + c_2(\mathcal{T}_{P_{12}}) = c_2(\xi_{\mathbf{P}^2_{\mathbf{C}}}(A)) + c_2(\xi_{\mathbf{P}^2_{\mathbf{C}}}(B))$. Since the support of $\mathcal{T}_{P_{12}}$ is at least 2-codimensional, $c_2(\mathcal{T}_{P_{12}}) \leq 0$, which allows us to conclude. □

Since every mixed Hodge structure can be written as a successive extension of pure Hodge structures whose **R**-split level is 0, we have:

Corollary 2.14. *For each $H \in$* **R***-MHS, $\alpha(H) \geq 0$.*

In particular, it means that if $A \to H$ (resp. $H \to B$) is an injective (resp. surjective) morphism of mixed Hodge structures we have $\alpha(H) \geq \alpha(A)$ (resp. $\alpha(H) \geq \alpha(B)$).

Moreover, the **R**-split level generalises the notion of **R**-split mixed Hodge structure:

Proposition 2.15. *A mixed Hodge structure is* **R***-split if and only if its* **R***-split level is* 0*.*

Proof. We only have to prove the only if part. It is a consequence of Donaldson's theory on vector bundles. Consider a mixed Hodge structure whose **R**-split level is zero. The associated Rees bundle has vanishing Chern classes and hence, according to [7], is trivial. The action of the torus gives then a splitting compatible with all the filtrations.

□

2.3. **Examples of calculation of the R-split level.** According to Deligne (see [4], [5]), the cohomology groups of algebraic varieties, here separated schemes of finite type over $\mathbf{C}$, are endowed with natural and functorial mixed Hodge structures. Let X be such an algebraic variety. We will consider the $\mathbf{R}$-split levels of X,

$$\alpha_j(X) = \alpha((H^j(X, \mathbf{C}), W_\bullet, F^\bullet, \overline{F^\bullet})),$$

where $j \in \mathbf{Z}$, $\alpha_j(X)$ being associated to the mixed Hodge structure on the j^{th} group of cohomology. They are invariants of the variety depending on its algebraic (or analytic) class.

In this section we explicit the $\mathbf{R}$-split level of the mixed Hodge structures on the cohomology of some algebraic curves.

For certain varieties these invariants are trivial. The mixed Hodge structures of length lower than 2 are indeed $\mathbf{R}$-split. It is the case for pure Hodge structures and hence for the Hodge structures on the cohomology groups of smooth projective algebraic varieties or of compact Kähler varieties. The lengths of the mixed Hodge structures on the cohomology of weighted projective spaces (see [6]) and of varieties with logarithmic singularities (see [23]) are lower than 2, these mixed Hodge structures are therefore $\mathbf{R}$-split.

The construction of mixed Hodge structures is functorial. Let X be an algebraic variety, $\pi : X' \to X$ be a resolution of singularities, $j : X \to \overline{X}$ be a compactification and $S = \overline{X} \setminus X$. By [4], [5] we can find a compatible smooth compactification $\overline{j} : X' \to \overline{X}'$ and a morphism $\overline{\pi} : \overline{X}' \to \overline{X}$ making the square below cartesian:

$$\begin{tikzcd} X' \arrow[r, "\overline{j}"] \arrow[d, "\pi"] & \overline{X}' \arrow[d, "\overline{\pi}"] & \\ X \arrow[r, "j"] & \overline{X} & S. \arrow[l, "i"] \arrow[ul, "\overline{i}"'] \end{tikzcd}$$

By [5], Proposition (8.2.6), π induces an epimorphism of mixed Hodge structures on the cohomology, and j induces a monomorphism. We thus have, for each integer l,

$$\alpha_l(X) \geq \alpha_l(X') \text{ and } \alpha_l(X) \geq \alpha_l(\overline{X}).$$

Let us focus our attention on curves. The variety X' of the preceding construction is now the normalization of X. The mixed Hodge structures on 0^{th} and second cohomology groups are pure. Let us describe it on the first cohomology groups. This cohomology is given by the hypercohomology of the complex

$$[\mathcal{O}_{\overline{X}} \xrightarrow{d} \pi_* \Omega^1_{\overline{X}'}(\log S)].$$

The weight and Hodge filtrations on

$$H^1(X,\mathbf{C}) = \mathbf{H}^1([\mathcal{O}_{\overline{X}} \xrightarrow{d} \pi_*\Omega^1_{\overline{X}'}(\log S)])$$

are described by:

Lemma 2.16. *([5], Lemme (10.3.11))*

(i) $W^1(H^1(X,\mathbf{R})) = \mathrm{Im}(H^1(\overline{X},\mathbf{R}) \to H^1(X,\mathbf{R}))$ *and*
$W^0(H^1(X,\mathbf{R})) = \mathrm{Ker}(H^1(\overline{X},\mathbf{R}) \to H^1(\overline{X}',\mathbf{R}))$.

(ii) *The spectral sequence defined by the naive fitration of* $[\mathcal{O}_{\overline{X}} \xrightarrow{d} \pi_*\Omega^1_{\overline{X}'}(logS)]$ *degenerates into the Hodge filtration of* $H^*(X,\mathbf{C})$.

In order to compute the $\mathbf{R}$-split level of the mixed Hodge structures on the cohomology groups we will compute the *period matrix* using the preceding lemma and then look at the intersection between the sub-vector space this matrix determines and its conjugate with respect to the underlying real structure.

Curves of genus 0. Let us consider a first non trivial example: a non complete nodal curve of genus 0. Let (m_1, m_2, p_1, q_1) be four distinct points of $\mathbf{P}^1_{\mathbf{C}}$ and X be the non complete singular curve obtained by gluing p_1 and q_1 together and removing the m_i (for a justification of the identification of the points see [20], chapter IV, part 3). We have $S = \cup_i\{m_i\}$. Let u be a coordinate on $\mathbf{P}^1_{\mathbf{C}}\backslash\{\infty\}$.

Lemma 2.17. $F^1H^1(X,\mathbf{C}) =< \omega = (\frac{1}{u-m_1} - \frac{1}{u-m_2})du >$.

Since the mixed Hodge structure on the first cohomology group of such a curve is an extension of a Tate Hodge structure of weight 1 by a Tate Hodge structure of weight 0 (we have $h^{0,0} = h^{1,1} = 1$ and the other Hodge numbers are zero), the $\mathbf{R}$-split level is completely determined by the integer $s^{1,1}$. To know $s^{1,1}$ we have to compute the dimension of the intersection of the subspace F^1 with its conjugate with respect to the real structure on $H^1(X,\mathbf{R})$. The Hodge filtration induces a filtration on the dual of the first cohomology group using the isomorphism

$$H^1_{DR}(X,\mathbf{C}) \cong H^1_{Betti}(X,\mathbf{C}) \cong H_1(X,\mathbf{C})^* = (H_1(X,\mathbf{R})\otimes\mathbf{C})^*.$$

Let us choose a basis γ_0, γ_1 of $H_1(X,\mathbf{R})$. Let γ_0 be a positively oriented loop whose homology class is nonzero in X but vanish in $X \cup m_1$ and γ_1 be the loop formed by a path from p_1 to q_1 in X' by identifying these points.

We can now compute the coordinates of the Hodge filtration with respect to this basis:

$$\begin{aligned} < w, \gamma_0 > &= \int_{\gamma_0} w = 2\pi i, \text{ and,} \\ < w, \gamma_1 > &= \int_{\gamma_1} w = [\log(\frac{u-m_1}{u-m_2})]_{p_1}^{q_1} = \log(q_1, p_1, m_1, m_2), \end{aligned}$$

where (q_1, p_1, m_1, m_2) is the cross-ratio of the four points. Since the action of $PGL(1)$ on $\mathbf{P}^1_{\mathbf{C}}$ is transitive on the triples of points, we can always suppose that $m_1 = 0, m_2 = 1, p_1 = \infty$. X is hence completely determined by $q_1 \in \mathbf{C} \setminus \{0, 1\}$ and will be denoted by X_{q_1}. The intersection of F^1 with its conjugate is nonzero if and only if $\frac{1}{2\pi i}\log(\frac{q_1}{q_1-1})$ is real that is if and only if q_1 belongs to the real line $\mathcal{R}_{\frac{1}{2}}$ of complex numbers whose real part is $\frac{1}{2}$. So, when $q_1 \in \mathcal{R}_{\frac{1}{2}}$, we have $s^{1,1} = 1$. Otherwise $s^{1,1} = 0$. We thus obtain:

Proposition 2.18. *The* $\mathbf{R}$*-split level of the first cohomology group of the curve* X_{q_1} *is given by* $\begin{cases} \alpha_1(X_{q_1}) = 0 \text{ if } q_1 \in \mathcal{R}_{\frac{1}{2}}, \\ \alpha_1(X_{q_1}) = 1 \text{ otherwise.} \end{cases}$

Remark 2.19. When we consider a family of curves parametrized by $q_1 \in S$, where S is a variety (this data gives in fact a variation of mixed Hodge structure, see below, Example 4.3), the $\mathbf{R}$-split level jumps in real dimensions. It reflects the behaviour of the intersection between the vector spaces given by the family of Hodge filtrations, which are holomorphic, and the family of its conjugate filtrations, which are anti-holomorphic.

Curves of genus 1. Let $(m_1, ..., m_k, p_1, ..., p_l, q_1, ...q_l)$ be $k + 2l$ distinct points of a complete smooth curve of genus 1 and X be the curve obtained by gluing p_i and q_i for each $i \in [1, l]$ and removing the m_i's. The original smooth curve is isomorphic to the quotient of $\mathbf{C}$, endowed with the coordinate u, by the lattice $\mathbf{Z} + \tau\mathbf{Z}$, where $\tau \in \mathbf{Z}$ and $\mathrm{Im}(\tau) > 0$. To explicit the mixed Hodge structure on the first cohomology group we need the following result:

Proposition 2.20. [14] *The function* $\Psi : z \mapsto \sum_{i=1}^{i=m-1} \lambda_i \frac{d}{du} \log(\theta(u - a_i)) + C$*, where the* $\{\lambda_i\}_{i\in[1,m-1]}$ *and* C *are complex numbers such that* $\sum_{i=1}^{i=m-1} \lambda_i = 1$*, is* $\Lambda_{\mathbf{Z}}$*-periodic with simple poles at the points* $a_i + \frac{1}{2}(1+\tau)$ *and residus* λ_i*, where* θ *is the theta function on the elliptic curve given by* $\Lambda_{\mathbf{Z}}$*,* $\theta(u) = \sum_{n\in\mathbf{Z}} \exp(\pi\sqrt{-1}n^2u + 2\pi\sqrt{-1}n\tau)$*.*

Hence we can choose following generators for $F^1H^1(X, \mathbf{C})$ (notice that they all come from $F^1H^1(X', \mathbf{C})$): $\omega_0 = du$ and $\{\omega_i = d\log(\theta(u - m_i - \frac{1}{2}(1+\tau))/\theta(u - m_i - \frac{1}{2}(1+\tau)))\}_{i\in[1,k-1]}$. Let $\gamma_0, \gamma_1, ..., \gamma_k, \eta_1, ...\eta_l$ be generators of

$H_1(X,\mathbf{R})$ such that γ_0 is the Poincaré dual of ω_0, such that for each $j\in[1,k]$ γ_i is null-homologue in $X\cup m_i$, and, for each $j\in[1,l]$, η_j represents a loop obtained by identifying p_j with q_j. For $i\in[1,k-1]$, the integration of ω_i along η_j gives

$$<\omega_i,\eta_j>=\int_{\eta_j}\omega_i=\int_{p_j}^{q_j}\omega_i=\log\left(\frac{\theta(q_j-m_i-\frac{1}{2}(1+\tau))}{\theta(q_j-m_{i+1}-\frac{1}{2}(1+\tau))}/\frac{\theta(p_j-m_i-\frac{1}{2}(1+\tau))}{\theta(p_j-m_{i+1}-\frac{1}{2}(1+\tau))}\right).$$

We denote this number by $\log(\theta(m_i,m_{i+1},p_j,q_j))$. The $k\times(k+l+1)$-period matrix is thus given by:

$$M=\left(\begin{array}{cccccc|ccc} 1 & 0 & ... & ... & ... & 0 & 0 & ... & 0\\ \lambda_1 & 1 & -1 & 0 & ... & 0 & ... & ... & ...\\ ... & ... & ... & ... & ... & ... & ... & \log(\theta(m_i,m_{i+1},p_j,q_j)) & ...\\ \lambda_{k-1} & 0 & ... & 0 & 1 & -1 & ... & ... & ... \end{array}\right)$$

Proposition 2.21. *With the above notation, the* $\mathbf{R}$*-split level of the first cohomology group of* X *is given by the integer* $s^{1,1}=2k-rk_{\mathbf{C}}\begin{pmatrix} M\\ \overline{M}\end{pmatrix}$.

2.4. **Extension of mixed Hodge structures.** Let us consider extensions in the abelian category of mixed Hodge structures. We will only consider *separated* extensions of mixed Hodge structures that is congruence classes of extensions of the type

$$0\longrightarrow A\xrightarrow{i}H\xrightarrow{\pi}B\longrightarrow 0,$$

where the highest weight of A is less than than the lowest weight of B. For these extensions, it is shown in [1] that the abelian group $\mathrm{Ext}^1_{MHS}(B,A)$ is naturally isomorphic to a generalised torus, the 0^{th} Jacobian of the mixed Hodge structure $\mathrm{Hom}(B,A)$. When we forget the underlying integral structure we get the group

$$\mathrm{Ext}^1_{\mathbf{R}-MHS}(B,A)\cong\mathrm{Hom}_{\mathbf{R}-MHS}(B,A)_{\mathbf{C}}/F^0\mathrm{Hom}_{\mathbf{R}-MHS}(B,A)_{\mathbf{C}}.$$

Theorem 2.7 provides an isomorphism of groups

$$\mathrm{Ext}^1_{\mathbf{R}-MHS}(B,A)\cong\mathrm{Ext}^1_{Bun_{\mathbf{P}^1_0-\mathrm{sst},\mu=0}(\mathbf{P}^2_{\mathbf{C}}/\mathbf{T}^\tau)}(\xi_B,\xi_A).$$

By looking at the $\mathbf{R}$-split level of the extension, this isomorphism gives a stratification

$$\mathrm{Ext}^1_{\mathbf{R}-MHS}(B,A)=\bigsqcup_{\alpha(A)+\alpha(B)\leq\alpha\leq\alpha_{max}}\mathrm{Ext}^1_{\mathbf{R}-MHS}(B,A)_\alpha.$$

Note that α_{max} could be explicitely computed in combinatorial terms.

Extension of Tate's Hodge structures. Consider now extensions of Tate's Hodge structures $\mathrm{Ext}^1_{\mathbf{R}-MHS}(T\langle p\rangle,T\langle q\rangle)$ where $p>q$. The $\mathbf{R}$-split level of the mixed Hodge structure associated with an extension class could be else $\alpha=0$, for the class given by the split extension, or $\alpha=(p-q)^2$ for the other classes. In the latest case, the extension in the category

$Bun_{\mathbf{P}^1_0-\mathrm{sst},\mu=0}(\mathbf{P}^2_{\mathbf{C}}/\mathbf{T}^\tau)$ leads to the exact sequence in the category of coherent equivariant sheaves

$$0 \longrightarrow \xi_A \xrightarrow{i} \xi_H \xrightarrow{\pi} \xi_B \otimes \mathcal{I}_{P_{12}} \longrightarrow 0,$$

where ξ_A, ξ_B are trivial line bundles and $\mathcal{I}_{P_{12}}$ is the ideal sheaf corresponding to the zero-dimensional subscheme of length α, $[P_{12}]$. Let us compute $\mathrm{Ext}^1(\xi_B \otimes \mathcal{I}_{P_{12}}, \xi_A)$ in the category of coherent sheaves. The exact sequence for Ext groups associated with the $\mathcal{E}xt^q(\xi_B \otimes \mathcal{I}_{P_{12}}, \xi_A)$ has E_2 terms

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(\xi_B \otimes \mathcal{I}_{P_{12}}, \xi_A)) \Rightarrow \mathrm{Ext}^{p+q}(\xi_B \otimes \mathcal{I}_{P_{12}}, \xi_A)).$$

This leads to the exact sequence

$0 \to H^1({\xi_B}^{-1} \otimes \xi_A) \to \mathrm{Ext}^1(\xi_B \otimes \mathcal{I}_{P_{12}}, \xi_A) \to H^0(\mathcal{E}xt^1(\xi_B \otimes \mathcal{I}_{P_{12}}, \xi_A)) \to H^2({\xi_B}^{-1} \otimes \xi_A)$.

Since for a surface $\mathcal{E}xt^1(\mathcal{I}_{P_{12}}, \mathcal{O}_{\mathbf{P}^2_{\mathbf{C}}}) = \mathcal{O}_{[P_{12}]}$ and here $H^1({\xi_B}^{-1} \otimes \xi_A) = H^2({\xi_B}^{-1} \otimes \xi_A) = 0$, we have $\mathrm{Ext}^1(\xi_B \otimes \mathcal{I}_{P_{12}}, \xi_A) = H^0(\mathbf{P}^2_{\mathbf{C}}, \mathcal{O}_{[P_{12}]})$. Equivariant extension groups are given by the spectral sequence with E_2 terms

$$E_2^{p,q} = H^p(\mathbf{T}, \mathrm{Ext}^q(\mathcal{F}, \mathcal{G})) \Rightarrow \mathrm{Ext}^{p+q}_{\mathbf{T}}(\mathcal{F}, \mathcal{G}).$$

Since reductive groups do not have higher cohomology, we get

$$\mathrm{Ext}^n_{\mathbf{T}}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Ext}^n(\mathcal{F}, \mathcal{G})^{\mathbf{T}}.$$

Equivariant extensions thus correspond to the sections of $\mathcal{O}_{[P_{12}]}$ which are invariant by the action of the torus, that is, constant sections. According to [8], Theorem 8 p.37, an extension corresponding to an element η is free if and only if the section η generates the sheaf $H^0(\mathbf{P}^2_{\mathbf{C}}, \mathcal{O}_{[P_{12}]})$, namely, the natural map $\mathcal{O}_{\mathbf{P}^2_{\mathbf{C}}} \to \mathcal{O}_{[P_{12}]}$ is onto. Free $\mathbf{T}$-equivariant extensions are hence classified by $\mathbf{C}^*$ (the zero extension corresponding to the non free split extension).

Finally we have to look at the τ-invariant sections. P_{12} being invariant for τ they correspond to all the sections of $O_{[P_{12}]}$. Free $\mathbf{T}^\tau$-equivariant sections are classified by $\mathbf{C}^*$. We recover in this way the stratified decomposition of the first extension group

$$\mathrm{Ext}^1_{\mathbf{R}-MHS}(T\langle p\rangle, T\langle q\rangle) \cong \mathbf{C} = \{0\}_{\alpha=0} \cup \mathbf{C}^*_{\alpha=(p-q)^2}.$$

2.5. **Higher extensions.** Let us now consider the higher extension groups in the abelian category of real mixed Hodge structures (see [2] for definitions and settings). We compute these groups using, for each integer $n > 1$, the isomorphism

$$\mathrm{Ext}^n_{\mathbf{R}-MHS}(B, A) \cong \mathrm{Ext}^n_{Bun_{\mathbf{P}^1_0-\mathrm{sst},\mu=0}(\mathbf{P}^2_{\mathbf{C}}/\mathbf{T}^\tau)}(\xi_B, \xi_A).$$

Proposition 2.22. *Let A, B be two elements of $\mathbf{R}$-MHS. Then, for each $n > 1$,*

$$Ext^n_{\mathbf{R}-MHS}(B, A) = 0.$$

Proof. Let $U = \mathbf{P}^2_{\mathbf{C}} \backslash \{P_{12}\}$ and $i : U \to \mathbf{P}^2_{\mathbf{C}}$ be the inclusion morphism. We will show that the morphism it induces, $i^*: \mathrm{Ext}^n_{Bun_{\mathbf{P}^1_0-\mathrm{sst},\mu=0}(\mathbf{P}^2_{\mathbf{C}}/\mathbf{T}^\tau)}(\xi_B, \xi_A) \to \mathrm{Ext}^n_{Coh(U/\mathbf{T}^\tau)}(i^*\xi_B, i^*\xi_A)$, is injective. Let η, η' be two classes of n-extensions such that $i^*\eta = i^*\eta'$. We choose a representative for each class that we denote by $(H_\bullet, f)$ and $(H'_\bullet, g)$ for short, where $(H_\bullet, f)$ means that we have following exact sequence in $Bun_{\mathbf{P}^1_0-\mathrm{sst},\mu=0}(\mathbf{P}^2_{\mathbf{C}}/\mathbf{T}^\tau)$:

$$0 \longrightarrow \xi_A \xrightarrow{f_0} \xi_{H_1} \xrightarrow{f_1} \cdots \longrightarrow \xi_{H_n} \xrightarrow{f_n} \xi_B \longrightarrow 0 \ .$$

The congruence between $i^*\eta$ and $i^*\eta'$ is given by a morphism $\alpha_k : i^*\xi_{H_k} \to i^*\xi_{H'_k}$ for each integer $k \in [1, n]$ verifying $\alpha_{k+1} \circ i^* f_k = i^* g_k \circ \alpha_k$ for each $k \in [0, n]$, where $\alpha_0 = id_{\xi_A|_U}$ and $\alpha_{n+1} = id_{\xi_B|_U}$. All the morphisms α_k are morphism between holomorphic vector bundles over $\mathbf{P}^2_{\mathbf{C}} \backslash \{P_{12}\}$ and, P_{12} being 2-codimensional, can therefore be extended in an unique way to morphisms $\tilde{\alpha}_k$ on the whole space. We have to show that these extensions give a congruence in the category of bundles in which the cokernels have been modified. We therefore have to show the commutativity of each square. Suppose we have shown the squares are commutative up to level k. We can suppose f_k and g_k to be surjective; $\xi_{H_{k+1}}$ (resp. $\xi_{H'_{k+1}}$) is then the cokernel of f_{k-1} (resp. g_{k-1}) in $Bun_{\mathbf{P}^1_0-\mathrm{sst},\mu=0}(\mathbf{P}^2_{\mathbf{C}}/\mathbf{T}^\tau)$. We thus have a diagram

$$\begin{tikzcd}
\xi_{H_k} \arrow[r] \arrow[d, "\tilde{\alpha}_k"] & Coker(f_{k-1}) \arrow[r, "\nu"] \arrow[d, "\beta_k"] & \xi_{H_{k+1}} \arrow[d, dotted, "\tilde{\alpha}_{k+1}"] \\
\xi_{H'_k} \arrow[r] & Coker(g_{k-1}) \arrow[r, "\nu'"] & \xi_{H'_{k+1}},
\end{tikzcd}$$

in which the first square is commutative. Since $\tilde{\alpha}_k$ is the unique arrow making the second diagram commutative, the whole commutativity in $Bun_{\mathbf{P}^1_0-\mathrm{sst},\mu=0}(\mathbf{P}^2_{\mathbf{C}}/\mathbf{T}^\tau)$ is verified.

Let us now compute the extension groups in the category of coherent sheaves on U. We have, since $i^*\xi_B$ is locally free, $\mathrm{Ext}^n_{Coh(U)}(i^*\xi_B, i^*\xi_A) = H^n(U, i^*(\xi_B^* \otimes \xi_A))$. The long exact sequence of cohomology groups

$$\cdots \longrightarrow H^n_{P_{12}}(\mathbf{P}^2_{\mathbf{C}}, \xi_B^* \otimes \xi_A) \longrightarrow H^n(\mathbf{P}^2_{\mathbf{C}}, \xi_B^* \otimes \xi_A) \longrightarrow$$
$$\longrightarrow H^n(U, i^*(\xi_B^* \otimes \xi_A)) \longrightarrow \cdots \ ,$$

and the vanishing of the local cohomology groups give the isomorphism $H^n(U, i^*(\xi_B^* \otimes \xi_A)) \cong H^n(\mathbf{P}^2_{\mathbf{C}}, \xi_B^* \otimes \xi_A)$. Letting $C = B^* \otimes A$, where B^* is the mixed Hodge structure dual to B, we get $\xi_C \cong \xi_B^* \otimes \xi_A$. Let r be the lowest integer such that $W_r C \neq 0$. C can be written as an extension of $W_r C$ by $C/W_r C$ whose length is strictly lower than the length of C and

W_rC is a pure Hodge structure. In the category of coherent sheaves we have two exact sequences

$$0 \longrightarrow \xi_{W_rC} \xrightarrow{i} \xi_C \xrightarrow{\pi} \mathcal{C}oker(i) \longrightarrow 0,$$

and

$$0 \longrightarrow \mathcal{C}oker(i) \xrightarrow{\nu} \xi_{C/W_rC} \longrightarrow \mathcal{T} \longrightarrow 0\ ,$$

where the support of the sheaf $\mathcal{T}$ is included in P_{12}. Using the fact that ξ_{W_rC} is a direct sum of line bundles of degree zero, and hence $H^n(\mathbf{P}^2_{\mathbf{C}}, \xi_{W_rC}) = 0$ for $n > 0$, and the long exact sequence of cohomology groups associated with the exact sequence above we get, for each $n > 0$, $H^n(\mathbf{P}^2_{\mathbf{C}}, \xi_C) \cong H^n(\mathbf{P}^2_{\mathbf{C}}, \mathcal{C}oker(i))$. Since $H^n(\mathbf{P}^2_{\mathbf{C}}, \mathcal{T}) = 0$ for each $n > 0$, the long exact sequence of cohomology groups associated with the second exact sequence

$$\cdots \longrightarrow H^{n-1}(\mathbf{P}^2_{\mathbf{C}}, \mathcal{T}) \longrightarrow H^n(\mathbf{P}^2_{\mathbf{C}}, \mathcal{C}oker(i)) \longrightarrow H^n(\mathbf{P}^2_{\mathbf{C}}, \xi_{B/W_rB}) \longrightarrow \cdots$$

gives, for each $n > 1$, $H^n(\mathbf{P}^2_{\mathbf{C}}, \xi_C) \cong H^n(\mathbf{P}^2_{\mathbf{C}}, \mathcal{C}oker(i)) \cong H^n(\mathbf{P}^2_{\mathbf{C}}, \xi_{C/W_rC})$. We can iterate the decomposition until we have a pure Hodge structure, so, for each $n > 1$, $H^n(\mathbf{P}^2_{\mathbf{C}}, \xi_C) = 0$. We thus obtain

$$\begin{aligned} \mathrm{Ext}^n_{Coh(lU/\mathbf{T}^\tau)}(i^*\xi_B, i^*\xi_A) &\cong \mathrm{Ext}^n_{Coh(U)}(i^*\xi_B, i^*\xi_A)^{\mathbf{T}^\tau} \\ &\cong H^n(U, i^*(\xi_B^* \otimes \xi_A))^{\mathbf{T}^\tau} \\ &\cong H^n(\mathbf{P}^2_{\mathbf{C}}, \xi_B^* \otimes \xi_A)^{\mathbf{T}^\tau} \\ &\cong 0, \end{aligned}$$

which completes the proof. □

3. Relative Rees construction

Let k be an algebraically closed field of characteristic zero. In this section, S will be a smooth algebraic variety, namely a nonsingular scheme of finite type over k. By point we understand geometric point. Let $(\mathcal{V}, (\mathcal{F}_i^\bullet)_{i \in \{0,1,2\}})$ be a finite dimensional 3-filtered vector bundle formed by a vector bundle $\mathcal{V}$ on S and, for each $i \in \{0,1,2\}$, a decreasing, complete and finite filtration of $\mathcal{V}$ by subbundles of the form

$$\mathcal{V} = \mathcal{F}_i^{p_i} \supset \mathcal{F}_i^{p_i+1} \supset \ldots \supset \mathcal{F}_i^{p_i+k} \supset \mathcal{F}_i^{p_i+k+1} = \{0\}$$

for an integer p_i, where $k \geq 0$ and $\mathcal{F}_i^p \supset \mathcal{F}_i^{p+1}$ means that $\mathcal{F}_i^{p+1}$ is a subbundle of $\mathcal{F}_i^p$. In particular, for each point $s \in S$, we have a 3-filtered vector space $(\mathcal{V}(s), (\mathcal{F}_i^\bullet(s))_{i\in\{0,1,2\}})$; recall that for any coherent sheaf $\mathcal{E}$, $\mathcal{E}(s)$ denotes the k-vector space which is the fibre of $\mathcal{E}$ over s. We can apply the Rees construction to this 3-filtered vector space and get a Rees bundle on the projective plane

$$\xi(\mathcal{V}(s), \mathcal{F}_0^\bullet(s), \mathcal{F}_1^\bullet(s), \mathcal{F}_2^\bullet(s)).$$

In this section, we give a construction that encodes families of 3-filtered vector spaces on S by associating to them coherent sheaves on the product $\mathbf{P}_k^2 \times S$. It turns out that these sheaves are S-flat. We will next compare the fibres over the points of the base of the family S, which are only *a priori* equivariant coherent sheaves, with the Rees vector bundles obtained by making directly the construction with the 3-filtered vector space associated to each point.

As mentioned in the first section for the standard Rees construction, the relative Rees construction of a coherent sheaf on $\mathbf{P}_k^2 \times S$ can be effectuated by gluing the three descriptions obtained on the open $\mathbf{A}_i^2 \times S$, $i \in \{0,1,2\}$, by a construction involving the 2-filtered vector bundle on S, $(\mathcal{V}, \mathcal{F}_j^\bullet, \mathcal{F}_l^\bullet)$, where $\{i,j,l\} = \{0,1,2\}$ and $j < l$.

A relative Rees sheaf could be seen as the equivariant coherent sheaf associated to a relative Δ-family given by a 3-filtered vector bundle, where Δ is the fan giving the toric projective plane in the previous sections. We do not develop the formalism of relative Δ-families here.

3.1. **Relative Rees module.** Let us start with a finite dimensional 2-filtered vector bundle $(\mathcal{V}, \mathcal{F}^\bullet, \mathcal{G}^\bullet)$ on a smooth algebraic variety S. We will build a coherent sheaf on $\mathbf{A}^2 \times S$ associated to this data. We can first suppose S to be affine, $S = \operatorname{Spec} A$.

For each pair of integers (p,q), we define $\mathcal{F}^p \cap \mathcal{G}^q$ to be the kernel of the morphism of coherent sheaves

$$\mathcal{F}^p \to \mathcal{V}/\mathcal{G}^q \tag{3.1}$$

obtained by the composition of the injective morphism $\mathcal{F}^p \to \mathcal{V}$ with the surjective morphism $\mathcal{V} \to \mathcal{V}/\mathcal{G}^q$. Let us denote by $M^{p,q}$ the an A-module corresponding to the coherent sheaf $\mathcal{F}^p \cap \mathcal{G}^q$; we have $\mathcal{F}^p \cap \mathcal{G}^q \cong \tilde{M}^{p,q}$. Since the sheaf $\mathcal{F}^p \cap \mathcal{G}^q$ is torsion free, being a subsheaf of a torsion free sheaf, the A-module $M^{p,q}$ is torsion free.

For each pair of integers (p,q), the function from S to the ring of non-negative integers $\mathbf{Z}_+$,

$$s \mapsto \dim_k(\mathcal{F}^p(s) \cap \mathcal{G}^q(s)),$$

is lower semi-continuous. We denote by $U^{p,q}$ the open set on which it takes its generic value, and by $V = \cap_{p,q} U^{p,q}$ the intersection of these sets. Since the intersection is finite, V is open.

Definition 3.1. The *Rees A-module* associated to the 2-filtered vector bundle $(\mathcal{V}, \mathcal{F}^\bullet, \mathcal{G}^\bullet)$ on $S = \operatorname{Spec} A$ is the torsion free $A[u,v]$-module $R_S(\mathcal{V}, \mathcal{F}^\bullet, \mathcal{G}^\bullet)$ generated by the elements of the form $u^{-p}v^{-q} \otimes_k m^{p,q}$, for $m^{p,q} \in M^{p,q}$.

The associated torsion free coherent sheaf on $\mathbf{A}^2 \times S$ is called the *relative Rees sheaf* associated with the 2-filtered vector bundle $(\mathcal{V}, \mathcal{F}^\bullet, \mathcal{G}^\bullet)$ and is denoted by $\xi_S(\mathcal{V}, \mathcal{F}^\bullet, \mathcal{G}^\bullet)$.

Note that the definition coincides with the definition of the Rees sheaf on the affine plane associated with a pair of filtrations when $S = \mathrm{Spec}(k)$ is reduced to a point.

The relative Rees sheaf $\xi_S(\mathcal{V}, \mathcal{F}^\bullet, \mathcal{G}^\bullet)$ is naturally equivariant for the action of the torus $(\mathbf{G}_m)^2$ on $\mathbf{A}^2 \times S$ which reduces to the standard action on the first factor and is trivial on the second. In terms of comodule this action can be described as follows: if we let $(\mathbf{G}_m)^2 = \mathrm{Spec}\, B$, where $B = k[u^{\pm 1}, v^{\pm 1}]$, the structure of B-comodule on $A[u, v]$ is deduced by extension of scalars from the B-comodule structure on $k[u, v]$ made explicite in the first section. It is clear that, similarly to the Rees module case, it induces a B-comodule structure on the relative Rees module $R_S(\mathcal{V}, \mathcal{F}^\bullet, \mathcal{G}^\bullet)$ and hence an action of the torus on the corresponding coherent sheaf.

Proposition 3.2. *The coherent sheaf of $\mathcal{O}_{\mathbf{A}^2 \times S}$-modules $\xi_S(\mathcal{V}, \mathcal{F}^\bullet, \mathcal{G}^\bullet)$ is reflexive and, moreover, its at least 3-codimensional singularity locus is an at least 1-codimensional subset of the 2-codimensional variety $\{(0,0)\} \times S$.*

To prove this statement, we first need to define the saturation of a module.

Definition 3.3. Let A be a regular local ring, L a free A-module and M a torsion free submodule of L. The *saturation* of M in L with respect to the ideal $\mathfrak{a}$ of A is defined by

$$M_{\mathfrak{a}}^{sat} = \{l \in L | \mathfrak{a}^j . l \subset M \text{ for } j \gg 0\}.$$

Lemma 3.4. *Let A be a regular local ring with maximal ideal $\mathfrak{m}$, and M be a torsion free finitely generated A-module. Let $M_{\mathfrak{m}}^{sat}$ be its saturated in a free A-module with respect to $\mathfrak{m}$, then,*

$$M_{\mathfrak{m}}^{sat} = M \Rightarrow \mathrm{depth}_{\mathfrak{m}} M \geq 2.$$

Proof. Consider the short exact sequence associated to the embedding of M in a free module L. The associated long exact sequence of local cohomology at $\mathfrak{m}$ and the fact that L is free, and hence $H^i_{\mathfrak{m}}(L)$ for $i \geq 0$, implies that $H^i_{\mathfrak{m}}(M) \cong H^{i-1}_{\mathfrak{m}}(L/M)$ for all $i \geq 1$. Since $M^{sat}/M = H^0_{\mathfrak{m}}(L/M)$, we can deduce the result from the equivalence $(\mathrm{depth}_{\mathfrak{m}}(M) \geq j) \Leftrightarrow (\forall i < j, H^i_{\mathfrak{m}}(M) = 0)$ (see [11], III, Ex.3.4). □

Proof. (of Proposition 3.2) Recall that a coherent sheaf is reflexive if and only if it is torsion free and normal (see [10]). Since the $A[u, v]$-module $R_S(\mathcal{V}, \mathcal{F}^\bullet, \mathcal{G}^\bullet)$ is torsion free, according to Lemma 3.4, it will be sufficient to prove that its local rings are saturated. Let us consider the maximal ideal $\mathfrak{m} = \mathfrak{a}.(u,v)$. The Rees module is a submodule of the free module $L = A[u^{\pm 1}, v^{\pm 1}] \otimes V$. Let $l = u^p v^q \otimes b \in R_S(\mathcal{V}, \mathcal{F}^\bullet, \mathcal{G}^\bullet)^{sat}_{\mathfrak{m}}$ be a generator. There is an integer n such that $\mathfrak{m}^j.l \in R_S(\mathcal{V}, \mathcal{F}^\bullet, \mathcal{G}^\bullet)$ for all $j \geq n$. In particular, this means that $u^n.l = u^{n+p}v^q \otimes b \in R_S(\mathcal{V}, \mathcal{F}^\bullet, \mathcal{G}^\bullet)$ and hence $b \in F^{p+n} \cap G^q$. In the same way there exists an integer n' such that $b \in F^p \cap G^{q+n'}$. So, $b \in F^p \cap G^q$, and thus $l \in R_S(\mathcal{V}, \mathcal{F}^\bullet, \mathcal{G}^\bullet)$, which proves the first statement.

Remark now that the localisation of the Rees module at a geometric point that do not belong to $\{(0,0)\} \times S$ is isomorphic to the localisation of a Rees module defined by only one of the filtration, the other being trivial. The proof of this fact is similar to the proof of Lemma 1.3 (ii)(a): let $s \in \mathbf{A}^1 \setminus \{0\}$, let $\mathfrak{p}$ be the prime ideal corresponding to the closed subset $\{s\} \times \mathbf{A}^1$ of $\mathbf{A}^2$ and $f_s : \{s\} \times \mathbf{A}^1 \times S \to \mathbf{A}^2 \times S$ the morphism it determines. Note that, for each integer q, $\mathcal{G}^q/\mathcal{G}^{q+1}$ is a vector bundle on S. Then,

$$\xi_S(\mathcal{V}, \mathcal{F}^\bullet, \mathcal{G}^\bullet)|_{\{s\}\times\mathbf{A}^1\times S} = f_s^*\xi_S(\mathcal{V}, \mathcal{F}^\bullet, \mathcal{G}^\bullet) = \tilde{R},$$

where R is an $A[v]$-module given by

$$R = R_S(\mathcal{V}, \mathcal{F}^\bullet, \mathcal{G}^\bullet) \otimes_{A[u,v]} (A[u,v]/\mathfrak{p}.A[u,v]) \cong R_S(\mathcal{V}, \mathcal{G}^\bullet) = \\ \oplus_q (v^{-q} \otimes_k k^{\mathrm{rk}(\mathcal{G}^q/\mathcal{G}^{q+1})}).$$

This proves that $\xi_S(\mathcal{V}, \mathcal{F}^\bullet, \mathcal{G}^\bullet)|_{\{s\}\times\mathbf{A}^1\times S}$ is locally free. The restriction of the Rees sheaf to $(\mathbf{A}^2 \setminus \{(0,0)\}) \times S$ is thus locally free, which completes the proof. □

3.2. **Localisation.** Let $(\mathcal{V}, \mathcal{F}^\bullet, \mathcal{G}^\bullet)$ be a 2-filtered vector bundle on a smooth affine algebraic variety $S = \operatorname{Spec} A$ and $s \in S$ be a point whose corresponding maximal ideal is $\mathfrak{p}$. We let $X = \mathbf{A}^2 \times S$ and denote by $f : X \to S$ the projection onto the second factor. To each $s \in S$ is associated a morphism $f_s : X_s \to X$, where $X_s = \operatorname{Spec}(k(s)) \times_S X$.

We want to compare the fibre over s of the family of coherent sheaves given by the relative Rees construction, that is the equivariant coherent sheaf on $\mathbf{A}^2$, $\xi_S(\mathcal{V}, \mathcal{F}^\bullet, \mathcal{G}^\bullet)|_{X_s} = f_s^*\xi_S(\mathcal{V}, \mathcal{F}^\bullet, \mathcal{G}^\bullet)$, with the "expected" fibre, that is, the Rees equivariant locally free sheaf associated with the 2-filtered vector space obtained by localising at s the 2-filtered vector bundle, $\xi(\mathcal{V}(s), \mathcal{F}^\bullet(s), \mathcal{G}^\bullet(s))$.

To shorten the notations, we will denote by $R(\mathfrak{p}) = R(\mathcal{V}(s), \mathcal{F}^\bullet(s), \mathcal{G}^\bullet(s))$ the $k[u,v]$-module corresponding to locally free sheaf on $\mathbf{A}^2$, $\xi(\mathcal{V}(s), \mathcal{F}^\bullet(s), \mathcal{G}^\bullet(s))$, and by $R_S(\mathfrak{p})$ the $k[u,v]$-module $\Gamma(f_s^*\xi_S(\mathcal{V}, \mathcal{F}^\bullet, \mathcal{G}^\bullet))$ corresponding to the coherent sheaf $f_s^*\xi_S(\mathcal{V}, \mathcal{F}^\bullet, \mathcal{G}^\bullet)$.

Lemma 3.5. *With the above notations, there exists a short exact sequence of $k[u,v]$-modules,*

$$0 \longrightarrow R_S(\mathfrak{p}) \longrightarrow R(\mathfrak{p}) \longrightarrow R'(\mathfrak{p}) \longrightarrow 0\ , \tag{3.2}$$

in which $R'(\mathfrak{p})$ is a torsion module.

Moreover, if $\mathfrak{p}$ corresponds to a point $s \in V$, then $R'(\mathfrak{p}) = 0$.

Proof. Let us first remark that for each pair $(p,q) \in \mathbf{Z}^2$ we have an injective morphism of k-vector spaces $(\mathcal{F}^p \cap \mathcal{G}^q)(s) \to \mathcal{F}^p(s) \cap \mathcal{G}^q(s)$. $(\mathcal{F}^p \cap \mathcal{G}^q)(s)$ is indeed the kernel of the morphism of fibre over s given by the morphism of stalks $\mathcal{F}^p_s \to (\mathcal{V}/\mathcal{G}^q)_s$, when $\mathcal{F}^p(s) \cap \mathcal{G}^q(s)$ is the kernel of the morphism of vector spaces $\mathcal{F}^p(s) \to \mathcal{V}(s)/\mathcal{G}^q(s)$. Then, we have $R_S(\mathfrak{p}) = R_S \otimes_{A[u,v]} (A[u,v]/\mathfrak{p}A[u,v]) \cong R_S/\mathfrak{p}R_S$. The $k[u,v]$-module $R_S(\mathfrak{p})$ is hence isomorphic to the Rees module generated by the elements of the form $u^{-p}v^{-q} \otimes_k \overline{m}^{p,q}$ where $\overline{m}^{p,q} \in M^{p,q}/\mathfrak{p}M^{p,q} \cong (\mathcal{F}^p \cap \mathcal{G}^q)(s)$. This yields, using the previous remark, an injective morphism of $k[u,v]$-modules from $R_S(\mathfrak{p})$ to $R(\mathfrak{p})$ which maps the piece $u^{-p}v^{-q} \otimes_k M^{p,q}/\mathfrak{p}M^{p,q}$ to $u^{-p}v^{-q} \otimes_k (\mathcal{F}^p(s) \cap \mathcal{G}^q(s))$.

Let us show that the cokernel of this morphism, $R'(\mathfrak{p})$, is a torsion $k[u,v]$-module if its localisation at (0), $R'(\mathfrak{p})_0 = 0$, is zero. Let $\overline{m}' \in R'(\mathfrak{p})$, and let $\overline{m} = \sum_{p,q} u^{-p}v^{-q} \otimes_k \overline{m}^{p,q}$ be a preimage of $\overline{m}'$ in $R(\mathfrak{p})$, where the sum is finite. For each pair of integers (p,q) there exists a pair $(r_{p,q}, s_{p,q})$, $r_{p,q} \leq p$, $s_{p,q} \leq q$, such that $\overline{m}^{p,q} \in M^{r_{p,q},s_{p,q}}/\mathfrak{p}M^{r_{p,q},s_{p,q}}$. Then $u^{p-r_{p,q}}v^{q-s_{p,q}}.(u^{-p}v^{-q} \otimes_k \overline{m}^{p,q}) \in R_S(\mathfrak{p})$. Let us consider the element $P(u,v) = \prod_{p,q} u^{p-r_{p,q}}v^{q-s_{p,q}}$, which is well-defined since the product is finite. Then $P(u,v).\overline{m} \in R_S(\mathfrak{p})$ and hence $P(u,v).\overline{m}' = 0$ in $R'(\mathfrak{p})$.

Let $\mathfrak{p}$ be the maximal ideal corresponding to a point $s \in V$. Then there exists an open set containing s on which, for each $(p,q) \in \mathbf{Z}$, the rank of the morphism between vector bundles $\mathcal{F}^p \to \mathcal{V}/\mathcal{G}^q$ is constant. The restrictions of the coherent sheaves $\mathcal{F}^p \cap \mathcal{G}^q$ are hence locally free, and thus $(\mathcal{F}^p \cap \mathcal{G}^q)(s) \cong \mathcal{F}^p(s) \cap \mathcal{G}^q(s)$. The $k[u,v]$-modules $R_S(\mathfrak{p})$ and $R(\mathfrak{p})$ are then isomorphic. □

Lemma 3.6. *With the above notations, for each $s \in S$ with corresponding ideal $\mathfrak{p}$, we have $R_S(\mathfrak{p})^{**} \cong R(\mathfrak{p})$, where the isomorphism is an isomorphism of $k[u,v]$-modules compatible with the structure of B-comodule.*

Proof. We prove the statement for the canonically associated coherent sheaves on $\mathbf{A}^2_k$. Let

$$0 \longrightarrow \mathcal{F}_S(\mathfrak{p}) \longrightarrow \mathcal{F}(\mathfrak{p}) \longrightarrow \mathcal{T}(\mathfrak{p}) \longrightarrow 0$$

the exact sequence associated to (3.2). By Proposition 3.2, we know that $\mathrm{Supp}(\mathcal{T}(\mathfrak{p})) \subset \{(0,0)\}$. This yields an isomorphism $\mathcal{F}_S(\mathfrak{p})|_U \cong \mathcal{F}(\mathfrak{p})|_U$

where $U = \mathbf{A}^2 \setminus \{(0,0)\}$, and hence an isomorphism $\mathcal{F}_S(\mathfrak{p})^{**}|_U \cong \mathcal{F}(\mathfrak{p})^{**}|_U$. Since the dual of each coherent sheaf is reflexive, thus normal, it is determined by its restriction to each open whose complementary is at least 2-codimensional. In consequence $\mathcal{F}_S(\mathfrak{p})^{**} \cong \mathcal{F}(\mathfrak{p})^{**}$; the corresponding modules are then isomorphic and all the isomorphisms are compatible with the action. $\square$

3.3. **Gluing and relative Rees sheaves on the projective plane.** Consider now a 3-filtered vector bundle $(\mathcal{V}, (\mathcal{F}_i^\bullet)_{i\in\{0,1,2\}})$ on an algebraic variety S as above. Let $S = \amalg_{\beta\in B} S_\beta$ be a covering of the base space by affine schemes. Let $i \in \{0,1,2\}$ and $\beta \in B$. We make, on the open set $\mathbf{A}_i^2 \times S_\beta$, the Rees construction involving the 2-filtered vector bundle on S_β, $(\mathcal{V}_\beta, \mathcal{F}_j^\bullet, \mathcal{F}_l^\bullet)$, with $\{i,j,l\} = \{0,1,2\}$ and $j < l$, where $\mathcal{V}_\beta$ denotes the restriction of the vector bundle $\mathcal{V}$ to S_β.

Lemma 3.7. *With the above notations, the* $\mathbf{T}$*-equivariant reflexive sheaves on* $\mathbf{A}_i^2 \times S_\beta$*,* $\xi_{S_\beta}(\mathcal{V}_\beta, \mathcal{F}_j^\bullet, \mathcal{F}_l^\bullet)$*, can be glued together to form a* $\mathbf{T}$*-equivariant reflexive sheaf on* $\mathbf{P}_k^2 \times S$ *denoted by* $\xi_S(\mathcal{V}, \mathcal{F}_0^\bullet, \mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet)$ *and called the Rees sheaf associated to the* 3*-filtered vector bundle* $(\mathcal{V}, (\mathcal{F}_i^\bullet)_{i\in\{0,1,2\}})$.

Proof. Let (i,j), $i<j$, and (i',j'), $i'<j'$, be two different pairs of integers of $\{0,1,2\}$, and let l be the integer such that $\{l\} = \{i,j\} \cap \{i',j'\}$. h (respectively h') is the integer such that $\{i,j,h\} = \{0,1,2\}$ (respectively $\{i',j',h'\} = \{0,1,2\}$). Then, the restrictions of the relative Rees sheaves $\xi_{S_\beta}(\mathcal{V}_\beta, \mathcal{F}_i^\bullet, \mathcal{F}_j^\bullet)$ and $\xi_{S_\beta}(\mathcal{V}_\beta, \mathcal{F}_{i'}^\bullet, \mathcal{F}_{j'}^\bullet)$ to the open set $(\mathbf{A}_h^2 \times S_\beta) \cap (\mathbf{A}_{h'}^2 \times S_\beta) \cong \mathbf{G}_m \times \mathbf{A}^1 \times S_\beta$ are isomorphic, via φ_h^l and $\varphi_{h'}^l$, respectively, as equivariant coherent sheaves, to the pullback $f^*\xi_{S_\beta}(\mathcal{V}_\beta, \mathcal{F}_l^\bullet)$ of the relative Rees bundle on $\mathbf{A}^1 \times S_\beta$ associated to the filtered vector bundle on S_β, $(\mathcal{V}_\beta, \mathcal{F}_l^\bullet)$, by the projection $f : \mathbf{G}_m \times \mathbf{A}^1 \times S_\beta \to \mathbf{A}^1 \times S_\beta$. This gives a transition map $\Phi_{hh'}^l$, where $\{h,h',l\} = \{0,1,2\}$, on the open intersection $(\mathbf{A}_h^2 \cap \mathbf{A}_{h'}^2) \times S_\beta$. The three transition maps satisfy the cocycle relation since all the restrictions coincide on the intersection of the three standard open sets covering $\mathbf{P}_k^2 \times S_\beta$. This yields an equivariant sheaf $\xi_{S_\beta}(\mathcal{V}, \mathcal{F}_0^\bullet, \mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet)$ on $\mathbf{P}_k^2 \times S_\beta$. One can clearly glue these sheaves to form a sheaf on $\mathbf{P}_k^2 \times S$. $\square$

As a consequence of Proposition 3.2 we have:

Proposition 3.8. *The Rees sheaf associated to a* 3*-filtered vector bundle* $\mathcal{V}$ *on a smooth algebraic variety* S*,* $\xi_S(\mathcal{V}, \mathcal{F}_0^\bullet, \mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet)$*, is a* $\mathbf{T}$*-equivariant reflexive sheaf whose singularity locus is included in* $\{P_{01}, P_{02}, P_{12}\} \times S$.

Since the relative Rees sheaves are reflexive, they form flat families of torsion free sheaves.

Corollary 3.9. *The Rees sheaf $\xi_S(\mathcal{V},\mathcal{F}_0^\bullet,\mathcal{F}_1^\bullet,\mathcal{F}_2^\bullet)$ on $\mathbf{P}_k^2\times S$ associated to a 3-filtered vector bundle $\mathcal{V}$ on a smooth algebraic variety S is S-flat.*

Proof. Denote by ξ_S the sheaf $\xi_S(\mathcal{V},\mathcal{F}_0^\bullet,\mathcal{F}_1^\bullet,\mathcal{F}_2^\bullet)$ and by $\mathcal{E}$ its restriction to $U\times S$, where $U=\mathbf{P}_k^2\setminus\{P_{01},P_{02},P_{12}\}$; recall that $\mathcal{E}$ is locally free. Let $f:X=\mathbf{P}_k^2\times S\to S$ and let $\mathcal{O}_X(1)$ be a line bundle on X whose restriction to any fibre X_s is ample. It suffices to prove that for all m sufficiently large the sheaves $f_*(\xi_S(m))$ are locally free. Let V be an open set of S, then, for each integer m, since ξ_S is reflexive and hence determined by its restrictions to open sets whose complementary is at least 2-codimensional,

$$f_*(\xi_S(m))(V)=\xi_S(m)(\mathbf{P}_k^2\times V)\cong\xi_S(m)(U\times V)=f_*\mathcal{E}(m)(V).$$

Since $f_*\mathcal{E}(m)(V)$ is a free $\mathcal{O}_S(V)$-module, so is $f_*(\xi_S(m))(V)$. □

We are now able to compare the fibres of the relative Rees family to the Rees vector bundles obtained at each point from the 3-filtered vector spaces deduced from the 3-filtered vector bundle.

Proposition 3.10. *Let $(\mathcal{V},(\mathcal{F}_i^\bullet)_{i\in\{0,1,2\}})$ be a 3-filtered vector bundle on a smooth algebraic variety S. Let $\xi_S(\mathcal{V},\mathcal{F}_0^\bullet,F_1^\bullet,\mathcal{F}_2^\bullet)$ be the associated relative Rees sheaf on $\mathbf{P}_k^2\times S$. Then for each $s\in S$ we have an isomorphism of $\mathbf{T}$-equivariant locally free sheaves on $\mathbf{P}_k^2$*

$$\xi_S(\mathcal{V},\mathcal{F}_0^\bullet,F_1^\bullet,\mathcal{F}_2^\bullet)(s)^{**}\cong\xi(\mathcal{V}(s),\mathcal{F}_0^\bullet(s),\mathcal{F}_1^\bullet(s),\mathcal{F}_2^\bullet(s)).$$

Moreover, there exists an open set $V\subset S$, such that for each $s\in V$

$$\xi_S(\mathcal{V},\mathcal{F}_0^\bullet,\mathcal{F}_1^\bullet,\mathcal{F}_2^\bullet)(s)\cong\xi(\mathcal{V}(s),\mathcal{F}_0^\bullet(s),\mathcal{F}_1^\bullet(s),\mathcal{F}_2^\bullet(s)).$$

Proof. According to Lemma 3.5, by gluing the local descriptions of the fibres of the relative Rees sheaves and the Rees locally free sheaves of the fibres of the filtered vector bundle as in Lemma 3.7, we have, for each $s\in S$, the exact sequence

$$0\longrightarrow\xi_S(\mathcal{V},\mathcal{F}_0^\bullet,F_1^\bullet,\mathcal{F}_2^\bullet)(s)\longrightarrow\xi(\mathcal{V}(s),\mathcal{F}_0^\bullet(s),\mathcal{F}_1^\bullet(s),\mathcal{F}_2^\bullet(s))\longrightarrow$$
$$\longrightarrow\mathcal{T}(s)\longrightarrow 0\ ,$$

where $\mathrm{Supp}(\mathcal{T}(s))\subset\{P_{01},P_{02},P_{12}\}$. Then, Lemma 3.6 gives the first isomorphism.

Next, the relative Rees sheaf being S-flat by Corollary 3.9, the set

$$V=\{s\in S|\xi_S(\mathcal{V},\mathcal{F}_0^\bullet,F_1^\bullet,\mathcal{F}_2^\bullet)(s)\text{ is a locally free sheaf }\}$$

is an open subset of S. According to Lemma 3.5, for each $s\in V$, $\mathcal{T}(s)=0$. For each point of V we thus have the required isomorphism. □

Example 3.11. Let V be a 2-dimensional vector space over k, $\{e, f\}$ be a basis of V and $\mathcal{V}$ be the trivial vector bundle associated to V on $\mathbf{A}^2 = \operatorname{Spec} k[\lambda, \mu]$. Consider the three finite, complete and decreasing filtrations of $\mathcal{V}$ given by $\mathcal{F}_0^{-2} = V$, $\mathcal{F}_0^{-1} = \mathcal{F}_0^0 =< f >$, $\mathcal{F}_0^1 = \{0\}$, $\mathcal{F}_1^0 = V$, $\mathcal{F}_1^1 =< e + P(\lambda, \mu) f >$, $\mathcal{F}_1^2 = \{0\}$, and, $\mathcal{F}_2^0 = V$, $\mathcal{F}_2^1 =< e + Q(\lambda, \mu) f >$, $\mathcal{F}_2^2 = \{0\}$, where P, Q are two polynomials in λ and μ. One verifes that at each point of the base space the given filtrations are opposed. On $\mathbf{A}_0^2 \times \mathbf{A}^2$, the Rees $k[u, v, \lambda, \mu]$-module $R_{\mathbf{A}^2}(\mathcal{V}, \mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet)$ is generated by $\{u^{-1} \otimes (e + P(\lambda, \mu) f), v^{-1} \otimes (e + Q(\lambda, \mu) f), f\}$ (with $u = \frac{u_1}{u_0}$ and $v = \frac{u_2}{u_0}$). Consider the closed subset of $\mathbf{A}^2$, $Z = \{(\lambda, \mu) \in \mathbf{A}^2 | P(\lambda, \mu) = Q(\lambda, \mu)\}$. The singularity set of the relative Rees sheaf on $\mathbf{P}_k^2 \times \mathbf{A}^2$ is $P_{12} \times Z$. This is due to the fact that the dimension of the intersection of the two subvector spaces of V given at each point by $\mathcal{F}_1^1$ and $\mathcal{F}_2^1$ jumps at the points which belong to Z. Indeed, for each maximal ideal $\mathfrak{p}$ which corresponds to a point $s \in Z$, we have an exact sequence of $k[u, v, \lambda, \mu]$-modules

$$0 \to R_{\mathbf{A}^2}(\mathcal{V}, \mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet)(\mathfrak{p}) \to R_{\mathbf{A}^2}(\mathcal{V}, \mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet)(\mathfrak{p})^{**} \to R'(\mathfrak{p}) \to 0,$$

where $R_{\mathbf{A}^2}(\mathcal{V}, \mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet)(\mathfrak{p})^{**}$ is isomorphic to the free $k[u, v]$-module generated by $\{u^{-1} v^{-1} \otimes e, f\}$, which is exactly $R(\mathcal{V}(s), \mathcal{F}_1^\bullet(s), \mathcal{F}_2^\bullet(s))$, and $R'(\mathfrak{p})$ is the torsion $k[u, v]$-module $u^{-1} v^{-1} k[u, v] / (u^{-1}, v^{-1}) k[u, v]$.

Since the dimensions of the intersections of the subspaces given by the filtrations do not jump on the other charts $\mathbf{A}_i^2 \times \mathbf{A}^2$, $i \in \{1, 2\}$, the associated relative Rees sheaves are locally free. So, for each $s \in \mathbf{A}^2 \setminus Z$, $\xi_{\mathbf{A}^1}(\mathcal{V}, \mathcal{F}_0^\bullet, \mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet)(s)$ is locally free, and, if $s \in Z$, we have

$$0 \to \xi_{\mathbf{A}^2}(\mathcal{V}, \mathcal{F}_0^\bullet, \mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet)(s) \to \xi(\mathcal{V}(s), \mathcal{F}_0^\bullet(s), \mathcal{F}_1^\bullet(s), \mathcal{F}_2^\bullet(s)) \to \mathcal{T}_{P_{12}} \to 0,$$

where $\mathcal{T}_{P_{12}}$ is a skyscraper sheaf of rank one supported at $P_{12} \subset \mathbf{P}_k^2$.

3.4. **Analytic relative Rees sheaves.** Let $(\mathcal{V}, (\mathcal{F}_i^\bullet)_{i \in \{0,1,2\}})$ be a 3-filtered holomorphic vector bundle on a complex manifold S. One can perform the construction of the relative Rees sheaf on $\mathbf{P}_{\mathbf{C}}^2 \times S$ in an analytic context; here $\mathbf{P}_{\mathbf{C}}^2 = \operatorname{Proj} \mathbf{C}[u_0, u_1, u_2]$ is considered as a complex manifold. The coherent sheaves $\mathcal{F}_j^p \cap \mathcal{F}_l^q$, $j \neq l$, can be defined in the same manner and are analytic torsion free coherent sheaves. Next, on $\mathbf{C}_i^2 \times S$, where $\mathbf{C}_i^2 = \operatorname{Spec} \mathbf{C}[u, v]$ is considered as a complex manifold, with $j < l$, $\{i, j, l\} = \{0, 1, 2\}$, one defines the $\mathcal{O}_S[u, v]$-module

$$\textstyle\sum_{p,q} u^{-p} v^{-q} \mathcal{F}_j^p \cap \mathcal{F}_l^q.$$

This module gives a reflexive coherent sheaf on $\mathbf{C}_i^2 \times S$ which is equivariant for the action of $(\mathbf{C}^*)^2 = \operatorname{Spec} \mathbf{C}[t_j^{\pm 1}, t_l^{\pm 1}]$ on the first factor. One can glue the local descriptions to get a $\mathbf{T}$-equivariant analytic reflexive Rees sheaf $\xi_S(\mathcal{V}, \mathcal{F}_0^\bullet, F_1^\bullet, \mathcal{F}_2^\bullet)$ on $\mathbf{P}_{\mathbf{C}}^2 \times S$. Propositions 3.8 and 3.10 translate directly in this context.

Note that, by GAGA, the coherent sheaves parametrized by S are algebraic coherent sheaves on the projective plane. Indeed, for each $s \in S$, $\xi_S(\mathcal{V}, \mathcal{F}_0^\bullet, F_1^\bullet, \mathcal{F}_2^\bullet)|_{\mathbf{P}^2_{\mathbf{C}} \times \{s\}}$ is an analytic coherent sheaf on $\mathbf{P}^2_{\mathbf{C}}$, hence algebraic.

4. Variation of mixed Hodge structure

4.1. **Rees sheaf associated to a variation of mixed Hodge structure.** The aim of this section is to codify variations of mixed Hodge structure in terms of families of Rees bundles, namely relative Rees sheaves. We *do not* give an analogue of the connections underlying these objects *nor* take into account the polarizations.

In this context, we define a variation of mixed Hodge structure to be:

Definition 4.1. A *variation of* $\mathbf{R}$*-mixed Hodge structure* on a complex manifold S is an ordered triple $(\mathcal{H}_{\mathbf{R}}, \mathcal{W}_\bullet, \mathcal{F}^\bullet)$ consisting of a local system $\mathcal{H}_{\mathbf{R}}$ of free $\mathbf{R}$-modules of finite rank over the variety S and two filtrations: an increasing filtration $\mathcal{W}_\bullet$ of $\mathcal{H}_{\mathbf{R}}$ by local subsystems and a decreasing filtration $\mathcal{F}^\bullet$ by subbundles of the holomorphic vector bundle $\mathcal{H}_{\mathcal{O}} = \mathcal{H}_{\mathbf{R}} \otimes \mathcal{O}_S$ such that:

- (i) For each point $s \in S$, the fibre $(\mathcal{H}_{\mathbf{R}}(s), \mathcal{W}_\bullet(s), \mathcal{F}^\bullet(s), \overline{\mathcal{F}}^\bullet(s)) \in \mathbf{R}$-MHS, namely is a $\mathbf{R}$-mixed Hodge structure.
- (ii) The canonical connection ∇ corresponding to the local system $\mathcal{H}_{\mathbf{R}}$ satisfies, for each integer p, the condition
$$\nabla \mathcal{F}^p \subset \Omega^1_S \otimes \mathcal{F}^{p-1}.$$

This definition, and all the results of this section, can be stated in a complex algebraic context. In that case, the base of the variation is a smooth scheme over $\mathbf{C}$ and the Hodge filtration is algebraic.

A variation of $\mathbf{R}$-mixed Hodge structure can be described in terms of a morphism from the base to an appropriate classifying space M of mixed Hodge structures. Before giving its construction, after [3], we need to recall some facts about $\mathbf{R}$-split mixed Hodge structures. There is a canonical way to deform a mixed Hodge structure into one which is split over $\mathbf{R}$. Consider $(H_{\mathbf{R}}, W_\bullet, F^\bullet, \overline{F}^\bullet) \in \mathbf{R}$-MHS and its canonically associated splitting $\{I^{p,q}\}_{p,q}$. We associate to this Hodge structure the nilpotent Lie algebra
$$\mathfrak{l}^{-1,-1} = \{\delta \in \mathrm{End}(H_{\mathbf{C}}) | \delta(I^{p,q}) \subset \oplus_{j<p,l<q} I^{j,l}\}.$$
Since $\bar{\mathfrak{l}}^{-1,-1} = \mathfrak{l}^{-1,-1}$, this algebra admits the real form $\mathfrak{l}^{-1,-1}_{\mathbf{R}} = \mathrm{End}(H_{\mathbf{R}}) \cap \mathfrak{l}^{-1,-1}$.

Proposition 4.2. [3] *Let* $(H_{\mathbf{R}}, W_\bullet, F^\bullet, \overline{F}^\bullet)$ *be a* $\mathbf{R}$*-mixed Hodge structure, there exists a unique* $\delta \in \mathfrak{l}^{-1,-1}_{\mathbf{R}}$ *such that the mixed Hodge structure* $(H_{\mathbf{R}}, W_\bullet, e^{-i\delta} F^\bullet, e^{i\delta}\overline{F}^\bullet)$ *is* $\mathbf{R}$*-split.*

Moreover, the functor $(H_{\mathbf{R}}, W_\bullet, F^\bullet, \overline{F}^\bullet) \mapsto ((H_{\mathbf{R}}, W_\bullet, e^{-i\delta}F^\bullet, e^{i\delta}\overline{F}^\bullet), \delta)$ *establishes an equivalence of categories between the category of* **R***-mixed Hodge structures and the category whose objects are pairs formed by a* **R***-split mixed Hodge structure and an endomorphism of its associated real nilpotent Lie algebra* $\delta \in \mathfrak{l}_{\mathbf{R}}^{-1,-1}$*, and whose morphisms are morphisms of mixed Hodge structures commuting with the endomorphisms* δ*.*

Consider a variation of mixed Hodge structure $(\mathcal{H}_{\mathbf{R}}, \mathcal{W}_\bullet, \mathcal{F}^\bullet)$ on S as above and let $f^p = \mathrm{rk}_{\mathbf{C}}(\mathcal{F}^p)$ and $f^p_r = \mathrm{rk}_{\mathbf{C}}(\mathcal{F}^p_r)$, where the vector bundle $\mathcal{F}^p_r$ is the p^{th} stage of the Hodge filtration of the variation of pure Hodge structures of weight r induced by $\mathcal{F}^p$ on $\mathcal{W}_r/\mathcal{W}_{r-1}$.

From Lemma 2.4, a **R**-mixed Hodge structure is completely determined by its splitting $\{I^{p,q}\}_{p,q}$. The classifying space of **R**-mixed Hodge structures M can therefore be viewed as the set of equivalences of splittings.

Let V be a fibre of $\mathcal{H}_{\mathbf{C}}$ and $V_{\mathbf{R}}$ be the underlying real vector space. Let $W_0GL(V)$ be the group of automorphisms of V which preserve the weight filtration of V. Consider the kernel $W_{-1}GL(V)$ of the canonical morphism of groups

$$W_0GL(V) \xrightarrow{\pi} \textstyle\prod_r GL(Gr^W_r V).$$

The set of real splittings is a principal homogenous space under $W_0GL(V)_{\mathbf{R}} = W_0GL(V) \cap GL(V_{\mathbf{R}})$, and, according to Proposition 4.2, the set of all possible splittings is a principal homogenous space under

$$H = W_0GL(V)_{\mathbf{R}}.W_{-1}GL(V).$$

Let us choose a **R**-split splitting and let $F^\bullet_0$ be the corresponding filtration. Let P be the subgroup of H which preserves $F^\bullet_0$, then,

$$M \cong H/P,$$

the isomorphism depending on the choice of the reference **R**-split mixed Hodge structure.

The variation of mixed Hodge structure $(\mathcal{H}_{\mathbf{R}}, \mathcal{W}_\bullet, \mathcal{F}^\bullet)$ on S gives rise to a morphism

$$\varphi : \tilde{S} \to M$$

from the universal covering of S to the classifying space which gives a morphism

$$\varphi' : S \to \Gamma \setminus M,$$

where $\Gamma = \mathrm{Im}(\pi_1(S, \star) \to GL(V_{\mathbf{R}}))$ is the image of the monodromy homomorphism. By analogy with the case of variations of pure Hodge structure, the morphism φ' is often called the *period mapping*.

Each point of $s \in M$ gives a Hodge filtration $F_s^\bullet$ such that $(V_{\mathbf{R}}, W_\bullet, F_s^\bullet, \overline{F}_s^\bullet)$ is a $\mathbf{R}$-mixed Hodge structure. There is a universal filtered vector bundle $(\mathcal{V}, \mathcal{F}_{univ}^\bullet)$ on M such that, for each integer p and each point $s \in M$,

$$\mathcal{F}_{univ}^p(s) = F_s^p,$$

and such that the Hodge filtration of the variation of mixed Hodge structure is given by

$$\mathcal{F}^\bullet = \varphi^* \mathcal{F}_{univ}^\bullet.$$

Next, each point $(s,t) \in M \times M$ gives an ordered 3-filtered vector space $(V, W_\bullet, F_s^\bullet, F_t^\bullet)$ and we have a universal 3-filtered vector bundle on $M \times M$ of the form

$$(V \otimes_{\mathbf{C}} \mathcal{O}_{M\times M}, W^\bullet \otimes_{\mathbf{C}} \mathcal{O}_{M\times M}, \mathcal{F}_{univ,1}^\bullet, \mathcal{F}_{univ,2}^\bullet),$$

where $\mathcal{F}_{univ,i}^\bullet$, $i = 1, 2$ is the pullback on $M \times M$ of the universal Hodge filtration on M by the projection on the first and second factors. We construct the relative Rees sheaf associated to this 3-filtered vector bundle and get a $\mathbf{T}$-equivariant reflexive sheaf on $\mathbf{P}_{\mathbf{C}}^2 \times M \times M$,

$$\xi_{M\times M}(V \otimes_{\mathbf{C}} \mathcal{O}_{M\times M}, W^\bullet \otimes_{\mathbf{C}} \mathcal{O}_{M\times M}, \mathcal{F}_{univ,1}^\bullet, \mathcal{F}_{univ,2}^\bullet).$$

Next, we focus on the locus of opposed filtrations $M^{opp} \subset M \times M$ which is defined to be the set of points $(s,t) \in M \times M$ such that $(V, W_\bullet, F_s^\bullet, F_t^\bullet) \in \mathbf{C}$-MHS, namely such that $(W_\bullet, F_s^\bullet, F_t^\bullet)$ forms an ordered triple of opposed filtrations on V. This space can be considered as the classifying space of complex mixed Hodge structures associated to the data $\{(V, W_\bullet), \{f_r^p\}_{r,p}\}$. Notice that $(s,t) \in M^{opp} \Leftrightarrow (t,s) \in M^{opp}$.

Lemma 4.3. *The locus of opposed filtrations M^{opp} is an open submanifold of $M \times M$.*

Before giving the proof, let us remark that the property to be opposed for the filtrations, namely, for each integer r, the fact that $F_s^\bullet$ and $F_t^\bullet$ are r-opposed on $Gr_r^W V$, is detected on the graded pieces of the graded object $Gr_W V$. For each r there is a surjective map

$$\pi_r : M \to Fl(V_r),$$

where $Fl(V_r) = Fl(Gr_r^W V, f_r^n, ..., f_r^1)$ is the flag manifold which parametrizes filtrations by subvector spaces, $Gr_r^W V = F^0 \supset F^1 \supset ... \supset F^n \supset \{0\}$, such that, for each p, $\dim_{\mathbf{C}} F^p = f_r^p$. The flag manifold $Fl(V_r)$ is a projective variety. This map sends a point s, corresponding to $F_s^\bullet$, to the Hodge filtration $F_s^\bullet$ induces on $Gr_r^W V$. Let $n_r = \dim_{\mathbf{C}}(Gr_r^W V)$ and remark that $f_r^p = n_r - f_r^{r-p+1}$.

Proof. Consider the morphism

$$M \times M \xrightarrow{\prod_r(\pi_r\times\pi_r)} \prod_r(Fl(V_r) \times Fl(V_r)).$$

Let $(s,t) \in M \times M$. Then, $F_s^\bullet$ and $F_t^\bullet$ are r-opposed on $Gr_r^W V$ if and only if for each p,

$$F_s^p \oplus F_t^{r-p+1} = Gr_r^W V. \tag{4.1}$$

This condition is open in $\prod_r(Fl(V_r)\times Fl(V_r))$. Indeed, to each flag manifold $Fl(V_r)$ is associated a natural injection into a product of Grassmanians

$$Fl(V_r) \hookrightarrow \prod_p G(Gr_r^W V, f_r^p),$$

the image of which is a compact manifold. For each integers p, r, and each point $s \in M$, the image of s in $G(Gr_r^W V, f_r^p)$ can be described as a $n_r \times f_r^p$ matrix of maximal rank f_r^p. In this setting, (4.1) amounts to the fact that the $n_r \times n_r$ matrix formed by the $n_r \times f_r^p$ matrix which represents F_s^p in $G(Gr_r^W V, f_r^p)$ and the $n_r \times f_r^{r-p+1}$ which represents F^{r-p+1} in $G(Gr_r^W V, f_r^{r-p+1})$ has maximal rank. Let U_r^p be the open submanifold of points in $G(Gr_r^W V, f_r^p) \times G(Gr_r^W V, f_r^p)$ which verify this condition. Then, the set

$$U_r = (Fl(V_r) \times Fl(V_r)) \cap (\cap_p U_r^p)$$

is an open submanifold of $Fl(V_r) \times Fl(V_r)$. We thus get an open inclusion

$$j : M^{opp} = \cap_r(\pi_r \times \pi_r)^{-1}(U_r) \hookrightarrow M \times M. \tag{4.2}$$

$\square$

The construction of M^{opp} allows us to define a universal Rees sheaf on $\mathbf{P}^2_{\mathbf{C}} \times M^{opp}$ which parametrizes triples of opposed ordered filtrations, namely objects of $\mathbf{C}$-MHS corresponding to the data of linear algebra of the variation of mixed Hodge structure $(\mathcal{H}_{\mathbf{R}}, \mathcal{W}_\bullet, \mathcal{F}^\bullet)$ on S,

$$\xi_{M^{opp}} = j^*\xi_{M\times M}(V \otimes_{\mathbf{C}} \mathcal{O}_{M\times M}, W^\bullet \otimes_{\mathbf{C}} \mathcal{O}_{M\times M}, \mathcal{F}^\bullet_{univ,1}, \mathcal{F}^\bullet_{univ,2}).$$

Consider now the morphism $\varphi \times \overline{\varphi} : \tilde{S} \to M \times M$, where the conjugation is taken with respect to the real structure on $V_{\mathbf{C}}$ which comes from $V_{\mathbf{R}}$. This morphism is real analytic and takes values in M^{opp}; the image by $\varphi \times \overline{\varphi}$ of a point $s \in \tilde{S}$ is the $\mathbf{R}$-mixed Hodge structure $(V_{\mathbf{R}}, W_\bullet, F_s^\bullet, \overline{F}_s^\bullet)$.

The pullback of $\xi_{M^{opp}}$ on $\mathbf{P}^2_{\mathbf{C}} \times \tilde{S}$ by $id_{\mathbf{P}^2_{\mathbf{C}}} \times \varphi \times \overline{\varphi}$,

$$\begin{tikzcd}
(id_{\mathbf{P}^2_{\mathbf{C}}} \times \varphi \times \overline{\varphi})^* \xi_{M^{opp}} \arrow[r] \arrow[d] & \xi_{M^{opp}} \arrow[d] \\
\mathbf{P}^2_{\mathbf{C}} \times \tilde{S} \arrow[r, "id_{\mathbf{P}^2_{\mathbf{C}}} \times \varphi \times \overline{\varphi}"] \arrow[d, "p_{\tilde{S}}"'] & \mathbf{P}^2_{\mathbf{C}} \times M^{opp} \arrow[d, "p_{M^{opp}}"] \\
\tilde{S} \arrow[r, "\varphi \times \overline{\varphi}"] & M^{opp},
\end{tikzcd}$$

is a real analytic sheaf in the direction of $\tilde{S}$ and complex analytic in the direction of $\mathbf{P}^2_{\mathbf{C}}$, namely is a sheaf of $\mathcal{C}^{\infty}_{\tilde{S}}\mathcal{O}_{\mathbf{P}^2_{\mathbf{C}}}$-modules, where $\mathcal{C}^{\infty}_{\tilde{S}}\mathcal{O}_{\mathbf{P}^2_{\mathbf{C}}}$ denotes the sheaf of functions on $\mathbf{P}^2_{\mathbf{C}} \times M^{opp}$ which are holomorphic in the direction of $\mathbf{P}^2_{\mathbf{C}}$ and $\mathcal{C}^{\infty}$ in the direction of $\tilde{S}$.

Note that, according to GAGA, for each $s \in \tilde{S}$, the restriction $(id_{\mathbf{P}^2_{\mathbf{C}}} \times \varphi \times \overline{\varphi})^* \xi_{M^{opp}}|_{\mathbf{P}^2_{\mathbf{C}} \times \{s\}}$ is an algebraic coherent sheaf. The sheaf of $\mathcal{C}^{\infty}_{\tilde{S}}\mathcal{O}_{\mathbf{P}^2_{\mathbf{C}}}$-modules $(id_{\mathbf{P}^2_{\mathbf{C}}} \times \varphi \times \overline{\varphi})^* \xi_{M^{opp}}$ is called *the relative Rees sheaf associated to the variation of mixed Hodge structure* $(\mathcal{H}_{\mathbf{R}}, \mathcal{W}_{\bullet}, \mathcal{F}^{\bullet})$.

We recover the mixed Hodge structures of the variation in the following way:

Theorem 4.4. *Let $(\mathcal{H}_{\mathbf{R}}, \mathcal{W}_{\bullet}, \mathcal{F}^{\bullet})$ be a variation of mixed Hodge structure on S, M the corresponding classifying space and $\varphi : \tilde{S} \to M$ the associated morphism. We define $M^{opp} \subset M \times M$ as above and consider the universal $\mathbf{T}$-equivariant reflexive Rees sheaf $\xi_{M^{opp}}$ on $\mathbf{P}^2_{\mathbf{C}} \times M^{opp}$. Let $(id_{\mathbf{P}^2_{\mathbf{C}}} \times \varphi \times \overline{\varphi})^* \xi_{M^{opp}}$ be the associated relative Rees sheaf on $\mathbf{P}^2_{\mathbf{C}} \times \tilde{S}$.*

Then, for each $s \in \tilde{S}$, the fibre $(id_{\mathbf{P}^2_{\mathbf{C}}} \times \varphi \times \overline{\varphi})^* \xi_{M^{opp}}|_{\mathbf{P}^2_{\mathbf{C}} \times \{s\}}$ is a $\mathbf{T}^{\tau}$-equivariant torsion free $\mathbf{P}^1_0$-semistable sheaf of $\mathcal{O}_{\mathbf{P}^2_{\mathbf{C}}}$-modules whose double dual $((id_{\mathbf{P}^2_{\mathbf{C}}} \times \varphi \times \overline{\varphi})^* \xi_{M^{opp}}|_{\mathbf{P}^2_{\mathbf{C}} \times \{s\}})^{**}$ corresponds, via the equivalence of categories stated in Theorem 1.27, to the $\mathbf{R}$-MHS $(\mathcal{H}_{\mathbf{R}}(s), \mathcal{W}_{\bullet}(s), \mathcal{F}^{\bullet}(s), \overline{\mathcal{F}}^{\bullet}(s))$.

Proof. By construction, for each $s \in S$, one has the following isomorphism of equivariant torsion free coherent sheaves of $\mathcal{O}_{\mathbf{P}^2}$-modules

$$(id_{\mathbf{P}^2_{\mathbf{C}}} \times \varphi \times \overline{\varphi})^* \xi_{M^{opp}}|_{\mathbf{P}^2_{\mathbf{C}} \times \{s\}} \cong \xi_{M^{opp}}|_{\mathbf{P}^2_{\mathbf{C}} \times \{\varphi(s)\} \times \{\overline{\varphi(s)}\}} \cong \xi_{M \times M}|_{\mathbf{P}^2_{\mathbf{C}} \times \{\varphi(s)\} \times \{\overline{\varphi(s)}\}}.$$

This endows $(id_{\mathbf{P}^2_{\mathbf{C}}} \times \varphi \times \overline{\varphi})^* \xi_{M^{opp}}|_{\mathbf{P}^2_{\mathbf{C}} \times \{s\}}$ with a structure of $\mathbf{T}$-equivariant $\mathcal{O}_{\mathbf{P}^2}$-module first, and then with a structure of $\mathbf{T}^{\tau}$-equivariant $\mathcal{O}_{\mathbf{P}^2}$-module since the two latest filtrations, given by $\varphi(s)$ and $\overline{\varphi(s)}$, are conjugate.

Next, we show that the torsion free sheaves on the projective plane parametrized by $\xi_{M^{opp}}$ are $\mathbf{P}^1_0$-semistable. This is a consequence of the fact that the Hodge numbers are constant on $\tilde{S}$ since they are upper semi-continous and the sum is constant (see [24] for example). This fact implies that the torsion free sheaves $\mathcal{W}_r \cap \mathcal{F}^p_{univ}$ defined as in (3.1) are locally free. The corresponding relative Rees sheaves $\xi_{M^{opp}}(\mathcal{H}_{\mathbf{C}}, \mathcal{W}^\bullet, \mathcal{F}^\bullet_{univ,2})$ on $\mathbf{C}^2_1 \times M^{opp}$, where the Hodge filtration comes from the first factor in $M \times M$, and $\xi_{M^{opp}}(\mathcal{H}_{\mathbf{C}}, \mathcal{W}^\bullet, \mathcal{F}^\bullet_{univ,1})$ on $\mathbf{C}^2_2 \times M^{opp}$, are therefore locally free. This proves, using Proposition 3.8, that the singular locus of $\xi_{M^{opp}}$ is included in $P_{12} \times M^{opp}$ and hence that, for each $s \in \tilde{S}$, on U, where $U = \mathbf{P}^2_{\mathbf{C}} \setminus P_{12}$, holds the isomorphism

$$\xi_{M^{opp}}|_{U \times \{\varphi(s)\} \times \{\overline{\varphi(s)}\}} \cong \xi(\mathcal{H}_{\mathbf{C}}(s), W^\bullet, \mathcal{F}^\bullet_{univ}(\varphi(s)), \overline{\mathcal{F}}^\bullet_{univ}(\varphi(s)))|_U$$

because $\mathcal{F}^\bullet_{univ,1}(\varphi(s) \times \overline{\varphi(s)}) = \mathcal{F}^\bullet_{univ}(\varphi(s))$ and $\mathcal{F}^\bullet_{univ,2}(\varphi(s) \times \overline{\varphi(s)}) = \mathcal{F}^\bullet_{univ}(\overline{\varphi(s)}) = \overline{\mathcal{F}}^\bullet_{univ}(\varphi(s))$. So, the locally free sheaf appearing in the second member is the Rees sheaf on $\mathbf{P}^2_{\mathbf{C}}$ associated to the mixed Hodge structure on $\mathcal{H}_{\mathbf{C}}(s)$ indexed by s. This bundle is therefore $\mathbf{P}^1_0$-semistable, namely its restriction to $\mathbf{P}^1_0$ is a direct sum of line bundles of the same slope, and, since $\mathbf{P}^1_0 \subset U$, so is $\xi_{M^{opp}}|_{U \times \times \{\varphi(s)\} \times \{\overline{\varphi(s)}\}}$, which achieves the proof of the statement. □

Remark 4.5. Note that, when one considers an algebraic variation of mixed Hodge structure, the morphism $\varphi \times \overline{\varphi}$ is real algebraic; the relative Rees sheaf $(id_{\mathbf{P}^2_{\mathbf{C}}} \times \varphi \times \overline{\varphi})^* \xi_{M^{opp}}$ associated to the variation of mixed Hodge structure is hence a real algebraic sheaf whose fibres over $\tilde{S}$ are algebraic over $\mathbf{C}$.

4.2. **Semi-continuity of the R-split level, stratification.** As above, we consider a variation of mixed Hodge structure $(\mathcal{H}_{\mathbf{R}}, \mathcal{W}_\bullet, \mathcal{F}^\bullet)$ on a complex manifold S. Since the associated relative Rees sheaf is $\tilde{S}$-flat, the Hilbert polynomial of the fibres is constant. The Chern classes are therefore constant, the degree is zero and we denote the second Chern class by α_{max}. According to Theorem 4.4, there exists a dense open subset $U \subset S$ on which the $\mathbf{R}$-split level of the fibres equals α_{max}.

In this section we prove that the $\mathbf{R}$-split level is upper semi-continuous on the classifying space of complex mixed Hodge structures M^{opp} and hence yields a stratification of this space. Such a stratification induces a stratification of the universal covering of the base $\tilde{S}$.

Proposition 4.6. *The function*

$$\begin{aligned} \alpha : M^{opp} &\to \mathbf{Z} \\ s &\mapsto \alpha((\xi_{M^{opp}}|_{\mathbf{P}^2_{\mathbf{C}} \times \{s\}})^{**}) \end{aligned}$$

which associate to each point of the classifying space of mixed Hodge structure the **R***-split level of the complex mixed Hodge structure it parametrizes is upper semi-continuous.*

The statement of the above proposition is a direct consequence of the lemma below. Let, for each $s \in M^{opp}, f^{p,q}(s) = \dim_{\mathbf{C}}(\mathcal{F}^q_{univ,1}(s) \cap \mathcal{F}^p_{univ,2}(s))$.

Lemma 4.7. *With the above notation,*

(i) *For each pair of integer p, q, the function $M^{opp} \to \mathbf{Z}$, $s \mapsto f^{p,q}(s)$ is lower semi-continuous.*

(ii) *The function α decomposes into a sum $\alpha = \alpha_+ - \alpha_-$ where α_+ and α_- are non negative, α_+ is constant and α_- is lower semi-continuous.*

Proof. The first assumption is clear. Next, according to formula (2.2), we can let $\alpha_+ = \frac{1}{2}\sum_{p,q}(p+q)^2 h^{p,q}$ and $\alpha_- = \frac{1}{2}\sum_{p,q}(p+q)^2 s^{p,q}$. The first function is constant on M^{opp} since the Hodge numbers are constant. Then, remark that, for each p, q and each $s \in M^{opp}$,

$$s^{p,q}(s) = f^{p,q}(s) - f^{p+1,q}(s) - f^{p,q+1}(s) + f^{p+1,q+1}(s).$$

This gives, for each s,

$$\alpha_-(s) =$$
$$\frac{1}{2}\Big(\sum_{p,q}(2pq+4)f^{p,q}(s) + \sum_p(2p-1)f^{0,q}(s) + \sum_q(2q-1)f^{p,0}(s) - f^{0,0}(s)\Big),$$

which proves, using (i), that α_- is lower semi-continuous because if $p = 0$ or $q = 0$ then $f^{p,q}$ is constant. □

Next we will study the stratification induced by the **R**-split level. By *stratification* of a topological space (X, T), we mean an increasing filtration by closed subsets

$$X_0 \subset X_1 \subset \ldots \subset X_{max} = X.$$

The *strata* of the stratification are the locally closed subsets $X_i - X_{i-1}$. The upper semi-continuity of the **R**-split level implies the following statement.

Corollary 4.8. *The* **R***-split level α induces a stratification of the complex manifold M^{opp},*

$$M_0^{opp} \subset M_1^{opp} \subset \ldots \subset M^{opp}_{\alpha_{max}} = M^{opp},$$

where

$$s \in M^{opp}_\alpha \Leftrightarrow \alpha((\xi_{M^{opp}}|_{\mathbf{P}^2_{\mathbf{C}} \times \{s\}})^{**}) \le \alpha.$$

Now that we have a stratification of M^{opp}, we deduce a stratification of $\tilde{S}$. This is a consequence of the lemma below whose proof is immediate.

Lemma 4.9. *Let $f : (X, T_X) \to (Y, T_Y)$ be a continuous morphism between topological spaces. Suppose (Y, T_Y) to be stratified by closed subsets $\{Y_i\}_{i\in\mathbf{Z}}$. Then, by letting $X_i = f^{-1}(Y_i)$, one gets a stratification $\{X_i\}_{i\in\mathbf{Z}}$ of (X, T_X).*

Now, we apply this lemma to the morphism $\varphi \times \overline{\varphi}$ which is continuous from the complex manifold $\tilde{S}$ endowed with the real analytic topology to the complex manifold M^{opp} endowed with the usual topology. This stratification descends to S.

Corollary 4.10. *The stratification of the complex manifold M^{opp}, which results from the upper semi-continuity of the $\mathbf{R}$-split level, induces a stratification of $\tilde{S}$,*

$$\tilde{S}_0 \subset \tilde{S}_1 \subset ... \subset \tilde{S}_{\alpha'_{max}} = S,$$

where

$$s \in \tilde{S}_\alpha \Leftrightarrow \alpha(((id_{\mathbf{P}^2_{\mathbf{C}}} \times \varphi \times \overline{\varphi})^* \xi_{M^{opp}}|_{\mathbf{P}^2_{\mathbf{C}} \times \{s\}})^{**}) \leq \alpha.$$

Moreover, this stratification descends to a stratification of S

$$S_0 \subset S_1 \subset ... \subset S_{\alpha'_{max}} = S.$$

Proof. We only have to prove the last statement. It is a direct consequence of the fact that the image of the monodromy homomorphism is a subgroup of $GL(V_{\mathbf{R}})$. This implies that the action of Γ on M^{opp} does not change the relative position of the Hodge filtration and its conjugate and hence conserves the $\mathbf{R}$-split level. Denote by $\pi : \tilde{S} \to S$ the universal covering. By letting $(s \in S_\alpha) \Leftrightarrow (\pi^{-1}(s) \in \tilde{S}_\alpha)$, we thus get the induced filtration. □

4.3. **An example.** Consider two variations of Tate Hodge structure $T\langle i\rangle$, $i = p, q$, $p > q$, on a complex manifold S. Both variations are trivial variations of pure Hodge structure since Tate Hodge structures do not have moduli. Let e_i be a vector generating $T\langle i\rangle_{\mathbf{R}}$. Now, consider an extension of these variations, namely a variation of mixed Hodge structures of rank 2, $(\mathcal{H}_{\mathbf{R}}, \mathcal{W}_\bullet, \mathcal{F}^\bullet)$ such that the variation induced on $Gr_i^{\mathcal{W}}$ is $T\langle i\rangle$, $i = p, q$. The classifying space is totally described by the extensions. According to section 2.4, we have $M \cong \mathbf{C}$ whose coordinate z gives at each point the extension parameter which yields

$$\mathcal{F}^p_{univ}(z) = e_p + e_q z \subset \mathcal{H}_{\mathbf{C}} = \mathbf{C}e_p \oplus \mathbf{C}e_q.$$

By a discussion analogue to the one made in Example 3.11 (with $p = 2$, $q = 0$ and $\mathcal{F}_0^\bullet$ is the decreasing filtration associated to the weight filtration and $P(\lambda, \mu) = \lambda$, $Q(\lambda, \mu) = \mu$), one shows that the relative Rees sheaf $\xi_{M^{opp}}$ on $M^{opp} = M \times M$ is locally free on $\mathbf{P}^2_{\mathbf{C}} \times M^{opp} \setminus (P_{12} \times \Delta_M)$, where Δ_M is the diagonal of $M \times M$. The stratification by the $\mathbf{R}$-split level is given by

$$M_0^{opp} = \Delta_M \subset M^{opp}_{(q-p)^2} = M^{opp}.$$

The morphism to the classifying space $\varphi : S \to M$ induces a stratification of S

$$S_0 \subset S_{(q-p)^2} = S,$$

where S_0 is the subset of points s whose image by $\varphi \times \overline{\varphi}$ belongs to Δ_M, namely such that $\varphi(s) = \overline{\varphi}(s)$. For each point $s \in S_{(q-p)^2} \setminus S_0$ one has the exact sequence of coherent sheaves on $\mathbf{P}^2_{\mathbf{C}}$

$$0 \to \xi(T\langle q\rangle) \to (id_{\mathbf{P}^2} \times \varphi \times \overline{\varphi})^{**}\xi_{M^{opp}}(s) \to \xi(T\langle q\rangle) \to \mathcal{T}_{P_{12}} \to 0,$$

where $T_{P_{12}}$ is a skyscraper sheaf of length $(q-p)^2$.

This example describes, for $p = 2$ and $q = 0$, the variation of mixed Hodge structure given by the a family of non complete nodal curves on $S = \mathbf{P}^1 \setminus \{0, 1, \infty\}$ with coordinate z. Then the morphism $\varphi : S \to M$ is given by $z \mapsto \frac{1}{2\pi i}\log(\frac{z}{z-1})$. According to Proposition 2.18, the real analytic stratification is

$$\mathcal{R}_{\frac{1}{2}} \subset \mathbf{P}^1 \setminus \{0, 1, \infty\} \ .$$

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