THE ORTHOGONAL SUBCATEGORY PROBLEM IN HOMOTOPY THEORY

CARLES CASACUBERTA AND BORIS CHORNY

ABSTRACT. It is known that, in a locally presentable category, localization exists with respect to every set of morphisms, while the statement that localization with respect to every (possibly proper) class of morphisms exists in locally presentable categories is equivalent to a large-cardinal axiom from set theory. One proves similarly, on one hand, that homotopy localization exists with respect to sets of maps in every cofibrantly generated, left proper, simplicial model category $\mathcal{M}$ whose underlying category is locally presentable. On the other hand, as we show in this article, the existence of localization with respect to possibly proper classes of maps in a model category $\mathcal{M}$ satisfying the above assumptions is implied by a large-cardinal axiom called Vopěnka's principle, although we do not know if the reverse implication holds.

We also show that, under the same assumptions on $\mathcal{M}$, every endofunctor of $\mathcal{M}$ that is idempotent up to homotopy is equivalent to localization with respect to some class $\mathcal{S}$ of maps, and if Vopěnka's principle holds then $\mathcal{S}$ can be chosen to be a set. There are examples showing that the latter need not be true if $\mathcal{M}$ is not cofibrantly generated. The above assumptions on $\mathcal{M}$ are satisfied by simplicial sets and symmetric spectra over simplicial sets, among many other model categories.

INTRODUCTION

Locally presentable categories were introduced by Gabriel and Ulmer in [18]. This concept has proved to be very useful in category theory. Among other things, the orthogonal subcategory problem (asking if localization with respect to a given class of morphisms exists) has a positive solution

2000 *Mathematics Subject Classification.* Primary 55U35; Secondary 55P60, 18G55.

Key words and phrases. Model category, localization, large cardinal.

During the preparation of this paper, the second-named author was a fellow of the Marie Curie Training Site with reference no. HPMT-CT-2000-00075 of the European Commission, hosted by the CRM, Barcelona. The first-named author was supported by the Spanish Ministry of Education and Science under MEC-FEDER grant MTM2004-03629.

in locally presentable categories if the given class of morphisms is a set; see, e.g., [1, 1.37]. Moreover, if one assumes the validity of a suitable set-theoretical principle, then there is also a positive solution to this problem for every proper class of morphisms. In fact, Adámek, Rosický and Trnková proved in [2] that the statement that the orthogonal subcategory problem has a solution for every class of morphisms in locally presentable categories is equivalent to the weak Vopěnka principle, a large-cardinal principle that cannot be proved using the usual ZFC axioms (Zermelo–Fraenkel axioms with the axiom of choice).

Localizing with respect to sets of maps is a common technique in homotopy theory, as well as in other areas of Mathematics. However, localizing with respect to proper classes of maps is a more delicate issue, since the standard methods may fall into set-theoretical difficulties (see for instance [9], where positive results in equivariant homotopy theory involving localization with respect to proper classes of maps were obtained). Due to difficulties of this sort, it is still unknown whether the existence of arbitrary cohomological localizations of spaces can be proved or not using the ZFC axioms. An interesting step was made in [8], based on results from [1], by showing that Vopěnka's principle implies the existence of localization with respect to every proper class of maps in the category of simplicial sets. The existence of cohomological localizations follows of course as a special case. Vopěnka's principle is equivalent to the statement that the category of ordinals cannot be fully embedded into the category of graphs (where a graph is meant to be a binary relation). This statement has a place in the hierarchy of large-cardinal principles; see [1].

In this article we contribute further to the ongoing program of extending basic results from locally presentable categories to homotopy theory, which may perhaps give answers to other open problems under large-cardinal assumptions. In order to achieve this, one has to work in suitable model categories. Specifically, our results are stated in left proper, combinatorial, simplicial model categories. The term "combinatorial" means that the model category is cofibrantly generated and the underlying category is locally presentable (see [1] and [20] for the definitions of these concepts). This notion is due to J. H. Smith, who constructed (in unpublished work) localizations of combinatorial model category structures with respect to sets of maps. In [15], Dugger proved that every combinatorial model category is equivalent to a localization of a category of diagrams of simplicial sets, hence generalizing [1, 1.46]. Among many other examples, the model category of simplicial sets and the model category of symmetric spectra based on simplicial sets are combinatorial.

In Section 1 we show that Vopěnka's principle implies the existence of homotopy localization with respect to every class of maps in left proper, combinatorial, simplicial model categories. This fact can also be deduced, with a different argument, from results obtained by Rosický and Tholen in [24, §2]. Furthermore, under Vopěnka's principle, every such localization is equivalent to localization with respect to some set of maps.

Next, we address a closely related question, raised by Dror Farjoun in [11], asking if any functor L on simplicial sets that is idempotent up to homotopy is equivalent to localization with respect to some single map f. He himself showed in [12] that, if L is assumed to be, in addition, *continuous*, then it is indeed equivalent to localization with respect to a *proper class* of maps. This result was improved in [8] by showing that the assumption that L be continuous is unnecessary, and that, under Vopěnka's principle, the proper class of maps defining L can be replaced by a set (in the category of simplicial sets). Furthermore, it was shown that such a replacement of a class by a set cannot be done using only the ZFC axioms, since a counterexample was exhibited by means of another assumption (the nonexistence of measurable cardinals), which is relatively consistent with ZFC.

In Section 2 we show (without resorting to large-cardinal principles) that every homotopy idempotent functor L in a simplicial model category $\mathcal{M}$ is equivalent to localization with respect to a proper class of maps, assuming either that L is continuous or that $\mathcal{M}$ satisfies suitable hypotheses allowing to approximate any homotopy functor by a continuous functor. For this, one may assume that $\mathcal{M}$ is a simplicial model category that is proper, cofibrantly generated and stable (as in [23]), or left proper and either combinatorial or cellular (as in [14]). Furthermore, if one assumes that Vopěnka's principle is true and $\mathcal{M}$ is combinatorial, then, again, the proper class of maps defining L can be replaced by a set. In most cases of interest, such a set of maps can further be replaced by a single map (by taking the coproduct of all maps in the set), but not always, as we show by means of an example at the end of the paper.

In [10] an example was given of a homotopy idempotent functor in a locally presentable (but not cofibrantly generated) model category that fails to be a localization with respect to any set of maps. Namely, in the category of maps between simplicial sets with the model structure generated by the collection of orbits (as defined in [13]), the functor that sends every map to the final object (i.e., a map between two points) is not a localization with respect to any set of maps. Hence our results in Section 2 below are sharp.

Acknowledgements: Discussions with Mark Hovey, Jiří Rosický, Brooke Shipley and Jeff Smith are greatly appreciated.

1. Simplicial orthogonality

Model categories were introduced by Quillen in [22] and have recently been discussed in the books [16], [19], [20], [21], among many other places, with slight changes in the terminology and even in the assumptions. In this article we will assume that model categories are complete, cocomplete, and equipped with functorial factorizations. See [16, § 9], [20, § 7] or [21, § 1] for more details.

Although our main results are stated for *simplicial* model categories (for the definition, see for example [19, II.3] or [20, 9.1.5]), several of our steps require only the use of *homotopy function complexes*, as introduced in [17] and discussed in [20, Ch. 17] or [21, § 5]. Thus, for any given model category $\mathcal{M}$, we choose functorially a fibrant simplicial set $\mathrm{map}(X, Y)$ for each X and Y in $\mathcal{M}$, whose homotopy type is the same as the diagonal of the bisimplicial set $\mathcal{M}(X^*, Y_*)$ where $X^* \to X$ is a cosimplicial resolution of X and $Y \to Y_*$ is a simplicial resolution of Y. The homotopy type of $\mathrm{map}(X, Y)$ remains unchanged if X or Y are replaced by weakly equivalent objects. If $\mathcal{M}$ is a simplicial model category and $\mathrm{Map}(X, Y)$ denotes the simplicial set given as part of the structure of $\mathcal{M}$, then $\mathrm{Map}(QX, RY)$ is a good choice of a homotopy function complex, where Q is a cofibrant approximation functor and R is a fibrant approximation functor in $\mathcal{M}$.

Before discussing simplicial orthogonality in model categories by means of homotopy function complexes, we recall the following older concepts from category theory. If $\mathcal{C}$ is any category, an object X and a morphism $f\colon A \to B$ are called *orthogonal* (see [1] or [7] for details and motivation) if the induced function

$$\mathcal{C}(f, X)\colon \mathcal{C}(B, X) \longrightarrow \mathcal{C}(A, X)$$

is bijective. (We denote by $\mathcal{C}(X, Y)$ the set of morphisms from X to Y in $\mathcal{C}$.) If L is an endofunctor of $\mathcal{C}$ equipped with a natural transformation $\eta\colon \mathrm{Id} \to L$ such that $L\eta\colon L \to LL$ is an isomorphism and $\eta L = L\eta$, then L is called an *idempotent functor* or a *localization*. Then every object isomorphic to LX for some X is orthogonal to every morphism f such that Lf is an isomorphism, and these two classes determine each other by the orthogonality relation; that is, an object is isomorphic to LX for some X if and only if it is orthogonal to all morphisms f such that Lf is an isomorphism, and reciprocally.

As a special case, this terminology applies to the homotopy category $\mathrm{Ho}\mathcal{M}$ associated with any model category $\mathcal{M}$. Thus, orthogonality in $\mathrm{Ho}\mathcal{M}$ between an object X and a map $f\colon A \to B$ amounts to the condition that

$$[f, X]\colon [B, X] \longrightarrow [A, X] \tag{1}$$

be bijective, where $[X,Y]$ means, as usual, $\operatorname{Ho}\mathcal{M}(X,Y)$. Examples of idempotent functors in the homotopy category of simplicial sets, such as homological localizations, have been studied since several decades ago; see [4].

Throughout the extensive study of localizations undertaken since then in homotopy theory, a stronger notion of "simplicially enriched orthogonality" came to be considered. There is no widely agreed terminology for it yet. It was called *simplicial orthogonality* in [8] and *homotopy orthogonality* in [20, §17]. Thus, if $\mathcal{M}$ is any model category with a choice of homotopy function complexes, an object X and a map $f\colon A \to B$ will be called *simplicially orthogonal* or *homotopy orthogonal* (not to be confused with orthogonality in $\operatorname{Ho}\mathcal{M}$) if the induced map

$$\operatorname{map}(f,X)\colon \operatorname{map}(B,X) \longrightarrow \operatorname{map}(A,X) \tag{2}$$

is a weak equivalence of simplicial sets. Since there is a natural bijection between $\pi_0 \operatorname{map}(X,Y)$ and $[X,Y]$, homotopy orthogonality implies indeed orthogonality in the homotopy category $\operatorname{Ho}\mathcal{M}$.

The fibrant objects that are homotopy orthogonal to a given map f are usually called *f-local*. More generally, if $\mathcal{S}$ is any class of maps, the fibrant objects that are homotopy orthogonal to all the maps in $\mathcal{S}$ are called *$\mathcal{S}$-local*. We denote by $\mathcal{S}^{\mathrm{h}\perp}$ the closure under weak equivalences of the class of $\mathcal{S}$-local objects, and call it the *homotopy orthogonal complement* of $\mathcal{S}$. Similarly, for a class $\mathcal{D}$ of objects, we denote by $\mathcal{D}^{\mathrm{h}\perp}$ the class of maps that are homotopy orthogonal to all the objects in $\mathcal{D}$. In particular, the maps in $(\mathcal{S}^{\mathrm{h}\perp})^{\mathrm{h}\perp}$ are called *$\mathcal{S}$-local equivalences*, or shortly *$\mathcal{S}$-equivalences*.

The homotopy-theoretical version of the *orthogonal subcategory problem* asks if $\mathcal{S}^{\mathrm{h}\perp}$ is reflective in $\operatorname{Ho}\mathcal{M}$ for a model category $\mathcal{M}$ and a given class of maps $\mathcal{S}$ in $\mathcal{M}$ (that is, if a localization L exists in $\operatorname{Ho}\mathcal{M}$ such that the closure of the image of L under isomorphisms is precisely the class $\mathcal{S}^{\mathrm{h}\perp}$). One reason for using (2) instead of (1) as orthogonality relation is the fact that the answer to the orthogonal subcategory problem would too often be negative using (1). For instance, there is no localization in the homotopy category of simplicial sets onto the class of simply connected spaces. See [5] for a more elaborate counterexample.

In order to construct localizations in $\operatorname{Ho}\mathcal{M}$, one normally operates in the corresponding model category $\mathcal{M}$. The following terminology is commonly used. A *homotopy idempotent functor* is an endofunctor $L\colon \mathcal{M} \to \mathcal{M}$ preserving weak equivalences, taking fibrant values, and equipped with a natural transformation $\eta\colon \mathrm{Id} \to L$ (called a *coaugmentation*) which is idempotent up to homotopy; that is, for each object X, the morphisms $L\eta_X$ and η_{LX} from LX to LLX coincide in $\operatorname{Ho}\mathcal{M}$ and are weak equivalences. Thus L defines indeed a localization in $\operatorname{Ho}\mathcal{M}$.

If L is a homotopy idempotent functor such that LX is $\mathcal{S}$-local for all X and $\eta_X\colon X \to LX$ is an $\mathcal{S}$-equivalence for all X, where $\mathcal{S}$ is any class of maps, then it is said that L is a *homotopy localization with respect to the class* $\mathcal{S}$, or shortly an *$\mathcal{S}$-localization.* In this case, the class $\mathcal{S}^{\mathrm{h}\perp}$ is indeed the closure of the image of L under isomorphisms in $\mathrm{Ho}\mathcal{M}$. (In order to prove that every object Y in $\mathcal{S}^{\mathrm{h}\perp}$ is weakly equivalent to LX for some X, consider the coaugmentation $\eta_Y\colon Y \to LY$, which induces a weak equivalence $\mathrm{map}(LY, Y) \simeq \mathrm{map}(Y, Y)$, hence a bijection $[LY, Y] \cong [Y, Y]$ yielding a map $LY \to Y$ in $\mathrm{Ho}\mathcal{M}$ which is inverse to η_Y, so $Y \simeq LY$, as needed.)

It is well known that the orthogonal subcategory problem for a class of maps $\mathcal{S}$ in a model category $\mathcal{M}$ has a positive solution whenever $\mathcal{S}$ is a set and $\mathcal{M}$ satisfies certain assumptions, which vary slightly depending on the authors. We will call a model category *combinatorial* if it is cofibrantly generated and the underlying category is locally presentable. The definition of a locally presentable category can be found in [1] or [18], and the definition of a cofibrantly generated model category is contained, e.g., in [20]. The notion of properness is also discussed in [20].

Theorem 1.1. *Let $\mathcal{M}$ be a left proper, combinatorial, simplicial model category. For every set of maps $\mathcal{S}$ there is a homotopy localization with respect to $\mathcal{S}$.*

Proof. The core of the proof is in [3]. See [20] for an updated approach. □

As far as we know, there is no way to prove this when $\mathcal{S}$ is a proper class, not even for simplicial sets, using the ordinary axioms of set theory. In [8] it was shown that the statement of Theorem 1.1 holds for a proper class $\mathcal{S}$ in the category of simplicial sets using a suitable large-cardinal axiom (Vopěnka's principle). We now undertake a generalization of this fact to other model categories.

If $\mathcal{M}$ is a cofibrantly generated model category and $\mathcal{C}$ is a small category, then the *projective* model structure on the category $\mathcal{M}^{\mathcal{C}}$ of $\mathcal{C}$-indexed diagrams in $\mathcal{M}$ has objectwise weak equivalences and objectwise fibrations, while the *injective* model structure has objectwise weak equivalences and objectwise cofibrations. The projective model structure is discussed in [20, 11.6 and 11.7], where it is shown that, if the model category $\mathcal{M}$ is simplicial, then $\mathcal{M}^{\mathcal{C}}$ with the projective model structure is also simplicial.

Lemma 1.2. *Let $\mathcal{M}$ be a cofibrantly generated simplicial model category and $\mathcal{C}$ a small category. Suppose that A is a cofibrant diagram in the projective model category structure of $\mathcal{M}^{\mathcal{C}}$ and X is a fibrant object of $\mathcal{M}$. Then* $\mathrm{Map}(A, X)$ *is fibrant in the injective model structure on the category of $\mathcal{C}^{\mathrm{op}}$-diagrams of simplicial sets.*

Proof. We have to show that any commutative square

$$\begin{array}{ccc} C & \longrightarrow & \mathrm{Map}(A,X) \\ {\scriptstyle i}\big\downarrow{\scriptstyle\wr} & & \big\downarrow \\ D & \longrightarrow & *, \end{array}$$

where i is an objectwise trivial cofibration of $\mathcal{C}^{\mathrm{op}}$-diagrams of simplicial sets, admits a lift. By adjunction (as in [20, 18.3.9]), this problem is equivalent to finding a lift in the following commutative square in $\mathcal{M}$:

$$\begin{array}{ccc} A \otimes_{\mathcal{C}} C & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ A \otimes_{\mathcal{C}} D & \longrightarrow & *. \end{array}$$

This problem is equivalent, by another adjunction, to finding a lift in the following commutative square in $\mathcal{M}^{\mathcal{C}}$:

$$\begin{array}{ccc} \emptyset & \longrightarrow & X^D \\ \big\downarrow & & {\scriptstyle X^i}\big\downarrow{\scriptstyle\wr} \\ A & \longrightarrow & X^C. \end{array}$$

In the last square a lift exists, since A is projectively cofibrant and X^i is a projective trivial fibration. □

Recall that a partially ordered set I is called λ-*directed*, where λ is a regular cardinal, if every subset of I of cardinality smaller than λ has an upper bound.

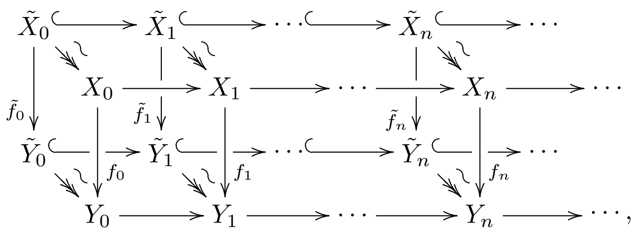
Lemma 1.3. *Let $\mathcal{D}$ be any class of objects in a combinatorial simplicial model category $\mathcal{M}$, and let $\mathcal{S}$ be its homotopy orthogonal complement. Then there exists a regular cardinal λ such that $\mathcal{S}$ is closed under λ-directed colimits in the category of maps of $\mathcal{M}$.*

Proof. Let C be a set of generating cofibrations for $\mathcal{M}$. Choose a regular cardinal λ such that any member of the set of domains and codomains of maps in C is λ-presentable (such a cardinal exists since $\mathcal{M}$ is locally presentable). Let I be any λ-directed partially ordered set, and suppose given a diagram $f\colon I \to \mathrm{Arr}\,\mathcal{M}$, where $\mathrm{Arr}\,\mathcal{M}$ is the category of maps in $\mathcal{M}$. Let us denote by $X\colon I \to \mathcal{M}$ the domain of f and by $Y\colon I \to \mathcal{M}$ the codomain, so f can also be seen as a map from X to Y in $\mathcal{M}^I$. Let us depict it, for simplicity, as a chain:

$$\begin{array}{ccccccccc} X_0 & \longrightarrow & X_1 & \longrightarrow & \cdots & \longrightarrow & X_n & \longrightarrow & \cdots \\ {\scriptstyle f_0}\big\downarrow & & {\scriptstyle f_1}\big\downarrow & & & & {\scriptstyle f_n}\big\downarrow & & \\ Y_0 & \longrightarrow & Y_1 & \longrightarrow & \cdots & \longrightarrow & Y_n & \longrightarrow & \cdots . \end{array} \tag{3}$$

Suppose that the maps f_i are in $\mathcal{S}$ for each $i \in I$. Since $\mathcal{M}$ is cocomplete, $\operatorname{Arr}\mathcal{M}$ is cocomplete as well, and we may consider the colimit of the diagram f. We need to show that the map $\operatorname{colim} f\colon \operatorname{colim} X \longrightarrow \operatorname{colim} Y$ is in $\mathcal{S}$.

Choose a cofibrant approximation $\tilde{f}\colon \tilde{X} \to \tilde{Y}$ to f using the projective model structure on $\mathcal{M}^I$, hence obtaining the following commutative diagram in $\mathcal{M}$:



where $\tilde{f}_i$ is a cofibrant approximation to f_i.

For every $Z \in \mathcal{D}$, let $\hat{Z}$ be a fibrant approximation to Z. The induced map

$$\operatorname{Map}(\operatorname{colim}\tilde{f}, \hat{Z})\colon \operatorname{Map}(\operatorname{colim}\tilde{Y}, \hat{Z}) \longrightarrow \operatorname{Map}(\operatorname{colim}\tilde{X}, \hat{Z})$$

can be written as a limit of a diagram of maps of simplicial sets

$$\lim \operatorname{Map}(\tilde{f}, \hat{Z})\colon \lim \operatorname{Map}(\tilde{Y}, \hat{Z}) \longrightarrow \lim \operatorname{Map}(\tilde{X}, \hat{Z}).$$

The I^{op}-diagrams of simplicial sets $\operatorname{Map}(\tilde{X}, \hat{Z})$ and $\operatorname{Map}(\tilde{Y}, \hat{Z})$ are fibrant in the injective model structure by Lemma 1.2. Therefore, their inverse limits are homotopy inverse limits (since the constant diagram of points is cofibrant in the injective model structure). Hence, $\operatorname{Map}(\operatorname{colim}\tilde{f}, \hat{Z})$ is a weak equivalence, as a map induced between homotopy inverse limits by levelwise weak equivalences $\operatorname{Map}(\tilde{f}_i, \hat{Z})$. This shows that $\operatorname{colim}\tilde{f}$ is in $\mathcal{S}$.

Now trivial fibrations in $\mathcal{M}$ are preserved under λ-directed colimits, since the set C of generating cofibrations has λ-presentable domains and codomains. From the commutative diagram

$$\begin{array}{ccc} \operatorname{colim}\tilde{X} & \overset{\sim}{\twoheadrightarrow} & \operatorname{colim} X \\ {\scriptstyle \operatorname{colim}\tilde{f}}\big\downarrow & & \big\downarrow{\scriptstyle \operatorname{colim} f} \\ \operatorname{colim}\tilde{Y} & \overset{\sim}{\twoheadrightarrow} & \operatorname{colim} Y \end{array}$$

we conclude that the map colim $\tilde{f}$ is a cofibrant approximation to the map colim f, since both colim $\tilde{X}$ and colim $\tilde{Y}$ are cofibrant in $\mathcal{M}$ (indeed, $\tilde{X}$ and $\tilde{Y}$ are cofibrant diagrams in $\mathcal{M}^I$, and the colimit functor $\mathcal{M}^I \to \mathcal{M}$ is left Quillen by [20, 11.6.8]). Hence, colim f is in $\mathcal{S}$, as claimed. □

The statement of Vopěnka's principle and enough motivation for its use in this context can be found in [1], [2], [8], and [24].

Lemma 1.4. *Suppose that Vopěnka's principle is true. Let $\mathcal{D}$ be any class of objects in a combinatorial simplicial model category $\mathcal{M}$, and let $\mathcal{S} = \mathcal{D}^{\mathrm{h}\perp}$. Then there exists a set of maps $\mathcal{X}$ such that $\mathcal{X}^{\mathrm{h}\perp} = \mathcal{S}^{\mathrm{h}\perp}$.*

Proof. By abuse of notation, we also denote by $\mathcal{S}$ the full subcategory of $\operatorname{Arr}\mathcal{M}$ generated by the class $\mathcal{S}$. Since $\mathcal{M}$ is locally presentable, $\operatorname{Arr}\mathcal{M}$ is also locally presentable. Then, assuming Vopěnka's principle, it follows from [1, Theorem 6.6] that $\mathcal{S}$ is bounded, i.e., it has a small dense subcategory. We have shown in Lemma 1.3 that there exists a regular cardinal λ such that $\mathcal{S}$ is closed under λ-directed colimits in the category $\operatorname{Arr}\mathcal{M}$. Hence, by [1, Corollary 6.18], the full subcategory generated by $\mathcal{S}$ in $\operatorname{Arr}\mathcal{M}$ is accessible. Thus, for a certain regular cardinal $\lambda_0 \geq \lambda$, the class $\mathcal{S}$ contains a set $\mathcal{X}$ of λ_0-presentable objects such that every object of $\mathcal{S}$ is a λ_0-directed colimit of objects of $\mathcal{X}$.

Since $\mathcal{X} \subset \mathcal{S}$, we have $\mathcal{X}^{\mathrm{h}\perp} \supset \mathcal{S}^{\mathrm{h}\perp}$ and $(\mathcal{X}^{\mathrm{h}\perp})^{\mathrm{h}\perp} \subset (\mathcal{S}^{\mathrm{h}\perp})^{\mathrm{h}\perp} = \mathcal{S}$. Our aim now is to show the reverse inclusion $(\mathcal{X}^{\mathrm{h}\perp})^{\mathrm{h}\perp} \supset \mathcal{S}$. By Lemma 1.3, $(\mathcal{X}^{\mathrm{h}\perp})^{\mathrm{h}\perp}$ is closed under λ-directed colimits. Hence $(\mathcal{X}^{\mathrm{h}\perp})^{\mathrm{h}\perp}$ is also closed under λ_0-directed colimits and every element of $\mathcal{S}$ is a λ_0-directed colimit of elements of $\mathcal{X}$. This concludes the proof, since $((\mathcal{X}^{\mathrm{h}\perp})^{\mathrm{h}\perp})^{\mathrm{h}\perp} = \mathcal{X}^{\mathrm{h}\perp}$. □

Theorem 1.5. *Let $\mathcal{M}$ be a left proper, combinatorial, simplicial model category. If Vopěnka's principle is assumed true, then for any (possibly proper) class of maps $\mathcal{S}$ there is a homotopy localization with respect to $\mathcal{S}$.*

Proof. By Lemma 1.4, there exists a set $\mathcal{X}$ of maps in $\mathcal{M}$ such that $\mathcal{X}^{\mathrm{h}\perp} = \mathcal{S}^{\mathrm{h}\perp}$. Then the homotopy localization with respect to $\mathcal{X}$, which exists by Theorem 1.1, is an $\mathcal{S}$-localization. □

Thus, the statement of Theorem 1.5 is a positive answer to the orthogonal subcategory problem for all classes of maps in sufficiently good model categories.

2. Idempotent functors and simplicial orthogonality

The next theorem is motivated by results of Dror Farjoun in [12]. We consider a model category $\mathcal{M}$ and assume, as in the beginning of the previous section, that a functorial choice of a homotopy function complex $\mathrm{map}(X, Y)$ for all X and Y has been made.

In what follows, if $f\colon A \to B$ is a map and X is an object, we keep denoting by $\mathrm{map}(f, X)$ the map of simplicial sets $\mathrm{map}(B, X) \to \mathrm{map}(A, X)$ induced by f. If $\eta\colon F \to G$ is a natural transformation between two functors and H is another functor, then $\eta H\colon FH \to GH$ denotes the natural transformation given by $(\eta H)_X = \eta_{HX}$ for every object X, and $H\eta\colon HF \to HG$ denotes the natural transformation given by $(H\eta)_X = H\eta_X$ for all X.

Theorem 2.1. *Let $\mathcal{M}$ be any model category. Let L be an endofunctor of the homotopy category* $\mathrm{Ho}\mathcal{M}$ *with the following properties:*

(a) *There is a natural transformation $\eta\colon \mathrm{Id} \to L$ in* $\mathrm{Ho}\mathcal{M}$ *such that $L\eta = \eta L$ and $L\eta\colon L \to LL$ is an isomorphism on all objects.*
(b) *There is a map $l_{X,Y}\colon \mathrm{map}(X, Y) \to \mathrm{map}(LX, LY)$ for all X, Y, which is natural in both variables up to homotopy.*
(c) $\mathrm{map}(\eta_X, LY) \circ l_{X,Y} \simeq \mathrm{map}(X, \eta_Y)$ *for all X and Y.*

Then the map

$$\mathrm{map}(\eta_X, LY)\colon\ \mathrm{map}(LX, LY) \longrightarrow \mathrm{map}(X, LY)$$

is a weak equivalence for all X, Y.

Proof. Let us write $Z = LY$ for simplicity. The assumption (a) says precisely that L is idempotent in the homotopy category $\mathrm{Ho}\mathcal{M}$. Hence, among other consequences of this fact, $\eta_Z\colon Z \to LZ$ is an isomorphism in $\mathrm{Ho}\mathcal{M}$. Then $\mathrm{map}(A, \eta_Z)$ is a weak equivalence of fibrant simplicial sets for every A, hence a homotopy equivalence. Choose a homotopy inverse

$$\xi_{A,Z}\colon\ \mathrm{map}(A, LZ) \longrightarrow \mathrm{map}(A, Z)$$

of $\mathrm{map}(A, \eta_Z)$ for each A. We claim that $\xi_{LX,Z} \circ l_{X,Z}$ is now a homotopy inverse of $\mathrm{map}(\eta_X, LY)$. The proof proceeds as in [6, Theorem 2.4]. On one hand, by the naturality of l,

$$\xi_{LX,Z} \circ l_{X,Z} \circ \mathrm{map}(\eta_X, Z) \simeq \xi_{LX,Z} \circ \mathrm{map}(L\eta_X, LZ) \circ l_{LX,Z}.$$

Then, using the fact that $L\eta = \eta L$ in $\mathrm{Ho}\mathcal{M}$ and assumption (c), we obtain

$$\begin{aligned} &\xi_{LX,Z} \circ \mathrm{map}(L\eta_X, LZ) \circ l_{LX,Z} \simeq \\ &\qquad \xi_{LX,Z} \circ \mathrm{map}(\eta_{LX}, LZ) \circ l_{LX,Z} \simeq \xi_{LX,Z} \circ \mathrm{map}(LX, \eta_Z) \simeq \mathrm{id}. \end{aligned}$$

On the other hand,

$$\mathrm{map}(\eta_X, Z) \circ \xi_{LX,Z} \circ l_{X,Z} \simeq \xi_{X,Z} \circ \mathrm{map}(X, \eta_Z) \circ \mathrm{map}(\eta_X, Z) \circ \xi_{LX,Z} \circ l_{X,Z}.$$

Since composition with η_X on the left and composition with η_Z on the right commute, we obtain

$$\begin{aligned}\xi_{X,Z} \circ \mathrm{map}(X, \eta_Z) \circ \mathrm{map}(\eta_X, Z) \circ \xi_{LX,Z} \circ l_{X,Z} \simeq \\ \xi_{X,Z} \circ \mathrm{map}(\eta_X, LZ) \circ \mathrm{map}(LX, \eta_Z) \circ \xi_{LX,Z} \circ l_{X,Z} \simeq \\ \xi_{X,Z} \circ \mathrm{map}(\eta_X, LZ) \circ l_{X,Z}.\end{aligned}$$

Finally, using (c) again,

$$\xi_{X,Z} \circ \mathrm{map}(\eta_X, LZ) \circ l_{X,Z} \simeq \xi_{X,Z} \circ \mathrm{map}(X, \eta_Z) \simeq \mathrm{id},$$

which completes the proof. □

Assumptions (b) and (c) in Theorem 2.1 need not be satisfied by arbitrary idempotent functors in Ho$\mathcal{M}$, not even by those derived from functors in $\mathcal{M}$. Recall that a functor F in a simplicial model category is called *simplicial* or *continuous* if it is equipped with natural maps of simplicial sets

$$l^F_{X,Y} \colon \mathrm{Map}(X, Y) \longrightarrow \mathrm{Map}(FX, FY)$$

preserving composition and identity; see [19, IX.1] or [20, 9.8]. A natural transformation $\zeta \colon F \to G$ of simplicial functors is called *simplicial* or *continuous* if

$$\mathrm{Map}(\zeta_X, GY) \circ l^G_{X,Y} = \mathrm{Map}(FX, \zeta_Y) \circ l^F_{X,Y} \tag{4}$$

for all X and Y; cf. [19, IX.1].

Thus, we may view conditions (b) and (c) in Theorem 2.1 as "continuity up to homotopy" of L and η, respectively. What we have shown is that continuity up to homotopy is sufficient for the validity of Dror Farjoun's result [12, Theorem 2.1]. In fact we have extended it to arbitrary model categories.

Now we use Proposition 6.4 in [23] to show that the assumptions (b) and (c) in Theorem 2.1 hold automatically in most cases of interest. Let $\mathcal{M}$ be any model category and let $s\mathcal{M}$ denote the category of simplicial objects over $\mathcal{M}$. The *canonical model structure* on $s\mathcal{M}$ is the one where every level equivalence is a weak equivalence, the cofibrations are the Reedy cofibrations, and the fibrant objects are the homotopically constant Reedy fibrant objects (see [23] for motivation and further details). This model structure need not exist; however, when it exists, $s\mathcal{M}$ is a simplicial model category that is Quillen equivalent to $\mathcal{M}$. Moreover, the simplicial model category structure on $s\mathcal{M}$ is unique up to simplicial Quillen equivalence.

Sufficient conditions for the existence of the canonical model structure in $s\mathcal{M}$ were given in [23], and other sufficient conditions can be found in [14]. Pointed model categories where the suspension functor and the loop functor are inverse equivalences on the homotopy category are called *stable*.

According to [23, Proposition 4.5], if $\mathcal{M}$ is a proper, cofibrantly generated, stable model category, then the canonical model structure on $s\mathcal{M}$ exists. Likewise, as shown in [14], if $\mathcal{M}$ is left proper and combinatorial, or left proper and cellular, then the canonical model structure on $s\mathcal{M}$ exists.

Theorem 2.2. *Let $\mathcal{M}$ be a cofibrantly generated simplicial model category where the canonical model category structure exists in $s\mathcal{M}$. Let L be an endofunctor of $\mathcal{M}$ equipped with a natural transformation $\eta\colon \mathrm{Id} \to L$ that renders L homotopy idempotent. Then L is a homotopy localization with respect to the class of maps η_X for all X.*

Proof. Let us denote by L' the simplicial approximation to L given by [23, Corollary 6.5]. Thus, $L'X = |Q\hat{L}\,\mathrm{Sing}\,RX|$ for each object X, where the notation is as follows. The singular functor Sing is defined as $(\mathrm{Sing}\,X)_n = X^{\Delta[n]}$ for all n; the realization functor $|-|$ is its left adjoint; $\hat{L}$ is the prolongation of L over $s\mathcal{M}$; R is a fibrant replacement functor in $\mathcal{M}$ and Q is a simplicial cofibrant replacement functor in $s\mathcal{M}$. By its construction, L' is a simplicial functor, since it is a composite of simplicial functors (see [23] for details), and there is a ziz-zag of weak equivalences between LX and $L'X$ for all X.

Although it is not explicitly stated in [23], if $\zeta\colon F \to G$ is any natural transformation of functors that preserve weak equivalences, then the above construction yields a natural transformation $\zeta'\colon F' \to G'$ which is itself simplicial. To prove this claim, note that, for each family of objects X_i in $\mathcal{M}$ indexed by a set I, the following diagram commutes:

$$\begin{array}{ccc}
\coprod_{i\in I} FX_i & \longrightarrow & F\left(\coprod_{i\in I} X_i\right) \\
\downarrow & & \downarrow \\
\coprod_{i\in I} GX_i & \longrightarrow & G\left(\coprod_{i\in I} X_i\right),
\end{array}$$

where the horizontal arrows are defined by applying F or G to the inclusions into the coproduct, and the vertical arrows are given by the natural transformation ζ. Now recall that for a simplicial set K and an object A in $s\mathcal{M}$, one defines $K \otimes A$ by $(K\otimes A)_n = \coprod_{s\in K_n} A_n$ (cf. [23]). Hence the above diagram yields the commutativity of the diagram

$$\begin{array}{ccc}
\Delta[n]\otimes \hat{F}A & \longrightarrow & \hat{F}(\Delta[n]\otimes A) \\
\downarrow & & \downarrow \\
\Delta[n]\otimes \hat{G}A & \longrightarrow & \hat{G}(\Delta[n]\otimes A),
\end{array}$$

for all n and every A in $s\mathcal{M}$, where $\hat{F}$ is the prolongation of F over $s\mathcal{M}$ and $\hat{G}$ is the prolongation of G. This implies that the following diagram (where the vertical arrows are now given by ζ') is also commutative:

$$\begin{array}{ccccc} \Delta[n]\otimes F'X & \longrightarrow & F'(\Delta[n]\otimes X) & \xrightarrow{F'\sigma} & F'Y \\ \downarrow & & \downarrow & & \downarrow \\ \Delta[n]\otimes G'X & \longrightarrow & G'(\Delta[n]\otimes X) & \xrightarrow{G'\sigma} & G'Y, \end{array}$$

for every n-simplex $\sigma\colon \Delta[n]\otimes X \to Y$ of $\mathrm{Map}(X,Y)$, where X and Y are any two objects of $\mathcal{M}$. This says precisely that ζ' is a simplicial natural transformation.

Thus, in our case, there is a simplicial natural transformation $\eta'\colon \mathrm{Id}' \to L'$ (where Id' need not be the identity). Therefore, although L' need not be a coaugmented functor in $\mathcal{M}$, it follows that L and η fulfill the conditions of Theorem 2.1 in the homotopy category $\mathrm{Ho}\mathcal{M}$, since L and L' define isomorphic functors in $\mathrm{Ho}\mathcal{M}$. More precisely, take $\mathrm{map}(X,Y) = \mathrm{Map}(QX,RY)$ as a homotopy function complex in $\mathcal{M}$. Then the map $l_{X,Y}$ required in (b) is the composite

$$\mathrm{Map}(QX,RY) \longrightarrow \mathrm{Map}(L'QX,L'RY) \longrightarrow \mathrm{Map}(L'QX,RL'RY),$$

where the first arrow is given by the continuity of L', followed by an equivalence

$$\mathrm{Map}(L'QX,RL'RY) \simeq \mathrm{Map}(QL'X,RL'Y)$$

given by the fact that L' preserves weak equivalences and takes cofibrant values since the realization functor preserves cofibrations; see [23].

The conclusion of Theorem 2.1 implies then that L is a homotopy localization with respect to the class of maps of the form η_X for all X. □

Now the results of the previous section yield the following answer to Dror Farjoun's problem in sufficiently good model categories.

Theorem 2.3. *Assuming Vopěnka's principle, any homotopy idempotent functor in a left proper, combinatorial, simplicial model category is an $\mathcal{X}$-localization for some set of maps $\mathcal{X}$.*

Proof. Under these assumptions, the canonical model structure exists in $s\mathcal{M}$ by [14]; cf. [23, Remark 3.8]. Therefore, Theorem 2.2 can be used and Lemma 1.4 completes the argument. □

This result applies to a useful case not previously established in the literature, namely to the stable homotopy category of Adams–Boardman, by using, for example, the model category of symmetric spectra based on simplicial sets.

In the model categories of simplicial sets or spectra, the set $\mathcal{X}$ of maps given by Theorem 2.3 can be replaced by a single map f, namely the coproduct $\coprod_{g\in\mathcal{X}} g$ of all maps in $\mathcal{X}$. In a general model category, one has to be more careful, in view of the next counterexample.

Consider the model category which is a product of two copies of the category of simplicial sets, i.e., the category of diagrams of simplicial sets over the discrete category with two objects, equipped with the projective model structure (where fibrations and weak equivalences are objectwise). Take $S = \{f, g\}$ for

$$f\colon (\emptyset,\emptyset) \longrightarrow (*,\emptyset) \quad \text{and} \quad g\colon (\emptyset,*) \longrightarrow (\emptyset, *\textstyle\coprod *)\,.$$

An object (X, Y) is S-local if and only if X and Y are fibrant, X is contractible and Y is either contractible or empty.

Suppose that there exists a map

$$h\colon (A,B) \longrightarrow (C,D)$$

such that any S-local object is also h-local, and vice versa. The object $(X,\emptyset)$ is h-local if and only if X is contractible. This condition implies that both B and D are empty; otherwise, for any simplicial set Z, either contractible or not, the object $(Z,\emptyset)$ would be h-local. But in this case any object (X,Y) with contractible X becomes h-local, hence the contradiction. Note however that, in order to ensure that every set of maps yields the same localization as their coproduct, it is enough to assume that the set of maps $X \to Y$ is nonempty for all X and Y in the model category under consideration.

References

[1] J. Adámek and J. Rosický, *Locally Presentable and Accessible Categories*, London Math. Soc. Lecture Note Ser. 189, Cambridge University Press, Cambridge, 1994.

[2] J. Adámek, J. Rosický, and V. Trnková, Are all limit-closed subcategories of locally presentable categories reflective?, in: *Categorical Algebra and its Applications (Louvain-la-Neuve, 1987)*, Lecture Notes in Math. 1348, Springer-Verlag, Berlin, Heidelberg, New York, 1988, pp. 1–18.

[3] A. K. Bousfield, Constructions of factorization systems in categories, *J. Pure Appl. Algebra* **9** (1976/77), no. 2, 207–220.

[4] A. K. Bousfield, The localization of spaces with respect to homology, *Topology* **14** (1975), 133–150.

[5] C. Casacuberta, On the rationalization of the circle, *Proc. Amer. Math. Soc.* **118** (1993), no. 3, 995–1000.

[6] C. Casacuberta, M. Golasiński, and A. Tonks, Homotopy localization of groupoids, *Forum Math.*, to appear.

[7] C. Casacuberta, G. Peschke, and M. Pfenniger, On orthogonal pairs in categories and localisation, in: *Adams Memorial Symposium on Algebraic Topology, vol. 1 (Manchester, 1990)*, London Math. Soc. Lecture Note Ser. 175, Cambridge University Press, Cambridge, 1992, pp. 211–223.

[8] C. Casacuberta, D. Scevenels, and J. H. Smith, Implications of large-cardinal principles in homotopical localizations, *Adv. Math.* **197** (2005), 120–139.
[9] B. Chorny, Localization with respect to a class of maps I – Equivariant localization of diagrams of spaces, *Israel J. Math.* **147** (2005), 93–140.
[10] B. Chorny, Localization with respect to a class of maps II – Equivariant cellularization and its applications, *Israel J. Math.* **147** (2005), 141–156.
[11] E. Dror Farjoun, Homotopy localization and v_1-periodic spaces, in: *Algebraic Topology; Homotopy and Group Cohomology (Sant Feliu de Guíxols, 1990)*, Lecture Notes in Math. 1509, Springer-Verlag, Berlin, Heidelberg, New York, 1992, pp. 104–113.
[12] E. Dror Farjoun, Higher homotopies of natural constructions, *J. Pure Appl. Algebra* **108** (1996), 23–34.
[13] E. Dror Farjoun, Homotopy and homology of diagrams of spaces, in: *Proceedings of the Conference on Algebraic Topology, Seattle 1985*, Lecture Notes in Math. 1286, Springer-Verlag, Berlin, Heidelberg, New York, 1987, pp. 93–134.
[14] D. Dugger, Replacing model categories with simplicial ones, *Trans. Amer. Math. Soc.* **353** (2001), no. 12, 5003–5027.
[15] D. Dugger, Combinatorial model categories have presentations, *Adv. Math.* **164** (2001), 177–201.
[16] W. G. Dwyer, P. S. Hirschhorn, D. M. Kan, and J. H. Smith, *Homotopy Limit Functors on Model Categories and Homotopical Categories*, Math. Surveys and Monographs 113, Amer. Math. Soc., Providence, RI, 2004.
[17] W. G. Dwyer and D. M. Kan, Function complexes in homotopical algebra, *Topology* **19** (1980), 427–440.
[18] P. Gabriel and F. Ulmer, *Lokal präsentierbare Kategorien*, Lecture Notes in Math. 221, Springer-Verlag, Berlin, Heidelberg, New York, 1971.
[19] P. G. Goerss, J. F. Jardine, *Simplicial Homotopy Theory*, Progress in Math. 174, Birkhäuser, Basel, 1999.
[20] P. S. Hirschhorn, *Model Categories and Their Localizations*, Math. Surveys and Monographs 99, Amer. Math. Soc., Providence, RI, 2003.
[21] M. Hovey, *Model Categories*, Math. Surveys and Monographs 63, Amer. Math. Soc., Providence, RI, 1999.
[22] D. G. Quillen, *Homotopical Algebra*, Lecture Notes in Math. 43, Springer-Verlag, Berlin, Heidelberg, New York, 1967.
[23] C. Rezk, S. Schwede, and B. Shipley, Simplicial structures on model categories and functors, *Amer. J. Math.* **123** (2001), 551–575.
[24] J. Rosický and W. Tholen, Left-determined model categories and universal homotopy theories, *Trans. Amer. Math. Soc.* **355** (2003), no. 9, 3611–3623.

Departament d'Àlgebra i Geometria, Universitat de Barcelona, Gran Via de les Corts Catalanes, 585, 08007 Barcelona, Spain
E-mail address: carles.casacuberta@ub.edu

Department of Mathematics, Middlesex College, The University of Western Ontario, London, Ontario N6A 5B7, Canada
E-mail address: bchorny2@uwo.ca