ON THE SET OF SUBRINGS WHICH ARE DIRECTED UNIONS OF ARTINIAN SUBRINGS

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Abstract. The paper contributes to the investigation of zero–dimensional rings which can be written as a directed union of Artinian subrings. We give conditions on $\mathcal{DU}(R)$ in order to be nonempty.

Introduction

Let R be a commutative ring with identity. We recall that R is reduced if $\bigcap_{P\in Min(R)} P = (0)$, where $Min(R)$ is the set of minimal prime ideals of R, and zero-dimensional if all prime ideals are maximal. The ring R is said to be von Neumann regular if for each element $r \in R$, there exists $r' \in R$ such that $r = r^2 r'$. It is easily seen that R is a von Neumann regular ring if and only if R is reduced and zero-dimensional. Initially we note that if R is the directed union of a family $\{R_\alpha\}_{\alpha\in A}$ of subrings, each of dimension at most n, then $dim(R) \leq n$; this follows from the fact that any chain $P_0 \subset P_1 \subset \cdots \subset P_k$ of prime ideals of R contracts to a chain of distinct primes on some R_α. Thus, a ring that is a directed union of Artinian subrings is zero-dimensional. Recently many authors have been interested in zero-dimensionality (see [6, 7, 9]). The purpose of this paper is to pursue the study of the problem of whether a von Neumann regular ring R is expressible as a directed union of Artinian subrings, raised by Gilmer and Heinzer in 1992 [3, Problem 42].

This fact leads us to consider the set of subrings of R which are directed unions of Artinian subrings of R denoted $\mathcal{DU}(R)$ and the family of Artinian subrings $\mathcal{A}(R)$. Also we use $Idem(R)$, $Spec(R)$, and $\mathcal{C}(R)$, respectively, to denote the set of idempotent elements of R, the set of prime ideals of R, and the set $\{char(R/M)$: M ranges over the maximal ideals of $R\}$.

The present paper considers the following questions:

(q_1) If $\mathcal{A}(R)$ is a nonempty set is it true that $\mathcal{DU}(R) \neq \emptyset$?

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(q_2) If $\mathcal{A}(R) \neq \emptyset$ under what conditions is the set $\mathcal{DU}(R)$ nonempty?

(q_3) Let R be a von Neumann regular ring, under what conditions is

R expressible as a directed union of Artinian subrings?
It is worthwhile recalling that any zero-dimensional ring R with finite spectrum is expressible as a directed union of Artinian subrings [7, Theorem 5.4]. This leads us to examine the case where spectrum is infinite, i.e., we are interested with rings which are expressible as a directed unions of Artinian subrings but are not Artinians.
We give a short overview of the paper. The first section is concerned with questions (q_1) and (q_2). We give a characterization of the set $\mathcal{DU}(R)$. Precisely, our main results of this section are Proposition 1.3, which gives the relationship between $\mathcal{A}(R)$ and $\mathcal{DU}(R)$, and Theorem 1.12 which shows that $\mathcal{DU}(R)$ can be a nonempty set and R is not a directed union of Artinian subrings. In the second section we give a necessary and sufficient condition for a von Neumann regular ring to be a directed union of Artinian subrings that is an answer to question (q_3). We probe this problem in connection with their families of residue fields say $\mathcal{F}(R) = \{\frac{R}{M} : M \text{ a maximal ideal of } R\}$.

1. Preliminaries and general results

Throughout this paper, all rings are assumed to be commutative with unit. Further, all ring-homomorphisms are unital. If R is a subring of a ring S, we assume that the unity of S belongs to R. Turning to the Artinian case, two key properties of an Artinian ring R that come into play are that $Spec(R)$ is finite and that R has only finitely many idempotents. The existence of Artinian subrings was among the primary focus of many mathematicians [6, 9, 10]. They were particularly interested in the question of whether a zero-dimensional ring R is expressible as a directed union of Artinian subrings. Naturally hereditarily zero-dimensional rings [1] are example. More generally each zero-dimensional ring with finite spectrum is a directed union of Artinian subrings. However not every zero-dimensional ring is expressible as a directed union of Artinian subrings. For instance, if $R = \prod_{i=1}^{\infty} \mathbb{Q}$ denotes the infinite product of copies $\mathbb{Q}$, the ring R is a zero-dimensional ring that is not a directed union of Artinian subrings.

This section was inspired by a desire to understand the relationship between $\mathcal{A}(R)$ and $\mathcal{DU}(R)$.
Let (R_j, f_{jk}) be a directed system of rings, indexed by a directed set $(I, \leq)$. Let $R = \bigcup_{j\in I} R_j$, together with the canonical maps $f_j : R_j \longrightarrow R$. The ring R is said to be a directed union of the R_j's if the f_{jk}'s are inclusion maps.

[1] A ring R is hereditarily zero-dimensional if all subrings of R are zero-dimensional.

Thus, directed unions can be treated by assuming all f_{jk} to be monomorphisms. Notice that R need not be Artinian even if each R_j is Artinian.
Next, we use the fact that the Krull dimension is preserved under integral extensions (cf.[4, (11.8)]). In particular, an integral extension ring of a zero-dimensional ring is zero-dimensional.

Lemma 1.1. *If S is integral over an Artinian ring A, then S is written as a directed union of Artinian subrings, and hence $\mathcal{DU}(S) \neq \emptyset$.*

Proof. we can write

$$S = \bigcup \{A[F] : \ F \text{ is a finite subset of } S\}.$$

The ring $A[F]$ is intermediate between A and S, so $\dim(A[F]) = 0$ and $A[F]$ is a finitely generated A-module. It follows that $A[F]$ is a Noetherian ring, and hence $A[F]$ is Artinian (see [2, Theorem 8.5]). Now, let F_1 and F_2 be two finite subsets of S, then $F = F_1 \cup F_2 \subset S$ and $A[F_1] \subseteq A[F]$ and $A[F_2] \subseteq A[F]$. It follows that $\{A[F] : \ F \text{ is a finite subset of } S\}$ is a directed family of Artinian subrings and hence $S = \bigcup \{A[F] : \ F \text{ is a finite subset of } S\}$ is a directed union of Artinian subrings. Thus, $\mathcal{DU}(S) \neq \emptyset$. □

In general, if R is von Neumann regular which is integral over a Noetherian subring, then R is a directed union of Artinian subrings.

Lemma 1.2. *Let R be a ring. Then*

(i) *If $\mathcal{DU}(R) \neq \emptyset$ then $\mathcal{C}(R)$ is finite.*
(ii) *If $\mathcal{C}(R)$ is infinite then $\mathcal{A}(R) = \emptyset$.*

Proof. (i) Let $S \in \mathcal{DU}(R)$, then $S = \bigcup_{i \in I} S_i$ is a directed union of Artinian subrings S_i. Then $S_i \in \mathcal{A}(R)$ and hence $\mathcal{A}(R) \neq \emptyset$. Therefore $\mathcal{C}(R)$ is finite [5, Proposition 1].
(ii) Assume that $\mathcal{A}(R) \neq \emptyset$, let $S \in \mathcal{A}(R)$, then $\mathcal{C}(R) \subseteq \mathcal{C}(S)$ and hence $\mathcal{C}(S)$ is infinite, a contradiction with S is Artinian ($\mathcal{C}(R)$ is finite). □

For the next results, we need the following definition.

Let $\{R_\alpha\}_{\alpha \in A}$ be a nonempty family of rings and $\prod_{\alpha \in A} R_\alpha$ their product. We frequently consider $\prod_{\alpha \in A} R_\alpha$ as the set of all functions $f : A \longrightarrow \bigcup_{\alpha \in A} R_\alpha$, such that $f(\alpha) \in R_\alpha$ for each $\alpha \in A$, with addition and multiplication defined pointwise: $(f+g)(\alpha) = f(\alpha)+g(\alpha)$ and $(fg)(\alpha) = f(\alpha)g(\alpha)$. In this perspective, the direct sum ideal of $\prod_{\alpha \in A} R_\alpha$, denoted $\bigoplus_{\alpha \in A} R_\alpha$, is the set of functions $f \in \prod_{\alpha \in A} R_\alpha$ that are finitely nonzero (i.e., $\{\alpha \in A : f(\alpha) \neq 0 \text{ in } R_\alpha\}$ is finite). We use $\mathbb{Z}$ and $\mathbb{N}$, respectively, to denote the set of integers and the set of natural numbers.

Proposition 1.3. *If $R = \bigoplus_{j=1}^{n} R_j$ is the direct product of zero-dimensional rings R_j, then the following conditions are equivalent.*

(i) $\mathcal{A}(R) \neq \emptyset$;
(ii) $\mathcal{A}(R_j) \neq \emptyset$, *for each* $j = 1, \ldots, n$;
(iii) $\mathcal{DU}(R_j) \neq \emptyset$, *for each* $j = 1, \ldots, n$;
(iv) $\mathcal{DU}(R) \neq \emptyset$.

We use the following lemmas to prove Proposition 1.3.

Lemma 1.4. *Let R be a zero-dimensional ring with only finitely many idempotent elements, then R is expressible as a directed union of Artinian subrings.*

In order to prove this result, we need the following lemmas.

Lemma 1.5. *Any zero–dimensional ring R with only finitely many idempotents is semi-quasilocal.*

Proof. Let $M_1, \ldots, M_{r+1}$ be distinct maximal ideals of R. Let $x \in M_{r+1} \diagdown (\cup_{i=1}^{r} M_i)$, since $\dim R = 0$ by [8, Theorem 3.1], there exists $t \in \mathbb{Z}^+$ and e an idempotent element of R such that $x^t R = eR$. Hence $e \in M_{r+1} \diagdown (\cup_{i=1}^{r} M_i)$. It follows that if R has n maximal ideals, it has at least $n-1$ idempotents. Therefore, R is necessarily semi-quasilocal. □

Lemma 1.6. [7, Corollary 5.5] *If R is a zero–dimensional semi-quasilocal ring then there exists an Artinian subring of R.*

Proof of Lemma 1.4. An immediate consequence of Lemma 1.5 and Lemma 1.6. □

Proof of Proposition 1.3.

(i) ⇔ (ii). Let $S \in \mathcal{A}(R)$ and e_j be the idempotent element of R associated with $\{j\}$. Then $S[e_1, \ldots, e_n]$ is an integral extension of S. Moreover, $S[e_1, \ldots, e_n]$ is finitely generated over S and hence $S[e_1, \ldots, e_n] = Se_1 \oplus \cdots \oplus Se_n$ is Artinian. Therefore, Se_j is an Artinian subring of R_j, for each j. Conversely, if $S_j \in \mathcal{A}(R_j)$ for each j, then $\oplus_{j=1}^{n} S_j$ is an Artinian subring of R.

(ii) ⇔ (iii). Let $j_o \in \{1, \ldots, n\}$ and suppose that $\mathcal{A}(R_{j_o}) \neq \emptyset$. Let $A \in \mathcal{A}(R_{j_o})$ and $Idem(R_{j_o}) = \{e_l\}_{l \in L}$ be the set of all idempotent elements of R_{j_o}. Two cases are then possible:

<u>Case 1:</u> If $\{e_l\}_{l \in L}$ is a finite set, by Lemma 1.4, R_{j_o} is a directed union of Artinian subrings and hence $\mathcal{DU}(R) \neq \emptyset$.

<u>Case 2:</u> If $\{e_l\}_{l \in L}$ is infinite, then $A[\{e_l\}_{l \in L}]$ is integral over A. According to Lemma 1.1, $A[\{e_l\}_{l \in L}]$ is a directed union of Artinian subrings. Consequently, $\mathcal{DU}(R) \neq \emptyset$. Conversely, if $S = \bigcup_{i \in I} S_i \in \mathcal{DU}(R_{j_o})$ for some

$j_o \in \{1,\dots,n\}$, then $S_i \in \mathcal{A}(R_{j_o})$ and hence $\mathcal{A}(R_{j_o}) \neq \emptyset$.
(iii) $\Leftrightarrow$ (iv). Let $S_j \in \mathcal{DU}(R_j)$ for each $j = 1,\dots,n$ and $S = \bigoplus_{j=1}^{n} S_j$. To show that $S \in \mathcal{DU}(R)$ it suffices to show it for $n = 2$. Now suppose that $S = S_1 \oplus S_2$, where $S_1 = \bigcup_{i\in J} V_i$ and $S_2 = \bigcup_{k\in K} W_k$ are directed unions of Artinian subrings. It is easy to see that $S = S_1 \oplus S_2 = \bigcup_{(i,k)\in J\times K}(V_i \oplus W_k)$ and $\{V_i \oplus W_k\}_{(i,k)\in J\times K}$ is a directed family. We notice also that $V_i \oplus W_k$ is Artinian for each $(i,k) \in J \times K$. It follows that $S = \bigcup_{(i,k)\in J\times K}(V_i \oplus W_k)$ is a directed union of Artinian subrings. Conversely, let $W = \bigcup_{j\in I} W_j \in \mathcal{DU}(R)$ and e_i be the idempotent element of R associated with $\{i\}$. Then $We_i = \bigcup_{j\in I} W_j e_i$. From $(i) \Leftrightarrow (ii)$ $W_j e_i \in \mathcal{A}(R_i)$ and if $W_l \subseteq W_s$ then $W_l e_i \subseteq W_s e_i$ and hence the family $\{W_j e_i\}_{j\in J}$ is directed. Thus, $We_i \in \mathcal{DU}(R_i)$.

□

Proposition 1.3 does not hold for infinite direct products as the following example shows.

Example 1.7. Let $R = \prod_{i=1}^{\infty} \frac{\mathbb{Z}}{p_i\mathbb{Z}}$, where $\{p_i\}_{i=1}^{\infty}$ is an infinite family of distinct prime integers. Clearly, $\mathcal{A}(\frac{\mathbb{Z}}{p_i\mathbb{Z}}) \neq \emptyset$ (respectively, $\mathcal{DU}(R) \neq \emptyset$) for each i. However, since $\{p_i : i = 1,2,\dots\} \subseteq \mathcal{C}(R)$, we have that $\mathcal{C}(R)$ is infinite. Thus, by [10, Theorem 2.1], $\mathcal{A}(R) = \emptyset$ (respectively, $\mathcal{DU}(R) = \emptyset$). This means that R has no Artinian subring.

Proposition 1.8. *Let R be a ring and $\{L_\lambda\}_{\lambda\in\Lambda}$ its residue fields. Then $\mathcal{A}(\prod_{\lambda\in\Lambda} L_\lambda) \neq \emptyset$ if and only if $\mathcal{DU}(\prod_{\lambda\in\Lambda} L_\lambda) \neq \emptyset$.*

Proof. Suppose that $\mathcal{DU}(\prod_{\lambda\in\Lambda} L_\lambda) \neq \emptyset$, let $S \in \mathcal{DU}(\prod_{\lambda\in\Lambda} L_\lambda)$ then $S = \bigcup_{i\in I} S_i$ is a directed union of Artinian subrings, and hence $S_i \in \mathcal{A}(S) \subseteq \mathcal{A}(\prod_{\lambda\in\Lambda} L_\lambda)$. Thus, $\mathcal{A}(\prod_{\lambda\in\Lambda} L_\lambda) \neq \emptyset$. Conversely, assume that $\mathcal{A}(\prod_{\lambda\in\Lambda} L_\lambda) \neq \emptyset$, then $\bigcup_{\lambda\in\Lambda} \mathcal{C}(L_\lambda) \subseteq \mathcal{C}(\prod_{\lambda\in\Lambda} L_\lambda)$ is a finite set of prime numbers. In other words, $\bigcup_{\lambda\in\Lambda} \mathcal{C}(L_\lambda) = \{p_1,\dots,p_k\}$, where p_i is a prime number for each $i = 1,\dots,k$. We can write $\Lambda = \Lambda_1 \cup \dots \cup \Lambda_k$ as a finite partition, where $\Lambda_i = \{\lambda \in \Lambda / char(L_\lambda) = p_i\}$, for each $i = 1,\dots,k$, and hence $\prod_{\lambda\in\Lambda} L_\lambda = \prod_{\lambda\in\Lambda_1} L_\lambda \bigoplus \prod_{\lambda\in\Lambda_2} L_\lambda \bigoplus \dots \bigoplus \prod_{\lambda\in\Lambda_k} L_\lambda$. Without loss of generality, we assume that $\Lambda = \Lambda_i$, for some $i \in \{1,\dots,k\}$. In other words, we suppose that $char(L_\lambda) = p_i$ for each $\lambda \in \Lambda$. In this case, $\mathbb{Z}_{p_i}^{w} \subseteq \prod_{\lambda\in\Lambda} L_\lambda$, where w is the cardinality of Λ, $\mathbb{Z}_{p_i}^{w}$ is the infinite direct product of copies of $\mathbb{Z}_{p_i}$, and $\mathbb{Z}_{p_i}$ is the prime field of characteristic p_i. Since $\varinjlim \mathbb{Z}_{p_i}^{(i)} \subsetneq \mathbb{Z}_{p_i}^{\omega}$, where $\mathbb{Z}_{p_i}^{(i)} = \{\{x_j\}_{j=1}^{\infty} \in \mathbb{Z}_{p_i}^{w_0} : x_{i-1} = x_i = \dots\}$ is a subring of R, and $\mathbb{Z}_{p_i}^{(i)} \simeq \mathbb{Z}_{p_i}^{i}$, the product of i copies of $\mathbb{Z}_{p_i}$, an Artinian von Neumann regular ring. Therefore, the direct limit $\varinjlim \mathbb{Z}_{p_i}^{(i)} \in \mathcal{DU}(\mathbb{Z}_{p_i}^{\omega})$ and $\mathcal{DU}(\mathbb{Z}_{p_i}^{\omega}) \subseteq \mathcal{DU}(\prod_{\lambda\in\Lambda} L_\lambda)$, we must have $\mathcal{DU}(\prod_{\lambda\in\Lambda} L_\lambda) \neq \emptyset$. □

Remark 1.9. If $\mathcal{C}(R)$ is finite, then $\mathcal{DU}(\prod_{\lambda\in\Lambda} L_\lambda) \neq \emptyset$. This follows from the fact that the finiteness of $\mathcal{C}(R)$ implies that $\mathcal{A}(\prod_{\lambda\in\Lambda} L_\lambda) \neq \emptyset$.

Proposition 1.10. *Let R be a zero-dimensional ring and $N(R)$ its nilradical. Then the following conditions are equivalent:*

(i) $\mathcal{DU}(R) \neq \emptyset$;
(ii) $\mathcal{DU}(\frac{R}{N(R)}) \neq \emptyset$.

Proof. (i) $\Rightarrow$ (ii). Let $\varphi : R \twoheadrightarrow \frac{R}{N(R)}$ be the canonical projection, and $T \in \mathcal{DU}(R)$. Then $\varphi(T)$ is a directed union of Artinian subrings of $\frac{R}{N(R)}$. Indeed, if $T = \bigcup_{i\in I} T_i$ is a directed union of Artinian subrings, then it is not difficult to see that $\varphi(T) = \bigcup_{i\in I} \varphi(T_i)$, where $\varphi(T_i) = \frac{T_i}{T_i\cap N(R)} = \frac{T_i}{N(T_i)}$. Notice that if $T_i \subseteq T_j$ then $\frac{T_i}{N(T_i)} \subseteq \frac{T_j}{N(T_j)}$, in other words, $\varphi(T_i) \subseteq \varphi(T_j)$ and it is easy to see that $\frac{T_i}{N(T_i)} \in \mathcal{A}(\frac{R}{N(R)})$ for each $i \in I$. Therefore, $\varphi(T) = \bigcup_{i\in I} \varphi(T_i)$ is a directed union of Artinian subrings. Thus, $\mathcal{DU}(\frac{R}{N(R)}) \neq \emptyset$.
(ii) $\Rightarrow$ (i). Suppose that $\mathcal{DU}(\frac{R}{N(R)}) \neq \emptyset$ and let $S = \bigcup_{i\in I} S_i \in \mathcal{DU}(\frac{R}{N(R)})$ then $S_i \in \mathcal{A}(\frac{R}{N(R)})$. Let $\sigma : R \twoheadrightarrow \frac{R}{N(R)}$ the canonical projection. Let $i_0 \in I$ and $R_{i_0} = \sigma^{-1}(S_{i_0})$, the inverse image of S_{i_0}. It is not difficult to show that $\frac{R_{i_0}}{N(R_{i_0})} \simeq S_{i_0}$ and hence R_{i_0} is a semi-quasilocal zero-dimensional subring of R. By [7, Corollary 5.5], R_{i_0} is a directed union of Artinian subrings. In other words, $R_{i_0} \in \mathcal{DU}(R)$. □

Now we observe that if R is a directed union of Artinian subrings, then $\mathcal{DU}(R)$ is a nonempty set. This raises the question of what the relationship is between $\mathcal{DU}(R) \neq \emptyset$ and the property that R is a directed union of Artinian subrings.

Example 1.11. Let $\Omega = \mathbb{Q}(X)$ be a simple transcendental extension of $\mathbb{Q}$, where $\mathbb{Q}$ denotes the field of rational numbers. Let $R = \Omega^{\mathbb{N}}$ be a countable direct product of copies of Ω. We consider $\Omega^{(i)} = \{\{x_j\}_{j=1}^{\infty} \in \Omega^{\mathbb{N}} : x_{i-1} = x_i = \dots\}$ a subring of R. It is easy to see that $\Omega^{(i)} \simeq \Omega^i$, the finite product of i copies of Ω, an Artinian von Neumann regular ring. It follows that $\mathcal{A}(R) \neq \emptyset$, and $\varinjlim \Omega^{(i)} = T \subseteq \Omega^{\mathbb{N}}$. The ring T is a directed union of Artinian subrings, in other words, $T \in \mathcal{DU}(R)$. However, R is not a directed union of Artinian subrings: indeed, consider $y = (y_i)_{i\in\mathbb{N}^*} \in R$ such that $y_i \neq y_j$ for $i \neq j$. Then, for each $i \in \mathbb{N}^*$, $y \notin \Omega^{(i)}$. It follows that $\varinjlim \Omega^{(i)} \subsetneq \Omega^{\mathbb{N}}$. Hence $\mathcal{A}(R) \neq \emptyset$ does not imply that R is a directed union of Artinian subrings.

Let R be a ring and $\{R_\alpha\}_{\alpha\in A}$ an infinite family of nonzero rings such that R is, up to isomorphism, a subring of each R_α. We use R^* to denote the diagonal imbedding of R in $\prod_{\alpha\in A} R_\alpha$, that is $R^* = \varphi(R)$, where $\varphi : R \hookrightarrow \prod_{\alpha\in A} R_\alpha$ is a monomorphism defined by $\varphi(x) = \{x_\alpha\}_{\alpha\in A}$ such that $x_\alpha = x$ for each $\alpha \in A$.

Theorem 1.12. *Let R be a zero–dimensional ring and $\mathcal{F}(R) = \{L_i\}_{i\in I}$ its set of residue fields. Assume that for each pair $(i,j) \in I^2$ we have $L_j \cap L_k = L \notin \mathcal{F}(R)$. Then*

(1) $\mathcal{DU}(R) \neq \emptyset$.
(2) *R is not a directed union of Artinian subrings.*

To prove this result, we need the following Lemma.

Lemma 1.13. [11, Proposition 2.1] *Let R be a Von Neumann regular ring and $R \subset T \subset \prod_{\lambda\in\Lambda} \frac{R}{M_\lambda}$ such that $T = \bigcup_{i\in I} T_i$ is a directed union of Artinian subrings and $\{M_\lambda\}_{\lambda\in\Lambda} = Spec(R)$. Then $\mathcal{F}(R) = \mathcal{F}(T)$.*

Proof of Theorem 1.12. (1) According to Lemma 1.2, we assume that $\mathcal{C}(R)$ is finite, otherwise, $\mathcal{DU}(R) = \emptyset$. Since $\mathcal{C}(R) = \{p_1, \ldots, p_l\}$ is a finite set of prime integers, we can write $I = I_1 \cup \cdots \cup I_l$ as a partition of I where $I_j = \{i \in I/char(L_i) = p_j\}$ for $j = 1, \ldots, l$. Let $T_j = \prod_{i\in I_j} L_i$ for each $j = 1, \ldots, l$. Thus $\prod_{i=1}^{l} T_i$ is isomorphic to $\prod_{i\in I} L_i$. For $j = 1, \ldots, l$, let $\mathbb{Z}_{p_j}$ be the prime subfield of characteristic p_j and $\mathbb{Z}_{p_j}^{I_j}$ the direct product of copies $\mathbb{Z}_{p_j}$. Then $\prod_{j=1}^{l} \mathbb{Z}_{p_j}^{I_j}$ is a subring of $\prod_{i\in I} L_i$. By [7, Theorem 6.7], $\prod_{j=1}^{l} \mathbb{Z}_{p_j}^{I_j} = \bigcup_{k\in A} S_k$ is a directed union of Artinian subrings. Let $R_k = R \cap S_k$ is a von Neumann regular subring of R, for each $k \in I$. Since each R_k is a subring of S_k and $|Idem(R_k)| \leq |Idem(S_k)|$, it follows that each R_k is Artinian. The family $\{R_k\}_{k\in A}$ is directed because the family $\{S_k\}_{k\in A}$ is so. It follows that $\bigcup_{k\in A} R_k \subseteq R$ is a directed union of Artinian subrings. Thus $\mathcal{DU}(R) \neq \emptyset$, and hence $\mathcal{DU}(R) \neq \emptyset$.
(2) Since $\mathcal{C}(R) = \{p_1, \ldots, p_l\}$ and $\mathcal{F}(R) = \mathcal{F}_1(R) \cup \mathcal{F}_2(R) \cup \cdots \cup \mathcal{F}_l(R)$, where $\mathcal{F}_j(R) = \{F \in \mathcal{F}(R)/char(F) = p_j\}$, to prove that the condition of theorem 1.12 is satisfied for $\mathcal{F}(R)$ it suffices to show that it is satisfied for each $\mathcal{F}_j(R)$. In this case we suppose that $\mathcal{F}(R) = \mathcal{F}_j(R)$ for some $j \in \{1, \ldots, l\}$. Now, assume that $L \notin \mathcal{F}_j(R)$. Let $\mathcal{S}$ be the subring of $\prod_{j\in I} L_j$ consisting of eventually constant sequences. Thus $\mathcal{S} \simeq L^* + J$, the L-subalgebra of $\prod_{j\in I} L_j$ generated by the direct sum ideal $J = \bigoplus_{i\in I} L_i$, where L^* is the diagonal imbedding of L in $\prod_{j\in I} L_j$. First, we claim that $\mathcal{S}$ is the maximal subring of $\prod_{j\in I} L_j$ with respect to being a directed union of Artinian subrings. Let $T = \bigcup_{j\in J} T_j$ be a subring of $\prod_{\alpha\in A} L_\alpha$ that is a directed union of Artinian subrings, and let $t = \{t_\alpha\}_{\alpha\in A} \in T$. There exists $j_o \in J$ such that $t \in T_{j_o}$ and T_{j_o} is a finite product of fields. Hence

$t \in \mathcal{S}$. We claim that $\mathcal{F}(\mathcal{S}) = \{L\} \bigcup \{L_j\}_{j\in I}$. Let $p_i : T = \prod_{j\in I} L_j \to L_i$ be the canonical projection and $p_{i|\mathcal{S}}$ its restriction on $\mathcal{S}$, which is a surjective homomorphism. We have that $Kerp_{i|\mathcal{S}} = (1 - e_i)T \bigcap \mathcal{S} = (1 - e_i)\mathcal{S} = M_i$, with e_i the primitive idempotent with support $\{i\}$, and $\mathcal{S}/M_i \simeq L_i$ for each $i \in I$. Also, J is a maximal ideal of $\mathcal{S}$ and $\mathcal{S}/J \simeq (L)^* \simeq L$. Thus $\{J\} \cup \{M_i\}_{i\in I} \subseteq Max(\mathcal{S})$. Let $P \in Spec(\mathcal{S})$, if $J \subseteq P$, then $J = P$. If $J \nsubseteq P$ then $e_i \notin P$, for some $i \in I$, and hence $1 - e_i \in P$. Therefore, $M_i \subseteq P$ and $P = M_i$. Consequently, $Max(\mathcal{S}) = \{J\} \bigcup \{M_i\}_{i\in I}$. Thus $\mathcal{F}(\mathcal{S}) = \{L\} \bigcup \{L_j\}_{j\in I}$. If R is a directed union of Artinian subrings, then $R \subset \mathcal{S}$. By Lemma 1.13, $\mathcal{F}(R) = \mathcal{F}(\mathcal{S})$, a contradiction to the fact that $L \notin \mathcal{F}(R)$. □

Corollary 1.14. *Let R be a ring and S is a multiplicative closed subset of R.*

(i) *If R is a directed union of Artinian subrings, then $S^{-1}R$ so is.*
(ii) *If $\mathcal{DU}(R) \neq \emptyset$, then $\mathcal{DU}(S^{-1}R) \neq \emptyset$.*
(iii) *We have $\mathcal{DU}(R(X)) \neq \emptyset$ if $\mathcal{DU}(R) \neq \emptyset$, where $R(X)$ is the Nagata ring and X is an indeterminate over R.*

Proof. (i) Suppose that $R = \bigcup_{\alpha\in A} R_\alpha$ is a directed union of Artinian subrings and $S_\alpha = S \cap R_\alpha$ be a multiplicative closed subset of R_α. It is not difficult to show that $S^{-1}R = \bigcup_{\alpha\in A} S_\alpha R_\alpha$. Since R_α is Artinian, the localization $S_\alpha^{-1}R_\alpha$ is also Artinian. The family $\{S_\alpha^{-1}R_\alpha\}_{\alpha\in A}$ is directed since $\{R_\alpha\}_{\alpha\in A}$ so is. It follows that $S^{-1}R = \bigcup_{\alpha\in A} S_\alpha R_\alpha$ is a directed union of Artinian subrings.
(ii) Suppose that $\mathcal{DU}(R) \neq \emptyset$, let $T = \bigcup_{j\in J} T_j \in \mathcal{DU}(R)$ and $U = S \cap T$ be a multiplicative closed subset of T. By (i), $U^{-1}T \subseteq S^{-1}R$ is a directed union of Artinian subrings. It follows that $\mathcal{DU}(S^{-1}R) \neq \emptyset$. As $R(X) = S^{-1}R[X]$, where $S = R[X] \setminus \bigcup\{MR[X] : M \text{ is a maximal ideal of } R\}$, we have $\mathcal{DU}(R(X)) \neq \emptyset$ if $\mathcal{DU}(R) \neq \emptyset$.
(iii) If $R = \bigcup_{i\in I} R_i$ is a directed union of Artinian subrings, then $R(X) = \bigcup_{i\in I} R_i(X)$. Since each R_i is Nœtherian, by [12, (6.17)], $R_i(X)$ is also Nœtherian and each $R_i(X)$ is zero-dimensional as each R_i is zero-dimensional (cf.[1, Proposition 1.21]). By [2, Theorem 8.5], $R_i(X)$ is an Artinian ring for each $i \in I$. The family $\{R_i(X)\}_{i\in I}$ is directed because so is the family $\{R_i\}_{i\in I}$. Then $R(X)$ is a directed union of Artinian subrings.

□

2. Von Neumann Regular rings as a directed union of Artinian subrings

Let R be a von Neumann regular ring and $\{M_i\}_{i\in I}$ its spectrum. Since R is reduced, we have $\bigcap_{i\in I} M_i = (0)$ and hence the homomorphism $\varphi : R \to \prod_{i\in I} \frac{R}{M_i}$, defined by $\varphi(x) = x + M_i$, is injective. This allows us to view R as a subring of $\prod_{i\in I} \frac{R}{M_i}$. We identify x with its image $\{x_i\}_{i\in I} \in \prod_{i\in I} \frac{R}{M_i}$. Finally, we denote $F_i = \frac{R}{M_i}$ for each $i \in I$. We assume that there is a field Ω containing each F_i. We can always make this assumption if the F_i have the same characteristic.

Given $x \in R$, let $x_i \in \Omega$ be the coset $x + M_i \in F_i$, but viewed as an element of Ω. We identify x with its image $\{x_i\}_{i\in I} \in \Omega^I$, the infinite direct product of copies of Ω. Given $y = \{y_i\}_{i\in I} \in \Omega^I$, let $||y|| = \{y_i \ : \ i \in I\} \subseteq \Omega$. Finally, we put

$$\mathcal{S} = \{y \in \Omega^I : \ ||y|| \text{ is finite}\}$$

Proposition 2.1. *With the notation and assumptions above, we have the following:*

(1) *$\mathcal{S}$ is von Neumann regular.*
(2) *$\mathcal{S}$ is a directed union of Artinian subrings.*
(3) *If $R \subset \mathcal{S}$ then R is a directed union of Artinian subrings.*

To prove this result, we require the following two lemmas.

Lemma 2.2. *Let $\{S_\alpha\}_{\alpha\in A}$ be a family of von Neumann regular subrings of a von Neumann regular ring T. Then $R := \bigcap_{\alpha\in A} S_\alpha$ is von Neumann regular.*

Proof. Given $x \in R$, let y be the unique element of T satisfying $xyx = x$ and $yxy = y$. For the existence, choose any z such that $xzx = x$ and put $y := zxz$. For the uniqueness, just note that uniqueness is clear for fields, so it is true locally and therefore globally. The existence and uniqueness, applied to each S_α, shows that $y \in S_\alpha$ for each α. Thus $y \in R$ as desired. □

Lemma 2.3. *Let R and S be von Neumann regular rings, with R a subring of S.*

(i) *If S is a directed union of Artinian subrings, so is R.*
(ii) *If $\mathcal{DU}(S) \neq \emptyset$, then $\mathcal{DU}(R) \neq \emptyset$.*

Proof. (i) Write $S = \bigcup_{i\in I} S_i$ as the directed union of Artinian rings S_i. Since $R = \bigcup_{i\in I} R_i$ is the directed limit of the rings $R_i = R\cap S_i$, it will suffice to show that each R_i is Artinian. By Lemma 2.2, we know at least that R_i is von Neumann regular. Since R_i has only finitely many idempotents (as $R_i \subseteq S_i$), R_i is a direct product of finitely many fields. □

Proof of Proposition 2.1. (1) Given $x \in \mathcal{S}$, define $y \in \Omega^I$ by letting $y_i = x_i^{-1}$ if $x_i \neq 0$, and $y_i = 0$ if $x_i = 0$. Then $y \in \mathcal{S}$ and $xyx = x$.
(2) Given a partition $\mathcal{P}$ of I into pairwise disjoint sets $J_1, \ldots, J_n$, let $\mathcal{A}_\mathcal{P}$ be the subring of Ω^I consisting of those elements $\{y_i\}_{i\in I}$ that are constant on each set J_r in the partition, that is, $i, j \in J_r$ implies $y_i = y_j$. Then $\mathcal{A}_\mathcal{P} \simeq K_1 \times \cdots \times K_n$, where $K_r = \bigcap_{i\in J_r} F_i$. Thus each $\mathcal{A}_\mathcal{P}$ is Artinian. The family $\{\mathcal{A}_\mathcal{P}, \mathcal{P}$ is a partition of I$\}$ is directed. Indeed, If $\mathcal{P} = \{J_i\}_{i=1}^l$ and $\mathcal{Q} = \{I_j\}_{j=1}^k$ are finite partitions of I, then $\mathcal{P} \cap \mathcal{Q} = \{J_i \cap I_k / 1 \leq j \leq l,\ \ 1 \leq i \leq k\}$ is a finite partition of I, and $\mathcal{A}_\mathcal{P} \cup \mathcal{A}_\mathcal{Q} \subseteq \mathcal{A}_{\mathcal{P}\cap\mathcal{Q}}$. Since $\mathcal{S}$ is the union of the rings $\mathcal{A}_\mathcal{P}$, (2) is proved.
(3) This follows from (1), (2) and Lemma 2.3. □

Here is a verification that $\mathbb{Q}^\mathbb{N}$ is not a direct limit of Artinian rings.

Lemma 2.4. *Let K be a field, let I be a non-empty index set, and put $\Pi = K^I$. We identify K with the subfield of Π consisting of constant functions in Π. Then K is a maximal subfield of Π.*

Proof. Suppose L is a subfield of Π that properly contains K. Choose two elements $x, y \in L$ that are linearly independent over K. Choose $i \in I$, and put $x_i = x(i)$ and $y(i) = y_i$. Then the i^{th} coordinate of $z := y_i x - x_i y$ is 0, whence z is a non-unit of Π. Since L is a field and $z \in L$, $z = 0$. This forces $y_i = 0$ (since x and y are linearly independent). But then y is not a unit, and this means that $y = 0$, again contradicting linear independence. □

Now let A be an Artinian subring of $\Pi := \mathbb{Q}^I$, where I is any index set. Let $e_1, \ldots, e_n$ be the primitive idempotents of A, with $e_1 + \cdots + e_n = 1$. These idempotents give a corresponding partition $\mathcal{P} = S_1 \cup \cdots \cup S_n$ of $\mathbb{N}$, where S_i is the support of e_i. The field Ae_i is a subfield of $\Pi e_i \cong \mathbb{Q}^{S_i}$. Therefore Ae_i contains the prime subfield $\mathbb{Q}$ of $\mathbb{Q}^{S_i}$ (consisting of functions that are constant on S_i), and by the Lemma Ae_i consists precisely of the constant functions on S_i. It follows that A consists of the functions in Π that are constant on *each* set S_i.
Given a partition $\mathcal{P}$ of I into a finite number of pairwise disjoint sets, let $A_\mathcal{P}$ be the subring of $\mathbb{Q}^I$ consisting of functions that are constant on each set in $\mathcal{P}$. Of course $A_\mathcal{P}$ is isomorphic to $\mathbb{Q}^n$, where n is the number of sets in the partition. In particular, $A_\mathcal{P}$ is Artinian. We have proved the following:

Proposition 2.5. *The Artinian subrings of $\mathbb{Q}^I$ are precisely the rings of the form $A_\mathcal{P}$, where $\mathcal{P}$ is some finite partition of I.*

Corollary 2.6. *If I is infinite, then $\mathbb{Q}^I$ is not a direct limit of Artinian subrings.*

Lemma 2.7. *Let R be a zero-dimensional ring with finite spectrum, then R is expressible as a finite product of zero-dimensional quasi-local subrings.*

Proof. Let $Spec(R) = \{M_i\}_{i=1}^{n}$ be the spectrum of R. Let $S_{M_i}(0)$ to denote $Ker\varphi_i$ for each $i = 1, ..., n$, where $\varphi_i : R \rightarrow R_{M_i}$ and $\varphi_i(r) = \frac{r}{1}$, is the canonical homomorphism. Since $Rad(S_{M_i}(0)) = M_i$, $S_{M_i}(0)$ is a primary ideal. Note that $\cap_{i=1}^{n} S_{M_i}(0) = (0)$ and $S_{M_i}(0) + S_{M_j}(0) = R$ for each $i \neq j$ in { 1,..., n}. Therefore, $R \simeq \frac{R}{\cap_{i=1}^{n} S_{M_i}(0)}$. By the Chinese remainder Theorem, $R \simeq \prod_{i=1}^{n} \frac{R}{S_{M_i}(0)}$, where $\frac{R}{S_{M_i}(0)}$ is quasi-local and zero-dimensional, for $i = 1, ..., n$. □

Lemma 2.8. *Let R be a von Neumann regular ring. Then the following conditions are equivalent.*

(i) *R is Artinian;*
(ii) *R is a finite product of fields;*
(iii) *R is Noetherian.*

Proof. We suppose that R is von Neumann regular and Artinian then by [2, Corollary 8.2], $Spec(R)$ is finite and hence $R = R_1 \oplus \cdots \oplus R_n$, where each R_i is a quasi-local and zero-dimensional ring, for $i = 1, \ldots, n$. Since R is von Neumann regular, each R_i is a von Neumann regular ring. As each R_i is quasi-local, [9, Theorem 3.1], R_i is a field for each $i = 1, \ldots, n$. It follows that R is a finite product of fields. The rest of the proof is straightforward since every subring of a reduced ring is reduced. □

Proposition 2.9. *Let R be a von Neumann regular ring and $\mathcal{F}(R) = \{F_\lambda\}_{\lambda \in \Lambda}$ its residue fields. If there exists $k \in \mathbb{N} \setminus \{0\}$ such that the set $\{\lambda \in \Lambda \ : \ |F_\lambda| > k\}$ is finite, then $\mathcal{DU}(R) \neq \emptyset$.*

Proof. Since R is a reduced ring, we have $\bigcap_{\lambda \in \Lambda} M_\lambda = (0)$. Then the homomorphism $\varphi : R \longrightarrow \prod_{\lambda \in \Lambda} R/M_\lambda$ defined by $\varphi(r) = \{r_\lambda\}_{\lambda \in \Lambda}$, where $r_\lambda \equiv \overline{r}$ Mod M_λ, is injective. In this case we regard R as a subring of $\prod_{\lambda \in \Lambda} F_\lambda$, where $F_\lambda = \frac{R}{M_\lambda}$ for each $\lambda \in \Lambda$. As $\{\lambda \in \Lambda \ : \ |F_\lambda| > k\}$ is finite for some $k \in \mathbb{N}^*$, by [7, Theorem 6.7] $\prod_{\lambda \in \Lambda} F_\lambda$ is a directed union of Artinian subrings. By Proposition 1.3, R is also a directed union of Artinian subrings and hence $\mathcal{DU}(R) \neq \emptyset$. □

Throughout this paper, we assume that for each $k \in \mathbb{N}^*$ the set $\{\lambda \in \Lambda \ : |F_\lambda| > k\}$ is infinite.

Remark 2.10. (1) From Proposition 2.9, if F is a field, then the infinite direct product $\prod_{\alpha \in A} F$ is a directed union of Artinian subrings if and only if F is finite.
(2) Let L be a field, then $\mathcal{DU}(\prod_{\alpha \in A} L) \neq \emptyset$.

Theorem 2.11. *Let R be a von Neumann regular ring with $\mathcal{F}(R) = \{L_\alpha\}_{\alpha\in A}$. Then $\mathcal{S} = \{\ r = \{r_\alpha\}_{\alpha\in A} \in \prod_{\alpha\in A} L_\alpha\ :\ ||r|| \text{ is finite}\}$ is a directed union of Artinian subrings if and only if for each subset B of A, there exists a partition $B = \bigcup_{i=1}^{s} B_i$, where each family $\{L_\alpha :\ \alpha \in B_i\}$ has a common subfield, up to isomorphism.*

Proof. Assume that $\mathcal{S}$ is a directed union of Artinian subrings and let B be a subset of A. Let $f \in \mathcal{S}$, then f belongs to a finite product of fields and hence it has only finitely many distinct components $f_1,\dots,\ f_t$. Now, let $A_i = \{\alpha \in A : f(\alpha) = f_i\}$, for each $i = 1,\dots,t$, so $A = \bigcup_{i=1}^{t} A_i$ is a partition of A which satisfies that $\bigcap_{\alpha\in A_j} L_\alpha$ is a field, up to isomorphism, for each $j = 1,\dots,t$. So we only consider $B = \bigcup\{A_i \bigcap B :\ 1 \le j \le t\ \}$, with $A_k \bigcap B \neq \emptyset$ for each $k = 1,\dots,t$. Furthermore, $\{L_\alpha : \alpha \in A_k \cap B\}$ has a common subfield, up to isomorphism. Conversely, to show that $\mathcal{S}$ is a directed union of Artinian subrings, we need only to show that for each f in $\mathcal{S}$ the set $\{f(\alpha) : \alpha \in A\} = \{f_1,\ \dots,\ f_t\}$ is contained in a finite product of fields. For, let $f \in \mathcal{S}$, and consider the partition $A_f = \bigcup_{i=1}^{n} A_i$ associated with f, i.e., $A_i = \{\alpha \in A :\ f(\alpha) = f_i\}$ and $i = 1,\dots,\ t$. Let $f_i^* = (f_i,\dots,\ f_i,\dots) \in \prod_{\alpha\in A_i} L_\alpha$, by hypothesis, there exists a partition $A_i = \cup_{k=1}^{s_i} I_k$, $i = 1,\dots,\ k$, such that each family of fields $\mathcal{F}_k = \{L_\alpha :\ \alpha \in I_k\}$ has a common subfield K_k^i (which need not belong to $\mathcal{F}_k$). Thus, $f_i^* \in \prod_{k=1}^{s_i} K_k^{i*}$, where K_k^{i*} denotes the diagonal imbedding of K_k^i in $\prod_{\alpha\in I_k} L_\alpha$. Consequently, $f \in \prod_{i=1}^{t}(\prod_{k=1}^{s_i} K_k^{i*})$ and hence f belongs to a finite product of fields. □

Corollary 2.12. *A von Neumann regular ring R is a directed union of Artinian subrings if and only if $R \subseteq \mathcal{S}$.*

Corollary 2.13. *A von Neumann regular ring R is imbeddable in a directed union of Artinian subrings if and only if R has the same property.*

Example 2.14. (1) Let $\{p\} \cup \{q_i\}_{i\in\mathbb{N}^*}$ be an infinite family of distinct positive prime integers. Let $\mathcal{F} = \{GF(p)\} \bigcup \{GF(p^{q_i})\}_{i=1}^{\infty}$ be a family of finite Galois fields, and α an element such that $\alpha^2 = 1$ with $\alpha \notin GF(p)$. Let $\mathcal{L} = GF(p)(\alpha)$ be a simple algebraic extension of $GF(p)$. We denote φ_i the imbedding of $GF(p)$ in $GF(p^{q_i})$ for each $i \in \mathbb{N}^*$. Let $\varphi = \prod_{i=1}^{\infty} \varphi_i$, $T = \prod_{i=1}^{\infty} GF(p^{q_i})$ and $I = \bigoplus_{i=1}^{\infty} GF(p^{q_i})$ the direct sum ideal of T. We denote $GF(p^) * = \varphi(GF(p)) = R_o$ the diagonal imbedding of $GF(p)$ in T. Let $\mathcal{S}_1 = R_o + I$, since $\mathcal{S}_1$ is a subring of T and $\dim(\mathcal{S}_1)$=0, the ring $\mathcal{S}_1$ is a von Neumann regular ring. We have shown in the proof of Theorem 2.11 that $\mathcal{F}(\mathcal{S}_1) = \{GF(p)\} \bigcup \{GF(p^{q_i}) : i \in \mathbb{N}^*\}$. From $\mathcal{F}(\mathcal{S}_1)$ we construct a direct union of Artinian subrings. Let $S_1 = GF(p^{q_1}) \times GF(p)^* \simeq GF(p^{q_1}) \times GF(p); ...;\ S_i = GF(p^{q_1}) \times ... \times GF(p^{q_i}) \times GF(p)^* \simeq GF(p^{q_1}) \times ... \times GF(p^{q_i}) \times$

$GF(p)$. We have that $S_j \subset S_{j+1}$, for each positive integer $j \in \mathbb{N}^*$. Therefore, $\mathcal{S}_1 = \bigcup_{i \in \mathbb{N}^*} S_i$ is an increasing union of Artinian subrings. We can generalize this example to the case where $\mathcal{F}(R) = \mathcal{F}$ has a partition $\mathcal{F} = \bigcup_{i=1}^{n} \mathcal{F}_i$ into finite nonempty subfamilies $\mathcal{F}_i$ where all elements of $\mathcal{F}_i$ have the same characteristic, and every element of $\mathcal{F}_i$ has a common subfield.
(2) Let $R = GF(p)(\alpha) + \mathcal{J}$, the $\mathcal{L}-$subalgebra of $T_1 = \prod_{i=1}^{\infty} \mathcal{L}(\zeta_i)$ generated by the direct sum ideal $\mathcal{J} = \oplus_{i=1}^{\infty} \mathcal{L}(\zeta_i)$, where ζ_i is a p^q-primitive root of unity and q is a prime integer. For each $i \in \mathbb{Z}^+$, let $\phi_i : \mathcal{L} \to \mathcal{L}(\zeta_i)$ be the field-homomorphism taking α to ζ_i. Let $\phi = \{\phi_i\}_{i=1}^{\infty} : T_1 \to T_1$, a ring-homomorphism. Let $R_1 = \phi(R)$. Being isomorphic to R, the ring R_1 is a directed union of Artinian subrings. We observe that the element $\zeta = \{\zeta_i\}_{i=1}^{\infty} \in R_1$ which is not in $\mathcal{S}$ but is contained in $\phi(\mathcal{S})$ ($\mathcal{S}$ is the maximum among all subrings of T_1 that are directed union of Artinian subrings).

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