EXAMPLES OF RETRACTS IN A FREE GROUP THAT ARE NOT THE FIXED SUBGROUP OF ANY GROUP OF AUTOMORPHISMS

LAURA CIOBANU AND WARREN DICKS

Abstract. Let F be a free group of rank at least three. We show that some retracts of F previously studied by Martino-Ventura are not equal to the fixed subgroup of any group of automorphisms of F. This shows that, in F, there exist subgroups that are equal to the fixed subgroup of some set of endomorphisms but are not equal to the fixed subgroup of any set of automorphisms. Moreover, we determine the Galois monoids of these retracts, where, by the Galois monoid of a subgroup H of F, we mean the monoid consisting of all endomorphisms of F that fix H.

1. Two Galois connections

Throughout, let F be a free group and let H be a subgroup of F.

1.1. **Notation.** For $x, y \in F$, $\bar{x}$ denotes x^{-1}, $[x, y]$ denotes $xy\bar{x}\bar{y}$, and we write $x \sim y$ if x and y are conjugate in F; thus, $x \sim \bar{y}xy$.

Let $\mathrm{End}(F)$ denote the monoid of endomorphisms of F. Let $\mathrm{End}_H(F)$ denote the submonoid consisting of all endomorphisms of F which fix every element of H.

Let $\mathrm{Aut}(F)$ denote the group of invertible elements in $\mathrm{End}(F)$, that is, the group of automorphisms of F. Let $\mathrm{Aut}_H(F)$ denote the subgroup consisting of all automorphisms of F which fix every element of H.

For any subset S of $\mathrm{End}(F)$, let $\mathrm{Fix}(S)$ denote the set of elements of F that are fixed by every element of S.

An F-*retraction* is an idempotent element of $\mathrm{End}(F)$, and an F-*retract* is the image, or set of fixed elements, of an F-retraction. □

We think of $\mathrm{Aut}_H(F)$ as a 'Galois group' and $\mathrm{End}_H(F)$ as a 'Galois monoid'.

2000 *Mathematics Subject Classification.* 20E05, 20E36.
Key words and phrases. free group, automorphism, retract, fixed subgroup.

Now $\mathrm{Aut}_{(-)}(F)$ is a function from the set of subgroups of F to the set of subsets of $\mathrm{Aut}(F)$, and $\mathrm{Fix}(-)$ is a function in the reverse direction. This pair of functions is a Galois connection, and the images of the functions are the sets of closed subsets. Thus H is $\mathrm{Aut}(F)$-*closed* (in F) if $H = \mathrm{Fix}(S)$ for some subset S of $\mathrm{Aut}(F)$, and the $\mathrm{Aut}(F)$-*closure* of H (in F) is $\mathrm{Fix}(\mathrm{Aut}_H(F))$, the smallest $\mathrm{Aut}(F)$-closed subgroup of F containing H.

Replacing Aut with End everywhere in the previous paragraph gives another Galois connection. Notice that all F-retracts are $\mathrm{End}(F)$-closed.

The $\mathrm{End}(F)$-closure of H, $\mathrm{Fix}(\mathrm{End}_H(F))$ is a subgroup of the $\mathrm{Aut}(F)$-closure of H, $\mathrm{Fix}(\mathrm{Aut}_H(F))$. In general, the relation between the two closures is not well understood. Enric Ventura [4, Theorem 3.9] showed that if the rank of F is at most two, then $\mathrm{Fix}(\mathrm{End}_H(F)) = \mathrm{Fix}(\mathrm{Aut}_H(F))$. We now show that if the rank of F is greater than two, then there exist examples where

$$\mathrm{Fix}(\mathrm{End}_H(F)) \neq \mathrm{Fix}(\mathrm{Aut}_H(F)).$$

1.2. **Hypotheses.** Let F be a free group of rank three, and let $\{a, b, c\}$ be a basis of F.

Here, we denote an element α of $\mathrm{End}(F)$ by the triple $(a\alpha, b\alpha, c\alpha)$. Notice that we write endomorphisms on the right of their arguments. □

1.3. **Example.** Suppose that Hypotheses 1.2 hold.

Let $d = ba[c, b]\bar{a} \in F$ and let $H = \langle a, d\rangle = \langle a, ba[c, b]\rangle \leq F$.

Let $\psi = (a, d, 1) \in \mathrm{End}(F)$ and let $\phi = (a, b, cb) \in \mathrm{Aut}(F)$. Notice that, for each $n \in \mathbb{Z}$, $\phi^n = (a, b, cb^n)$ and $\phi^n\psi = (a, d, d^n)$.

Using normal-form arguments, we find, in Corollary 3.3 below, that

$$\mathrm{End}_H(F) = \{\phi^n, \phi^n\psi \mid n \in \mathbb{Z}\} = \langle\phi\rangle \cup \langle\phi\rangle\psi \quad \text{and, hence,} \quad \mathrm{Aut}_H(F) = \langle\phi\rangle.$$

It is then straightforward to verify the following.

Every non-invertible element of $\mathrm{End}_H(F)$ is an F-retraction with image H.

The multiplication in $\mathrm{End}_H(F)$ is described by the monoid presentation

$$\mathrm{End}_H(F) = \langle \phi, \bar{\phi}, \psi \mid \psi\phi = \psi^2 = \psi, \bar{\phi}\phi = \phi\bar{\phi} = 1\rangle_{\text{monoid}}.$$

Let $K = \langle a, b, cb\bar{c}\rangle$. The two closures of H are

$$\mathrm{Fix}(\mathrm{Aut}_H(F)) = \mathrm{Fix}(\{\phi\}) = K \quad \text{and} \quad \mathrm{Fix}(\mathrm{End}_H(F)) = \mathrm{Fix}(\{\psi\}) = H.$$

Here, $K = H * \langle b\rangle \neq H$.

The closure of $\{\psi\}$ in $\mathrm{End}(F)$ is $\mathrm{End}_{\mathrm{Fix}(\{\psi\})}(F) = \langle\phi\rangle \cup \langle\phi\rangle\psi$. □

There is another type of behaviour we want to mention.

1.4. **Example.** Suppose that Hypotheses 1.2 hold.

Let $d = ba[c^2, b]\bar{a} \in F$ and let $H = \langle a, d\rangle = \langle a, ba[c^2, b]\rangle \leq F$.

Let $\psi = (a, d, 1) \in \mathrm{End}(F)$ and let $\phi = (a, b, cb) \in \mathrm{Aut}(F)$. Notice that, for each $n \in \mathbb{Z}$, $\phi^n\psi = (a, d, d^n)$.

In Corollary 3.4 below, we find that

$$\mathrm{End}_H(F) = \{1, \phi^n\psi \mid n \in \mathbb{Z}\} = \{1\} \cup \langle\phi\rangle\psi \quad \text{and, hence,} \quad \mathrm{Aut}_H(F) = \{1\}.$$

It is then straightforward to verify the following.

Every non-identity element of $\mathrm{End}_H(F)$ is an F-retraction with image H.

The multiplication in $\mathrm{End}_H(F)$ is described by the monoid presentation

$$\mathrm{End}_H(F) = \langle\{\phi^n\psi \mid n \in \mathbb{Z}\} \mid \{\phi^n\psi \cdot \phi^m\psi = \phi^n\psi \mid m, n \in \mathbb{Z}\}\rangle_{\mathrm{monoid}}.$$

The two closures of H are

$$\mathrm{Fix}(\mathrm{Aut}_H(F)) = \mathrm{Fix}(\{1\}) = F \quad \text{and} \quad \mathrm{Fix}(\mathrm{End}_H(F)) = \mathrm{Fix}(\{\psi\}) = H.$$

Here, $F \neq H$.

The closure of $\{\psi\}$ in $\mathrm{End}(F)$ is $\mathrm{End}_{\mathrm{Fix}(\{\psi\})}(F) = \{1\} \cup \langle\phi\rangle\psi$. □

By adjoining a free-group free factor simultaneously to H and to F, one obtains examples where F has arbitrary rank greater than two.

2. Relation with work of Martino-Ventura

2.1. **Definitions.** We say that H is *one-auto fixed* (in F) if $H = \mathrm{Fix}(S)$ for some one-element subset S of $\mathrm{Aut}(F)$. Thus a one-auto-fixed subgroup is $\mathrm{Aut}(F)$-closed.

We define *one-endo fixed* analogously. Thus an F-retract is one-endo fixed, and a one-endo-fixed subgroup is $\mathrm{End}(F)$-closed. □

Ventura [4, Theorem 3.9] showed that, if the rank of F is at most two, then the four concepts, one-endo fixed, $\mathrm{End}(F)$-closed, one-auto fixed, and $\mathrm{Aut}(F)$-closed, all coincide.

Now suppose that the rank of F is finite and greater than two.

It is not known whether or not the $\mathrm{Aut}(F)$-closed subgroups are just the one-auto-fixed subgroups, nor whether or not the $\mathrm{End}(F)$-closed subgroups are just the one-endo-fixed subgroups.

In [2, Corollary 3.4], A. Martino and E. Ventura showed that if H is $\mathrm{Aut}(F)$-closed then H is a free factor of some one-auto-fixed subgroup K of F. By [2, Proposition 5.4], there exists an example where H is not $\mathrm{Aut}(F)$-closed but H is a free factor of some one-auto-fixed subgroup K of F. We find it interesting that the latter phenomenon appears unbidden in Example 1.3, above.

Later, Martino-Ventura [3] gave a family of ingeniously chosen F-retracts none of which is one-auto fixed, thus showing that not all one-endo-fixed subgroups are one-auto fixed. Explicitly, they show the following.

Suppose that Hypotheses 1.2 hold, let i, j, k be integers, and let

$$H_{i,j,k} = \langle a, ba^i c^j bc^k \bar{b}\rangle.$$

Then [3, Proposition 18] shows that $H_{i,j,k}$ is an F-retract, but that $H_{i,j,k}$ is one-auto fixed if and only if $ijk = 0$. The six-tuple (i, j, k, a, b, c) in our notation corresponds to $(-s, r, t, b, cb^s, a)$ in their notation.

What we do in this article is to describe elementary techniques that give satisfactorily complete information about $H_{1,1,-1}$ and $H_{1,2,-2}$. It is straightforward to check that the same techniques apply to $H_{i,j,-j}$, for $ij \neq 0$, and to verify that no new types of behaviour arise.

For $k \neq -j$ and $ijk \neq 0$, we have been unable to calculate the $\mathrm{Aut}(F)$-closure of $H_{i,j,k}$.

3. Some Galois monoids

Our argument is concentrated in the next result.

3.1. **Lemma.** *Suppose that Hypotheses* 1.2 *hold. For all* $d \in \langle c\rangle$ *and all* $x \in F$*, the following implications hold.*

(i) *If* $\bar{d}\bar{a}bdx \sim \bar{a}\bar{x}bx$ *in* F*, then* $x \in \{1, b\bar{d}\bar{b}ad\bar{a}\}$.
(ii) *If* $x \sim \bar{a}\bar{x}ba[d, b]x$ *in* F*, then* $x \in \{b, ba[d, b]\bar{a}\}$.

Proof. (i). Suppose that

$$\bar{d}\bar{a}bdx \sim \bar{a}\bar{x}bx. \tag{1}$$

Let $B = \{a, b, c, \bar{a}, \bar{b}, \bar{c}\}$. We view the elements of F as reduced monoid words in B.

Write $x = b^m ya^n$ where m, $n \in \mathbb{Z}$, $y \in F$, y does not begin with b or $\bar{b}$, and y does not end with a or $\bar{a}$.

By abelianizing (1) we see that the derived subgroup of F, denoted F', contains $x = b^m ya^n$; thus

$$yF' = \bar{a}^n \bar{b}^m F' \tag{2}$$

in F/F', a free abelian group with basis $\{aF', bF', cF'\}$.

Also, notice that (1) can be rewritten as

$$\bar{d}\bar{a}bdb^m ya^n \sim \bar{a}\bar{y}by, \tag{3}$$

and that the right-hand side of (3) is now cyclically reduced.

Let $|y|_a$ denote the number of times a occurs in the expression of y as a reduced monoid word in B, and similarly for the other five elements of B. On applying $|-|_a$ to (3) we see that

$$0 + |y|_a + \max\{n, 0\} \geq 0 + |y|_{\bar{a}} + 0 + |y|_a\,, \tag{4}$$

and, moreover, equality holds in (4) if and only if no a is cancelled in the cyclic reduction of the left-hand side of (3). Now (4) amounts to $\max\{n,0\} \geq |y|_{\bar{a}}$, and, by (2), it is clear that equality holds. Thus $|y|_{\bar{a}} = \max\{n,0\}$, and no a is cancelled in the cyclic reduction of the left-hand side of (3).

Similarly, by applying $|{-}|_{\bar{a}}$ to (3), we find that $|y|_a = \max\{-n,0\}$.

Also, by applying $|{-}|_b$ and $|{-}|_{\bar{b}}$ to (3), we find that $|y|_{\bar{b}} = \max\{m,0\}$, $|y|_b = \max\{-m,0\}$, and no b is cancelled in the cyclic reduction of the left-hand side of (3).

We now consider five non-pairwise-disjoint cases.

Case 1: $m = n = 0$.

Here $|y|_a = |y|_{\bar{a}} = |y|_b = |y|_{\bar{b}} = 0$. Thus, each side of (3) has no occurrence of a or $\bar{b}$, has a unique occurrence of $\bar{a}$, and has a unique occurrence of b. In each side, we can equate the cyclic subword between $\bar{a}$ and b and we find that $1 = \bar{y}$. Hence $x = 1$, as desired.

Case 2: $m \leq -1$.

Here $|y|_b = -m$ and $|y|_{\bar{b}} = 0$. Write $y = y'by''$ as a reduced monoid word such that y' has no occurrence of b, and, hence, y'' has $-m-1$ occurrences of b. Notice that $y' \neq 1$ since y does not begin with b. In each side of (3) there are $-m+1$ occurrences of b followed, cyclically, by $-m$ occurrences of $\bar{b}$. In each side, we take the first of the $-m+1$ occurrences of b as the terminating point. For the left-hand side we get $y''a^n\bar{d}\bar{a}bdb^my'b$, that is, $y''a^n\bar{d}\bar{a}bd\bar{b}^{-m}y'b$. For the right-hand side we get $y\bar{a}\bar{y}b$, that is, $y\bar{a}\bar{y}''\bar{b}\bar{y}'b$. Since there is no cancellation of b or $\bar{b}$, we can equate the cyclic subwords between the last $\bar{b}$ and the first b, and we find that $y' = \bar{y}'$. Hence $y' = 1$, which is a contradiction, as desired.

Case 3: $m \geq 1$.

Here $|y|_{\bar{b}} = m$ and $|y|_b = 0$. In each side of (3), there are $m+1$ occurrences of b followed, cyclically, by m occurrences of $\bar{b}$. In each side, we take the last of the $m+1$ occurrences of b as the terminating point, and find that

$$ya^n\bar{d}\bar{a}bdb^m = y\bar{a}\bar{y}b.$$

Rearranging, we find that $y = \bar{b}^{m-1}\bar{d}\,\bar{b}ad\bar{a}^{n+1}$. Since y does not begin with b or $\bar{b}$, and does not end with a or $\bar{a}$, we see that $m = 1$, $n = -1$, $y = \bar{d}\,\bar{b}ad$, and $x = bd\bar{d}\,\bar{b}ad\bar{a}$, as desired.

Case 4: $n \leq -1$.

This is similar to Case 3. Here $|y|_a = -n$ and $|y|_{\bar{a}} = 0$. In each side of (3) there are $-n+1$ occurrences of $\bar{a}$ followed, cyclically, by $-n$ occurrences of a. In each side, we take the first of the $-n+1$ occurrences of $\bar{a}$ as the starting point, and find that $\bar{a}^{-n}\bar{d}\bar{a}bdb^m y = \bar{a}\bar{y}by$. Thus $y = \bar{b}^{m-1}\bar{d}\bar{b}ad\bar{a}^{n+1}$, and, as in Case 3, $x = b\bar{d}\bar{b}ad\bar{a}$, as desired.

Case 5: $n \geq 1$.

This is similar to Case 2. Here $|y|_a = 0$ and $|y|_{\bar{a}} = n$. Write $y = y'\bar{a}y''$ as a reduced monoid word such that y'' has no occurrence of $\bar{a}$, and, hence, y' has $n-1$ occurrences of $\bar{a}$. Notice that $y'' \neq 1$ since y does not end with $\bar{a}$. In each side of (3) there are $n+1$ occurrences of $\bar{a}$ followed, cyclically, by n occurrences of a. In each side, we take the last of the $n+1$ occurrences of $\bar{a}$ as the starting point, and find $\bar{a}y''a^n\bar{d}\bar{a}bdb^m y' = \bar{a}\bar{y}by = \bar{a}\bar{y}''a\bar{y}'by$. Equating the cyclic subwords between the last $\bar{a}$ and the first a, we find that $y'' = \bar{y}''$. Hence $y'' = 1$, which is a contradiction, as desired.

This completes the proof of (i).

(ii). Suppose that $x \sim \bar{a}\bar{x}ba[d,b]x$. Let $y := \bar{b}x$. Then $x = by$ and

$$by \sim \bar{a}\bar{y}adb\bar{d}y.$$

Applying $\phi := (a, \bar{d}\bar{a}bd, c) \in \mathrm{Aut}(F)$ and letting $z := y\phi$, we find that

$$\bar{d}\bar{a}bdz \sim \bar{a}\bar{z}bz.$$

By (i), $z \in \{1, b\bar{d}\bar{b}ad\bar{a}\}$. Applying $\phi^{-1} = (a, adb\bar{d}, c)$, we see that

$$y = z\phi^{-1} \in \{1, a[d,b]\bar{a}\}.$$

Left multiplying by b we find that $x = by \in \{b, ba[d,b]\bar{a}\}$, as desired. □

We can now calculate some Galois monoids.

3.2. **Theorem.** *Suppose that Hypotheses* 1.2 *hold. Let j be a positive integer, let $d = ba[c^j, b]\bar{a}$, and let $H = \langle a, d\rangle = \langle a, ba[c^j, b]\rangle$. For all $(x, y, z) \in \mathrm{End}_H(F)$, the following hold.*

(i) $y \in \{d, b\}$.
(ii) *If $y = d$ then there exists an integer n such that $z = d^n$.*
(iii) *If $y = b$ and $j = 1$ then there exists an integer n such that $z = cb^n$.*
(iv) *If $y = b$ and $j \geq 2$ then $z = c$.*

Proof. (i). Here $x = a$ and $yx[z^j, y] = ba[c^j, b]$. Hence $yaz^jy\bar{z}^j\bar{y} = ba[c^j, b]$ and $z^jy\bar{z}^j = \bar{a}\bar{y}ba[c^j, b]y$. Thus $y \sim \bar{a}\bar{y}ba[c^j, b]y$. By Lemma 3.1(ii),

$$y \in \{b, ba[c^j, b]\bar{a}\} = \{b, d\}.$$

This proves (i).

(ii). Let $y = d$.

Let C denote the centralizer of z^j in F.

Now

$$z^j d\bar{z}^j = \bar{a}\bar{d}ba[c^j, b]d = \bar{a}(a[c^j, b]\bar{a}\bar{b})ba[c^j, b]d = d.$$

Hence $d \in C$.

To show that $z \in \langle d \rangle$, we may assume that $z \neq 1$. Recall that $C = \langle z' \rangle$ for some $z' \in F$; see [1, Proposition I.2.19]. (A trivial normal-form argument shows that there exists some $z' \in F$ such that z' is not a proper power and $z^j = z'^n$ for some positive integer n. An almost-as-trivial normal-form argument shows that $C = \langle z' \rangle$. Alternatively, C is a free (sub)group with a non-trivial center, and hence cyclic.)

Thus $d = z'^m$ for some integer m. Since $bF' = dF' = z'^m F' = (z'F')^m$ in F/F', a free abelian group with basis $\{aF', bF', cF'\}$, we see that $m = \pm 1$. Thus $d = z'^{\pm 1}$.

Now $z \in C = \langle z' \rangle = \langle d \rangle$. This proves (ii).

(iii) and (iv). Let $y = b$.

Then $z^j b \bar{z}^j = \bar{a}\bar{b}ba[c^j, b]b = c^j b \bar{c}^j$. Hence $\bar{c}^j z^j$ commutes with b. A trivial normal-form argument shows that there exists $n \in \mathbb{Z}$ such that $\bar{c}^j z^j = b^n$.

If $j = 1$, then $z = cb^n$. This proves (iii).

If $j \geq 2$, then the equation $c^j b^n = z^j$ clearly implies that $n = 0$ and $z = c$. This proves (iv). □

The case of Theorem 3.2 where $j = 1$ gives the key part of Example 1.3.

3.3. **Corollary.** *Suppose that Hypotheses* 1.2 *hold, let* $d = ba[c, b]\bar{a}$*, and let* $H = \langle a, d \rangle = \langle a, ba[c, b] \rangle$*. If* $(x, y, z) \in \mathrm{End}_H(F)$*, then there exists* $n \in \mathbb{Z}$ *such that either* $(x, y, z) = (a, b, cb^n)$ *or* $(x, y, z) = (a, d, d^n)$*.*

Conversely, all these endomorphisms fix a *and* d*, and, hence, fix* H*.* □

The case of Theorem 3.2 where $j = 2$ gives the key part of Example 1.4.

3.4. **Corollary.** *Suppose that Hypotheses* 1.2 *hold, let* $d = ba[c^2, b]\bar{a}$*, and let* $H = \langle a, d \rangle = \langle a, ba[c^2, b] \rangle$*. If* $(x, y, z) \in \mathrm{End}_H(F)$*, then either* $(x, y, z) = (a, b, c)$ *or there exists* $n \in \mathbb{Z}$ *such that* $(x, y, z) = (a, d, d^n)$*.*

Conversely, all these endomorphisms fix a *and* d*, and, hence, fix* H*.* □

Acknowledgments

The first-named author was supported by Marie Curie Intra-European Fellowship 515027 while a guest at the Centre de Recerca Matemàtica. She is grateful for the hospitality she received there.

The research of the second-named author was funded by the DGI (Spain) through Project BFM2003-06613.

References

1. Roger C. Lyndon and Paul E. Schupp, *Combinatorial Group Theory,* Ergeb. Math. Grenzgeb. **89**, Springer-Verlag, Berlin, 1977.
2. A. Martino and E. Ventura, *On automorphism-fixed subgroups of a free group,* J. Algebra **230**(2000), 596–607.
3. A. Martino and E. Ventura, *Examples of retracts in free groups that are not the fixed subgroup of any automorphism,* J. Algebra **269**(2003), 735–747.

4. Enric Ventura, *On fixed subgroups of maximal rank,* Comm. in Algebra **25**(1997), 3361–3375.

Laura Ciobanu, Department of Mathematics, University of Auckland, Private Bag 92019, Auckland, New Zealand

E-mail addresses: ciobanu@math.auckland.ac.nz, LCiobanu@crm.es

Warren Dicks, Departament de Matemàtiques, Universitat Autònoma de Barcelona, E-08193 Bellaterra (Barcelona), Spain

E-mail address: dicks@mat.uab.es

URL: http://mat.uab.es/∼dicks/