EMBEDDING THEOREMS OF FUNCTION CLASSES, III

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Abstract. In this paper we obtain the necessary and sufficient conditions for embedding results of different function classes. The main result is a criterion for embedding theorems for the so-called generalized Weyl-Nikol'skii class and the generalized Lipschitz class. To define the Weyl-Nikol'skii class, we use the concept of a (λ, β)-derivative, which is a generalization of the derivative in the sense of Weyl. As corollaries, we give estimates of norms and moduli of smoothness of transformed Fourier series.

1. Introduction

History of the question. One of the main problems of constructive approximation[1] is in finding a relationship between differential properties of a function and its structural or constructive characteristics. This topic started to develop more than a century ago and in many cases the research was conducted as follows; authors considered a given functional class and by investigating the properties of its elements obtained embedding theorems with other functional classes. We recommend the articles by A. Pinkus [Pi] and V. V. Zhuk and G. I. Natanson [Zh-Na] for a historical review.

The following three classical results gave rise to development of new areas within the approximation theory:

$$\text{(A)} \quad f^{(r)} \in \text{Lip}\alpha \Longleftrightarrow E_n(f) = O\left(\frac{1}{n^{r+\alpha}}\right) \qquad (0<\alpha<1, r \in \mathbf{Z}_+),$$

$$\text{(B)} \quad f^{(r)} \in \text{Lip}\alpha \Longleftrightarrow \omega_{r+1}\Big(f, \frac{1}{n}\Big) = O\left(\frac{1}{n^{r+\alpha}}\right) \quad (0<\alpha<1, r \in \mathbf{Z}_+),$$

$$\text{(C)} \quad f \in \text{Lip}\alpha \Longrightarrow \widetilde{f} \in \text{Lip}\alpha \qquad (0<\alpha<1).$$

2000 *Mathematics Subject Classification.* 46E35, 26A16, 26A33, 42A45.

Key words and phrases. Embedding theorems, Lipschitz class, Weyl-Nikol'skii class, (λ, β)-derivative, Moduli of smoothness of fractional order.

[1]The concept became well known through Bersntein's paper [Be,S,4, V. 2, p.295-300; p. 349-360].

Result (A) was proved[2] by D. Jackson (1911, [Ja]) in the necessity part, and by S. Bernstein (1912, [Be,S,1], [Be,S,2]) and Ch. de la Vallée-Poussin (1919, [Va]) in the sufficiency part.

The theorems of this type are called direct and inverse theorems for trigonometric approximation. Direct theorems for $L_p, 1 \le p \le \infty$ (see the review [Zh-Na]) are written as follows:

$$E_n(f)_p \le C(k)\, \omega_k\Big(f, \frac{1}{n+1}\Big)_p, \qquad k, n \in \mathbf{N}, \tag{1}$$

$$E_n(f)_p \le \frac{C(k)}{n^r}\, \omega_k\Big(f^{(r)}, \frac{1}{n+1}\Big)_p, \qquad k, n, r \in \mathbf{N}. \tag{2}$$

Inverse theorems for $L_p, 1 \le p \le \infty$ (see the review [Zh-Na]):

$$\omega_k\Big(f, \frac{1}{n+1}\Big)_p \le \frac{C(k)}{(n+1)^k} \sum_{\nu=0}^{n} (\nu+1)^{k-1} E_\nu(f)_p, \quad k, n \in \mathbf{N}, \tag{3}$$

$$\omega_k\Big(f^{(r)}, \frac{1}{n+1}\Big)_p \le \tag{4}$$
$$C(k)\left(\frac{1}{(n+1)^k}\sum_{\nu=0}^{n}(\nu+1)^{k+r-1}E_\nu(f)_p + \sum_{\nu=n+1}^{\infty}(\nu+1)^{r-1}E_\nu(f)_p\right),$$
$$k, n \in \mathbf{N}.$$

Here and further, the best trigonometric approximation $E_n(f)_p$ and the modulus of smoothness $\omega_k(f,\delta)_p$ are defined by

$$E_n(f)_p = \min\Big(\|f - T\|_p ; T \in \mathbf{T}_n\Big),$$
$$\mathbf{T}_n = span\,\{\cos mx, \sin mx : |m| \le n\}$$

and

$$\omega_k(f,\delta)_p = \sup_{|h|\le\delta} \left\| \triangle_h^k f(x) \right\|_p, \tag{5}$$

$$\triangle_h^k f(x) = \triangle_h^{k-1}(\triangle_h f(x)) \qquad \text{and} \qquad \triangle_h f(x) = f(x+h) - f(x)$$

respectively.

Note that the theorems on existence of the r-th derivative of f from a given space have been initiated by Bernstein [Be,S,1]. He proved that the condition $\sum_{\nu=1}^{\infty} \nu^{r-1} E_\nu(f)_\infty < \infty$ implies $f^{(r)} \in C$. Later, for $L_p\,(1 \le p \le \infty)$, the following results were obtained (see the review [Zh-Na] and the

[2]See [Be,S,3] for a detailed review of the question before the 30-ths XX century.

paper by O.V. Besov [Be,O]). For convenience, we write these results in terms of the Besov space $B^r_{p,\theta}$ and the Sobolev space W^r_p:

$$\begin{aligned} B^r_{p,1} \subset W^r_p \subset B^r_{p,\infty} \qquad & p = 1, \infty, \\ B^r_{p,p} \subset W^r_p \subset B^r_{p,2} \qquad & 1 < p \le 2, \\ B^r_{p,2} \subset W^r_p \subset B^r_{p,p} \qquad & 2 \le p < \infty. \end{aligned}$$

Result (B) was proved by A. Zygmund (1945, [Zy,1]). He was one of the first to use the modulus of smoothness concept of integer order introduced by S. Bernstein in 1912 ([Be,S,1]). At present, the moduli of smoothness properties are well-studied ([Jo], [Zh-Na]) and the result (B) follows from inequalities (see [De-Lo, Chapters 2 and 6], [Jo-Sc]):

$$\omega_{k+r}\Big(f, \frac{1}{n}\Big)_p \le \frac{C(k,r)}{n^r} \omega_k\Big(f^{(r)}, \frac{1}{n}\Big)_p, \qquad k, r, n \in \mathbf{N} \tag{6}$$

$$\omega_k\Big(f^{(r)}, \frac{1}{n}\Big)_p \le C(k,r) \sum_{\nu=n+1}^{\infty} \nu^{r-1} \omega_{k+r}\Big(f, \frac{1}{\nu}\Big)_p, \; k, r, n \in \mathbf{N}. \tag{7}$$

Comparing the last two inequalities and inequalities (2) and (4) we see that from (6) and (7), using (1) and (3), it is easy to get (2) and (4).

Result (C) was proved by I.I. Privalov (1919, [Pr]). The most complete version of the inequality, from which embedding (C) follows, was obtained by A. Zygmund ([Zy,1]) and N.K. Bary and S.B. Stechkin ([Ba-St]) and is the following one ($p = 1, \infty$)

$$\omega_k\Big(\widetilde{f}^{(r)}, \frac{1}{n}\Big)_p \le C(k,r) \left(n^{-k} \sum_{\nu=1}^{n} \nu^{k+r-1} \omega_k\Big(f, \frac{1}{\nu}\Big)_p + \sum_{\nu=n+1}^{\infty} \nu^{r-1} \omega_k\Big(f, \frac{1}{\nu}\Big)_p \right), \tag{8}$$

$k, n \in \mathbf{N}, r \in \mathbf{Z}_+$.

Finally, we note the paper by G.H. Hardy and J.S. Littlewood [Ha-Li] in which seemingly, for the first time, some problems were formulated and solved in the same setting as in the present paper. For the historical aspects of this approach see also [Og].

Embedding theorems for functional classes. The results (A) - (C) as well as their generalizations mentioned above can be written as the embedding theorems of the following functional classes:

$$\begin{aligned} W^r_p &= \Big\{ f \in L_p : f^{(r)} \in L_p \Big\}, \\ \widetilde{W}^r_p &= \Big\{ f \in L_p : \widetilde{f}^{(r)} \in L_p \Big\}, \\ W^r_p H_\alpha[\varphi] &= \Big\{ f \in W^r_p : \omega_\alpha \left(f^{(r)}, \delta \right)_p = O[\varphi(\delta)] \Big\}, \end{aligned}$$

$$\widetilde{W}_p^r H_\alpha[\phi] = \left\{ f \in \widetilde{W}_p^r : \omega_\alpha \left(\widetilde{f}^{(r)}, \delta \right)_p = O[\phi(\delta)] \right\},$$
$$W_p^r E[\xi] = \left\{ f \in W_p^r : E_n \left(f^{(r)} \right)_p = O[\xi(1/n)] \right\}.$$

We will study more general classes for which W_p^r, $\widetilde{W}_p^r$, $W_p^r H_\alpha[\varphi]$, $\widetilde{W}_p^r H_\alpha[\phi]$, $W^r E_p[\xi]$ are particular cases.

Transformed Fourier series. Let $L_p = L_p[0, 2\pi]$ $(1 \le p < \infty)$ be a space of 2π-periodic measurable functions for which $|f|^p$ is integrable, and $L_\infty \equiv C[0, 2\pi]$ be the space of 2π-periodic continuous functions with $\|f\|_\infty = \max\{|f(x)|, 0 \le x \le 2\pi\}$.

Let summable function $f(x)$ have the Fourier series

$$f(x) \sim \sigma(f) := \frac{a_0(f)}{2} + \sum_{\nu=1}^{\infty} (a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x) \equiv \sum_{\nu=0}^{\infty} A_\nu(f, x). \tag{9}$$

By *the transformed Fourier series* of (9) we mean the series

$$\sigma(f, \lambda, \beta) := \sum_{\nu=1}^{\infty} \lambda_\nu \left[a_\nu \cos\left(\nu x + \frac{\pi\beta}{2} \right) + b_\nu \sin\left(\nu x + \frac{\pi\beta}{2} \right) \right],$$

where $\beta \in \mathbf{R}$ and $\lambda = \{\lambda_n\}$ is a given sequence of positive numbers.

Studies of the transformed Fourier series are naturally related to the problems of Fourier multipliers theory (see [Be-Lö], [Be-Il-Ni], [Zy,2, Vol 1, Chapter III]), summability methods (see [Bu-Ne, Chapter 1.2], [Zy,2, Vol 1, Chapter III]) and[3] so-called the fractional Sobolev classes or the Weyl classes.

We call the class

$$W_p^{\lambda,\beta} = \left\{ f \in L_p : \quad \exists\, g \in L_p, \quad g(x) \sim \sigma(f, \lambda, \beta) \right\}$$

the Weyl class. It is because $\lambda_n = n^r, r > 0$ and $\beta = r$ the class $W_p^{\lambda,\beta}$ coincides with the class W_p^r, which is defined in the term of fractional derivatives $f^{(r)}$ in the Weyl sense ([Zy,2, Vol. 2, Chapter XII]). In the case $\lambda_n = n^r, r > 0$ and $\beta = r + 1$ the class $W_p^{\lambda,\beta}$ coincides with the class $\widetilde{W}_p^r$. We call the function $g(x) \sim \sigma(f, \lambda, \beta)$ the (λ, β)-derivative of the function $f(x)$ and denote it by $f^{(\lambda,\beta)}(x)$.

Generalized Weyl-Nikolskii class. In the definition of this functional class we use *the modulus of smoothness* concept $\omega_\alpha(f, \delta)_p$ of fractional[4] order of a

[3]See also references to [Ba,2, §13, Chapter II].

[4]The term "fractional" can be found in earlier papers ([Bu-Dy-Gö-St] and [Ta]) which used this definition. As in the case of fractional derivatives, the positive number α that defines the modulus order is not necessarily rational.

function $f(x) \in L_p$, i.e.,

$$\omega_\alpha(f,\delta)_p = \sup_{|h|\le\delta} \|\triangle_h^\alpha f(x)\|_p \,,$$

where

$$\triangle_h^\alpha f(x) = \sum_{\nu=0}^{\infty}(-1)^\nu \binom{\alpha}{\nu} f(x+(\alpha-\nu)h), \qquad \alpha > 0$$

is the α-th difference[5] of a function f with step h at the point x. It is clear that for $\alpha \in \mathbf{N}$ this definition is the same as (5).

Let Φ_α $(\alpha > 0)$ be the class of functions $\varphi(\delta)$, defined and non-negative on $(0,\pi]$ such that

1. $\varphi(\delta) \to 0$ $(\delta \to 0)$,
2. $\varphi(\delta)$ is non-decreasing,
3. $\delta^{-\alpha}\varphi(\delta)$ is non-increasing.

For such $\alpha > 0, \varphi \in \Phi_\alpha$ and $\lambda = \{\lambda_n\}$ *the generalized Weyl-Nikolskii class* is defined similarly to the classes $W_p^r H^\alpha[\varphi]$ and $\widetilde{W}_p^r H^\alpha[\phi]$:

$$W_p^{\lambda,\beta} H_\alpha[\varphi] = \Big\{ f \in W_p^{\lambda,\beta} : \ \omega_\alpha\left(f^{(\lambda,\beta)},\delta\right)_p = O\left[\varphi(\delta)\right], \delta \to +0 \Big\}.$$

It is clear that if $\lambda_n = n^r, r > 0$ and $\beta = r$, then $W_p^{\lambda,\beta}H_\alpha[\varphi] \equiv W_p^r H_\alpha[\varphi]$ and if $\lambda_n = n^r, r > 0$ and $\beta = r+1$, then $W_p^{\lambda,\beta}H_\alpha[\varphi] \equiv \widetilde{W}_p^r H_\alpha[\varphi]$.

In the case $\lambda_n \equiv 1$ and $\beta = 0$ the class $W_p^{\lambda,\beta}H_\alpha[\varphi]$ coincides with *the generalized Lipschitz class* H_α^φ, i.e.,

$$H_\alpha^p[\varphi] = \Big\{ f \in L_p : \omega_\alpha(f,\delta)_p = O\left[\varphi(\delta)\right] \quad \delta \to +0 \Big\}.$$

In particular,

$$\mathrm{Lip}\,(\gamma, L_p) \equiv H_1^p\left[\delta^\gamma\right] = \Big\{ f \in L_p : \omega_1(f,\delta)_p = O\left[\delta^\gamma\right] \quad \delta \to +0 \Big\}.$$

The problem setting and the structure of the paper. In this paper, we obtain embedding theorems for the Weyl class $W_p^{\lambda,\beta}$, for the generalized Weyl-Nikolskii class $W_p^{\lambda,\beta}H_\alpha[\varphi]$ and for the generalized Lipschitz class $H_\gamma^p[\omega]$. We show a relation between the parameters α and γ depending on the behavior of the sequence $\{\lambda_n\}$ and on the metric L_p.

The remainder of the paper is organized as follows. In section 2 we present the main theorem. Sections 3 and 4 contain the proofs of the sufficiency and necessity parts of the main theorem respectively. In section 5 we provide several corollaries. In particular, we describe the difference in results for metrics $L_p, 1 < p < \infty$ and $L_p, p = 1, \infty$. The estimates

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[5] As usual, $\binom{\beta}{\nu} = \frac{\beta(\beta-1)\cdots(\beta-\nu+1)}{\nu!}$ for $\nu > 1$, $\binom{\beta}{\nu} = \beta$ for $\nu = 1$, and $\binom{\beta}{\nu} = 1$ for $\nu = 0$.

$\omega_\gamma(f^{(r)},\delta)_p$ and $\omega_\gamma(\widetilde{f}^{(r)},\delta)_p$ are written in terms of $\omega_\beta(f,\delta)_p$ for different values of r,γ and β. The concluding remarks are given in section 6.

2. Embedding theorems for generalized Lipschitz and Weyl-Nikolskii classes

For $\lambda=\{\lambda_n\}_{n\in\mathbf{N}}$ we define $\triangle\lambda_n:=\lambda_n-\lambda_{n+1}$; $\triangle^2\lambda_n:=\triangle(\triangle\lambda_n)$.

Theorem 1. *Let $\theta=\min(2,p)$, $\alpha\in\mathbf{R}_+$, $\beta\in\mathbf{R}$, and $\lambda=\{\lambda_n\}$ be a non-decreasing sequence of positive numbers. Let ρ be a non-negative number such that the sequence $\{n^{-\rho}\lambda_n\}$ is non-increasing.*

I. *Let $1<p<\infty$. Then*

$$H^p_{\alpha+\rho}[\omega]\subset W^{\lambda,\beta}_p \iff \sum_{n=1}^{\infty}\left(\lambda^\theta_{n+1}-\lambda^\theta_n\right)\omega^\theta\left(\frac{1}{n}\right)<\infty, \tag{10}$$

$$H^p_{\alpha+\rho}[\omega]\subset W^{\lambda,\beta}_p H_\alpha[\varphi] \iff$$
$$\Bigg\{n^{-\alpha\theta}\sum_{\nu=1}^{n}\nu^{(\rho+\alpha)\theta}\left(\nu^{-\rho\theta}\lambda^\theta_\nu-(\nu+1)^{-\rho\theta}\lambda^\theta_{\nu+1}\right)\omega^\theta\left(\frac{1}{\nu}\right)$$
$$+\sum_{\nu=n+2}^{\infty}\left(\lambda^\theta_{\nu+1}-\lambda^\theta_\nu\right)\omega^\theta\left(\frac{1}{\nu}\right)+\lambda^\theta_{n+1}\omega^\theta\left(\frac{1}{n+1}\right)\Bigg\}^{\frac{1}{\theta}}$$
$$= O\left[\varphi\left(\frac{1}{n+1}\right)\right], \tag{11}$$

$$W^{\lambda,\beta}_p\subset H^p_{\alpha+\rho}[\omega] \iff \frac{1}{\lambda_n}=O\left[\omega\left(\frac{1}{n}\right)\right], \tag{12}$$

$$W^{\lambda,\beta}_p H_\alpha[\varphi]\subset H^p_{\alpha+\rho}[\omega] \iff \frac{\varphi\left(\frac{1}{n}\right)}{\lambda_n}=O\left[\omega\left(\frac{1}{n}\right)\right]. \tag{13}$$

II. *Let $p=1$ or $p=\infty$.*

(a) *If $\triangle^2\lambda_n\ge 0$ or $\triangle^2\lambda_n\le 0$, then*

$$H^p_{\alpha+\rho}[\omega]\subset W^{\lambda,\beta}_p \iff |\cos\frac{\beta\pi}{2}|\sum_{n=1}^{\infty}(\lambda_{n+1}-\lambda_n)\,\omega\left(\frac{1}{n}\right)$$
$$+\ |\sin\frac{\beta\pi}{2}|\sum_{n=1}^{\infty}\lambda_n\frac{\omega\left(\frac{1}{n}\right)}{n}<\infty; \tag{14}$$

and if, additionally, for some $\tau>0$ the following inequality holds,

$$\triangle^2\left(\frac{\lambda_n}{n^r}\right)\ge 0 \quad \textit{with} \quad r=\rho+\tau\, sign\left|\sin\frac{(\beta-\rho)\pi}{2}\right|,$$

then

$$H^p_{\alpha+r}[\omega]\subset W^{\lambda,\beta}_p H_\alpha[\varphi] \iff$$

$$\begin{aligned}\iff\quad & n^{-\alpha}\sum_{\nu=1}^{n}\nu^{r+\alpha}\left(\nu^{-r}\lambda_\nu-(\nu+1)^{-r}\lambda_{\nu+1}\right)\omega\left(\frac{1}{\nu}\right)\\ + \quad & |\cos\frac{\beta\pi}{2}|\sum_{\nu=n+2}^{\infty}(\lambda_{\nu+1}-\lambda_\nu)\,\omega\left(\frac{1}{\nu}\right)\\ + \quad & |\sin\frac{\beta\pi}{2}|\sum_{\nu=n+2}^{\infty}\lambda_\nu\frac{\omega\left(\frac{1}{\nu}\right)}{\nu}+\lambda_{n+1}\omega\left(\frac{1}{n+1}\right)\\ = \quad & O\left[\varphi\left(\frac{1}{n+1}\right)\right]. \qquad (15)\end{aligned}$$

(b) *If for* $\beta = 2k$, $k \in \mathbf{Z}$, *the condition* $\triangle^2(1/\lambda_n) \geq 0$ *holds, and for* $\beta \neq 2k$, $k \in \mathbf{Z}$, *conditions* $\triangle^2(1/\lambda_n) \geq 0$ *and* $\sum\limits_{\nu=n+1}^{\infty}\frac{1}{\nu\lambda_\nu} \leq \frac{C}{\lambda_n}$ *are fulfilled, then*

$$W_p^{\lambda,\beta} \subset H^p_{\alpha+\rho}[\omega] \quad\iff\quad \frac{1}{\lambda_n} = O\left[\omega\left(\frac{1}{n}\right)\right]; \qquad (16)$$

and if, additionally, for some $\tau > 0$ *the following inequality holds,*

$$\triangle^2\left(\frac{n^r}{\lambda_n}\right) \geq 0 \quad or \quad \triangle^2\left(\frac{n^r}{\lambda_n}\right) \leq 0 \quad with \quad r = \rho + \tau\, sign\left|\sin\frac{(\beta-\rho)\pi}{2}\right|,$$

then

$$W_p^{\lambda,\beta}H_\alpha[\varphi] \subset H^p_{\alpha+r}[\omega] \quad\iff\quad \frac{\varphi\left(\frac{1}{n}\right)}{\lambda_n} = O\left[\omega\left(\frac{1}{n}\right)\right]. \qquad (17)$$

3. Proof of sufficiency in Theorem 1.

We will use the following notations.
Let a function $f(x) \in L$ have Fourier series (9). Then $S_n(f)$ denotes the n-th partial sum of (9), $V_n(f)$ denotes the de la Vallée-Poussin mean and $K_n(x)$ is the Fejér kernel, i.e.,

$$S_n(f) = \sum_{\nu=0}^{n} A_\nu(x), \quad V_n(f) = \frac{1}{n}\sum_{\nu=n}^{2n-1} S_\nu(f),$$
$$K_n(x) = \frac{1}{n+1}\sum_{\nu=0}^{n}\left(\frac{1}{2}+\sum_{m=1}^{\nu}\cos mx\right).$$

The following lemmas play the central role in the proof of Theorem 1.

Lemma 3.1. *Let* $f(x) \in L_p$, $1 \leq p \leq \infty$, *and* $\alpha > 0$. *Then*

$$\begin{aligned}C_1(p,\alpha)\omega_\alpha\Big(f,\frac{1}{n}\Big)_p \leq & \Big(n^{-\alpha}\left\|V_n^{(\alpha)}(f,x))\right\|_p + \|f(x)-V_n(f,x)\|_p\Big)\\ \leq & C_2(p,\alpha)\omega_\alpha\Big(f,\frac{1}{n}\Big)_p. \qquad (18)\end{aligned}$$

If $f(x) \in L_p$, $1 < p < \infty$, *then*

$$\begin{aligned} C_1(p,\alpha)\omega_\alpha\Big(f,\frac{1}{n}\Big)_p &\le \left(n^{-\alpha}\left\|S_n^{(\alpha)}(f,x))\right\|_p + \|f(x) - S_n(f,x)\|_p\right) \\ &\le C_2(p,\alpha)\omega_\alpha\Big(f,\frac{1}{n}\Big)_p. \end{aligned} \tag{19}$$

Proof of Lemma 3.1. The estimate from above $\omega_\alpha\Big(f,\frac{1}{n}\Big)_p$ follows from the inequality (see [Bu-Dy-Gö-St]) $\omega_\alpha\Big(T_n,\frac{1}{n}\Big)_p \le C(p,\alpha)n^{-\alpha}\left\|T_n^{(\alpha)}\right\|_p$, where T_n is a trigonometric polynomial of order n. We get

$$\begin{aligned} \omega_\alpha\Big(f,\frac{1}{n}\Big)_p &\le C(p,\alpha)\left(\omega_\alpha\Big(T_n,\frac{1}{n}\Big)_p + \|f - T_n\|_p\right) \\ &\le C(p,\alpha)\left(n^{-\alpha}\left\|T_n^{(\alpha)}\right\|_p + \|f - T_n\|_p\right). \end{aligned}$$

Now we will estimate $\omega_\alpha\Big(f,\frac{1}{n}\Big)_p$ from below. We need the generalized Nikol'skii-Stechkin inequality (see [Ta]) $n^{-\alpha}\left\|T_n^{(\alpha)}\right\|_p \le C(p,\alpha)\omega_\alpha\Big(T_n,\frac{1}{n}\Big)_p$ and the generalized Jackson inequality (see [Bu-Dy-Gö-St])

$$E_n(f)_p \le C(\alpha)\ \omega_\alpha\left(f,\frac{1}{n+1}\right)_p. \tag{20}$$

It is well known that the Vallée-Poussin mean is the nearly best approximant, i.e.,

$$\|f - V_n(f)\|_p \le CE_n(f)_p. \tag{21}$$

Then,

$$\begin{aligned} & n^{-\alpha}\left\|V_n^{(\alpha)}(f,x)\right\|_p + \|f(x) - V_n(f,x)\|_p \\ \le\ & C(p,\alpha)\left(\omega_\alpha\Big(V_n,\frac{1}{n}\Big)_p + E_n(f)_p\right) \\ \le\ & C(p,\alpha)\left(\omega_\alpha\Big(f,\frac{1}{n}\Big)_p + \omega_\alpha\Big(f - V_n,\frac{1}{n}\Big)_p\right) \le C(p,\alpha)\omega_\alpha\Big(f,\frac{1}{n}\Big)_p, \end{aligned}$$

and (18) is proved. Using

$$\|f - S_n(f)\|_p \le C(p)\, E_n(f)_p \tag{22}$$

for $1 < p < \infty$, we will have (19) similarly. This completes the proof of Lemma 3.1.

We note that (18) and (19) are realization results (see paper [Di-Hr-Iv] by Z. Ditzian, V. H. Hristov, K. G. Ivanov).

Lemma 3.2. ([St,S]). *Let $f(x) \in L_p$, $p = 1, \infty$, and let $\sum_{k=1}^{\infty} k^{-1} E_k(f)_p < \infty$ be true. Then $\tilde{f}(x) \in L_p$ and*

$$E_n(\tilde{f})_p \le C \left(E_n(f)_p + \sum_{k=n+1}^{\infty} k^{-1} E_k(f)_p \right), \qquad n \in \mathbf{N}.$$

Lemma 3.3. *Let $p = 1, \infty$ and $\{\lambda_n\}$ be monotonic concave (or convex) sequence. Let*

$$\begin{aligned} T_n(x) &= \sum_{\nu=0}^{n} a_\nu \cos \nu x + b_\nu \sin \nu x, \\ T_n(\lambda, x) &= \sum_{\nu=0}^{n} \lambda_\nu \left(a_\nu \cos \nu x + b_\nu \sin \nu x \right). \end{aligned}$$

Then for $M > N \ge 0$ one has

$$\| T_M(\lambda, x) - T_N(\lambda, x) \|_p \le \mu(M, N) \| T_M(x) - T_N(x) \|_p ,$$

where $\mu(M, N) =$

$$\begin{cases} 2M(\lambda_M - \lambda_{M-1}) + \lambda_{N+1} - (N+1)(\lambda_{N+2} - \lambda_{N+1}), & \text{if } \lambda_n \uparrow\ (n\uparrow),\ \triangle^2 \lambda_n \ge 0; \\ 2\lambda_M + (N+1)(\lambda_{N+2} - \lambda_{N+1}) - \lambda_{N+1}, & \text{if } \lambda_n \uparrow (n\uparrow),\ \triangle^2 \lambda_n \le 0; \\ (N+1)(\lambda_{N+1} - \lambda_{N+2}) + \lambda_{N+1}, & \text{if } \lambda_n \downarrow (n\uparrow),\ \triangle^2 \lambda_n \ge 0. \end{cases}$$

Proof of Lemma 3.3. Applying two times Abel's transformation we write

$$\| T_M(\lambda, x) - T_N(\lambda, x) \|_p = \frac{1}{\pi} \Big\| \int_{-\pi}^{\pi} (T_M - T_N)(x+u) \sum_{\nu=N+1}^{M} \lambda_\nu \cos \nu u \, du \Big\|_p =$$

$$= \frac{1}{\pi} \Big\| \int_{-\pi}^{\pi} (T_M - T_N)(x+u) \Big\{ \sum_{\nu=N+1}^{M-2} (\lambda_\nu - 2\lambda_{\nu+1} + \lambda_{\nu+2})(\nu+1) K_\nu(u) +$$

$$+ (\lambda_{N+2} - \lambda_{N+1})(N+1) K_N(u) +$$

$$+ (\lambda_{M-1} - \lambda_M) M K_{M-1}(u) \Big\} du + \lambda_M (T_M - T_N) \Big\|_p \le$$

$$\le \| T_M(x) - T_N(x) \|_p \cdot$$

$$\cdot \Big\{ \sum_{\nu=N+1}^{M-2} |\lambda_\nu - 2\lambda_{\nu+1} + \lambda_{\nu+2}| (\nu+1) + |\lambda_{M-1} - \lambda_M| M + \lambda_M \Big\}$$

$$=: \| T_M(x) - T_N(x) \|_p I(M, N).$$

First we estimate $I(M,N)$ in the case $\lambda_n \uparrow (n\uparrow)$, $\triangle^2\lambda_n \geq 0$. We have

$$\begin{aligned} I(M,N) &= \sum_{\nu=N+1}^{M-2}(\lambda_\nu - 2\lambda_{\nu+1}+\lambda_{\nu+2})\,(\nu+1)+(\lambda_M-\lambda_{M-1})\,M+\lambda_M \\ &= -(N+1)\,(\lambda_{N+2}-\lambda_{N+1})+(\lambda_{N+1}-\lambda_{M-1}) \\ &\quad +\lambda_M+(2M-1)\,(\lambda_M-\lambda_{M-1}) \\ &= -(N+1)\,(\lambda_{N+2}-\lambda_{N+1})+\lambda_{N+1}+2M\,(\lambda_M-\lambda_{M-1})\,. \end{aligned}$$

If $\lambda_n \uparrow (n\uparrow)$, $\triangle^2\lambda_n \leq 0$, then

$$\begin{aligned} I(M,N) &= (N+1)\,(\lambda_{N+1}-\lambda_{N+2})+(\lambda_{M-1}-\lambda_{N+1})+\lambda_M \\ &\quad +(\lambda_M-\lambda_{M-1})\,. \end{aligned}$$

Finally, if $\lambda_n \downarrow (n\uparrow)$, $\triangle^2\lambda_n \geq 0$, then

$$I(M,N)=(N+1)(\lambda_{N+1}-\lambda_{N+2})+\lambda_{N+1}.$$

This completes the proof of Lemma 3.3.

Lemma 3.4. *Suppose* $p=1,\infty$.
Let $T_{2^n,\,2^{n+1}}(x)=\sum\limits_{\nu=2^n}^{2^{n+1}}\Big(c_\nu\cos\nu x+d_\nu\sin\nu x\Big)$; *then*

$$C_1\left\|\widetilde{T}_{2^n,\,2^{n+1}}(x)\right\|_p \leq \left\|T_{2^n,\,2^{n+1}}(x)\right\|_p \leq C_2\left\|\widetilde{T}_{2^n,\,2^{n+1}}(x)\right\|_p. \tag{23}$$

Proof of Lemma 3.4. We rewrite $T_{2^n,\,2^{n+1}}(x)$ in the following way

$$T_{2^n,\,2^{n+1}}(x)=\sum_{\nu=2^n}^{2^{n+1}}\frac{1}{\nu}\Big(\nu c_\nu\cos\nu x+\nu d_\nu\sin\nu x\Big).$$

Applying Lemma 3.3 and the Bernstein inequality we have

$$\begin{aligned} \left\|T_{2^n,\,2^{n+1}}(x)\right\|_p &\leq C\frac{1}{2^n}\left\|\sum_{\nu=2^n}^{2^{n+1}}\Big(\nu c_\nu\cos\nu x+\nu d_\nu\sin\nu x\Big)\right\|_p \\ &= C\frac{1}{2^n}\left\|\left(\sum_{\nu=2^n}^{2^{n+1}}-d_\nu\cos\nu x+c_\nu\sin\nu x\right)'\right\|_p \\ &\leq C\left\|\widetilde{T}_{2^n,\,2^{n+1}}(x)\right\|_p. \end{aligned}$$

The same reasoning for $\widetilde{T}_{2^n,\,2^{n+1}}(x)$ works for the left-hand inequality of (23). The proof is now complete.

Sufficiency in (10) - (17).

I. $1 < p < \infty$. In this case, if $\lambda_n \equiv 1$, the Riesz inequality ([Zy,2, V. 1, p. 253]) $\|\tilde{f}\|_p \le C(p)\|f\|_p$ implies

$$\|f^{(\lambda,\beta)}\|_p \le C(p,\beta)\|f\|_p. \tag{24}$$

Henceforth, $C(s,t,\cdots)$ will be positive constants that are dependent only on $s,t,\cdots$ and may be different in different formulas.

Let the series in the right part of (10) be convergent and $f \in H^p_{\alpha+\rho}[\omega]$. We will use the following representation

$$\lambda^\theta_{2^n} = \lambda^\theta_1 + \sum_{\nu=2}^{n+1}\left(\lambda^\theta_{2^{\nu-1}} - \lambda^\theta_{2^{\nu-2}}\right).$$

Applying the Minkowski's inequality, we get $\Big($ here and further $\triangle_1 := A_1(f,x), \triangle_{n+2} := \sum\limits_{\nu=2^n+1}^{2^{n+1}} A_\nu(f,x)$, where $A_\nu(f,x)$ is from (9) $\Big)$

$$\begin{aligned} I_1 &:= \left\{\int_0^{2\pi}\left[\sum_{n=1}^\infty \lambda^2_{2^{n-1}}\triangle^2_n\right]^{\frac{p}{2}}dx\right\}^{\frac{\theta}{p}} \\ &\le C(p)\left(\lambda^\theta_1\left\{\int_0^{2\pi}\left[\sum_{n=1}^\infty \triangle^2_n\right]^{\frac{p}{2}}dx\right\}^{\frac{\theta}{p}} + \right. \\ &\quad \left. + \sum_{s=2}^\infty\left(\lambda^\theta_{2^{s-1}} - \lambda^\theta_{2^{s-2}}\right)\left\{\int_0^{2\pi}\left[\sum_{n=s}^\infty \triangle^2_n\right]^{\frac{p}{2}}dx\right\}^{\frac{\theta}{p}}\right)^{\frac{1}{\theta}}. \end{aligned} \tag{25}$$

By the Littlewood-Paley theorem ([Zy,2, V. II, p. 233]) and (22), we write

$$I_1 \le C(p)\left\{\lambda^\theta_1\|f\|^\theta_p + \sum_{s=1}^\infty\left(\lambda^\theta_{2^s} - \lambda^\theta_{2^{s-1}}\right)E^\theta_{2^{s-1}}(f)_p\right\}^{\frac{1}{\theta}}. \tag{26}$$

Then, both the generalized Jackson inequality (20) and $f \in H^p_{\alpha+\rho}[\omega]$ imply $I_1 < \infty$. Thus, there exists a function $g \in L_p$ with Fourier series

$$\sum_{n=1}^\infty \lambda_{2^{n-1}}\triangle_n, \tag{27}$$

and also $\|g\|_p \le C(p)I_1$. We rewrite series (27) in the form of $\sum\limits_{n=1}^\infty \gamma_n A_n(f,x)$, where $\gamma_i := \lambda_i$, $i = 1,2$ and $\gamma_\nu := \lambda_{2^n}$ for $2^{n-1}+1 \le \nu \le 2^n$ $(n = 2,3,\cdots)$.

Further, we write the series

$$\sum_{n=1}^{\infty} \lambda_n A_n(f,x) = \sum_{n=1}^{\infty} \gamma_n \Lambda_n A_n(f,x), \tag{28}$$

where $\Lambda_1 = \Lambda_2 = 1$, $\Lambda_\nu := \lambda_\nu/\gamma_n = \lambda_\nu/\lambda_{2^n}$ for $2^{n-1}+1 \le \nu \le 2^n$ $(n = 2,3,\cdots)$. The sequence $\{\Lambda_n\}$ satisfies the conditions of the Marcinkiewicz multiplier theorem ([Zy,2, V.II, p. 232]), i.e., series (28) is the Fourier series of a function $f^{(\lambda,0)} \in L_p$, $\|f^{(\lambda,0)}\|_p \le C(p)\|g\|_p$. Then, inequalities (20), (24) and (26) imply

$$\begin{aligned} \|f^{(\lambda,\beta)}\|_p &\le C(p,\beta)\left\{\lambda_1^\theta \|f\|_p^\theta + \sum_{s=1}^{\infty} E_{2^{s-1}}^\theta(f)_p \sum_{n=2^{s-1}}^{2^s-1}\left(\lambda_{n+1}^\theta - \lambda_n^\theta\right)\right\}^{\frac{1}{\theta}} \qquad (29) \\ &\le C(p,\beta,\alpha,\rho)\left\{\lambda_1^\theta \|f\|_p^\theta + \sum_{n=1}^{\infty}\left(\lambda_{n+1}^\theta - \lambda_n^\theta\right)\omega_{\alpha+\rho}^\theta\left(f,\frac{1}{n}\right)_p\right\}^{\frac{1}{\theta}}, \end{aligned}$$

i.e., the sufficiency in (10) has been proved.

Let the inequality in (11) hold, and $f \in H_{\alpha+\rho}^p[\omega]$. Let us prove $f \in W_p^{\lambda,\beta} H_\alpha[\varphi]$. First we estimate $\omega_\alpha(f^{(\lambda,\beta)}, \frac{1}{n})_p$. By Lemma 3.1,

$$\omega_\alpha\left(f^{(\lambda,\beta)},\frac{1}{n}\right)_p \le C(p,\alpha)\Big(\|f^{(\lambda,\beta)} - S_n(f^{(\lambda,\beta)})\|_p + n^{-\alpha}\|S_n^{(\alpha)}(f^{(\lambda,\beta)})\|_p\Big). \tag{30}$$

Using (29) for the function $(f - S_n)$, we have ($[a]$ is the integer part of a)

$$\begin{aligned} &\|f^{(\lambda,\beta)} - S_n(f^{(\lambda,\beta)})\|_p \le \\ &\le C(p,\beta,\alpha,\rho)\left\{\lambda_{n+1}^\theta \|f - S_n\|_p^\theta + E_{\left[\frac{n}{2}\right]}^\theta(f)_p \sum_{s=1}^{2n}\left(\lambda_{s+1}^\theta - \lambda_s^\theta\right)\right. \\ &\left. + \sum_{s=n+1}^{\infty}\left(\lambda_{s+1}^\theta - \lambda_s^\theta\right) E_{\left[\frac{s}{2}\right]}^\theta(f)_p\right\}^{\frac{1}{\theta}} \\ &\le C(p,\beta,\alpha,\rho)\left\{\lambda_{n+1}^\theta \omega_{\alpha+\rho}^\theta\left(f,\frac{1}{n}\right)_p + \sum_{\nu=n+1}^{\infty}\left(\lambda_{\nu+1}^\theta - \lambda_\nu^\theta\right)\omega_{\alpha+\rho}^\theta\left(f,\frac{1}{\nu}\right)_p\right\}^{\frac{1}{\theta}}. \end{aligned} \tag{31}$$

Further, we estimate the second item of (30). Let m be an integer such that $2^m \le n+1 < 2^{m+1}$. We will use here the representation

$$2^{-s\rho\theta}\lambda_{2^s}^\theta = 2^{-(m+1)\rho\theta}\lambda_{2^{m+1}}^\theta + \sum_{\nu=s}^{m}\left(2^{-\nu\rho\theta}\lambda_{2^\nu}^\theta - 2^{-(\nu+1)\rho\theta}\lambda_{2^{\nu+1}}^\theta\right).$$

Then, using Lemmas 3.1, 3.3, we can follow the way of proof in (25)-(29). Then, one can obtain

$$\begin{aligned} n^{-\alpha}\|S_n^{(\alpha)}\left(f^{(\lambda,\beta)}\right)\|_p &\leq \\ &\leq C(p,\beta,\alpha,\rho)\Bigg\{\lambda_{n+1}^{\theta}\omega_{\alpha+\rho}^{\theta}(f,\frac{1}{n})_p \\ &+ n^{-\alpha\theta}\sum_{\nu=1}^{n}\left(\nu^{-\rho\theta}\lambda_\nu^\theta-(\nu+1)^{-\rho\theta}\lambda_{\nu+1}^\theta\right)\nu^{(\rho+\alpha)\theta}\omega_{\alpha+\rho}^{\theta}\left(f,\frac{1}{\nu}\right)_p\Bigg\}^{\frac{1}{\theta}}. \end{aligned} \quad (32)$$

We use (31), (32) and the right part of (11) to obtain $f \in W_p^{\lambda,\beta}H_\alpha[\varphi]$.

Now we prove that conditions $\frac{1}{\lambda_n} = O\left[\omega\left(\frac{1}{n}\right)\right]$ and $\frac{\varphi\left(\frac{1}{n}\right)}{\lambda_n} = O\left[\omega\left(\frac{1}{n}\right)\right]$ are sufficient for $W_p^{\lambda,\beta} \subset H_{\alpha+\rho}^p[\omega]$ and $W_p^{\lambda,\beta}H_\alpha[\varphi] \subset H_{\alpha+\rho}^p[\omega]$, respectively.

From the properties of the sequence $\{\lambda_n\}$, using the Littlewood-Paley and the Marcinkiewicz multiplier theorem, we get

$$\begin{aligned} \omega_{\alpha+\rho}\left(f,\frac{1}{n}\right)_p &\leq \\ &\leq C(p,\alpha,\rho)\Big(\|f-S_n(f)\|_p+n^{-(\alpha+\rho)}\|S_n^{(\alpha+\rho)}(f)\|_p\Big) \\ &\leq C(p,\beta,\alpha,\rho)\Big(\lambda_n^{-1}\|f^{(\lambda,\beta)}-S_n\left(f^{(\lambda,\beta)}\right)\|_p+\lambda_n^{-1}n^{-\alpha}\|S_n^{(\alpha)}(f^{(\lambda,\beta)})\|_p\Big). \end{aligned}$$

Then, by Lemma 3.1, the following inequalities are true

$$\begin{aligned} \omega_{\alpha+\rho}\left(f,\frac{1}{n}\right)_p &\leq C(p,\beta,\alpha,\rho)\lambda_n^{-1}\omega_\alpha\left(f^{(\lambda,\beta)},\frac{1}{n}\right)_p \leq \\ &\leq C(p,\beta,\alpha,\rho)\lambda_n^{-1}\|f^{(\lambda,\beta)}\|_p. \end{aligned}$$

Thus, the first part implies sufficiency in (13) and the second implies sufficiency in (12).

II. $p = 1$ or $p = \infty$. Let the series in (14) be convergent, and let $f \in H_{\alpha+\rho}^p[\omega]$. Consider the series

$$\begin{aligned} &\cos\frac{\pi\beta}{2}V_1(\lambda,f)-\sin\frac{\pi\beta}{2}\widetilde{V_1}(\lambda,f) \quad (33) \\ &+\sum_{n=1}^{\infty}\Bigg\{\cos\frac{\pi\beta}{2}\left(V_{2^n}(\lambda,f)-V_{2^{n-1}}(\lambda,f)\right)-\sin\frac{\pi\beta}{2}\left(\widetilde{V_{2^n}}(\lambda,f)-\widetilde{V_{2^{n-1}}}(\lambda,f)\right)\Bigg\}, \end{aligned}$$

where $V_1(\lambda,f) := \lambda_1 A_1(f,x)$,

$$V_n(\lambda,f) := \sigma(\lambda,V_n(f)) =$$

$$= \sum_{m=1}^{n} \lambda_m A_m(f,x) + \sum_{m=n+1}^{2n-1} \lambda_m \left(1 - \frac{m-n}{n}\right) A_m(f,x)\,(n \geq 2).$$

Let $M > N > 0$. From the inequality $\|f - V_n(f)\|_p \leq CE_n(f)_p$ and the Jackson inequality (20), and using the properties of $\{\lambda_n\}$ and the outline of Lemma 3.3, we get

$$A :=$$

$$\left\|\sum_{n=N}^{M}\left[\cos\frac{\pi\beta}{2}\left(V_{2^{n+1}}(\lambda,f) - V_{2^n}(\lambda,f)\right) - \sin\frac{\pi\beta}{2}\left(\widetilde{V_{2^{n+1}}}(\lambda,f) - \widetilde{V_{2^n}}(\lambda,f)\right)\right]\right\|_p$$

$$\leq \sum_{n=N}^{M}\left[|\cos\frac{\pi\beta}{2}|\,\|V_{2^{n+1}}(f) - V_{2^n}(f)\|_p\left(\sum_{m=2^n}^{2^{n+2}-1}|\triangle^2\lambda_m|(m+1) + 2^{n+2}|\triangle\lambda_{2^{n+2}}|\right)\right.$$

$$\left.+|\sin\frac{\pi\beta}{2}|\left\|\widetilde{V_{2^{n+1}}}(f) - \widetilde{V_{2^n}}(f)\right\|_p\left(\sum_{m=2^n}^{2^{n+2}-1}|\triangle^2\lambda_m|(m+1) + 2^{n+2}|\triangle\lambda_{2^{n+2}-1}|\right)\right]$$

$$+|\cos\frac{\pi\beta}{2}|\left\|\sum_{n=N}^{M}\lambda_{2^{n+2}}(V_{2^{n+1}} - V_{2^n})(f)\right\|_p + |\sin\frac{\pi\beta}{2}|\left\|\sum_{n=N}^{M}\lambda_{2^{n+2}}\left(\widetilde{V_{2^{n+1}}} - \widetilde{V_{2^n}}\right)(f)\right\|_p$$

$$\leq C\left\{\lambda_{2^N}\left(|\cos\frac{\pi\beta}{2}|E_{2^N}(f)_p + |\sin\frac{\pi\beta}{2}|E_{2^N}(\tilde{f})_p\right)\right.$$

$$\left.+\sum_{n=2^N-1}^{\infty}(\lambda_{n+1} - \lambda_n)\left(|\cos\frac{\pi\beta}{2}|\omega_{\alpha+\rho}\left(f,\frac{1}{n}\right)_p + |\sin\frac{\pi\beta}{2}|\omega_{\alpha+\rho}\left(\tilde{f},\frac{1}{n}\right)_p\right)\right\}. \tag{34}$$

To complete the proof of the sufficiency part in (14), we apply Lemma 3.2, inequality (3) (see [Ta] for the case $k > 0$), and inequality (20). Then the convergence of series in (14) and $f \in H^p_{\alpha+\rho}[\omega]$ imply that there exists $\varphi \in L_p$ such that series (33) converges to φ in L_p. Let us show that $\sigma(\varphi) = \sigma(f^{(\lambda,\beta)})$. If F_n is the n-th partial sum of (33), then, say for cosine coefficients,

$$a_n(\varphi) = a_n(\varphi - F_{N+n}) + a_n(F_{N+n}) = a_n(\varphi - F_{N+n}) + a_n(f^{(\lambda,\beta)}),$$

and

$$a_n(\varphi - F_{N+n}) = \frac{1}{\pi}\int_{-\pi}^{\pi}(\varphi - F_{N+n})(x)\cos nx\,dx$$

$$\leq C(p)\,\|\varphi - F_{N+n}\|_p \longrightarrow 0\ (N \to \infty).$$

This completes the proof of the sufficiency part of (14).

Let $f \in H^p_{\alpha+r}[\omega]$ and the condition in the right part of (15) hold. We will estimate from above $\omega_\alpha\left(f^{(\lambda,\beta)}, \frac{1}{n}\right)_p$. By Lemma 3.1,

$$\omega_\alpha\left(f^{(\lambda,\beta)}, \frac{1}{n}\right)_p \le C(\alpha)\left(\left\|f^{(\lambda,\beta)} - V_n(f^{(\lambda,\beta)})\right\|_p + n^{-\alpha}\left\|V_n^{(\alpha)}(f^{(\lambda,\beta)})\right\|_p\right).$$

Let us show that

$$\begin{aligned}
&\left\|f^{(\lambda,\beta)} - V_n(f^{(\lambda,\beta)})\right\|_p \le \\
&\le C(\beta,\alpha,r)\left(\lambda_n\omega_{\alpha+r}\left(f, \frac{1}{n}\right)_p + |\cos\frac{\pi\beta}{2}|\sum_{\nu=n+1}^{\infty}(\lambda_{\nu+1}-\lambda_\nu)\,\omega_{\alpha+r}\left(f, \frac{1}{\nu}\right)_p\right. \\
&\left. + |\sin\frac{\pi\beta}{2}|\sum_{\nu=n+1}^{\infty}\frac{\lambda_\nu}{\nu}\omega_{\alpha+r}\left(f, \frac{1}{\nu}\right)_p\right). \qquad (35)
\end{aligned}$$

We have already shown that $A = \left\|\sum_{n=N}^{M}\left(V_{2^{n+1}}(f^{(\lambda,\beta)}) - V_{2^n}(f^{(\lambda,\beta)})\right)\right\|_p$. Then for $2^m \le n < 2^{m+1}$

$$\begin{aligned}
\left\|f^{(\lambda,\beta)} - V_n(f^{(\lambda,\beta)})\right\|_p &\le \left\|V_n(f^{(\lambda,\beta)}) - V_{2^{m+1}}(f^{(\lambda,\beta)})\right\|_p \\
&+ \left\|\sum_{\nu=m+1}^{\infty}\left(V_{2^\nu}(f^{(\lambda,\beta)}) - V_{2^{\nu+1}}(f^{(\lambda,\beta)})\right)\right\|_p \\
&=: I_1 + I_2.
\end{aligned}$$

By Lemma 3.4, we have

$$\begin{aligned}
I_1 &= \left\|\cos\frac{\pi\beta}{2}\left(V_n(\lambda,f) - V_{2^{m+1}}(\lambda,f)\right) - \sin\frac{\pi\beta}{2}\left(\widetilde{V_n}(\lambda,f) - \widetilde{V_{2^{m+1}}}(\lambda,f)\right)\right\|_p \\
&\le \left\|V_n(\lambda,f) - V_{2^{m+1}}(\lambda,f)\right\|_p + \left\|\widetilde{V_n}(\lambda,f) - \widetilde{V_{2^{m+1}}}(\lambda,f)\right\|_p \\
&\le C\left\|V_n(\lambda,f) - V_{2^{m+1}}(\lambda,f)\right\|_p
\end{aligned}$$

and, by Lemma 3.3, we get $I_1 \le \lambda_n\omega_{\alpha+r}\left(f, \frac{1}{n}\right)_p$.

Secondly

$$\begin{aligned}
I_2 &\le |\cos\frac{\pi\beta}{2}|\left\|\sum_{\nu=m+1}^{\infty}\left(V_{2^\nu}(\lambda,f) - V_{2^{\nu+1}}(\lambda,f)\right)\right\|_p \\
&+ |\sin\frac{\pi\beta}{2}|\left\|\sum_{\nu=m+1}^{\infty}\left(\widetilde{V_{2^\nu}}(\lambda,f) - \widetilde{V_{2^{\nu+1}}}(\lambda,f)\right)\right\|_p.
\end{aligned}$$

As in (34), we have

$$\left\| \sum_{\nu=N}^{M} \left(V_{2^{\nu+1}} - V_{2^{\nu}}\right)(\lambda, f) \right\|_p \le C \left(\lambda_{2^N} \omega_{\alpha+r}\left(f, \frac{1}{2^N}\right)_p + \right.$$

$$\left. + \sum_{\nu=2^N+1}^{\infty} \left(\lambda_{\nu+1} - \lambda_{\nu}\right) \omega_{\alpha+r}\left(f, \frac{1}{\nu}\right)_p \right);$$

$$\left\| \sum_{\nu=N}^{M} \left(\widetilde{V_{2^{\nu}}}(\lambda, f) - \widetilde{V_{2^{\nu+1}}}(\lambda, f) \right) \right\|_p \le \left\| \sum_{\nu=N}^{M} \lambda_{2^{\nu+2}} \left(\widetilde{V_{2^{\nu}}}(f) - \widetilde{V_{2^{\nu+1}}}(f) \right) \right\|_p$$

$$+ \sum_{\nu=N}^{M} \left\| \widetilde{V_{2^{\nu}}}(f) - \widetilde{V_{2^{\nu+1}}}(f) \right\|_p \left(\sum_{m=2^{\nu}}^{2^{\nu+2}-1} |\triangle^2 \lambda_m|(m+1) + 2^{\nu+2} |\triangle \lambda_{2^{\nu+2}}| \right)$$

$$=: I_{21} + I_{22}.$$

By Lemma 3.2,

$$\begin{aligned}
I_{21} &\le \lambda_{2^{N+1}} \left\| \widetilde{V_{2^{M+1}}}(f) - \widetilde{V_{2^N}}(f) \right\|_p \\
&+ \sum_{\nu=N}^{M} \left(\lambda_{2^{\nu+2}} - \lambda_{2^{\nu+1}}\right) \left\| \widetilde{V_{2^{\nu+1}}}(f) - \widetilde{V_{2^{\nu}}}(f) \right\|_p \\
&\le C \left(\lambda_{2^{N+1}} \sum_{\nu=N}^{\infty} E_{2^{\nu}}(f)_p + \sum_{\nu=N}^{\infty} \left(\lambda_{2^{\nu+2}} - \lambda_{2^{\nu+1}}\right) \sum_{s=\nu}^{\infty} E_{2^s}(f)_p \right) \\
&\le C \sum_{\nu=2^N}^{\infty} \frac{\lambda_\nu}{\nu} \omega_{\alpha+r}\left(f, \frac{1}{\nu}\right)_p, \\
I_{22} &\le C \sum_{\nu=N}^{M} \left(\lambda_{2^{\nu+3}} - \lambda_{2^{\nu-1}}\right) E_{2^{\nu}}(\tilde{f})_p \le C \sum_{\nu=2^N}^{\infty} \frac{\lambda_\nu}{\nu} \omega_{\alpha+r}\left(f, \frac{1}{\nu}\right)_p,
\end{aligned}$$

and (35) follows.

Repeating the argument in (34), we estimate $n^{-\alpha} \left\| V_n^{(\alpha)}(f^{(\lambda,\beta)}) \right\|_p$. By Lemma 3.3 and by inequalities (20) and (21), we write

$$\begin{aligned}
\left\| V_n^{(\alpha)}(f^{(\lambda,\beta)}) \right\|_p &\le C(\beta, \alpha, r) \left(n^{\alpha} \lambda_n \omega_{\alpha+r}\left(f, \frac{1}{n}\right)_p + \right. \qquad (36) \\
&+ \left. \sum_{\nu=1}^{n} \left(\frac{\lambda_\nu}{\nu^r} - \frac{\lambda_{\nu+1}}{(\nu+1)^r} \right) \nu^{\alpha+r} \omega_{\alpha+r}\left(f, \frac{1}{\nu}\right)_p \right).
\end{aligned}$$

By means of (35) and (36) we get $f \in W_p^{\lambda,\beta} H_\alpha[\varphi]$, because of the condition in (15).

Let us prove (16). Let $f \in W_p^{\lambda,\beta}$. To establish $f \in H_{\alpha+\rho}^p(\omega)$, we shall first estimate $E_n(f)_p$ for the case $\sin \frac{\pi\beta}{2} \neq 0$. Repeating the reasoning in (34), by (21), we get ($2^m \leq n < 2^{m+1}$)

$$\begin{aligned} E_n(f)_p &\leq C \sum_{\nu=m}^{\infty} \frac{1}{\lambda_{2^\nu}} \left\| \left(V_{[2^{\nu-1}]} - V_{2^\nu} \right) (f^{(\lambda,\beta)}) \right\|_p \\ &\leq C E_{[2^{m-1}]}(f^{(\lambda,\beta)})_p \sum_{\nu=m}^{\infty} \frac{1}{\lambda_{2^\nu}} \leq C(\rho, p) \frac{\|f^{(\lambda,\beta)}\|_p}{\lambda_n}. \end{aligned}$$

If $\sin \frac{\pi\beta}{2} = 0$, it is easy to see that

$$E_n(f)_p \leq \frac{C}{\lambda_n} E_n(f^{(\lambda,\beta)})_p \leq \frac{C}{\lambda_n} \|f^{(\lambda,\beta)}\|_p.$$

Substituting the bound for $E_n(f)_p$ into (3) and using the fact that $n^\rho \lambda_n^{-1} \uparrow$ $(n \uparrow)$, we can write

$$\omega_{\alpha+\rho}\left(f, \frac{1}{n}\right)_p \leq C(\alpha, \rho) \frac{1}{n^{\alpha+\rho}} \sum_{\nu=0}^{n} \nu^{\alpha+\rho-1} E_{\nu-1}(f)_p \leq \frac{C(\alpha, \rho)}{\lambda_n} = O\left[\omega\left(\frac{1}{n}\right)\right],$$

i.e., $f \in H_{\alpha+\rho}^p(\omega)$. This completes the proof of the sufficiency part in (16).

Let the right part of (17) be true, and $f \in H_{\alpha+\rho}^p[\omega]$. First let us prove that

$$\left\| V_n^{(\alpha+r)}(f) \right\|_p \leq C(\alpha, r) \frac{n^r}{\lambda_n} \left\| V_{2n}^{(\alpha)}(f^{(\lambda,\beta)}) \right\|_p. \tag{37}$$

If $\beta = \rho + 2m$, and therefore, $r = \rho$, then $V_n^{(\alpha+r)}(f) = V_n^{(\alpha)}\left(\{\frac{\nu^r}{\lambda_\nu}\}, f^{(\lambda,\beta)}\right)$ and, by Lemma 3.3

$$\left\| V_n^{(\alpha+r)}(f) \right\|_p \leq C(\alpha, r) \frac{n^r}{\lambda_n} \left\| V_n^{(\alpha)}(f^{(\lambda,\beta)}) \right\|_p \leq C(\alpha, r) \frac{n^r}{\lambda_n} \left\| V_{2n}^{(\alpha)}(f^{(\lambda,\beta)}) \right\|_p.$$

If $\beta \neq \rho + 2m$, and therefore, $r > \rho$, then by Lemma 3.4

$$\begin{aligned} &\left\| V_{2^n}^{(\alpha+r)}(f) \right\|_p \leq \\ &\leq \sum_{\nu=0}^{n} \left\| V_{2^\nu}^{(\alpha+r)}(f) - V_{2^{\nu-1}}^{(\alpha+r)}(f) \right\|_p + \left\| V_1^{(\alpha+r)}(f) \right\|_p \\ &\leq C \sum_{\nu=0}^{n} \frac{2^{\nu r}}{\lambda_{2^\nu}} \left\| V_{2^\nu}^{(\alpha)}(f^{(\lambda,\beta)}) - V_{2^{\nu-1}}^{(\alpha)}(f^{(\lambda,\beta)}) \right\|_p + \frac{1}{\lambda_1} \left\| V_1^{(\alpha)}(f^{(\lambda,\beta)}) \right\|_p \end{aligned}$$

$$\begin{aligned} &\leq C\left\|V_{2^{n+1}}^{(\alpha)}(f^{(\lambda,\beta)})\right\|_p \left(\sum_{\nu=0}^{n}\frac{2^{\nu r}}{\lambda_{2^\nu}}+\frac{1}{\lambda_1}\right)\\ &\leq C(\alpha,r)\frac{2^{nr}}{\lambda_{2^n}}\left\|V_{2^{n+1}}^{(\alpha)}(f^{(\lambda,\beta)})\right\|_p. \end{aligned}$$

Thus, by (37) and the estimate $E_n(f)_p \leq \frac{C}{\lambda_n}E_{\left[\frac{n}{4}\right]}(f^{(\lambda,\beta)})_p$, we can write

$$\begin{aligned} \omega_{\alpha+r}\left(f,\frac{1}{n}\right)_p &\leq C(\alpha,r)\left(n^{-(\alpha+r)}\left\|V_n^{(\alpha+r)}(f)\right\|_p+E_n(f)_p\right)\\ &\leq \frac{C(\alpha,r)}{\lambda_n}\left(n^{-\alpha}\left\|V_n^{(\alpha)}(f^{(\lambda,\beta)})\right\|_p+E_{\left[\frac{n}{4}\right]}(f^{(\lambda,\beta)})_p\right)\\ &\leq \frac{C(\alpha,r)}{\lambda_n}\omega_\alpha\left(f^{(\lambda,\beta)},\frac{1}{n}\right)_p = O\left[\frac{\varphi\left(\frac{1}{n}\right)}{\lambda_n}\right] = O\left[\omega\left(\frac{1}{n}\right)\right]. \end{aligned}$$

Thus, $\frac{\varphi\left(\frac{1}{n}\right)}{\lambda_n} = O\left[\omega\left(\frac{1}{n}\right)\right]$ is sufficient for $W_p^{\lambda,\beta}H_\alpha[\varphi] \subset H_{\alpha+r}^p[\omega]$.

4. Proof of necessity in Theorem 1.

First we define the trigonometric polynomials $\tau_{n+1}(x)$:

$$\tau_{n+1}(x) = \sum_{j=1}^{n+1}\alpha_j^n \sin jx, \qquad \text{where} \quad \alpha_j^n = \begin{cases} \frac{j}{n+2}, & 1 \leq j \leq \frac{n+2}{2}\\ 1-\frac{j}{n+2}, & \frac{n+2}{2} \leq j \leq n+1. \end{cases}$$

We will use the following lemmas as well as Lemmas 3.1-3.2.

Lemma 4.1. ([Te]). *Let $f(x) \in L_1$ have the Fourier series (9). Then*

$$E_n(f)_1 \geq C\left|\sum_{\nu=n+1}^{\infty}\frac{b_\nu}{\nu}\right|.$$

Lemma 4.2. ([Zy,2, V. 1, p. 215]) *Let $1 \leq p < \infty$.*
(a) *If a function $f(x) \in L_p$ has the Fourier series $\sum\limits_{\nu=1}^{\infty}(a_\nu \cos 2^\nu x + b_\nu \sin 2^\nu x)$, then*

$$\left\{\sum_{\nu=1}^{\infty}\left(a_\nu^2+b_\nu^2\right)\right\}^{\frac{1}{2}} \leq C\|f\|_p.$$

(b) *Let $a_n, b_n (n \in \mathbf{N})$ be real numbers such that $\sum\limits_{\nu=1}^{\infty}\left(a_\nu^2+b_\nu^2\right) < \infty$. Then $\sum\limits_{\nu=1}^{\infty}(a_\nu \cos 2^\nu x + b_\nu \sin 2^\nu x)$ be the Fourier series of a function $f(x) \in L_p$,* *and*

$$\|f\|_p \le C\left\{\sum_{\nu=1}^{\infty}\left(a_\nu^2+b_\nu^2\right)\right\}^{\frac{1}{2}}.$$

Lemma 4.3. ([Ba,2, Vol.2, p. 269]). *Let* $f(x) \in L_\infty$ *have the Fourier series* $\sum_{\nu=1}^{\infty}(a_\nu \cos 2^\nu x + b_\nu \sin 2^\nu x)$, *where* $a_\nu, b_\nu \ge 0$. *Then*

$$C_1 \sum_{\xi=n}^{\infty}(a_\xi+b_\xi) \le E_{2^n-1}(f)_\infty \le C_2 \sum_{\xi=n}^{\infty}(a_\xi+b_\xi).$$

We need the following definitions. Let $\omega(\cdot) \in \Phi_\alpha$.
We call ψ a $Q_{\alpha,\theta}(\omega)$-*sequence* if

$$0 \;<\; \psi_n \;\le\; n^\alpha \omega\left(\frac{1}{n}\right), \quad \psi_n \uparrow (n\uparrow) \tag{38}$$

$$C_1\,\omega\left(\frac{1}{n}\right) \;\le\; \left\{\sum_{\nu=n}^{\infty}\nu^{-\alpha\theta-1}\psi_\nu^\theta\right\}^{\frac{1}{\theta}} \le C_2\,\omega\left(\frac{1}{n}\right). \tag{39}$$

We call ε a $q_{\alpha,\theta}(\omega)$-*sequence* if

$$0 \;<\; \varepsilon_n \le \omega\left(\frac{1}{n+1}\right), \quad \varepsilon_n \downarrow (n\uparrow) \tag{40}$$

$$C_1\,\omega\left(\frac{1}{n+1}\right) \;\le\; \left\{(n+1)^{-\alpha\theta}\sum_{\nu=1}^{n+1}\nu^{\alpha\theta-1}\varepsilon_\nu^\theta\right\}^{\frac{1}{\theta}} \le C_2\,\omega\left(\frac{1}{n+1}\right). \tag{41}$$

Necessity in (10) - (17).

We prove the necessity part by constructing corresponding examples. The proof is in eight steps.

I. $1<p<\infty$. Step 1. Let us show the necessity part in (10).
Let $\omega(\cdot) \in \Phi_{\alpha+\rho}$ and $\theta = \min(2,p)$. We will construct a sequence ψ that will be a $Q_{\alpha+\rho,\theta}(\omega)$-sequence.
Let us assume that we have chosen integers $1 = n_1 < n_2 < \cdots < n_s$. Then, we define n_{s+1} as minimum of integers $N > n_s$ such that

$$\omega\left(\frac{1}{N}\right) < \frac{1}{2}\,\omega\left(\frac{1}{n_s}\right) \le \omega\left(\frac{1}{N-1}\right).$$

We set

$$\psi_n = \begin{cases} n_s^{\rho+\alpha}\omega\left(\frac{1}{n_s}\right), & \text{if} \quad n_s \le n < n_{s+1},\\ 0, & \text{if} \quad n = 0.\end{cases}$$

It is easy to see that this sequence is required.

Let $H^p_{\alpha+\rho}[\omega] \subset W^{\lambda,\beta}_p$, and let the series in (10) be divergent. By means of properties of sequence $\{\psi_n\}$, we have

$$\begin{aligned}\infty &= \sum_{\nu=1}^{\infty} \left(\lambda^\theta_{\nu+1} - \lambda^\theta_\nu\right) \omega^\theta \left(\frac{1}{\nu}\right) \\ &\le C(\alpha,\rho,\theta) \sum_{\nu=1}^{\infty} \left(\lambda^\theta_{\nu+1} - \lambda^\theta_\nu\right) \sum_{m=\nu}^{\infty} m^{-(\alpha+\rho)\theta-1} \psi^\theta_m \\ &\le C(\alpha,\rho,\theta) \sum_{\nu=1}^{\infty} \lambda^\theta_\nu \nu^{-(\alpha+\rho)\theta-1} \psi^\theta_\nu.\end{aligned}$$

Step 1(a): $2 \le p < \infty$. We consider the series

$$\sum_{\nu=1}^{\infty} 2^{-\nu(\alpha+\rho)} \left(\psi^2_{2^\nu} - \psi^2_{2^{\nu-1}}\right)^{\frac{1}{2}} \cos 2^\nu x. \tag{42}$$

Since

$$\begin{aligned}\sum_{\nu=1}^{\infty} 2^{-2\nu(\alpha+\rho)} \left(\psi^2_{2^\nu} - \psi^2_{2^{\nu-1}}\right) &\le \sum_{\nu=1}^{\infty} \left(\psi^2_{2^\nu} - \psi^2_{2^{\nu-1}}\right) \sum_{\xi=\nu}^{\infty} 2^{-2\xi(\alpha+\rho)} \\ &\le \sum_{\nu=1}^{\infty} 2^{-2\nu(\alpha+\rho)} \psi^2_{2^\nu} \le C\omega^2(1),\end{aligned} \tag{43}$$

then, by Zygmund's Lemma 4.2, series (42) is the Fourier series of a function $f_1(x) \in L_p$. By Lemmas 3.1 and 4.2,

$$\begin{aligned}&C(\alpha,\rho)\omega_{\alpha+\rho}\Big(f_1, \frac{1}{2^n}\Big)_p \le \\ &\le 2^{-n(\alpha+\rho)} \left(\sum_{\nu=1}^{n} a^2_\nu 2^{2(\alpha+\rho)\nu}\right)^{\frac{1}{2}} + \left(\sum_{\nu=n+1}^{\infty} a^2_\nu\right)^{\frac{1}{2}} =: I_1 + I_2,\end{aligned}$$

where $a_\nu = 2^{-(\alpha+\rho)} \left(\psi^2_{2^\nu} - \psi^2_{2^{\nu-1}}\right)^{\frac{1}{2}}$. Repeating the argument in (43), we get

$$I_1 \le 2^{-n(\alpha+\rho)} \psi_{2^n} \le \omega\Big(\frac{1}{2^n}\Big),$$

because of (38) and

$$I_2 \le C(\alpha,\rho) \left(\sum_{\nu=n+1}^{\infty} 2^{-2\nu(\alpha+\rho)} \psi^2_{2^\nu}\right)^{\frac{1}{2}} \le C(\alpha,\rho)\omega\Big(\frac{1}{2^n}\Big),$$

because of (39).

Thus, $f_1(x) \in H^p_{\alpha+\rho}[\omega]$. Then from our assumption, $f_1(x) \in W_p^{\lambda,\beta}$. On the other hand,

$$\left\|f_1^{(\lambda,\beta)}\right\|_p \geq C(\alpha,\rho,\theta)\left(\sum_{\nu=1}^{\infty}\lambda_\nu^2\nu^{-2(\alpha+\rho)-1}\psi_\nu^2\right)^{\frac{1}{2}} = \infty.$$

This contradiction proves the convergence of series in (10).
Step 1(b): $1 < p \leq 2$. Consider series[6]

$$\psi_1\cos x + \sum_{\nu=1}^{\infty}2^{-\nu(\alpha+\rho)}2^{\nu(\frac{1}{p}-1)}\left(\psi_{2^\nu}^p - \psi_{2^{\nu-1}}^p\right)^{\frac{1}{p}}\sum_{\mu=2^{\nu-1}+1}^{2^\nu}\cos\mu x. \qquad (44)$$

Using Jensen inequality $\left(\sum\limits_{n=1}^{\infty}a_n^\alpha\right)^{1/\alpha} \leq \left(\sum\limits_{n=1}^{\infty}a_n^\beta\right)^{1/\beta}$ ($a_n \geq 0$ and $0 < \beta \leq \alpha < \infty$), we write

$$\begin{aligned}
&\int_0^{2\pi}\left[\sum_{\nu=1}^{\infty}\left(2^{-\nu(\alpha+\rho)}2^{\nu(\frac{1}{p}-1)}\left(\psi_{2^\nu}^p - \psi_{2^{\nu-1}}^p\right)^{\frac{1}{p}}\sum_{\mu=2^{\nu-1}+1}^{2^\nu}\cos\mu x\right)^2\right]^{\frac{p}{2}}dx\\
\leq\ &\int_0^{2\pi}\left[\sum_{\nu=1}^{\infty}2^{-\nu p(\alpha+\rho)}2^{\nu(1-p)}\left(\psi_{2^\nu}^p - \psi_{2^{\nu-1}}^p\right)\left|\sum_{\mu=2^{\nu-1}+1}^{2^\nu}\cos\mu x\right|^p\right]dx\\
\leq\ &C(p)\sum_{\nu=1}^{\infty}\left(\psi_{2^\nu}^p - \psi_{2^{\nu-1}}^p\right)2^{-\nu p(\alpha+\rho)} \leq C(p)\omega^p(1),
\end{aligned}$$

because of $C_1(p)2^{\nu(p-1)} \leq \left\|\sum\limits_{\mu=2^{\nu-1}+1}^{2^\nu}\cos\mu x\right\|_p^p \leq C_2(p)2^{\nu(p-1)}$. By the Littlewood-Paley theorem, there exists a function $f_2 \in L_p$ with Fourier series (44). One can see that $f_2 \in H^p_{\alpha+\rho}[\omega]$. Then $f_2 \in W_p^{\lambda,\beta}$. On the other hand, Paley's theorem on Fourier coefficients [Zy,2, V.2, p. 121] implies that for $f_2 \in L_p$

$$\begin{aligned}
&\left\|f_2^{(\lambda,\beta)}\right\|_p^p \geq\\
\geq\ &C(p)\sum_{\nu=1}^{\infty}2^{-\nu p(\alpha+\rho)}2^{\nu(1-p)}\left(\psi_{2^\nu}^p - \psi_{2^{\nu-1}}^p\right)\sum_{\mu=2^{\nu-1}+1}^{2^\nu}\lambda_\mu^p\mu^{p-2}\\
\geq\ &C(\alpha,\rho,p)\sum_{\nu=1}^{\infty}\left(\psi_{2^\nu}^p - \psi_{2^{\nu-1}}^p\right)\sum_{\xi=\nu}^{\infty}\left(2^{-\xi(\alpha+\rho)p}\lambda_{2^\xi}^p - 2^{-(\xi+1)(\alpha+\rho)p}\lambda_{2^{\xi+1}}^p\right)
\end{aligned}$$

[6] Series of this type was considered in [Ti,M,1].

$$\geq \quad C_1(\alpha,\rho,p)\sum_{\nu=1}^{\infty}\psi_\nu^p\lambda_\nu^p\nu^{-p(\alpha+\rho)-1}-C_2(\alpha,\rho,p)\psi_1^p\lambda_2^p 2^{-p(\alpha+\rho)-1} \;=\; \infty.$$

This contradiction shows that the series in the right part of (10) converges. This completes the proof of the necessity part of (10).

Step 2. Let us prove the necessity in (11) for $2\leq p<\infty$.

We notice that by Lemmas 3.1 and 4.2, we have for $f(x)\sim\sum\limits_{\nu=1}^{\infty}(a_\nu\cos 2^\nu x+b_\nu\sin 2^\nu x)$

$$\omega_\alpha\left(f,\frac{1}{2^m}\right)_p \asymp \left(2^{-2m\alpha}\sum_{\nu=1}^{m}(a_\nu^2+b_\nu^2)2^{2\nu\alpha}\right)^{\frac12}+\left(\sum_{\nu=m+1}^{\infty}(a_\nu^2+b_\nu^2)\right)^{\frac12}. \quad (45)$$

Let $\omega(\cdot)\in\Phi_{\alpha+\rho}$. One can construct[7] a sequence ε such that ε is a $q_{\alpha+\rho,\,\theta}(\omega)$-sequence. We consider for this case

$$\varepsilon_0+\left(\varepsilon_1^2-\varepsilon_2^2\right)^{\frac12}\cos x+\sum_{\nu=1}^{\infty}\left(\varepsilon_{2^\nu}^2-\varepsilon_{2^{\nu+1}}^2\right)^{\frac12}\cos 2^\nu x. \quad (46)$$

Repeating the argument for series (42) we obtain that series (46) is the Fourier series of a function $f_3\in L_p$. Since $E_{2^{n-1}}(f_3)_p\leq C(p)\varepsilon_{2^n}$, then by (3), (41) implies $f_3\in H^p_{\alpha+\rho}[\omega]$. We define $f_{13}:=f_1+f_3$. We have $f_{13}\in H^p_{\alpha+\rho}[\omega]\subset W_p^{\lambda,\beta}H_\alpha[\varphi]$. It is easy to see from (45) that

$$C(\alpha,\beta)\,\omega_\alpha\Big(f_{13}^{(\lambda,\beta)},\frac{1}{n+1}\Big)_p\geq\omega_\alpha\Big(f_1^{(\lambda,\beta)},\frac{1}{n+1}\Big)_p+\omega_\alpha\Big(f_3^{(\lambda,\beta)},\frac{1}{n+1}\Big)_p. \quad (47)$$

Let us estimate $\omega_\alpha\Big(f_1^{(\lambda,\beta)},\frac{1}{n+1}\Big)_p$. By (45), using the properties of the sequence $\{\psi_\nu\}$, we have ($2^m\leq n+1<2^{m+1}$)

$$\begin{aligned}
&\omega_\alpha^2\left(f_1^{(\lambda,\beta)},\frac{1}{n+1}\right)_p\geq\\
&\geq\quad C(\alpha,\beta)\sum_{\nu=m}^{\infty}\psi_{2^\nu}^2 2^{-2\nu(r+\alpha)}\left(\sum_{k=m}^{\nu}(\lambda_{2^k}^2-\lambda_{2^{k-1}}^2)+\lambda_{2^{m-1}}^2\right)\\
&\geq\quad C(\alpha,\beta)\left(\lambda_{2^m}^2\omega^2\left(\frac{1}{2^m}\right)+\sum_{k=m}^{\infty}(\lambda_{2^k}^2-\lambda_{2^{k-1}}^2)\omega^2\left(\frac{1}{2^k}\right)\right)\\
&\geq\quad C(\alpha,\beta)\left(\lambda_{n+1}^2\omega^2\left(\frac{1}{n+1}\right)+\sum_{\nu=n+2}^{\infty}\left(\lambda_{\nu+1}^2-\lambda_\nu^2\right)\omega^2\left(\frac{1}{\nu}\right)\right). \quad (48)
\end{aligned}$$

[7]See, for example, [Ge].

Now we estimate $\omega_\alpha\Big(f_3^{(\lambda,\beta)}, \frac{1}{n+1}\Big)_p$. By (45), we have ($2^m \le n+1 < 2^{m+1}$):

$$\begin{aligned}
&\omega_\alpha^2 \left(f_3^{(\lambda,\beta)}, \frac{1}{n+1}\right)_p \ge \\
&\ge \quad C(\alpha,\beta) 2^{-2m\alpha} \sum_{\nu=0}^{m} 2^{2\nu\alpha} \lambda_{2^\nu}^2 \left(\varepsilon_{2^\nu}^2 - \varepsilon_{2^{\nu+1}}^2\right) \\
&\ge \quad C_1(\alpha,\beta) 2^{-2m\alpha} \sum_{\nu=0}^{m} 2^{2\nu\alpha} \lambda_{2^\nu}^2 \varepsilon_{2^\nu}^2 - C_2(\alpha,p) \lambda_{2^{m+1}}^2 \varepsilon_{2^{m+1}}^2. \qquad (49)
\end{aligned}$$

The Jackson inequality implies

$$\begin{aligned}
\omega_\alpha^2 \left(f_3^{(\lambda,\beta)}, \frac{1}{n+1}\right)_p &\ge C(\alpha) E_{2^m-1}^2 \left(f_3^{(\lambda,\beta)}\right)_p \\
&\ge C(\alpha,\beta) \sum_{\nu=m}^{\infty} \lambda_{2^\nu}^2 \left(\varepsilon_{2^\nu}^2 - \varepsilon_{2^{\nu+1}}^2\right) \\
&\ge C(\alpha,\beta) \lambda_{2^m}^2 \varepsilon_{2^m}^2. \qquad (50)
\end{aligned}$$

Both estimates (49) and (50) imply

$$\omega_\alpha \left(f_3^{(\lambda,\beta)}, \frac{1}{n+1}\right)_p \ge C(\alpha,\beta) \left((n+1)^{-2\alpha} \sum_{\nu=1}^{n+1} \lambda_\nu^2 \nu^{2\alpha-1} \varepsilon_\nu^2\right)^{\frac{1}{2}}. \qquad (51)$$

Using (41) and $\nu^{-\rho}\lambda_\nu \downarrow$, we get

$$\begin{aligned}
&\lambda_{n+1}^2 \omega^2 \left(\frac{1}{n+1}\right) + \\
&+ \quad (n+1)^{-2\alpha} \sum_{\nu=1}^{n+1} \nu^{2(\rho+\alpha)} \omega^2 \left(\frac{1}{\nu}\right) \left(\lambda_\nu^2 \nu^{-2\rho} - \lambda_{\nu+1}^2 (\nu+1)^{-2\rho}\right) \\
&\le \quad C(\alpha,\rho) (n+1)^{-2\alpha} \sum_{\nu=1}^{n+1} \nu^{2\alpha-1} \lambda_\nu^2 \varepsilon_\nu^2. \qquad (52)
\end{aligned}$$

Combining estimates (47), (48), (51), (52), and $\omega_\alpha\Big(f_{13}^{(\lambda,\beta)}, \frac{1}{n+1}\Big)_p = O\left[\varphi\left(\frac{1}{n}\right)\right]$, we arrive at the condition in the right part of (11).

Step 3. Let us prove the necessity in (11) for $1 < p < 2$. The proof for $1 < p \le 2$ is similar to $2 \le p < \infty$. The only difference is that we use Paley's theorem on Fourier coefficients instead of Zygmund's theorem. In this case we consider the sum of $f_2(x)$ and the following function

$$\varepsilon_0 + (\varepsilon_1^p - \varepsilon_2^p)^{\frac{1}{p}} \cos x + \sum_{\nu=0}^{\infty} 2^{\nu\left(\frac{1}{p}-1\right)} \left(\varepsilon_{2^{\nu+1}}^p - \varepsilon_{2^{\nu+2}}^p\right)^{\frac{1}{p}} \sum_{\mu=2^\nu+1}^{2^{\nu+1}} \cos \mu x. \quad (53)$$

Step 4. To prove the necessity in (12) and (16), we consider the general case $1 \le p \le \infty$. Let Φ be the class of all decreasing null-sequences. It is clear that

$$\frac{1}{\lambda_n} = O\left[\omega\left(\frac{1}{n}\right)\right] \qquad \Longleftrightarrow \qquad \forall\ \gamma = \{\gamma_n\} \in \Phi \qquad \frac{\gamma_n}{\lambda_n} = O\left[\omega\left(\frac{1}{n}\right)\right].$$

Let us assume that $\frac{\gamma_n}{\lambda_n} = O\left[\omega\left(\frac{1}{n}\right)\right]$ does not hold for all $\gamma \in \Phi$ and $W_p^{\lambda,\beta} \subset H_{\alpha+\rho}^p[\omega]$. Then there exist $\gamma = \{\gamma_n\} \in \Phi$ and $\{C_n \uparrow \infty\}$ such that $\frac{\gamma_{m_n}}{\lambda_{m_n}} \ge C_n \omega\left(\frac{1}{m_n}\right)$. Further, we choose a subsequence $\{m_{n_k}\}$ such that $\frac{m_{n_{k+1}}}{m_{n_k}} \ge 2$ and $\gamma_{m_{n_k}} \le 2^{-k}$. Consider the series

$$\sum_{k=0}^{\infty} \frac{\gamma_{m_{n_k}}}{\lambda_{m_{n_k}}} \cos(m_{n_k}+1)x. \quad (54)$$

Since $\sum_{k=0}^{\infty} \frac{\gamma_{m_{n_k}}}{\lambda_{m_{n_k}}} \le \frac{1}{\lambda_{m_{n_0}}} \sum_{k=0}^{\infty} \frac{1}{2^k} < \infty$, there exists a function $f_4 \in L_p$ with Fourier series (54). Because $\sum_{k=0}^{\infty} \gamma_{m_{n_k}} \le \sum_{k=0}^{\infty} \frac{1}{2^k} < \infty$, we have $f_4^{(\lambda,\beta)} \in L_p$, i.e., $f_4 \in W_p^{\lambda,\beta}$.

On the other hand, by (20) and $E_{n-1}(f)_p \ge C\left(|a_n| + |b_n|\right)$,

$$\begin{aligned} \omega_{\alpha+\rho}\left(f_4, \frac{1}{m_{n_k}}\right)_p &\ge C(\alpha,\rho) E_{m_{n_k}}(f_4)_p \ge C(\alpha,\rho)\frac{\gamma_{m_{n_k}}}{\lambda_{m_{n_k}}} \\ &\ge C(\alpha,\rho) C_{n_k} \omega\left(\frac{1}{m_{n_k}}\right), \end{aligned}$$

i.e., $f_4 \notin H_{\alpha+\rho}^p[\omega]$. This contradiction implies that the condition $\frac{1}{\lambda_n} = O\left[\omega\left(\frac{1}{n}\right)\right]$ is necessary for $W_p^{\lambda,\beta} \subset H_{\alpha+\rho}^p[\omega]$.

Step 5. To prove the necessity in (13) and (17), we verify that for any $\rho > 0$ and for any $1 \le p \le \infty$,

$$W_p^{\lambda,\beta} H_\alpha[\varphi] \subset H_{\alpha+\rho}^p[\omega] \Longrightarrow \frac{\varphi\left(\frac{1}{n}\right)}{\lambda_n} = O\left[\omega\left(\frac{1}{n}\right)\right]. \quad (55)$$

First we remark that

$$\frac{\varphi\left(\frac{1}{n}\right)}{\lambda_n} = O\left[\omega\left(\frac{1}{n}\right)\right] \Longleftrightarrow \forall\ \gamma = \{\gamma_n\} \in \Phi \qquad \frac{\gamma_n \varphi\left(\frac{1}{n}\right)}{\lambda_n} = O\left[\omega\left(\frac{1}{n}\right)\right].$$

Assume that the relation in the right hand side of (55) does not hold. Then there exist $\gamma=\{\gamma_n\}\in\Phi$ and $\{C_n\uparrow\infty\}$ such that $\frac{\gamma_{m_n}\varphi\left(\frac{1}{m_n}\right)}{\lambda_{m_n}}\geq C_n\omega\left(\frac{1}{m_n}\right)$. We choose a subsequence $\{m_{n_k}\}$ such that $\frac{m_{n_{k+1}}}{m_{n_k}}\geq 2$ and $\gamma_{m_{n_k}}\leq 2^{-k}$. Because $\sum\limits_{k=0}^{\infty}\gamma_{m_{n_k}}\varphi\left(\frac{1}{m_n}\right)\leq\varphi\left(\frac{1}{m_0}\right)\sum\limits_{k=0}^{\infty}\frac{1}{2^k}<\infty$, there exists a function $f_5\in L_p$ with Fourier series

$$\sum_{k=0}^{\infty}\gamma_{m_{n_k}}\varphi\left(\frac{1}{m_{n_k}}\right)\cos(m_{n_k}+1)x. \tag{56}$$

For $m_{n_k}\leq n<m_{n_{k+1}}$, by Lemmas 3.1 and 4.3, we have

$$\begin{aligned}\omega_\alpha\left(f_5,\frac{1}{n}\right)_p &\leq C\omega_\alpha\left(f_5,\frac{1}{n}\right)_\infty\\ &\leq C\left(n^{-\alpha}\sum_{s=0}^{k}\gamma_{m_{n_s}}\varphi\left(\frac{1}{m_{n_s}}\right)m_{n_s}^{\alpha}+\sum_{s=k+1}^{\infty}\gamma_{m_{n_s}}\varphi\left(\frac{1}{m_{n_s}}\right)\right)\\ &\leq C\left(\varphi\left(\frac{1}{n}\right)\sum_{s=0}^{k}\gamma_{m_{n_s}}+\varphi\left(\frac{1}{n}\right)\sum_{s=k+1}^{\infty}\gamma_{m_{n_s}}\right)\leq C\varphi\left(\frac{1}{n}\right).\end{aligned}$$

Therefore, $f_5\in H_\alpha^p[\varphi]$, i.e., setting $\frac{1}{\lambda}:=\left\{\frac{1}{\lambda_1},\frac{1}{\lambda_2},\frac{1}{\lambda_3},\cdots\right\}$, we have $f_5^{(\frac{1}{\lambda},-\beta)}\in W_p^{\lambda,\beta}H_\alpha[\varphi]$.

However, we have

$$\begin{aligned}\omega_{\alpha+\rho}\left(f_5^{(\frac{1}{\lambda},-\beta)},\frac{1}{m_{n_k}}\right)_p &\geq CE_{m_{n_k}}(f_5^{(\frac{1}{\lambda},-\beta)})_p\geq C\frac{\gamma_{m_{n_k}}\varphi\left(\frac{1}{m_{n_k}}\right)}{\lambda_{m_{n_k}}}\\ &\geq CC_{n_k}\omega\left(\frac{1}{m_{n_k}}\right),\end{aligned}$$

i.e., $f_5^{(\frac{1}{\lambda},-\beta)}\notin H_{\alpha+\rho}^p[\omega]$. This contradicts our assumption. The proof of the necessity part in (12)-(13) and (16)-(17) is now complete.

II. $p=1$ or $p=\infty$. Step 6. Let us prove the necessity in (14). Let $H_{\alpha+\rho}^p[\omega]\subset W_p^{\lambda,\beta}$ and the series in (14) be divergent.

Step 6(a): $\sin\frac{\beta\pi}{2}\neq 0$. Then the divergence of the series in (14) is equivalent to the divergence of the series $\sum\limits_{n=1}^{\infty}\frac{\lambda_n}{n}\omega\left(\frac{1}{n}\right)$.

Let $p=1$. We take a sequence ε which is a $q_{\alpha+\rho,1}(\omega)$-sequence and consider the series

$$\sum_{\nu=1}^{\infty}(\varepsilon_\nu-\varepsilon_{\nu+1})K_\nu(x). \tag{57}$$

This series is convergent in L_1 (see [Ge]) to a function $f_6(x)$, $E_n(f_6)_1 = O(\varepsilon_n)$. By means of (3), (41), we get $f_6 \in H^p_{\alpha+\rho}[\omega] \subset W_p^{\lambda,\beta}$. One can rewrite (57) in the following form

$$\sum_{\nu=1}^{\infty} a_\nu \cos \nu x, \qquad \text{where} \qquad a_\nu = \varepsilon_\nu - \nu \sum_{j=\nu}^{\infty} \frac{\varepsilon_j - \varepsilon_{j+1}}{j+1}.$$

By Lemma 4.1,

$$\begin{aligned} \left\| f_6^{(\lambda,\beta)} \right\|_1 &\geq C(\beta) \sum_{\nu=1}^{\infty} \frac{\lambda_\nu}{\nu} a_\nu = C(\beta) \left(\sum_{\nu=1}^{\infty} \frac{\lambda_\nu}{\nu} \varepsilon_\nu - \sum_{\nu=1}^{\infty} \lambda_\nu \sum_{j=\nu}^{\infty} \frac{\varepsilon_j - \varepsilon_{j+1}}{j+1} \right) \\ &= C(\beta) \left(\sum_{\nu=1}^{\infty} \frac{\lambda_\nu}{\nu} \varepsilon_\nu - \sum_{\nu=1}^{\infty} (a_\nu - a_{\nu+1}) \lambda_\nu \right) \\ &\geq C_1(\beta) \sum_{\nu=1}^{\infty} \frac{\lambda_\nu}{\nu} \varepsilon_\nu - C_2(\beta) \left(\lambda_1 a_1 + \sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n) a_n \right). \end{aligned}$$

Using (40) and (41), we get

$$\begin{aligned} \sum_{\nu=1}^{\infty} \frac{\lambda_\nu}{\nu} \varepsilon_\nu &\leq C(\rho) \sum_{\nu=1}^{\infty} \frac{\lambda_\nu}{\nu} \omega\left(\frac{1}{\nu}\right) \leq C(\rho) \sum_{\nu=1}^{\infty} \lambda_\nu \nu^{-(\alpha+\rho)-1} \sum_{m=1}^{\nu} m^{\alpha+\rho-1} \varepsilon_m \\ &= C(\rho) \sum_{m=1}^{\infty} m^{\alpha+\rho-1} \varepsilon_m \sum_{\nu=m}^{\infty} \frac{\lambda_\nu}{\nu^\rho} \nu^{-\alpha-1} \leq C(\rho,\alpha) \sum_{m=1}^{\infty} \frac{\lambda_m}{m} \varepsilon_m. \end{aligned}$$

Therefore,

$$\left\| f_6^{(\lambda,\beta)} \right\|_1 \geq \tag{58}$$
$$\geq C_1(\alpha,\rho,\beta) \sum_{\nu=1}^{\infty} \lambda_\nu \nu^{-1} \omega\left(\frac{1}{\nu}\right) - C_2(\alpha,\rho,\beta) \left(\lambda_1 a_1 + \sum_{\nu=1}^{\infty} (\lambda_{\nu+1} - \lambda_\nu) a_\nu \right).$$

On the other hand, using monotonicity of $\{a_\nu\}$ and Lemma 4.1, we have

$$\begin{aligned} C(\beta,\rho) \left\| f_6^{(\lambda,\beta)} \right\|_1 &\geq C(\rho) \left(\lambda_1 a_1 + \sum_{\nu=1}^{\infty} \frac{\lambda_\nu a_\nu}{\nu} \right) \\ &\geq C(\rho) \left(\lambda_1 a_1 + \sum_{\nu=0}^{\infty} \lambda_{2^\nu} a_{2^\nu} \right) \\ &\geq \lambda_1 a_1 + \sum_{\nu=0}^{\infty} a_{2^\nu} (\lambda_{2^{\nu+1}} - \lambda_{2^\nu}) \end{aligned}$$

$$\geq \quad \lambda_1 a_1 + \sum_{\nu=1}^{\infty} (\lambda_{\nu+1} - \lambda_\nu) a_\nu. \tag{59}$$

From (58) and (59) we get

$$\left\| f_6^{(\lambda,\beta)} \right\|_1 \geq C(\alpha,\rho,\beta) \sum_{\nu=1}^{\infty} \lambda_\nu \nu^{-1} \omega\left(\frac{1}{\nu}\right) = \infty.$$

This contradiction implies the convergence of series in (14).

Let now $p = \infty$. Define the function (see [Ba,1]) $f_7(x) = \sum\limits_{\nu=1}^{\infty} \varepsilon_\nu \nu^{-1} \sin \nu x$, where ε is a $q_{\alpha+\rho,\,1}(\omega)$-sequence. We have $E_n(f_7)_\infty \leq C\varepsilon_{n+1}$. Using (3) and (41), we get $f_7 \in H^p_{\alpha+\rho}[\omega] \subset W_p^{\lambda,\beta}$. On the other hand,

$$\begin{aligned}
\infty = \left\| f_7^{(\lambda,\beta)} \right\|_\infty \quad &\geq \quad C(\beta) \sum_{\nu=0}^{\infty} 2^{\nu(\alpha+\rho)} \varepsilon_{2^\nu} \frac{\lambda_{2^\nu}}{2^{\nu(\alpha+\rho)}} \\
&\geq \quad C(\alpha,\rho,\beta) \sum_{\nu=0}^{\infty} 2^{\nu(\alpha+\rho)} \varepsilon_{2^\nu} \sum_{m=\nu}^{\infty} \frac{\lambda_{2^m}}{2^{m(\alpha+\rho)}} \\
&\geq \quad C(\alpha,\rho,\beta) \sum_{\nu=1}^{\infty} \lambda_\nu \nu^{-1} \omega\left(\frac{1}{\nu}\right).
\end{aligned}$$

This implies the convergence of the series in (14).

Step 6(b): $\sin \frac{\beta\pi}{2} = 0$. Let the series in (14) be divergent. We will consider only the non-trivial case $\rho > 0$. Let ε be a $q_{\alpha+\rho,\,1}(\omega)$-sequence. By means of the properties $\{\lambda_n\}$, we have

$$\begin{aligned}
&\sum_{\nu=2}^{\infty} (\lambda_{\nu+1} - \lambda_\nu)\, \omega\left(\frac{1}{\nu}\right) \\
&\leq C(\alpha,\rho) \sum_{s=0}^{\infty} (\varepsilon_{2^s} - \varepsilon_{2^{s+1}}) \left[\sum_{m=0}^{s} 2^{m(\alpha+\rho)} \sum_{\nu=m}^{s} 2^{-\nu(\alpha+\rho)} (\lambda_{2^\nu} - \lambda_{2^{\nu-1}}) \right. \\
&\left. + \sum_{m=0}^{s} 2^{m(\alpha+\rho)} \sum_{\nu=s+1}^{\infty} 2^{-\nu(\alpha+\rho)} (\lambda_{2^\nu} - \lambda_{2^{\nu-1}}) \right] \\
&\leq C(\alpha,\rho) \sum_{\nu=0}^{\infty} (\varepsilon_\nu - \varepsilon_{\nu+1})\, \lambda_\nu.
\end{aligned} \tag{60}$$

Let $p = 1$. The series

$$\sum_{\nu=1}^{\infty} (\varepsilon_\nu - \varepsilon_{\nu+1})\, \tau_\nu(x) \tag{61}$$

converges ([Ge]) to a $f_8 \in L_1$, and $E_n(f_8)_1 \le C\varepsilon_{n+1}$. Then $f_8 \in H^p_{\alpha+\rho}[\omega] \subset W_p^{\lambda,\beta}$. We rewrite (61) in the following way

$$\sum_{\nu=1}^{\infty} b_\nu \sin \nu x, \qquad \text{where}$$

$$b_\nu = \sum_{j=\nu}^{2\nu-2} \left(1 - \frac{\nu}{j+1}\right)(\varepsilon_\nu - \varepsilon_{\nu+1}) + \sum_{j=2\nu-1}^{\infty} \frac{\nu}{j+1}(\varepsilon_\nu - \varepsilon_{\nu+1}).$$

By Lemma 4.1, we write

$$\left\| f_8^{(\lambda,\beta)} \right\|_1 \ge C(\beta) \sum_{\nu=1}^{\infty} \lambda_\nu \frac{b_\nu}{\nu} \ge C(\beta) \sum_{\nu=0}^{\infty} \lambda_\nu (\varepsilon_\nu - \varepsilon_{\nu+1}).$$

This contradicts the divergence of the series in (60). Thus, the series in (14) converges.

Let $p = \infty$. Let ψ be a $Q_{\alpha+\rho,1}$-sequence. Then, we define

$$f_9(x) = \psi_1 \cos x + \sum_{\nu=1}^{\infty} 2^{-\nu(\rho+\alpha)} (\psi_{2^\nu} - \psi_{2^{\nu-1}}) \cos 2^\nu x.$$

Similarly, as for f_1 in the case $2 \le p < \infty$, it is easy to see that $f_9 \in H^p_{\alpha+\rho}[\omega] \subset W_p^{\lambda,\beta}$. By Lemma 4.3, we have

$$\begin{aligned} \left\| f_9^{(\lambda,\beta)} \right\|_\infty &\ge C(\beta) \left(\lambda_1 \psi_1 + \sum_{\nu=1}^{\infty} \lambda_{2^\nu} 2^{-\nu(\rho+\alpha)} (\psi_{2^\nu} - \psi_{2^{\nu-1}}) \right) \\ &\ge C(\beta) \sum_{\nu=1}^{\infty} \lambda_\nu \nu^{-(\rho+\alpha)-1} \psi_\nu \\ &\ge C(\beta) \sum_{\nu=1}^{\infty} (\lambda_{\nu+1} - \lambda_\nu)\, \omega(1/\nu) = \infty, \end{aligned}$$

this contradicts $f_9 \in W_p^{\lambda,\beta}$. This completes the proof of the necessity in (14).

Step 7. We will prove the necessity in (15) for the case $\sin \frac{\pi\beta}{2} \ne 0$. Let $H^p_{\alpha+r}[\omega] \subset W_p^{\lambda,\beta} H_\alpha[\varphi]$. Let ε be a $q_{\alpha+r,\,1}(\omega)$-sequence. So, (41) holds for $\alpha + r$ instead of α and 1 instead of θ. Since $\sin \frac{\pi\beta}{2} \ne 0$,

$$\begin{aligned} J \;:=\; & \lambda_{n+1} \omega\left(\frac{1}{n+1}\right) + n^{-\alpha} \sum_{\nu=1}^{n} \nu^{r+\alpha} \left(\nu^{-r}\lambda_\nu - (\nu+1)^{-r}\lambda_{\nu+1}\right) \omega\left(\frac{1}{\nu}\right) \\ + \; & |\cos \frac{\beta\pi}{2}| \sum_{\nu=n+2}^{\infty} (\lambda_{\nu+1} - \lambda_\nu)\, \omega\left(\frac{1}{\nu}\right) + |\sin \frac{\beta\pi}{2}| \sum_{\nu=n+2}^{\infty} \lambda_\nu \frac{\omega\left(\frac{1}{\nu}\right)}{\nu} \end{aligned}$$

$$\leq C(\alpha,\beta,r)\left(\sum_{\nu=n+1}^{\infty}\lambda_\nu\frac{\varepsilon_\nu}{\nu}+n^{-\alpha}\sum_{\nu=1}^{n}\lambda_\nu\varepsilon_\nu\nu^{\alpha-1}\right)=:C(\alpha,\beta,r)\,(J_1+J_2)\,. \quad (62)$$

We will use several times the following evident inequalities:

$$\omega_\alpha\Big(f,\frac{1}{n}\Big)_p \asymp \omega_\alpha\Big(f_+,\frac{1}{n}\Big)_p+\omega_\alpha\Big(f_-,\frac{1}{n}\Big)_p, \quad \text{where} \quad f_\pm(x):=\frac{f(x)\pm f(-x)}{2}. \quad (63)$$

Step 7(a): $p=\infty$ and $\cos\frac{\pi\alpha}{2}\neq 0$. Then by Lemma 3.1, we write $\Big(\ f_{7+}^{(\lambda,\beta)}:=\big(f_7^{(\lambda,\beta)}\big)_+\ \Big)$

$$\begin{aligned}\omega_\alpha\Big(f_7^{(\lambda,\beta)},\frac{1}{n}\Big)_p &\geq \omega_\alpha\Big(f_{7+}^{(\lambda,\beta)},\frac{1}{n}\Big)_p\geq C(\alpha)n^{-\alpha}\left\|V_n^{(\alpha)}(f_{7+}^{(\lambda,\beta)})(\cdot)\right\|_p\\ &\geq C(\alpha)n^{-\alpha}\left|V_n^{(\alpha)}(f_{7+}^{(\lambda,\beta)})(0)\right|_p\geq C(\alpha,\beta)J_2. \quad (64)\end{aligned}$$

By (20) and $\sum\limits_{k=2n}^{\infty}a_k\leq 4E_n(f)_\infty$ (see [Ba,1]), we have

$$\omega_\alpha\Big(f_7^{(\lambda,\beta)},\frac{1}{n}\Big)_p\geq C(\alpha)E_{[\frac{n}{2}]}(f_{7+}^{(\lambda,\beta)})_p\geq C(\alpha,\beta)J_1. \quad (65)$$

We have already proved that $f_7\in H^p_{\alpha+r}[\omega]\subset W_p^{\lambda,\beta}H_\alpha[\varphi]$. Collecting inequalities (64), (65), and (62), we arrive at the right-hand side of (15).

Step 7(b): $p=\infty$ and $\cos\frac{\pi\alpha}{2}=0$. If $\cos\frac{\pi\beta}{2}\neq 0$, then we use (65) and

$$\omega_\alpha\Big(f_7^{(\lambda,\beta)},\frac{1}{n}\Big)_p \geq C(\alpha)n^{-\alpha}\left\|V_n^{(\alpha)}(f_{7-}^{(\lambda,\beta)})(\cdot)\right\|_p\geq C(\alpha,\beta)J_2.$$

If $\cos\frac{\pi\beta}{2}=0$, then $f_7=\pm f_{7+}$ and

$$\omega_\alpha\Big(f_7^{(\lambda,\beta)},\frac{1}{n}\Big)_p\geq C(\alpha)E_{[\frac{n}{2}]}(f_{7+}^{(\lambda,\beta)})_p\geq C(\alpha,\beta)J_1.$$

To obtain the estimate of J_2, we define

$$f_{10}(x)=\frac{\varepsilon_0}{2}+(\varepsilon_1-\varepsilon_2)\cos x+\sum_{\nu=1}^{\infty}(\varepsilon_{2^\nu}-\varepsilon_{2^{\nu+1}})\cos 2^\nu x.$$

It is clear that $E_n(f_{10})_p\leq\varepsilon_{n+1}$. Then, by (41), we have $f_{10}\in H^p_{\alpha+r}[\omega]\subset W_p^{\lambda,\beta}H_\alpha[\varphi]$. Using $\omega_\alpha\left(f,\frac{1}{n}\right)_p\geq C(\alpha)n^{-\alpha}\left\|V_n^{(\alpha)}(f)\right\|_p$ and Lemma 4.3, we write $(2^m\leq n+1<2^{m+1})$

$$\omega_\alpha\left(f_{10}^{(\lambda,\beta)},\frac{1}{2^m}\right)_p\geq$$

$$\geq \quad C(\alpha,\beta)2^{-m\alpha}\sum_{\nu=0}^{m}\lambda_{2^\nu}2^{\nu\alpha}\left(\varepsilon_{2^\nu}-\varepsilon_{2^{\nu+1}}\right)$$
$$\geq \quad C_1(\alpha,\beta,r)2^{-m\alpha}\sum_{\nu=0}^{m}\lambda_{2^\nu}2^{\nu\alpha}\varepsilon_{2^\nu}-C_2(\alpha,\beta,r)\lambda_{2^{m+1}}\varepsilon_{2^{m+1}}. \quad (66)$$

Concurrently, $\omega_\alpha\left(f_{10}^{(\lambda,\beta)},\frac{1}{2^m}\right)_p \geq C(\alpha,\beta)\lambda_{2^{m+1}}\varepsilon_{2^{m+1}}$. Then, we get

$$\omega_\alpha\left(f_{10}^{(\lambda,\beta)},\frac{1}{n}\right)_p \geq C(\alpha,\beta,r)J_2, \quad (67)$$

and by (63),

$$\begin{aligned} C(\alpha,\beta,r)\Big[J_1+J_2\Big] \quad &\leq \quad \omega_\alpha\left(f_{10}^{(\lambda,\beta)},\frac{1}{2^m}\right)_p+\omega_\alpha\left(f_{7}^{(\lambda,\beta)},\frac{1}{2^m}\right)_p \\ &\asymp \quad \omega_\alpha\left((f_7+f_{10})^{(\lambda,\beta)},\frac{1}{n}\right)_p = O\left[\varphi\left(\frac{1}{n}\right)\right]. \end{aligned}$$

The necessity in (15) follows.
Step 7(c): $p=1$ and $\cos\frac{\pi\alpha}{2}\neq 0$. We use the function f_6. We have $f_6\in H^p_{\alpha+r}[\omega]$ and by Lemma 4.1,

$$\begin{aligned} \omega_\alpha\Big(f_6^{(\lambda,\beta)},\frac{1}{n}\Big)_1 \quad &\geq \quad C(\alpha)E_n(f_{6-}^{(\lambda,\beta)})_p \geq C(\alpha,\beta)\sum_{\nu=n+1}^{\infty}\lambda_\nu\frac{a_\nu}{\nu} \\ &\geq \quad C(\alpha,\beta)\left(\sum_{\nu=n+1}^{\infty}\lambda_\nu\frac{\varepsilon_\nu}{\nu}-a_{n+1}\lambda_n-\sum_{\nu=n+1}^{\infty}\left(\lambda_\nu-\lambda_{\nu-1}\right)a_\nu\right). \end{aligned}$$

On the other hand,

$$\omega_\alpha\Big(f_6^{(\lambda,\beta)},\frac{1}{n}\Big)_1 \geq C(\alpha,\beta,r)\left(a_{n+1}\lambda_n+\sum_{\nu=n+1}^{\infty}\left(\lambda_\nu-\lambda_{\nu-1}\right)a_\nu\right).$$

The last two inequalities imply $\omega_\alpha\big(f_6^{(\lambda,\beta)},\frac{1}{n}\big)_1 \geq C(\alpha,\beta,r)J_1$. Also,

$$\omega_\alpha\Big(f_6^{(\lambda,\beta)},\frac{1}{n}\Big)_p \quad \geq \quad C(\alpha)n^{-\alpha}\left\|V_n^{(\alpha)}(f_{6-}^{(\lambda,\beta)})(\cdot)\right\|_p \geq C(\alpha,\beta)J_2. \quad (68)$$

Step 7(d): $p=1$ and $\cos\frac{\pi\alpha}{2}=0$. If $\cos\frac{\pi\beta}{2}\neq 0$, then we use $\omega_\alpha\big(f_6^{(\lambda,\beta)},\frac{1}{n}\big)_1 \geq C(\alpha,\beta,r)J_1$ and

$$\omega_\alpha\Big(f_6^{(\lambda,\beta)},\frac{1}{n}\Big)_p \quad \geq \quad C(\alpha)n^{-\alpha}\left\|V_n^{(\alpha)}(f_{6+}^{(\lambda,\beta)})(\cdot)\right\|_p \geq C(\alpha,\beta)J_2.$$

If $\cos\frac{\pi\beta}{2}=0$, we consider f_6+f_8. Using Lemmas 3.1 and 4.1, we have

$$\begin{aligned}\omega_\alpha\left(f_8^{(\lambda,\beta)},\frac{1}{n}\right)_1 &\geq C(\alpha,\beta)n^{-\alpha}\sum_{\nu=1}^{n}\lambda_\nu\nu^{\alpha-1}b_\nu\\ &\geq C_1(\alpha,\beta,r)n^{-\alpha}\sum_{\nu=1}^{n}\lambda_\nu\nu^{\alpha-1}\varepsilon_{\nu-1}-C_2(\alpha,\beta,r)\lambda_n\varepsilon_n.\end{aligned}$$

Since $\omega_\alpha\left(f_8^{(\lambda,\beta)},\frac{1}{n}\right)_1\geq C(\alpha,\beta,r)\lambda_n\varepsilon_n$, then $\omega_\alpha\left(f_8^{(\lambda,\beta)},\frac{1}{n}\right)_1\geq C(\alpha,\beta,r)J_2$. Thus

$$C(\alpha,\beta,r)\Big[J_1+J_2\Big]\leq\omega_\alpha\left(\left(f_6+f_8\right)^{(\lambda,\beta)},\frac{1}{n}\right)_p=O\left[\varphi\left(\frac{1}{n}\right)\right],$$

i.e., the necessity in (15) follows.

Step 8. We will prove the necessity in (15) for the case $\sin\frac{\pi\beta}{2}=0$. Let $H^p_{\alpha+r}[\omega]\subset W_p^{\lambda,\beta}H_\alpha[\varphi]$ and ε be a $q_{\alpha+r,1}(\omega)$-sequence. Since $\sin\frac{\pi\beta}{2}=0$, we have from (41)

$$\begin{aligned}\lambda_{n+1}\omega\left(\frac{1}{n+1}\right) &+ n^{-\alpha}\sum_{\nu=1}^{n}\nu^{r+\alpha}\left(\nu^{-r}\lambda_\nu-(\nu+1)^{-r}\lambda_{\nu+1}\right)\omega\left(\frac{1}{\nu}\right)\\ &+ |\cos\frac{\beta\pi}{2}|\sum_{\nu=n+2}^{\infty}\left(\lambda_{\nu+1}-\lambda_\nu\right)\omega\left(\frac{1}{\nu}\right)\\ &\leq C(\alpha,\beta,r)\left(\sum_{\nu=n+1}^{\infty}\lambda_\nu\left(\varepsilon_\nu-\varepsilon_{\nu+1}\right)+n^{-\alpha}\sum_{\nu=1}^{n}\lambda_\nu\varepsilon_\nu\nu^{\alpha-1}\right)\\ &=: C(\alpha,\beta,r)\left(J_3+J_4\right).\end{aligned}\tag{69}$$

Step 8(a): $p=\infty$ and $\cos\frac{\pi\alpha}{2}\neq0$. By the Jackson inequality and Lemma 4.3, we have

$$\omega_\alpha\left(f_{10}^{(\lambda,\beta)},\frac{1}{2^m}\right)_p\geq C(\alpha,\beta)\sum_{\nu=m}^{\infty}\lambda_{2^\nu}\left(\varepsilon_{2^\nu}-\varepsilon_{2^{\nu+1}}\right).\tag{70}$$

We also note that by Lemma (4.3), (67) holds for all $\alpha>0$. This and (70) give $\omega_\alpha\left(f_{10}^{(\lambda,\beta)},\frac{1}{n+1}\right)_p\geq C(\alpha,\beta)\left(J_3+J_4\right)$. Using condition (69) and $f_{10}\in H^p_{\alpha+r}[\omega]\subset W_p^{\lambda,\beta}H_\alpha[\varphi]$, we arrive at the relation in the right part of (15).

Step 8(b): $p=\infty$ and $\cos\frac{\pi\alpha}{2}=0$. Then we consider f_{10} and $f_{11}:=\widetilde{f_{10}}$. It is clear that $f_{11}\in L_p$ and $f_{10}+f_{11}\in H^p_{\alpha+r}[\omega]$. Nevertheless, $\omega_\alpha\left(f_{10}^{(\lambda,\beta)},\frac{1}{n+1}\right)_p\geq C(\alpha,\beta)J_3$, $\omega_\alpha\left(f_{11}^{(\lambda,\beta)},\frac{1}{n+1}\right)_p\geq C(\alpha,\beta)J_4$, and

$\omega_\alpha\left(f_{10}^{(\lambda,\beta)},\frac{1}{n+1}\right)_p+\omega_\alpha\left(f_{11}^{(\lambda,\beta)},\frac{1}{n+1}\right)_p\asymp\omega_\alpha\left((f_{10}+f_{11})^{(\lambda,\beta)},\frac{1}{n+1}\right)_p.$

Step 8(c): $p=1$ and $\cos\frac{\pi\alpha}{2}\neq 0$. Since $f_8^{(\lambda,\beta)}(x)\sim\pm\sum\limits_{\nu=1}^{\infty}\lambda_\nu b_\nu\sin\nu x$, by Lemmas 3.1 and 4.1, we write (see Step 7(d))

$$\omega_\alpha\Big(f_8^{(\lambda,\beta)},\frac{1}{n}\Big)_p\geq C(\alpha)n^{-\alpha}\left\|V_n^{(\alpha)}(f_{8-}^{(\lambda,\beta)})(\cdot)\right\|_p\geq C(\alpha,\beta)J_4. \tag{71}$$

Also, by Lemma 4.1 and the Jackson inequality (20), we have

$$\begin{aligned}\omega_\alpha\left(f_8^{(\lambda,\beta)},\frac{1}{n}\right)_1 &\geq C(\alpha,\beta)\sum_{\nu=n+1}^{\infty}\lambda_\nu\frac{b_\nu}{\nu}\\ &\geq C(\alpha,\beta)\sum_{\nu=n+1}^{\infty}\lambda_\nu\sum_{j=2\nu-1}^{\infty}\frac{\varepsilon_j-\varepsilon_{j+1}}{j+1}\\ &\geq C(\alpha,\beta,r)\sum_{j=4n-1}^{\infty}(\varepsilon_j-\varepsilon_{j+1})\,\lambda_\nu.\end{aligned}$$

Using the properties of modulus of smoothness, we get (15).

Step 8(d): $p=1$ and $\cos\frac{\pi\alpha}{2}=0$. We use that $f_{12}:=\widetilde{f_8}\in L_1$ and $E_n(f_{12})_p\leq C\varepsilon_{n+1}$ (see [Ge]). Then $f_8+f_{12}\in H^p_{\alpha+r}[\omega]$ and $\omega_\alpha\left(f_8^{(\lambda,\beta)},\frac{1}{n}\right)_1\geq C(\alpha,\beta)J_3$, $\omega_\alpha\left(f_{12}^{(\lambda,\beta)},\frac{1}{n}\right)_1\geq C(\alpha,\beta)J_4$. This completes the proof of Theorem 1.

5. Corollaries. Estimates of transformed Fourier series.

Theorem 1 actually provides estimates of the norms and moduli of smoothness of the transformed Fourier series, i.e., the estimates of $\|\varphi\|_p$ and $\omega_\alpha(\varphi,\delta)_p$, where $\varphi\sim\sigma(f,\lambda)$, by $\omega_\gamma(f,\delta)_p$. Analyzing Theorem 1, one can see that the following two conditions play a crucial role. The first is the behavior of the transforming sequence $\{\lambda_n\}$ and the second is the alternation between L_p, $1<p<\infty$ and L_p, $p=1,\infty$.

We will investigate in detail some important examples for L_p, $1<p<\infty$ and for L_p, $p=1,\infty$, separately.

1. The case $1<p<\infty$.

Theorem 2. *Let $1<p<\infty$, $\theta=\min(2,p)$ $\tau=\max(2,p)$, $\alpha\in\mathbf{R}_+$, and $\lambda=\{\lambda_n\}$ be a non-decreasing sequence of positive numbers. Let ρ be a non-negative number such that the sequence $\{n^{-\rho}\lambda_n\}$ is non-increasing.*

I. *If for $f \in L_p^0$ the series*

$$\sum_{n=1}^{\infty} \left(\lambda_{n+1}^{\theta} - \lambda_n^{\theta}\right) \omega_{\alpha+\rho}^{\theta}\left(f, \frac{1}{n}\right)_p$$

converges, then there exists a function $\varphi \in L_p^0$ with Fourier series $\sigma(f, \lambda)$, and

$$\|\varphi\|_p \leq$$

$$\leq C(p, \lambda, \alpha, \rho) \left\{ \lambda_1^{\theta} \|f\|_p^{\theta} + \sum_{n=1}^{\infty} \left(\lambda_{n+1}^{\theta} - \lambda_n^{\theta}\right) \omega_{\alpha+\rho}^{\theta}\left(f, \frac{1}{n}\right)_p \right\}^{\frac{1}{\theta}}, \tag{72}$$

$$\omega_{\alpha}\left(\varphi, \frac{1}{n+1}\right)_p \leq$$

$$\leq C(p, \lambda, \alpha, \rho) \Bigg\{ n^{-\alpha\theta} \sum_{\nu=1}^{n} \nu^{(\rho+\alpha)\theta} \left(\nu^{-\rho\theta} \lambda_{\nu}^{\theta} - (\nu+1)^{-\rho\theta} \lambda_{\nu+1}^{\theta}\right) \omega_{\alpha+\rho}^{\theta}\left(f, \frac{1}{\nu}\right)_p$$

$$+ \sum_{\nu=n+2}^{\infty} \left(\lambda_{n+1}^{\theta} - \lambda_n^{\theta}\right) \omega_{\alpha+\rho}^{\theta}\left(f, \frac{1}{\nu}\right)_p + \lambda_{n+1}^{\theta} \omega_{\alpha+\rho}^{\theta}\left(f, \frac{1}{n}\right)_p \Bigg\}^{\frac{1}{\theta}}. \tag{73}$$

II. *If for $f \in L_p^0$ there exists a function $\varphi \in L_p$ with Fourier series $\sigma(f, \lambda)$, then*

$$\left\{ \lambda_1^{\tau} \|f\|_p^{\tau} + \sum_{n=1}^{\infty} \left(\lambda_{n+1}^{\tau} - \lambda_n^{\tau}\right) \omega_{\alpha+\rho}^{\tau}\left(f, \frac{1}{n}\right)_p \right\}^{\frac{1}{\tau}} \leq C(p, \lambda, \alpha, \rho) \|\varphi\|_p, \tag{74}$$

$$\Bigg\{ n^{-\alpha\tau} \sum_{\nu=1}^{n} \nu^{(\rho+\alpha)\tau} \left(\nu^{-\rho\tau} \lambda_{\nu}^{\tau} - (\nu+1)^{-\rho\tau} \lambda_{\nu+1}^{\tau}\right) \omega_{\alpha+\rho}^{\tau}\left(f, \frac{1}{\nu}\right)_p + \tag{75}$$

$$\sum_{\nu=n+2}^{\infty} \left(\lambda_{n+1}^{\tau} - \lambda_n^{\tau}\right) \omega_{\alpha+\rho}^{\tau}\left(f, \frac{1}{\nu}\right)_p + \lambda_{n+1}^{\tau} \omega_{\alpha+\rho}^{\tau}\left(f, \frac{1}{n}\right)_p \Bigg\}^{\frac{1}{\tau}}$$

$$\leq C(p, \lambda, \alpha, \rho) \omega_{\alpha}\left(\varphi, \frac{1}{n+1}\right)_p,$$

$$\omega_{\alpha+\rho}\left(f, \frac{1}{n}\right)_p \leq C(p, \lambda, \alpha, \rho) \frac{\|\varphi\|_p}{\lambda_n}, \tag{76}$$

$$\omega_{\alpha+\rho}\left(f, \frac{1}{n}\right)_p \leq C(p, \lambda, \alpha, \rho) \frac{\omega_{\alpha}\left(\varphi, \frac{1}{n}\right)_p}{\lambda_n}. \tag{77}$$

Inequalities (72)-(73) and (76)-(77) were proved in Theorem 1 (see the sufficiency in part **I**). To prove (74)-(75), we use similar reasoning, applied to the theorems by Littlewood-Paley, Marcinkiewicz, and the Minkowski's inequality.

As an important corollary of Theorem 2,

Corollary 1. *Let* $1 < p < \infty$, $\theta = \min(2,p)$ $\tau = \max(2,p)$. *Then, for any* $k, r > 0$,

$$C_1 \left\{ \sum_{\nu=n+1}^{\infty} \nu^{r\tau-1} \omega_{k+r}^{\tau} \left(f, \frac{1}{\nu} \right)_p \right\}^{\frac{1}{\tau}} \leq \omega_k \left(f^{(r)}, \frac{1}{n} \right)_p \tag{78}$$

$$\leq C_2 \left\{ \sum_{\nu=n+1}^{\infty} \nu^{r\theta-1} \omega_{k+r}^{\theta} \left(f, \frac{1}{\nu} \right)_p \right\}^{\frac{1}{\theta}},$$

where $C_1 = C_1(p,k,r), C_2 = C_2(p,k,r), n \in \mathbf{N}$.

The last two inequalities are an improvement compared to (6) and (7). Indeed, by properties of the modulus of smoothness and by the Jensen inequality,

$$n^r \omega_{k+r} \left(f, \frac{1}{n} \right)_p \leq C(k,r) \left\{ \sum_{\nu=n+1}^{\infty} \nu^{r\tau-1} \omega_{k+r}^{\tau} \left(f, \frac{1}{\nu} \right)_p \right\}^{\frac{1}{\tau}};$$

$$\left\{ \sum_{\nu=n+1}^{\infty} \nu^{r\theta-1} \omega_{k+r}^{\theta} \left(f, \frac{1}{\nu} \right)_p \right\}^{\frac{1}{\theta}} \leq C(k,r) \sum_{\nu=n+1}^{\infty} \nu^{r-1} \omega_{k+r} \left(f, \frac{1}{\nu} \right)_p.$$

Example. Let $\psi(t) = t^r \ln^{-A}(1/t)$ and $2 \leq p < \infty, \frac{1}{2} < A < 1$. For $\omega_{k+r}(f,t)_p \asymp \psi(t)$, (6) and (7) give only $C \ln^{-A}(1/t) \leq \omega_k \left(f^{(r)}, t \right)_p$, and (78) gives $C_1 \ln^{-A+1/p}(1/t) \leq \omega_k \left(f^{(r)}, t \right)_p \leq C_2 \ln^{-A+1/2}(1/t)$, which is superior.

Proof of Corollary 1 is obvious from (73) and (75) with $r = \rho$, because if $f \in L_p$, $1 < p < \infty$, then $f^{(r)} \sim \sigma(f, \lambda)$ for $\{\lambda_n = n^r\}$.

From (6), (7) and (78), one can see that it is natural to estimate $\omega_\alpha \left(f^{(\gamma)}, \delta \right)_p$ by $\omega_{\alpha+r}(f,t)_p$. Further analysis allows the distinction between three different types of such estimates. It will be convenient for us to write inequalities in integral form:

1. $\gamma = r$ (see Corollary 1)[8]

$$\left\{\int_0^\delta t^{-r\tau-1}\omega_{r+\alpha}^\tau(f,t)_p dt\right\}^{\frac{1}{\tau}} \ll \omega_\alpha(f^{(r)},\delta)_p \ll \left\{\int_0^\delta t^{-r\theta-1}\omega_{r+\alpha}^\theta(f,t)_p dt\right\}^{\frac{1}{\theta}}; \tag{79}$$

2. $\gamma = r-\varepsilon,\ 0<\varepsilon<r$ (see Theorem 2 for $\rho = r$ and $\lambda_n = n^{r-\varepsilon}$):

$$\left\{\int_0^\delta t^{-(r-\varepsilon)\tau-1}\omega_{r+\alpha}^\tau(f,t)_p dt + \delta^{\alpha\tau}\int_\delta^1 t^{-(r-\varepsilon+\alpha)\tau-1}\omega_{r+\alpha}^\tau(f,t)_p dt\right\}^{\frac{1}{\tau}} \ll$$

$$\ll \omega_\alpha(f^{(r-\varepsilon)},\delta)_p, \tag{80}$$

$$\omega_\alpha(f^{(r-\varepsilon)},\delta)_p \ll$$

$$\ll \left\{\int_0^\delta t^{-(r-\varepsilon)\theta-1}\omega_{r+\alpha}^\theta(f,t)_p dt + \delta^{\alpha\theta}\int_\delta^1 t^{-(r-\varepsilon+\alpha)\theta-1}\omega_{r+\alpha}^\theta(f,t)_p dt\right\}^{\frac{1}{\theta}}; \tag{81}$$

3. $\gamma = r+\varepsilon,\ 0<\varepsilon<\alpha$, (see [Po-Si-Ti,1]):

$$\left\{\int_0^\delta t^{-(r+\varepsilon)\tau-1}\omega_{r+\alpha}^\tau(f,t)_p\, dt\right\}^{\frac{1}{\tau}} \ll \delta^{\alpha-\varepsilon}\left\{\int_\delta^1 t^{-(\alpha-\varepsilon)\theta-1}\omega_\alpha^\theta(f^{(r+\varepsilon)},t)_p\, dt\right\}^{\frac{1}{\theta}}, \tag{82}$$

$$\delta^{\alpha-\varepsilon}\left\{\int_\delta^1 t^{-(\alpha-\varepsilon)\tau-1}\omega_\alpha^\tau(f^{(r+\varepsilon)},t)_p dt\right\}^{\frac{1}{\tau}} \ll \left\{\int_0^\delta t^{-(r+\varepsilon)\theta-1}\omega_{r+\alpha}^\theta(f,t)_p dt\right\}^{\frac{1}{\theta}}. \tag{83}$$

One can consider a more general sequence than $\{\lambda_n = n^\gamma\}$, $\gamma>0$. Let us define $\{\Lambda_n(s) := \Lambda(s,\frac{1}{n})\}$, where

$$\Lambda(s,t) = \Lambda(s,r,t) = \left(\int_t^1 \xi(u)du + t^{-rs}\int_0^t u^{rs}\xi(u)du\right)^{\frac{1}{s}}, \tag{84}$$

and non-negative function $\xi(u)$ on $[0,1]$ is such that $u^{rs}\xi(u)$ is summable.

The sequence $\{\Lambda_n = \Lambda_n(s)\}$ is often considered a transforming sequence.
Example. 1. $\{\Lambda_n = n^\gamma\}$, $0<\gamma<r$; 2. $\{\Lambda_n = n^r\ln^{-A} n\}$, $A>0$; 3. $\{\Lambda_n = \ln^A n\}$, $A>0$.

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[8] Here and further $\tau = \max(2,p), \theta = \min(2,p)$. We will write $A_1 \ll A_2$, if $A_1 \le CA_2$, $C\ge 1$. Also, if $A_1 \ll A_2$ and $A_2 \ll A_1$, then $A_1 \asymp A_2$.

Properties of $\Lambda(s,t)$: $\Lambda(s,t)$ is non-increasing on t, $\Lambda(s,r,t)t^r$ is non-decreasing on t. By these properties and Theorem 2, one can obtain the following estimates, which are more general than (80)–(81) (see also [Po-Si,1]).

Theorem 3. *Let* $1<p<\infty$, $\theta=\min(2,p)$ $\tau=\max(2,p)$, $\alpha,r>0$, *and* $\Lambda=\{\Lambda_n(s):=\Lambda(s,\frac{1}{n})\}$, *where* $\Lambda(s,t)$ *and* $\xi(t)$ *were denoted above.*
I. *If for* $f\in L_p$ *the integral*

$$I_1:=\left\{\int_0^1 \xi(t)\ \omega_{\alpha+r}^s\left(f,t\right)_p\,dt\right\}^{\frac{1}{s}}$$

is finite for $s=\theta$, *then there exists a function* $\varphi\in L_p$ *with Fourier series* $\sigma(f,\Lambda)$, $\|\varphi\|_p\le C(p,\xi,\alpha,r,s)I_1$, *and*

$$\begin{aligned}\omega_\alpha\left(\varphi,\delta\right)_p \quad &\le \quad C(p,\xi,\alpha,r,s)\left\{\int_0^\delta t^{-rs-1}\int_0^t u^{rs}\xi(u)\,du\,\omega_{r+\alpha}^s(f,t)\,dt\right.\\ &+\quad \left.\delta^{\alpha s}\int_\delta^1 t^{\alpha s-1}\int_t^1 \xi(u)\,du\,\omega_{r+\alpha}^s(f,t)_p\,dt\right\}^{\frac{1}{s}} =: C\,I_2. \qquad (85)\end{aligned}$$

II. *If for* $f\in L_p$ *there exists a function* $\varphi\in L_p$ *with Fourier series* $\sigma(f,\Lambda)$, *then for* $s=\tau$ $\|\varphi\|_p\ge C(p,\xi,\alpha,r,s)I_1$, *and* $\omega_\alpha\left(\varphi,\delta\right)_p\ge C(p,\xi,\alpha,r,s)I_2$.

In a similar way we can generalize inequalities (82)–(83).

Theorem 4. *Let* $1<p<\infty$, $\theta=\min(2,p)$ $\tau=\max(2,p)$, $\alpha,r>0$, *and* $\Lambda=\{\Lambda_n(s):=\Lambda(s,\frac{1}{n})\}$, *where* $\Lambda(s,t)$ *and* $\xi(t)$ *were denoted above.*
I. *If for* $f\in L_p$ *the integral*

$$I_3:=\left\{\int_0^1 \xi(t)\ \omega_{\alpha+r}^s\left(f,t\right)_p\,dt\right\}^{\frac{1}{s}}$$

is finite for $s=\theta$, *then there exists a function* $\varphi\in L_p$ *with Fourier series* $\sigma(f,\Lambda)$, $\|\varphi\|_p\le C(p,\xi,\alpha,r,s)I_3$, *and for any* $\alpha_1>\alpha$

$$\begin{aligned}&I_4(\tau,s)=:\\ &\quad =:\left\{\delta^{(r+\alpha)\tau}\Lambda^\tau(s,r,\delta)\int_\delta^1\frac{t^{-(r+\alpha)\tau-1}}{\Lambda^\tau(s,r,t)}\omega_{r+\alpha_1}^\tau(\varphi,t)_p\,dt\right\}^{\frac{1}{\tau}}\\ &\quad \le C(p,\xi,\alpha,\alpha_1,r,s)\end{aligned}$$

$$\times \left[\Lambda(s,r,\delta)\omega_{r+\alpha}(f,\delta)_p + \left\{ \int\limits_0^\delta t^{-r\theta-1} \int\limits_0^t u^{r\tau}\xi(u)du\omega_{r+\alpha}^\theta(f,t)_p dt \right\}^{\frac{1}{\theta}} \right]$$

$$=: CI_5(\theta) \qquad (86)$$

II. *If for $f \in L_p$ there exists a function $\varphi \in L_p$ with Fourier series $\sigma(f,\Lambda)$, then for $s = \tau$ and for any $\alpha_1 > \alpha$ $\|\varphi\|_p \geq C(p,\lambda,\alpha,r,s)I_3$ and $I_4(s,\theta) \geq C(p,\xi,\alpha,\alpha_1,r,s)I_5(s)$.*

2. The case $\mathbf{p = 1, \infty}$.

Estimates $\omega_\alpha(\varphi,t)_p$ by $\omega_{r+\alpha}(f,t)_p$ for this case follow from Theorem 1 (see part **II**). We will write only the commonly used estimates of $\omega_\alpha(f^{(r)},t)_p$ and $\omega_\alpha(\tilde{f}^{(r)},t)_p$ by $\omega_{r+\alpha}(f,t)_p$.

Corollary 2. *If $p = 1, \infty$, then (6), (7) are true for any $k, r > 0$.*

As a corollary of Theorem 1 for $\{\lambda_n = n^\rho\}$, $\rho \geq 0$ and $\beta = \rho + 1$,

Corollary 3. *Let $p = 1, \infty$. One has*

$$H^p_{\alpha+\rho}[\omega] \subset \widetilde{W}^\rho_p \Longleftrightarrow \sum_{n=1}^\infty n^{\rho-1}\omega\left(\frac{1}{n}\right) < \infty.$$

For $p = 1, \rho = 0$ and $\alpha = 1$ Corollary 3 gives the answer for the question by F. Moricz [Mo]. Also, we mention the papers [Be,S,1], [Ha-Sh], [St,S], where the embedding results were obtained in the necessity part.

Corollary 4. *Let $p = 1, \infty$, and $r, \alpha, \varepsilon > 0$.*

I. *If for $f \in L_p$ the series $\sum\limits_{\nu=1}^\infty \nu^{r-1}\omega_{r+\alpha+\varepsilon}\left(f, \frac{1}{\nu}\right)_p$ converges, then there exists $\tilde{f}^{(r)} \in L_p$, and*

$$\omega_\alpha\left(\widetilde{f}^{(r)}, \frac{1}{n}\right)_p \leq$$

$$\leq C(r,\alpha,\varepsilon)\left(n^{-\alpha}\sum_{\nu=1}^n \nu^{r+\alpha-1}\omega_{r+\alpha+\varepsilon}\left(f,\frac{1}{\nu}\right)_p + \sum_{\nu=n+1}^\infty \nu^{r-1}\omega_{r+\alpha+\varepsilon}\left(f,\frac{1}{\nu}\right)_p\right),$$

$n \in \mathbf{N}$.

II. *If for $f \in L_p$ there exists $\tilde{f}^{(r)} \in L_p$, then*

$$\omega_{r+\alpha+\varepsilon}\left(f,\frac{1}{n}\right)_p \leq \frac{C(r,\alpha,\varepsilon)}{n^r}\omega_\alpha\left(\tilde{f}^{(r)},\frac{1}{n}\right)_p, \qquad n \in \mathbf{N}. \qquad (87)$$

Using direct and inverse approximation theorems, we can rewrite inequality in **I** in the following equivalent form (compare to [Ba-St]):

Corollary 5. *Let $p = 1, \infty$, and $r, \alpha > 0$. If for $f \in L_p$ the series $\sum\limits_{\nu=1}^{\infty} \nu^{r-1} E_\nu(f)_p$ converges, then there exists $\tilde{f}^{(r)} \in L_p$, and*

$$\omega_\alpha\Big(\widetilde{f}^{(r)}, \frac{1}{n}\Big)_p \le C(r,\alpha,\varepsilon)\left(n^{-\alpha}\sum_{\nu=1}^{n}\nu^{r+\alpha-1}E_\nu(f)_p + \sum_{\nu=n+1}^{\infty}\nu^{r-1}E_\nu(f)_p\right),$$

$n \in \mathbf{N}$.

6. Final remarks

1. The Weyl class $W_p^{\lambda,\beta}$ coincides with the class of functions from $L(0,2\pi)$ such that their Fourier series can be presented in the following form

$$\frac{a_0(f)}{2} + \sum_{\nu=1}^{\infty}\frac{1}{\pi\lambda_\nu}\int_0^{2\pi}\psi(x-t)\cos\Big(\nu t - \frac{\pi\beta}{2}\Big)\,dt, \quad \psi(x) \in L^0.$$

Further, consider the case when

$$\sum_{\nu=1}^{\infty}\frac{1}{\lambda_\nu}\cos\Big(\nu t - \frac{\pi\beta}{2}\Big)$$

is the Fourier series of a summable function $D_{\lambda,\beta}(t)$. For example, it is so if $\{\lambda_\nu \uparrow \infty\}$ $(n \uparrow)$, and $\sum\limits_{\nu=1}^{\infty}\frac{1}{\nu\lambda_\nu} < \infty$. Then the elements of $W_p^{\lambda,\beta}$ can differ only by the mean value from functions f, which have the following representation by convolution,

$$f(x) = \frac{1}{\pi}\int_0^{2\pi}\psi(x-t)D_{\lambda,\beta}(t)\,dt, \quad \psi(x) \in L^0.$$

Here, ψ coincides almost everywhere with $f^{(\lambda,\beta)}$. See, for example, [Bu-Ne, Ch. 11]. A representation by convolution was first considered in [Sz].

2. The generalization of $W_p^r E[\xi]$ is the class

$$W_p^{\lambda,\beta}E[\xi] = \Big\{f \in W_p^{\lambda,\beta}: \ E_n\left(f^{(\lambda,\beta)}\right)_p = O\left[\xi(1/n)\right]\Big\}.$$

We do not consider in this paper the embedding results between $W_p^{\lambda,\beta}E[\xi]$, $W_p^{\lambda,\beta}H_\alpha[\varphi]$ and $E[\varphi] \equiv W_p^{\{1\},0}E[\varphi]$. We only notice that some results of such types follow from direct and inverse theorems (1)-(4). Also, we mention the papers [Ha-Li], [Og], [Si-Ti], and [St,A].

3. The results of sections 2 and 5 can be generalized to a multidimensional case. We only write here the following estimates for the mixed modulus of smoothness $\omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_p$ of a function f (in the L_p metric) of orders α_1

and α_2 $(\alpha_1, \alpha_2 > 0)$ with respect to the variables x_1 and x_2, respectively (see, for example, [Po-Si-Ti,1]).

Theorem 5. *Let $f(x_1, x_2) \in L_p^0$, $1 < p < \infty$, $\theta = \min(2, p), \tau = \max(2, p)$, and let α_1, α_2, $r_1, r_2 > 0$,*
I. *If*

$$J_1(\theta) := \left(\int_0^1 \int_0^1 t_1^{-r_1\theta-1} t_2^{-r_2\theta-1} \omega^\theta_{r_1+\alpha_1, r_2+\alpha_2}(f, t_1, t_2)_p \, dt_1 \, dt_2 \right)^{\frac{1}{\theta}} < \infty,$$

then f has a mixed derivative in the sense of Weyl $f^{(r_1,r_2)} \in L_p^0$,

$$\|f^{(r_1,r_2)}\|_p \le C(p, r_1, r_2) J_1(\theta),$$

and

$$\begin{aligned} &\omega_{\alpha_1,\alpha_2}(f^{(r_1,r_2)}, \delta_1, \delta_2)_p \le \\ &\le \quad C(p, \alpha_1, \alpha_2, r_1, r_2) \left(\int_0^{\delta_1}\int_0^{\delta_2} t_1^{-r_1\theta-1} t_2^{-r_2\theta-1} \omega^\theta_{r_1+\alpha_1, r_2+\alpha_2}(f, t_1, t_2)_p dt_1 dt_2 \right)^{\frac{1}{\theta}} \\ &=: \quad C\, J_2(\theta). \end{aligned}$$

II. *If f has a mixed derivative in the sense of Weyl $f^{(r_1,r_2)} \in L_p^0$, then*

$$J_1(\tau) \le C(p, r_1, r_2) \|f^{(r_1,r_2)}\|_p,$$

and

$$J_2(\tau) \le C(p, \alpha_1, \alpha_2, r_1, r_2) \omega_{\alpha_1,\alpha_2}(f^{(r_1,r_2)}, \delta_1, \delta_2)_p.$$

Theorem 6. *Let $f(x_1, x_2) \in L_p^0$, $p = 1, \infty$, and let $\alpha_1, \alpha_2, r_1, r_2 > 0$.*
I. *If*

$$J_3 := \int_0^1 \int_0^1 t_1^{-r_1-1} t_2^{-r_2-1} \omega_{r_1+\alpha_1, r_2+\alpha_2}(f, t_1, t_2)_p \, dt_1 \, dt_2 < \infty,$$

then f has a mixed derivative in the sense of Weyl $f^{(r_1,r_2)} \in L_p^0$,

$$\|f^{(r_1,r_2)}\|_p \le C(r_1, r_2) J_3,$$

and

$$\begin{aligned} &\omega_{\alpha_1,\alpha_2}(f^{(r_1,r_2)}, \delta_1, \delta_2)_p \le \\ &\qquad \le C(\alpha_1, \alpha_2, r_1, r_2) \int_0^{\delta_1}\int_0^{\delta_2} t_1^{-r_1-1} t_2^{-r_2-1} \omega_{r_1+\alpha_1, r_2+\alpha_2}(f, t_1, t_2)_p \, dt_1 \, dt_2. \end{aligned}$$

II. *If f has a mixed derivative in the sense of Weyl $f^{(r_1,r_2)} \in L_p^0$, then*

$$\omega_{r_1+\alpha_1,r_2+\alpha_2}(f,\delta_1,\delta_2)_p \le C(\alpha_1,\alpha_2,r_1,r_2)\delta_1^{r_1}\delta_2^{r_2}\|f^{(r_1,r_2)}\|_p,$$

and

$$\omega_{r_1+\alpha_1,r_2+\alpha_2}(f,\delta_1,\delta_2)_p \le C(\alpha_1,\alpha_2,r_1,r_2)\delta_1^{r_1}\delta_2^{r_2}\omega_{\alpha_1,\alpha_2}(f^{(r_1,r_2)},\delta_1,\delta_2)_p.$$

See [Po-Si-Ti,1] for the estimates of transformed series in a multidimensional case.

4. In view of inequalities (79) and (80) - (83), the problem of finding the estimates of $\omega_\alpha(\varphi,t)_p$ by $\omega_{\alpha+r}(f,t)_p$ arises such as in the case $\varphi \sim \sigma(f,\lambda)$ with $\lambda_n = n^r \ln^A n$. If $A<0$ (that is the analogue of the case $\lambda_n = n^{r-\varepsilon}$), then estimates $\omega_\alpha(\varphi,t)_p$ follow from Theorem 3. For example, if $p=2$ and $\varphi \sim \sigma(f, n^r \ln^A n)$, $A<0$, then

$$\omega_\beta^2(\varphi,\delta)_2 \asymp \int_0^\delta \frac{t^{-2r-1}}{\ln^{2|A|}\left(\frac{2}{t}\right)}\omega_{r+\beta}^2(f,t)_2\,dt + \delta^{2\beta}\int_\delta^1 \frac{t^{-2(r+\beta)-1}}{\ln^{1+2|A|}\left(\frac{2}{t}\right)}\omega_{r+\beta}^2(f,t)_2\,dt. \tag{88}$$

Note that the only difference between (88) and (80)-(81) is related to the replacement of $n^{-\varepsilon}$ by $\ln^A n$.

The case $A>0$ (that is the analogue of the case $\lambda_n = n^{r+\varepsilon}$) is interesting. For $p=2$ and $\varphi \sim \sigma(f, n^r \ln^A n)$, $A>0$ we have

$$\delta^{2\beta}\ln^{2A}\left(\frac{2}{\delta}\right)\int_\delta^1 \frac{t^{-2\beta-1}}{\ln^{1+2A}\left(\frac{2}{t}\right)}\omega_\beta^2(\varphi,t)_2\,dt + \omega_\beta^2(\varphi,\delta)_2 \asymp$$
$$\asymp \int_0^\delta t^{-2r-1}\ln^{2A}\left(\frac{2}{t}\right)\omega_{r+\beta}^2(f,t)_2\,dt. \tag{89}$$

Comparing this with the estimates (82)-(83) one can remark that the new item $\omega_\beta^2(\varphi,\delta)_2$ has appeared in (89). Thus, this case has essential distinctions. See for detail [Po-Si-Ti,2], [Ti,S,1].

5. Defining the class $W_p^{\lambda,\beta}H_\alpha[\varphi]$ we assumed that $\varphi \in \Phi_\alpha$. This restriction is natural for a majorant of the modulus of smoothness of order α (see [Ti,S,3]).

6. In Theorem 1 (item II) we used the inequality $\sum\limits_{\nu=n+1}^{\infty}\frac{1}{\nu\lambda_\nu} \le \frac{C}{\lambda_n}$. We remark that it is equivalent to the following condition: there exists $\varepsilon>0$ such that the sequence $\{n^{-\varepsilon}\lambda_n\}$ is almost increasing, i.e., $n^{-\varepsilon}\lambda_n \le Cm^{-\varepsilon}\lambda_m$, $C \ge 1$, $n \le m$. This and other equivalence results can be found in [Ba-St] and [Ti,S,2].

7. Acknowledgements

The first named author acknowledges financial support from Russian Fund of Basic Research (project 03-01-00080). The second named author acknowledges financial support from the European Commission (Contract MIF1-CT-2004-509465). The paper was written while they were staying at the Centre de Recerca Matematica (CRM) in Barcelona, Spain. The authors are grateful to the staff of CRM and in particular to its director, Manuel Castellet, for their hospitality.

References

[Ba,1] N. K. Bari, *On best approximation of two conjugate functions by trigonometric polynomials* // Izv. Akad. Nauk SSSR. Ser. Mat., 19, 285–302, 1955.

[Ba,2] N. K. Bary, *A treatise on trigonometric series.* Vol. I, II. Pergamon Press, 1964.

[Ba-St] N. K. Bari; S. B. Stečkin, *Best approximations and differential properties of two conjugate functions* // Trudy Moskov. Mat. Obšč., 5, 483–522, 1956.

[Be-Lö] J. Bergh, J. Löfström, *Interpolation spaces. An introduction.* Springer-Verlag, 1976.

[Be,S,1] S. Bernstein, *On the best approximation of continuous functions by polynomials of a given degree* // Comm. Soc. Math. Kharkow, Ser. 2, V. 13, 49–194, 1912; reprinted in Collected works (see [Be,S,4]), Vol. I, pp. 11–104; for French transl. see [Be,S,2].

[Be,S,2] S. Bernstein, *Sur l'orde de la meilleure approximation des fonctions continues par les polynomes de degré donné* // Mém. Cl. Sci. Acad. Roy. Belg., 4, 1–103, 1912. Available online at `http://www.math.technion.ac.il/hat/fpapers/ber2.pdf`

[Be,S,3] S. Bernstein, *Sur les recherches recentes relatives à la meilleure approximation des fonctions continues par des polynômes.* [Proc. 5. Intern. Math. Congr.] 1, 256–266, 1913; reprinted in Collected works (see [Be,S,4]), Vol. II, pp. 112–124. Available online at `http://www.math.technion.ac.il/hat/fpapers/ber.pdf`

[Be,S,4] S. N. Bernstein, *Collected works.* Vol. I and II. Izdat. Akad. Nauk, Moscow, 1952, 1954.

[Be,O] O. V. Besov, *Investigation of a class of function spaces in connection with imbedding and extension theorems* // Trudy. Mat. Inst. Steklov, 60, 42–81, 1961.

[Be-Il-Ni] O. V. Besov, V. P. Il'in, S. M. Nikol'skii, *Integral representations of functions and imbedding theorems.* Vol. I, II, Washington, D.C., 1978, 1979.

[Bu-Ne] P. L. Butzer, R.J. Nessel, *Fourier analysis and approximation.* Vol. 1: One-dimensional theory. Academic Press, N.Y.-L., 1971.

[Bu-Dy-Gö-St] P.L. Butzer, H. Dyckhoff, E. Göerlich, R.L. Stens, *Best trigonometric approximation, fractional order derivatives and Lipschitz classes* // Can. J. Math., 29, 781–793, 1977.

[De-Lo] R. A. DeVore, G. G. Lorentz, *Constructive approximation.* Berlin: Springer-Verlag, 1993.

[Di-Hr-Iv] Z. Ditzian, V. H. Hristov, K. G. Ivanov, *Moduli of smoothness and K-functionals in L_p,* $0 < p < 1$ // Constr. Approx., 11, no. 1, 67–83, 1995.

[Ge] V. È. Geĭt, *The accuracy of certain inequalities in approximation theory* // Math. Notes, 10, 768-776, 1971-72; Transl. from Mat. Zametki, 10, 571–582, 1971.

[Ha-Li] G.H. Hardy, J.S. Littlwood, *Some properties of fractional integrals* // Math. Z., 27, 565–606, 1928.

[Ha-Sh] M. Hasson, O. Shisha, *On the condition* $\sum_{n=1}^{\infty} n^{p-1} E_n^*(f) < \infty$ // J. Approx. Theory, 39, no. 4, 389–393, 1983.

[Ja] D. Jackson, *Über die Genauigkeit der Annaheurung Stetiger Funktionen durch ganze rationale Funtionen gegebenen Grades und trigonometrischen Summen gegebenen Ordnung.* Diss., Göttingen, 1911.

[Jo] H. Johnen, *Inequalities connected with the moduli of smoothness* // Mat. Vesnik, 9(24), 289–303, 1972.

[Jo-Sc] H. Johnen, K. Scherer, *On the equivalence of the K-functional and moduli of continuity and some applications.* Constr.Theory Funct. Several Variables. [Proc. Conf. Oberwolfach 1976], Lect. Notes Math., 571, 119–140, 1977.

[Mo] F. Moricz, *A quantitative version of the Young test for the convergence of conjugate series* // J. Approx. Theory, 81, 207–216, 1995.

[Og] I. I. Ogieveckiĭ, *Integration and differentiation of fractional order of periodic functions and the constructive theory of functions.* Studies of Modern Problems of Constructive Theory of Functions, Fizmatgiz, Moscow, 159–164, 1961.

[Pi] A. Pinkus, *Negative theorems in approximation theory* // Amer. Math. Monthly, 110, no. 10, 900–911, 2003.

[Po-Si,1] M.K. Potapov, B.V. Simonov *On the interrelation of the generalized Besov-Nikol'skii and Weyl-Nikol'skii classes of functions* // Analysis Mathematica, V. 22, 299–316, 1996.

[Po-Si-Ti,1] M. K. Potapov, B. V. Simonov, S. Yu. Tikhonov, *On the Besov and the Besov-Nikol'skii classes and estimates for mixed moduli of smoothness of fractional derivatives* // Proc. Steklov Inst. Math., 243, 234–246, 2003; Transl. from Tr. Mat. Inst. Steklova, 243, 244–256, 2003.

[Po-Si-Ti,2] M. K. Potapov, B. V. Simonov, S. Yu. Tikhonov, *Transformation of Fourier series by means of power and slowly varying sequences* // Math. Notes, 77, no. 1, 90–107, 2005; Transl. from Mat. Zametki, 77, no. 1, 99-116, 2005.

[Pr] I.I. Privalov *Sur les fonctions conjuguées* // Bull. Soc. Math. France, V.44, 100–103, 1916.

[Si-Ti] B. V. Simonov, S. Yu. Tikhonov, *On embeddings of functional classes defined by constructive characteristics* // To appear in Banach Center Publ.

[St,S] S. B. Stečkin, *On best approximation of conjugate functions by trigonometric polynomials* // Izv. Akad. Nauk SSSR, 20, 197–206, 1956.

[St,A] A. I. Stepanets, *Inverse theorems for the approximation of periodic functions* // Ukraïn. Mat. Zh. 47, no. 9 (1995), 1266–1273; translation in Ukrainian Math. J. 47, no. 9 (1996), 1441–1448.

[Sz] B. Sz.-Nagy, *Über gewisse Extremalfragen bei transformierten trigonometrischen Entwicklungen.* I // Periodischer Fall. Ber. Verh. Sächs. Akad. Leipzig, 90, 103–134, 1938.

[Ta] R. Taberski, *Differences, moduli and derivatives of fractional orders* // Commentat. Math., V.19, 389–400, 1976-1977.

[Te] S. A. Telyakovskiĭ, *Lower bounds of the integral modulus of continuity of a function in terms of its Fourier coefficients* // Math. Notes, 52, N 5, 1149-1153, 1992; Transl. from Mat. Zametki, 52, N 5, 107-112, 1992.

[Ti,S,1] S. Tikhonov, *Moduli of smoothness and the interrelation of some classes of functions.* Function Spaces, Interpolation Theory and Related Topics. [Proc. of the Conf. on Function Spaces, Interpolation Theory And Related Topics in Honour of Jaak Peetre on his 65th Birthday, August 17-22, 2000.] W. de Gruyter. Berlin., 413–423, 2002.

[Ti,S,2] S. Tikhonov, *On generalized Lipschitz classes and Fourier series* // Z. Anal. Anwendungen, 23, no. 4, p. 745–764, 2004.

[Ti,S,3] S. Tikhonov, *On moduli of smoothness of fractional order* // Real Analysis Exchange, V. 30, no. 2, 2005.

[Ti,M,1] M. F. Timan, *The imbedding of the $L_p^{(k)}$ classes of functions* // Izv. Vysš. Učebn. Zaved. Matematika, no. 10(149), 61–74, 1974.

[Va] Ch.J. Vallée-Poussin, *Lecons sur l'approximation des fonctions d'une variable reelle.* Paris, 1919.

[Zh-Na] V.V. Zhuk; G.I. Natanson, *S. N. Bernstein and direct and converse theorems of constructive function theory* // American Mathematical Society. Transl., Ser. 2, 205, 59-82, 2002; Transl. from Proceedings of the St. Petersburg Mathematical Society, Vol. 8, 70–95, 2001.

[Zy,1] A. Zygmund, *Smooth functions* // Duke Math. J., 12, 47–76, 1945.

[Zy,2] A. Zygmund, *Trigonometric series.* Vol. I, II. Third edition. Cambridge, 2002.

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