FINITENESS OF HILBERT FUNCTIONS AND BOUNDS FOR CASTELNUOVO-MUMFORD REGULARITY OF INITIAL IDEALS

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Abstract. Bounds for the Castelnuovo-Mumford regularity and Hilbert coefficients are given in terms of the arithmetic degree (if the ring is reduced) or in terms of the defining degrees. From this it follows that there exists only a finite number of Hilbert functions associated to reduced algebras over an algebraically closed field with a given arithmetic degree and dimension. A good bound is also given for the Castelnuovo-Mumford regularity of initial ideals which depends neither on term orders nor on the coordinates, and holds for any infinite field.

Introduction

In the famous book SGA6, Kleiman proved that given two positive integers e and d, there exists only a finite number of Hilbert functions associated to reduced and equidimensional K-algebras S over an algebraically closed field such that $\deg S \leq e$ and $\dim S = d$ (see [K, Corollary 6.11]). An easier and eleganter proof of this result can be found in a recent paper by M. Rossi, N. V. Trung and G. Valla [RTV2]. Moreover, the paper [RTV2] gives a rather general approach to derive the finiteness of Hilbert functions. It is shown that this problem (for a certain class of ideals) is equivalent to the boundness of the Castelnuovo-Mumford regularity and the embedding dimension (see [RTV2, Theorem 2.3]). The first main purpose of this paper is to extend Kleiman's result to reduced K-algebras. A key point is to find a suitable invariant to replace the degree. Of course, a so-called extended degree is a choice, see [RVV, Corollary 4.4], but such an invariant is very big. It turns out that in our situation one can take the so-called arithmetic

1991 *Mathematics Subject Classification.* 13D45, 13D40, 13P10.

Key words and phrases. Castelnuovo-Mumford regularity, local cohomology, Hilbert function, Hilbert polynomial, initial ideal.

The author was supported in part by the National Basic Research Program (Vietnam). The final preparation of the article was done during his stay at the Centre de Recerca Matemàtica (Spain)).

degree - a notion which maybe reflects better the complexity of ideals than the usual degree (see [BM, Section 3] and [V, Chapter 9]).

Theorem 0.1. *Given two positive integers a and d. Assume that K is an algebraically closed field. Then there exists only a finite number of Hilbert functions associated to reduced K-algebras S such that $\operatorname{adeg} S \leq a$ and $\dim S = d$.*

Note that the above result does not hold for an arbitrary algebra (however see [RVV, Corollary 4.4] and [RTV2, Theorem 3.1] for a possible generalization). As mentioned above, the main point in the proof of Theorem 0.1 is to bound the Castelnuovo-Mumford regularity. We will establish the following result:

Theorem 0.2. *Let K be an arbitrary field. Assume that $S = R/I$ is a reduced ring of dimension at least two. Then*

$$\operatorname{reg} I \leq (\frac{e(e-1)}{2} + \operatorname{adeg} I)^{2^{d-2}}.$$

Applied to the case of reduced and equidimensional algebras, the bound of Theorem 0.2 is better than the one given in [RTV2, Theorem 3.1 and Lemma 3.3]. In view of the Eisenbud-Goto conjecture, the above bound is too big. However it is a first explicit bound in terms of the arithmetic degree. In order to prove Theorem 0.2, as in [K], [BM] and [RTV2], we proceed by induction on the dimension. However, there is a different point: we bound this invariant and the length of graded components of certain local cohomology modules simultaneously (see Theorem 1.5 and also Theorem 2.5).

The above technique can be also used to estimate the Castelnuovo-Mumford regularity of arbitrary homogeneous ideals in terms of the maximal degree Δ of minimal generators of $I \subset R = K[x_1, ..., x_n]$. If K is any field of zero characteristic, from Giusti's paper [Gi] it follows that $\operatorname{reg}(I) \leq (2\Delta)^{2^{n-2}}$. Bayer and Mumford suggested that this bound holds in any characteristic (see the comment after Theorem 3.7 in [BM]). Not long ago, G. Caviglia and E. Sbarra proved that this is indeed the case:

$$\operatorname{reg} I \leq (\Delta^c + \Delta c - c + 1)^{2^{d-1}},$$

(see [CS, Corollary 2.6]). In this paper we will give a completely different proof for this result (see Theorem 2.1).

The next problem we are interested in is to give good bounds for the Castelnuovo-Mumford regularity of initial ideals $\operatorname{in}(I)$ with respect to *any term order* and in *any coordinates*. Inspired by a result of Chardin and Moreno-Sosias, it was shown in [HH] that if R/I is a Cohen-Macaulay ring of multiplicity $e \geq 2$ and $d \geq 2$, then $\mathrm{reg}(\mathrm{in}(I)) \leq e^{2^{d-1}}/2^{2^{d-2}}$. Long before that M. Giusti [Gi] showed that $\mathrm{reg}\,\mathrm{in}(I) \leq (2\Delta)^{2^{n-1}}$ if the coordinates are chosen generically and the term order is the lexicographic order. The paper [CS] suggests that such a kind of bounds should hold for any $\mathrm{reg}(\mathrm{in}\, I)$. Our second main result confirms it:

Theorem 0.3. *Let K be an arbitrary field and $I \subset R = K[x_1, ..., x_n]$. With respect to any term order and any coordinates we have*

$$\mathrm{reg}(\mathrm{in}\, I) \leq (2\Delta^c)^{d2^{d-1}}.$$

Moreover, if R/I is a reduced algebra, then we also have

$$\mathrm{reg}(\mathrm{in}\, I) \leq (\mathrm{adeg}(I))^{(n-1)2^{d-1}}.$$

An immediate consequence of this theorem says that with respect to any term order and any coordinates the maximal degree of the reduced Gröbner base, with respect to any term order and any coordinates, is bounded by $(2\Delta^c)^{d2^{d-1}}$. In view of a remarkable example due to Mayr and Meyer, this bound is nearly the best possible (see, e.g., Example 3.9 and Proposition 3.11 in [BM]).

In order to prove this theorem we develop further the method in [HH]. Instead of initial ideals we consider a much bigger class: the class of all ideals J having the same Hilbert function as I. By doing this one can use Gotzmann's regularity theorem to bound $\mathrm{reg}\, J$ in terms of some data of I. Then, by virtue of Theorem 0.2 and the Caviglia-Sbarra theorem, we will see that the only thing left is to estimate the Hilbert coefficients e_i in terms of Δ or $\mathrm{adeg}(I)$ (see Lemmas 6.1 and 6.3). This problem is also of independent interest. Main steps to do it may be explained as follows. First, using a recent result by Herzog, Popescu and Vladoiu [HPV] one can bound cohomological Hilbert functions (i.e. the length of graded components of local cohomology modules) in terms of the Castelnuovo-Mumford regularity. From that we get bounds for the Hilbert coefficients by the Castelnuovo-Mumford regularity (see theorems 4.1 and 4.6). The existence of such a bound was predicted by [RTV2, Theorem 2.3], and this approach is somewhat new, because usually one tries to estimate the latter invariant by the former ones (see, e.g., [K] and [BrS, Section 17.2]). However, it is a surprising fact, that the relationships of these invariants in theorems 4.1 and 4.6 are rather simple. Combining them with results on the Castelnuovo-Mumford regularity found earlier we get bounds for $|e_i|$ in terms of Δ or $\mathrm{adeg}(I)$ (see propositions 4.3 and 4.7). These bounds are huge: they are double exponential functions of i. But they are good enough for our application in proving Theorem 0.3. Furthermore, theorems 4.1 and 4.6 sometimes give really good bounds for $|e_i|$ if we already know a good estimation for the Castelnuovo-Mumford regularity (see corollaries 4.4 and 4.8).

We now give a brief content of the paper. In Section 1 we will prove Theorem 0.2, and reprove in Section 2 the Caviglia-Sbarra bound on the Castelnuovo-Mumford regularity of an arbitrary homogeneous ideal in terms of the degrees of its defining equations (Theorem 2.1). Section 3 is devoted to bounding Hilbert cohomological functions in terms of the Castelnuovo-Mumford regularity (see Theorem 3.4). Bounds on Hilbert coefficients will be given in Section 4. Putting results of the sections 1 and 4 together we are able to prove Theorem 0.1 without using [RTV2]. This will be done in Section 5. Theorem 0.3 will be proved in the last Section 6. We refer the readers to Eisenbud's book [E] for unexplained terminology.

Acknowledgement: The author would like to thank the Centre de Recerca Matemàtica (Spain) for financial support and hospitality during the final preparation of this article.

1. Bounds in terms of the arithmetic degree

Throughout this paper, if not otherwise stated, K is an arbitrary infinite field, $R = K[x_1, ..., x_n]$ is a polynomial ring and $I \subset R$ is a homogeneous ideal of dimension d. Let $c = n - d$. Note that c is the true codimension of I if I does not contain a linear form. Let $\mathfrak{m} = (x_1, ..., x_n)$ denote the maximal homogeneous ideal of R and set $S = R/I$. Let us recall some notions.

For an artinian $\mathbb{Z}$-graded module N, let

$$\mathrm{end}(N) = \max\{t;\ N_t \neq 0\}$$

(with the convention $\max \emptyset = -\infty$). Further, let

$$a_i(R/I) = \mathrm{end}(H^i_{\mathfrak{m}}(R/I)),$$

where $H^i_{\mathfrak{m}}(R/I)$ is the local cohomology module with the support in $\mathfrak{m}$ and $0 \leq i \leq d$. The *Castelnuovo-Mumford regularity* is the number

$$\mathrm{reg}(R/I) = \max\{a_i(R/I) + i;\ 0 \leq i \leq d\}.$$

Note that $\mathrm{reg}(I) = \mathrm{reg}(R/I) + 1$. Sometimes we also use the notation

$$\mathrm{reg}_k(R/I) = \max\{a_i(R/I) + i;\ k \leq i \leq d\}, \qquad (1)$$

where k is a non-negative integer.

Following Brodmann and Sharp [BrS], the function

$$h^i_S(t) := \ell(H^i_{\mathfrak{m}}(S)_t)$$

is called the i-th *Hilbert homological function* of S, where $\ell(.)$ denotes the dimension of a vector space over K. Let $H_S(t))$ and $P_S(t)$ denote the Hilbert function and the Hilbert polynomial of S, respectively. We will often use the Grothendieck-Serre formula

$$P_S(t) - H_S(t) = \sum_{i=0}^{d} (-1)^{i+1} h_S^i(t). \tag{2}$$

The arithmetic degree is defined as follows:

$$\operatorname{adeg} S = \operatorname{adeg} I = \sum_{\mathfrak{p} \in \operatorname{Ass}(R/I)} \ell(H^0_{\mathfrak{m}_\mathfrak{p}}(R/I)_\mathfrak{p}) e(\mathfrak{p})$$

(see [BM, Definition 3.4] and [V, Definition 9.13]). The number $\ell(H^0_{\mathfrak{m}_\mathfrak{p}}(R/I)_\mathfrak{p})$ is the multiplicity of the component $\mathfrak{p}$ with respect to I. In this definition $\mathfrak{p}$ runs over all associated primes of S, while the usual degree $\deg S$ can be computed by a similar formula, but the sum is only taken over primes of the highest dimension. Thus

$$\operatorname{adeg} S \geq \deg S,$$

and the equality holds if and only if S is a pure-dimensional ring. We also denote $\deg S$ by $e(S)$, or just by e.

In this section we prove Theorem 0.2. Since $\operatorname{reg} I$, e and $\operatorname{adeg} I$ do not change when replacing K by $K(u)$, where u is a new indeterminate, in order to prove the theorem we may assume that K is an infinite field as usual. We need some auxiliary results.

Lemma 1.1. *Let S be an one-dimensional Cohen-Macaulay ring. Then*

$$h_S^1(0) + \cdots + h_S^1(\operatorname{reg} S - 1) \leq e(e-1)/2.$$

Proof. Since $P_S(t) = e$, from the Grothendieck-Serre formula (2) we have

$$h_S^1(t) = e - H_S(t).$$

Let $r = \operatorname{reg} S$. Since S is a Cohen-Macaulay ring, its Hilbert-Poincare series can be written in the form

$$HP_S(z) := \sum_{i \geq 0} H_S(i) z^i = \frac{1 + h_1 z + \cdots + h_r z^r}{1 - z},$$

where $h_1, ..., h_r$ are positive integers (see, e.g., [V, p. 240]). From this it follows that

$$H_S(t) = 1 + h_1 + \cdots + h_t \geq t + 1$$

for all $t \leq r-1$. Moreover, under the Cohen-Macaulay assumption, $r \leq e-1$. Hence

$$h_S^1(0) + \cdots + h_S^1(\operatorname{reg} S - 1) \leq re - (1 + \cdots + r) = r(2e - r - 1)/2 \leq e(e-1)/2.$$

□

Lemma 1.2. *Assume that $S = R/I$ is a reduced ring of dimension at least two. Then*

$$h^1_S(-1) \leq \operatorname{adeg} I - e.$$

Proof. Since S is reduced, one may write $I = J \cap Q$, where J is the intersection of all associated primes of R/I of dimension at least 2, and Q is the intersection of all associated primes of R/I of dimension 1. By [HSV, Lemma 1] we have $h^1_{R/J}(-1) = 0$. Thus if $Q = R$, then $h^1_S(-1) = 0$. Assume that $Q \neq R$. Since $J \neq R$ and R/I has no embedded primes, $J+Q$ is an $\mathfrak{m}$-primary ideal, i.e. $\dim R/(J+Q) = 0$. The exact sequence

$$0 \to S \to R/J \oplus R/Q \to R/(J+Q) \to 0$$

implies

$$h^1_S(-1) = h^1_{R/J}(-1) + h^1_{R/Q}(-1) = h^1_{R/Q}(-1).$$

Note that $\deg R/Q = \operatorname{adeg} I - \operatorname{adeg} J \leq \operatorname{adeg} I - e$. Since R/Q is an one-dimensional ring, by the Grothendieck-Serre formula, we have

$$h^1_{R/Q}(-1) = \deg R/Q \leq \operatorname{adeg} I - e.$$

□

The proof of Theorem 0.2 is proceeded by induction. The next two lemmas allow us to do induction. The first one is concerning the behavior of the arithmetic degree by hyperplane section. It is more subtle than the usual degree, see [MVY]. However we have

Lemma 1.3. *Let $S = R/I$ be an arbitrary ring of dimension at least two and positive depth. Assume that x_n is chosen generically. Let $T = R/((I, x_n) : \mathfrak{m}^\infty)$ and $r = \operatorname{reg} T$. Then:*

(i) $\operatorname{reg} T \leq \operatorname{reg} S$.
(ii) $\operatorname{adeg} T \leq \operatorname{adeg} S$.

Proof. (i) Since x_n is generic, it is a regular element on S we have

$$\operatorname{reg} T = \operatorname{reg}_1 S/x_nS \leq \operatorname{reg} S/x_nS = \operatorname{reg} S.$$

(ii) For an R-module M and $r \geq -1$, let

$$\operatorname{adeg}_r(M) = \sum_{\mathfrak{p}\in \operatorname{Ass}(M),\ \dim R/\mathfrak{p}=r+1} \ell(H^0_{\mathfrak{m}_\mathfrak{p}}(M_\mathfrak{p})e(\mathfrak{p})$$

(see [BM, Definition 3.4]). Since x_n is generic, by the prime avoidance lemma, we may assume

$$x_n \notin \cup\{\mathfrak{p};\ \mathfrak{m} \neq \mathfrak{p} \in \operatorname{Ass}(S) \cup_{j\geq 1} \operatorname{Ass}(\operatorname{Ext}^{n-j}_R(S, R))\}.$$

By [MVY, Corollary 2.5] it follows that

$$\operatorname{adeg}_{r-1}(T) = \operatorname{adeg}_r(S) \ \text{ for all } r \geq 1.$$

Since S and T have no zero-dimensional component, we get

$$\begin{aligned}\operatorname{adeg} T &= \operatorname{adeg}_0(T) + \cdots + \operatorname{adeg}_{d-1}(T)\\ &= \operatorname{adeg}_1(S) + \cdots + \operatorname{adeg}_d(S) \le \operatorname{adeg} S.\end{aligned}$$

□

The first three statements of the next lemma is contained in [K, Proposition 1.4] (cf. also [RTV1, Theorem 1.4] and [RTV2, Theorem 1.3]). In order to make the paper more self-contained, we give here a sketch of the proof. The proof of (iii) here is also simpler.

Lemma 1.4. *Assume that $S = R/I$ is a reduced ring of dimension at least two. Assume that x_n is chosen generically. Let $T = R/((I, x_n) : \mathfrak{m}^\infty)$ and $r = \operatorname{reg} T$. Then T is also a reduced ring and we have*

(i) $\operatorname{reg}_2(S) \le r$ *(see the definition in (1)).*
(ii) $h^1_S(t) \ge h^1_S(t+1)$ *for all $t \ge r-1$.*
(iii) $\operatorname{reg} S \le r + h^1_S(r-1)$.
(iv) $h^1_S(t) \le h^1_T(0) + \cdots + h^1_T(t) + \operatorname{adeg} I - e$, *for all $t \ge 0$.*

Proof. Note that T can be considered as the homogeneous coordinate ring of a generic hyperplane section of the scheme $\operatorname{Proj}(S)$. Since K is an infinite ring and x_n is generic, by Bertini's theorem [FOV, Corollary 3.4.14] it follows that T is reduced.

The long exact sequence

$$\begin{aligned}0 \to H^0_{\mathfrak{m}}(S/x_nS)_t \to H^1_{\mathfrak{m}}(S)_{t-1} \to H^1_{\mathfrak{m}}(S)_t \xrightarrow{\varphi_t} H^1_{\mathfrak{m}}(S/x_nS)_t = H^1_{\mathfrak{m}}(T)_t \quad (3)\\ \to H^2_{\mathfrak{m}}(S)_{t-1} \to H^2_{\mathfrak{m}}(S)_t \to \cdots\end{aligned}$$

implies (i) and the short exact sequence

$$0 \to H^0_{\mathfrak{m}}(S/x_nS)_t \to H^1_{\mathfrak{m}}(S)_{t-1} \to H^1_{\mathfrak{m}}(S)_t \to 0$$

for all $t \ge r$. This yields (ii). If $h^1_S(t_0-1) \ge h^1_S(t_0)$ for some $t_0 \ge r+1$, we would have $h^0_{S/x_nS}(t_0) = 0$. Since $\operatorname{reg}_1(S/x_nS) = \operatorname{reg} T = r$, it then implies that $\operatorname{reg}(S/x_nS) \le t_0$. Hence $h^1_S(t_0) = h^1_S(t_0+1) = \cdots = 0$. Therefore $h^1_S(t)$ is strictly decreasing to zero when $t \ge r$, which implies (iii).

It remains to show (iv). From the exact sequence (3) we have

$$h^1_S(u) - h^1_S(u-1) = \ell(\operatorname{Im}(\varphi_u)) - h^0_{S/x_nS}(u) \le h^1_T(u)$$

for all $u \in \mathbb{Z}$. Summarizing these inequalities and using Lemma 1.2 we get

$$\begin{aligned}h^1_S(t) &\le h^1_T(0) + \cdots + h^1_T(t) + h^1_S(-1)\\ &\le h^1_T(0) + \cdots + h^1_T(t) + \operatorname{adeg} I - e\end{aligned}$$

for all $t \ge 0$. □

Theorem 0.2 is a part of the following result. By estimating the Castelnuovo-Mumford regularity and $h^1_S(t)$ simultaneously, we are able to do induction. If $a \in \mathbb{R}$, we denote by $[a]$ the largest integer not exceeding a.

Theorem 1.5. *Assume that $S = R/I$ is a reduced ring of dimension at least two. Let*

$$m = \frac{e(e-1)}{2} + \operatorname{adeg} I.$$

Then

(i) $\operatorname{reg} S \le m^{2^{d-2}} - 1$.

(ii) *For all $t \ge 0$, we have $h^1_S(t) \le m^{2^{d-2}} - e \cdot m^{[2^{d-3}]}$.*

Proof. We may assume x_n to be generic and choose T as in the previous lemma. By that lemma T is a reduced ring. Set $r = \operatorname{reg} T$.

Let $d = 2$. In order to show (ii), by Lemma 1.4(ii), we may assume that $t \le r - 1$. Note that T is a Cohen-Macaulay ring and $e(T) = e$. Then Lemma 1.4(iv) and Lemma 1.1 yield:

$$\begin{aligned} h^1_S(t) &\le h^1_T(0) + \cdots + h^1_T(t) + \operatorname{adeg} I - e \\ &\le h^1_T(0) + \cdots + h^1_T(r-1) + \operatorname{adeg} I - e \\ &\le \tfrac{e(e-1)}{2} + \operatorname{adeg} I - e = m - e. \end{aligned}$$

Using this inequality and the fact that $r \le e - 1$ (since T is a Cohen-Macaulay ring), by Lemma 1.4(iii) we get

$$\operatorname{reg} S \le e - 1 + m - e = m - 1.$$

Thus the case $d = 2$ is proven.

Let $d \ge 3$. Since $\dim T = d - 1$, $e(T) = e$, and $\operatorname{adeg} T \le \operatorname{adeg} S$ (by Lemma 1.3(ii)), the induction hypothesis gives

$$r \le m^{2^{d-3}} - 1, \tag{4}$$

and for all $t \ge 0$

$$h^1_T(t) \le m^{2^{d-3}} - e \cdot m^{[2^{d-4}]} \le m^{2^{d-3}} - e. \tag{5}$$

In order to prove (ii), again by Lemma 1.4(ii), we may assume that $t \le r-1$. Then, by Lemma 1.4(iv), for all $t \ge 0$ we have

$$\begin{aligned} h^1_S(t) &\le h^1_T(0) + \cdots + h^1_T(t) + \operatorname{adeg} I - e \\ &\le r(m^{2^{d-3}} - e) + \operatorname{adeg} I - e \quad \text{(by (5))} \\ &\le (m^{2^{d-3}} - 1)(m^{2^{d-3}} - e) + \operatorname{adeg} I - e \quad \text{(by (4))} \\ &= m^{2^{d-2}} - e \cdot m^{2^{d-3}} - m^{2^{d-3}} + \operatorname{adeg} I \\ &\le m^{2^{d-2}} - e \cdot m^{2^{d-3}}. \end{aligned}$$

To prove (i) we use (ii) and Lemma 1.4(iii) :

$$\operatorname{reg} S \le r + h^1_S(r-1) \le m^{2^{d-3}} - 1 + m^{2^{d-2}} - e \cdot m^{2^{d-3}} \le m^{2^{d-2}} - 1.$$

□

If K is an algebraically closed field, then a result of Gruson, Lazarfeld and Peskine [GLP] yields a better bound for the case $d = 2$ as shown in the following statement.

Proposition 1.6. *Let K be an algebraically closed field. Assume that R/I is a reduced ring of dimension two. Then* $\operatorname{reg} I \le \operatorname{adeg} I$.

Proof. Write $I = J \cap Q$ as in the proof of Lemma 1.2. Then we have an exact sequence:

$$0 \to H^0_{\mathfrak{m}}(R/J+Q)_t \to H^1_{\mathfrak{m}}(R/I)_t \to H^1_{\mathfrak{m}}(R/J)_t \oplus H^1_{\mathfrak{m}}(R/Q)_t \to 0,$$

and

$$H^2_{\mathfrak{m}}(R/I)_t \cong H^2_{\mathfrak{m}}(R/J)_t \oplus H^2_{\mathfrak{m}}(R/Q)_t.$$

By [GLP, Theorem 1.1] (see also Remark on p. 497 there), $\operatorname{reg} R/J \le e - 1 \le \operatorname{adeg} I - 1$. So we may assume that $Q \ne R$. Since R/Q is one-dimensional and reduced, it is a Cohen-Macaulay ring. Hence $\operatorname{reg} R/Q \le \operatorname{adeg} R/Q - 1 < \operatorname{adeg} I$. To complete the proof it suffices to show that

$$H^0_{\mathfrak{m}}(R/J+Q)_t = 0 \text{ for all } t \ge \operatorname{adeg} I - 1.$$

Since $\operatorname{reg} J \le e$, J is generated by elements of degree $\le e$. Hence one may choose an element $x \in J$ of degree e such that x does not belong to any prime in Q, i.e. x is a regular element on R/Q. Then we have

$$\operatorname{reg} R/(Q,x) = \operatorname{reg} R/Q + e - 1 \le \deg R/Q - 1 + e - 1 = \operatorname{adeg} I - 2.$$

Since $R/(Q,x)$ is a zero-dimensional ring, this means $(R/(Q,x))_t = 0$ for all $t \ge \operatorname{adeg} I - 1$. Using the inequality $\ell((R/J+Q)_t \le \ell((R/(Q,x))_t$ we get $H^0_{\mathfrak{m}}(R/J+Q)_t = (R/J+Q)_t = 0$ for all $t \ge \operatorname{adeg} I - 1$, as required. □

However we cannot use the above proposition for induction, because $h^1_S(t)$ can be much bigger than $\operatorname{reg} S$, i.e. a similar statement to Theorem 1.5(ii) does not work if were m taken to be $\operatorname{adeg} S$.

Example 1.7. Given $e \ge 6$ and $S = K[x^e, x^{e-1}y, xy^{e-1}, y^e]$. Then for $0 \le t \le e-2$ one can show that $h^1_S(t) = te + 1 - (t+1)^2$, while $\operatorname{reg} S = e-2$.

Theorem 0.2 does not hold if the ring R/I is not reduced.

Example 1.8. (see [V, Example 9.3.1]) Let $S = K[x,y,u,v]/((x,y)^2, xu^t + yv^t)$, $t \ge 1$. Then $\operatorname{adeg} S = e = 1$, while $\operatorname{reg} S = t$ can be arbitrarily large.

2. Bounds in terms of degrees of defining equations

In this section we study arbitrary homogeneous ideals. We will always write the degrees of polynomials in a minimal homogeneous basis of I in a decreasing sequence

$$\Delta := \delta_1 \ge \delta_2 \ge \cdots$$

and assume $\Delta \geq 2$. As mentioned in the introduction, G. Caviglia and E. Sbarra already proved that

$$\operatorname{reg} I \leq (\Delta^c + \Delta c - c + 1)^{2^{d-1}}$$

(see [CS, Corollary 2.6]). The purpose of this section is to prove the following theorem which is a slight improvement of the above result.

Theorem 2.1. *Let K be an arbitrary field and I be an arbitrary homogeneous ideal of dimension $d \geq 1$. Then*

$$\operatorname{reg} I \leq (\delta_1 \cdots \delta_c + \delta_1 + \cdots + \delta_c - c)^{2^{d-1}} < (2\Delta^c)^{2^{d-1}}.$$

The proof of [CS] uses properties of Borel-fixed ideals. The proof here is completely different and simpler than in [CS]. The main idea of the proof is similar to that of Theorem 0.2. If it is necessary, changing K by $K(u)$, we may assume that K is an infinite field, as usual. We need some technical lemmas. For short, set

$$\sigma = \delta_1 + \cdots + \delta_c - c \ \text{ and } \ \pi = \delta_1 \cdots \delta_c.$$

If $S = R/I$ we also write $\Delta = \Delta(S),\ \delta_1 = \delta_1(S), ...$ to emphasize their dependence on S (or I).The following result is well-known

Lemma 2.2. *If $\dim S = 0$, then $\operatorname{reg} S \leq \sigma$.*

The next result is a special case of [HH, Lemma 3]

Lemma 2.3. *Assume that $d = 1$. Then for all $t \geq 1$, $h^0_S(t) \leq \pi - 1$.*

Recall that an element $x \in \mathfrak{m}$ is called *filter regular* if $0 : \mathfrak{m}^\infty$ is of finite length.

Lemma 2.4. *Assume that $\dim S \geq 1$ and $x = x_n$ is a filter regular element on S. Let $T = S/xS$ and $r \geq \max\{\operatorname{reg} T,\ \Delta - 1\}$. Then*

(i) $\operatorname{reg}_1(S) \leq r$ *(see the definition in (1)).*
(ii) $h^0_S(t) \geq h^0_S(t+1)$ *for all $t \geq r$.*
(iii) $\operatorname{reg} S \leq r + h^0_S(r)$.
(iv) $h^0_S(t) \leq h^0_T(1) + \cdots + h^0_T(t)$, *for all $t \geq 1$.*

Proof. (i)-(iii) were shown in the proof of [BM, Proposition 3.8]. It follows from the following exact sequence

$$0 \to (0:x)_{t-1} \to H^0_{\mathfrak{m}}(S)_{t-1} \to H^0_{\mathfrak{m}}(S)_t \xrightarrow{\varphi_t} H^0_{\mathfrak{m}}(T)_t \to H^1_{\mathfrak{m}}(S)_{t-1} \to H^1_{\mathfrak{m}}(S)_t \to \cdots$$

For (iii) we need also the assumption $r \geq \Delta - 1$ in order to apply the regularity criterion of [BS, Theorem 1.10].

From the above exact sequence we have

$$h^0_S(u) - h^0_S(u-1) = \ell(\operatorname{Im}(\varphi_u)) - \ell((0:x)_{u-1}) \leq h^0_T(u)$$

for all $u \in \mathbb{Z}$. Since $h^0_S(0) = 0$, summarizing these inequalities we get (iv). □

Theorem 2.1 is a part of the following

Theorem 2.5. *Let $d \geq 1$. Then*

(i) $\operatorname{reg} S \leq (\sigma + \pi)^{2^{d-1}} - 1$.

(ii) *For all $t \geq 1$, we have $h^0_S(t) \leq (\sigma + \pi)^{2^{d-1}} - \sigma \cdot (\sigma + \pi)^{[2^{d-2}]}$.*

Proof. Keep the notation of Lemma 2.4. Let I' denote the image of I in $K[x_1, ..., x_n]/(x_n) \cong K[x_1, ..., x_{n-1}] =: R'$. Then $T \cong R'/I'$ and it is clear that $\sigma(I') \leq \sigma$ and $\pi(I') \leq \pi$.

First let $d = 1$. With the above remark, Lemma 2.3 yields $h^0_S(t) \leq \pi - 1$ for all $t \geq 1$. Thus (ii) holds. By Lemma 2.2, $\operatorname{reg} T \leq \sigma$. Since $\sigma \geq \Delta - 1$, applying Lemma 2.4(iii) to $r = \sigma$ we get $\operatorname{reg} S \leq \sigma + h^0_S(\sigma) \leq \sigma + \pi - 1$. Thus the case $d = 1$ is proven.

Now let $d \geq 2$. With the remark at the beginning of the proof and by the induction hypothesis we may assume that

$$\operatorname{reg} T \leq (\sigma + \pi)^{2^{d-2}} - 1,$$

and

$$h^0_T(t) \leq (\sigma + \pi)^{2^{d-2}} - \sigma \cdot (\sigma + \pi)^{[2^{d-3}]} \leq (\sigma + \pi)^{2^{d-2}} - \sigma.$$

Let $r = (\sigma + \pi)^{2^{d-2}} - 1$. Obviously $r \geq \Delta$. In order to prove (i), by Lemma 2.4(ii), we may assume that $t \leq r$. Then, by Lemma 2.4(ii) and (iv) and the induction hypothesis, we have

$$h^0_S(t) \leq r h^0_T(t) \leq [(\sigma+\pi)^{2^{d-2}} - 1][(\sigma+\pi)^{2^{d-2}} - \sigma] < (\sigma+\pi)^{2^{d-1}} - \sigma(\sigma+\pi)^{2^{d-2}}.$$

Thus (ii) is proven. Using this and Lemma 2.4(iii) we immediately get (i). □

Remark 2.6. The bound in Theorem 2.1 is nearly the best possible. It was shown that there is an ideal I, due to Mayr and Meyer, generated by $10n-6$ forms of degree at most 4 in $10n+1$ variables such that $\operatorname{reg}(I) > 4^{2^{n-1}} + 1$ (see, e.g., [BM, Example 3.9 and Proposition 3.11]).

3. Hilbert cohomological functions

In this section we give a bound on $h^i_S(t)$. First, we do this for Borel-fixed ideals. We need some notation and results from [HPV]. Let $I \neq 0$ be a monomial ideal. Denote by $G(I)$ the unique set of monomial generators of I. For a monomial u, let $m(u)$ be the maximal index of a variable appeared in u. Set

$$m(I) = \max\{m(u);\ u \in G(I)\}.$$

Recursively we define an ascending chain of monomial ideals

$$I = I_0 \subset I_1 \subset \cdots \subset I_{l+1} = R$$

as follows: let $I_0 = I$. Suppose I_j is already defined. If $I_j = R$, then the chain ends. Otherwise, let $n_j = m(I_j)$ and set

$$I_{j+1} = I_j : x_{n_j}^{\infty} := \cup_{k=1}^{\infty} I_j : x_{n_j}^k.$$

A stable ideal under the action of upper triangle matrices is called *Borel-fixed*. It is always a monomial ideal. If I is a Borel-fixed ideal, then $(x_1, ..., x_c)$ is the unique minimal associated prime of R/I (see [E, Corollary 15.25]). Hence in this case $n \geq n_0 > n_1 > \cdots > n_l = c$. For $j = 0, ..., l$, let $J_j \subset K[x_1, ..., x_{n_j}]$ be the monomial ideal with $G(I_j) = G(J_j)$. Denote by

$$J_j^{sat} = J_j : (x_1, ..., x_{n_j})^{\infty}$$

the *saturation* of J_j. Then by [HPV, Corollary 2.6] and local duality we have

Lemma 3.1. *Let $I \neq 0$ be a Borel-fixed ideal. Then $H_{\mathfrak{m}}^j(S) = 0$ if $j \notin \{n-n_0, ..., n-n_l\}$, and we have an isomorphism of $\mathbb{Z}$-graded R-modules:*

$$H_{\mathfrak{m}}^j(S) \cong (J_i^{sat}/J_i)[x_{n_i+1}^{-1}, ..., x_n^{-1}],$$

if $j = n - n_i$ for some $i = 0, ..., l$.

In the sequel, for a Borel-fixed ideal I let us denote

$$B := B(I) = \ell(R/(I, x_{c+1}, ..., x_n)) \tag{6}$$

For short, set $e = \deg(I)$. Note that $B \geq e$.

Lemma 3.2. *Let $I \neq 0$ be a Borel-fixed ideal. Then*
 (i) *$\ell(J_l^{sat}/J_l) = e$.*
 (ii) *For $i < l$ and all $t \geq 0$ we have*

$$\ell([J_i^{sat}/J_i]_t) \leq (B-1)\binom{t + n_i - c - 2}{n_i - c - 1}.$$

Proof. Let $M = J_i^{sat}/J_i$ and $R' = K[x_1, ..., x_c]$. Since $(x_1, ..., x_c)$ is the unique minimal associated prime of R/I, by the construction we have $I \subseteq I_i \subseteq (x_1, ..., x_c)$.

Let $i = l$. We have $J_l^{sat} = R'$ and $J_l = I : (x_{c+1}, ..., x_n)^{\infty}$. Hence

$$\ell(M) = \ell(R'/I : (x_{c+1}, ..., x_n)^{\infty}) = \ell((R/I)_{(x_{c+1},...,x_n)}) = e.$$

Let $i < l$. Set $R" = K[x_1, ..., x_{n_i}]$. By the definition $J_i = G(I_i)R"$. Hence $x_{c+1}, ..., x_{n_i}$ is a s.o.p. of $R"/J_i$, and $I \cap R' \subseteq J_i \cap R'$. This implies

$$\ell\left(\frac{R"}{(J_i, x_{c+1}, ..., x_{n_i})}\right) = \ell\left(\frac{R'}{J_i \cap R'}\right) \leq \ell\left(\frac{R'}{I \cap R'}\right) = \ell\left(\frac{R}{(I, x_{c+1}, ..., x_n)}\right) = B. \tag{7}$$

On the other side, the inclusion $J_i^{sat} \subseteq (x_1, ..., x_c)$ yields

$$\begin{aligned} \ell(M_t) &\leq \ell(\frac{(x_1, ..., x_c)R"}{J_i}) = \\ &= \ell(\frac{R"}{J_i}) - \ell(\frac{R"}{(x_1, ..., x_c)R"} = \ell(\frac{R"}{J_i}) - \binom{t + n_i - c - 1}{n_i - c - 1}. \end{aligned}$$

By [RVV, Proposition 2.4] and (7) we have

$$\begin{aligned} \ell(\tfrac{R"}{J_i}) &\leq (\ell(\tfrac{R"}{(J_i, x_{c+1}, ..., x_{n_i})}) - 1)\tbinom{t+n_i-c-2}{n_i-c-1} + \tbinom{t+n_i-c-1}{n_i-c-1} \\ &\leq (B-1)\tbinom{t+n_i-c-2}{n_i-c-1} + \tbinom{t+n_i-c-1}{n_i-c-1}. \end{aligned}$$

Hence $\ell(M_t) \leq (B-1)\binom{t+n_i-c-2}{n_i-c-1}$. □

Lemma 3.3. *Let $I \neq 0$ be a Borel-fixed ideal and $S = R/I$. Then*

(i) $h_S^0(t) \leq (B-1)\binom{t+d-2}{d-1}$ *for all* $t \geq 0$.

(ii) *For* $1 \leq j \leq d-1$ *and* $t \leq \operatorname{reg} S$*:*

$$h_S^j(t) \leq (B-1)\binom{\operatorname{reg} S + d - j - 2}{d - j - 1}\binom{\operatorname{reg} S - t}{j}.$$

(iii) *For* $t < \operatorname{reg} S$*:*

$$h_S^d(t) \leq e\binom{\operatorname{reg} S - t - 1}{d - 1} \leq B\binom{\operatorname{reg} S - t - 1}{d - 1}.$$

Proof. By virtue of Lemma 3.1, (i) is a special case of Lemma 3.2 (when $i = 0$). Let $j \geq 1$. By Lemma 3.1 we may assume that $j = n - n_i$ for some $i > 0$. Let $M = J_i^{sat}/J_i$. Lemma 3.1 implies

$$\begin{aligned} h_S^j(t) &= \sum_{u=1}^{\operatorname{end}(M)} \ell(M_u)\binom{u - j - t + j - 1}{j - 1} & (8) \\ &\leq [\max_{1\leq u\leq \operatorname{end}(M)} \ell(M_u)] \sum_{v=1}^{\operatorname{end}(M)-j-t} \binom{v + j - 1}{j} \\ &= [\max_{1\leq u\leq \operatorname{end}(M)} \ell(M_u)]\binom{\operatorname{end}(M) - t}{j}. & (9) \end{aligned}$$

(In the above calculation we set $\binom{a}{b} = 0$ if $b \geq 0$ and $a < b$.) Moreover, again by Lemma 3.1, $\operatorname{end}(M) = a_j(S) + j \leq \operatorname{reg} S$ (see also [HPV, Corollary 2.7]). Since $n_i - c = n - j - c = d - j$, Lemma 3.2(ii) yields

$$\begin{aligned} \max_{1\leq u\leq \operatorname{end}(M)} \ell(M_u) &\leq (B-1) \max_{1\leq u\leq \operatorname{reg} S} \binom{u + d - j - 2}{d - j - 1} \\ &= (B-1)\binom{\operatorname{reg} S + d - j - 2}{d - j - 1}. \end{aligned}$$

From this and (9) we get (ii).

Let $j = d$. Then $n_i = c$. From (8) we have

$$\begin{array}{ll} h^d_S(t) & \leq [\max_{1\leq u\leq \mathrm{end}(M)} \binom{u-t-1}{d-1}] \cdot \sum_{u=1}^{\mathrm{end}(M)} \ell(M_u) \\ & \leq \binom{\mathrm{reg}\, S-t-1}{d-1} \cdot \ell(M) \\ & = e\binom{\mathrm{reg}\, S-t-1}{d-1} \text{ (by Lemma 3.2(i))}. \end{array}$$

□

Now we can bound the Hilbert cohomological functions of an arbitrary homogeneous ideal I. Recall that the defining degrees of I are written in a decreasing sequence

$$\Delta := \delta_1 \geq \delta_2 \geq \cdots ,$$

and assume $\Delta \geq 2$.

In the proof of the following theorem, a result of Vasconcelos on the reduction number plays an essential role.

Theorem 3.4. *Let I be an arbitrary homogeneous ideal of R and $S = R/I$. Let*

$$b = \min\{\delta_1 \cdots \delta_c,\ (\mathrm{adeg}\, I)^c\}.$$

Then

(i) $h^0_S(t) \leq (b-1)\binom{t+d-2}{d-1}$ *for all* $t \geq 0$.

(ii) *For* $1 \leq j \leq d-1$ *and* $t \leq \mathrm{reg}\, S$*:*

$$h^j_S(t) \leq (b-1)\binom{\mathrm{reg}\, S+d-j-2}{d-j-1}\binom{\mathrm{reg}\, S-t}{j}.$$

(iii) *For* $t < \mathrm{reg}\, S$*:*

$$h^d_S(t) \leq e\binom{\mathrm{reg}\, S-t-1}{d-1} \leq b\binom{\mathrm{reg}\, S-t-1}{d-1}.$$

Proof. Let $\mathrm{Gin}\, I$ denote the generic initial ideal of I with respect to the reverse lexicographic order. Then $\mathrm{Gin}\, I$ is a Borel-fixed ideal. Moreover we may assume that the coordinates $x_1, .., x_n$ are chosen generically. By [BS, Lemma 2.2 and Theorem 2.4] we have

$$\ell(R/(I, x_{c+1}, ..., x_n)) = \ell(R/(\mathrm{Gin}\, I, x_{c+1}, ..., x_n)),$$

and

$$\mathrm{reg}(R/I) = \mathrm{reg}(R/\,\mathrm{Gin}\, I).$$

By Macaulay's theorem: $e(R/I) = e(R/\,\mathrm{Gin}\, I)$. Moreover, by [S, Theorem 2.4]

$$h^i_{R/I}(t) \leq h^i_{R/\,\mathrm{Gin}\, I}(t)$$

for all $i \geq 0$ and $t \in \mathbb{Z}$. Hence, the theorem immediately follows from the previous lemma if we can show that

$$B := \ell(R/(I, x_{c+1}, ..., x_n)) \leq b.$$

a) Let I' denote the image of I in $R' = R/(x_{c+1}, ..., x_n) \cong K[x_1, ..., x_c]$. It is a $(x_1, ..., x_c)$-primary ideal. Since I can be generated by elements of degrees $d_1 \leq \delta_1, d_2 \leq \delta_2, ...,$ I' contains a regular sequence consisting of forms $f_1, ..., f_c$ of degrees $d_1' \leq d_1 \leq \delta_1, ..., d_c' \leq d_c \leq \delta_c$. Hence

$$B = \ell(R'/I') \leq \ell(R'/(f_1, ..., f_c)) = d_1' \cdots d_c' \leq \delta_1 \cdots \delta_c.$$

b) Since $x_{c+1}, ..., x_n$ is a s.o.p. of $R/\operatorname{Gin}(I)$, it is also a s.o.p. of R/I. Hence it is a minimal reduction of the algebra R/I. By [V, Theorem 9.3.4]

$$x_i^{\operatorname{adeg}(I)} \in (I, x_{c+1}, ..., x_n), \quad \text{for all } i \geq 1.$$

This means $x_1^{\operatorname{adeg}(I)}, ..., x_c^{\operatorname{adeg}(I)}$ form a regular sequence in I'. The above argument gives $B \leq (\operatorname{adeg} I)^c$. ☐

Remark 3.5. i) In the above theorem we may replace b by $(\operatorname{reg} I)^c$ in order to get a bound for $h_S^j(t)$ which depends only on $\operatorname{reg} I$ and d, c. However, such a replacement is not useful in the application.

ii) Hilbert cohomological functions are of reverse polynomial type, i.e. for each $i \geq 0$ there is a polynomial $p_S^i(t)$ such that $h_S^i(t) = p_S^i(t)$ for all $t \ll 0$ (see [BrS, Theorem 17.1.9]). The number

$$\nu_S^i = \min\{t \in \mathbb{Z};\ h_S^i(t) \neq p_S^i(t)\} - 1$$

is called i-th cohomological postulation number of S (see [BrL]). Thus, if $H_\mathfrak{m}^i(S)$ is of finite length, then all graded components $H_\mathfrak{m}^i(S)_t$ vanish below ν_S^i. Brodmann and Lashgari proved that all $-\nu_S^i,\ i \leq d$, can be bounded by a polynomial (of huge degree) in the numbers $h_S^1(-1), ..., h_S^d(-d)$ (see [BrL, Theorem 4.6]). Combining their result with Theorem 3.4 we see that $-\nu_S^i$ can be bounded by a polynomial in $\operatorname{reg} S$. Thus, the number of "irregular" negative components of local cohomology modules is governed by the Castelnuovo-Mumford regularity.

4. Hilbert coefficients

Write the Hilbert polynomial in the form:

$$P_S(t) = e_0\binom{t+d-1}{d-1} - e_1\binom{t+d-2}{d-2} + \cdots + (-1)^{d-1}e_{d-1}.$$

Then $e_0, e_1, ..., e_{d-1}$ are called *Hilbert coefficients* of S. Note that $e_0 = e$. Sometimes we also write $e_i = e_i(S)$ to emphasize its dependence on S.

First we estimate $|e_i|$ in terms of the arithmetic degree. For the application later, the following result is formulated in a rather technical way.

Theorem 4.1. *Let I be an arbitrary homogeneous ideal. Assume that $x_{c+1},...,x_n$ are chosen generically. Let $T_d = R/(I : \mathfrak{m}^\infty)$, $T_{d-1}= R/((I,x_n):\mathfrak{m}^\infty),....$ Then*

(i) $|e_1| \leq (\operatorname{adeg} I)^c(\operatorname{reg} T_2 + 1) \leq (\operatorname{adeg} I)^c \operatorname{reg} I$.

(ii) *For $i \geq 2$,* $|e_i| \leq \frac{3}{2}(\operatorname{adeg} I)^c(\operatorname{reg} T_{i+1}+1)^i \leq \frac{3}{2}(\operatorname{adeg} I)^c(\operatorname{reg} I)^i$.

Proof. The second inequalities in both (i) and (ii) follow from Lemma 1.3(i). Let us prove the first ones. Set $T = T_d$. Let $H_T(t)$ denote the Hilbert function of T. Since S and T have the same Hilbert polynomial, $e_i = e_i(T)$ for all $i \geq 0$. From the Grothendieck-Serre formula

$$P_T(t) - H_T(t) = \sum_{i=0}^{d}(-1)^{i+1}h^i_T(t),$$

we get (setting $t = -1$):

$$(-1)^{d-1}e_{d-1} = C - D,$$

where

$$C = h^1_T(-1) + h^3_T(-1)\ \cdots,$$

and

$$D = h^2_T(-1) + h^4_T(-1)\ \cdots.$$

Hence

$$|e_{d-1}| \leq \max\{C,\ D\}.$$

For short, set $b = (\operatorname{adeg} I)^c$. If $d = 2$, then by Theorem 3.4 we have $C = h^1_T(-1) \leq (b-1)(\operatorname{reg} T + 1)$ and $D = h^2_T(-1) \leq b\cdot \operatorname{reg} T$. Therefore $|e_1| \leq b \cdot (\operatorname{reg} T_2 + 1)$.

Let $d \geq 3$. Since x_n is generic, it is a regular element on T. Since $P_{T_{d-1}}(t) = P_{T/x_nT}(t)$, we have $e_i = e_i(T) = e_i(T_{d-1})$ for all $i \leq d-2$. The corresponding sequence of rings constructed for T_{d-1} as above are exactly the rings $T_{d-1}, T_{d-2}, ..., T_1$. By Lemma 1.3(ii), $\operatorname{adeg} T_{d-1} \leq \operatorname{adeg} T$. Hence, by induction hypothesis, it remains to prove (ii) for $i = d-1$. Note that

$$\binom{v+u-1}{u} \leq v^u, \quad \text{and} \quad \binom{v+1}{u} \leq (v+1)\frac{v^{u-1}}{u!}.$$

Let $r = \operatorname{reg} T$. If $d = 2k+1$, where $k \geq 1$, then Theorem 3.4 yields

$$\begin{aligned} C &\leq (b-1)(r+1)r^{d-2}\{1+\tfrac{1}{3!}+\cdots+\tfrac{1}{(2k-1)!}\} + b\tfrac{r^{d-1}}{(2k)!}\\ &\leq b(r+1)r^{d-2}\{1+\tfrac{1}{3!}+\cdots+\tfrac{1}{(2k-1)!}+\tfrac{1}{(2k)!}\}\\ &\leq \tfrac{3}{2}b(r+1)^{d-1}, \end{aligned}$$

and

$$D \leq (b-1)(r+1)r^{d-2}\{\frac{1}{2!}+\cdots+\frac{1}{(2k)!}\} \leq (b-1)(r+1)^{d-1}.$$

Hence $|e_{d-1}| \leq \frac{3}{2}b(r+1)^{d-1}$.

The inequality in the case $d = 2k,\ k \geq 2$ can be shown similarly. □

Remark 4.2. In the above proof, if $h^1_T(-1) = 0$, then $C \leq (\operatorname{adeg} I)^c \cdot (r+1)^{d-1}$. Hence, if $h^1_{T_{i+1}}(-1) = 0$, then

$$|e_i| \leq (\operatorname{adeg} I)^c(\operatorname{reg} T_{i+1} + 1)^i \leq (\operatorname{adeg} I)^c(\operatorname{reg} I)^i.$$

Note that $\dim T_{i+1} = i + 1$. Combining Theorem 4.1 and Theorem 0.2 we get

Proposition 4.3. *Let S be a reduced ring of dimension at least two. Then*

(i) $|e_1| \leq (\operatorname{adeg} S)^c(\frac{e(e-1)}{2} + \operatorname{adeg} S)$.

(ii) $|e_i| \leq \frac{3}{2}(\operatorname{adeg} S)^c(\frac{e(e-1)}{2} + \operatorname{adeg} S)^{i2^{i-1}}$ *if* $i \geq 2$.

For further comments, let us recall

Eisenbud-Goto conjecture [EG]: Let K be an algebraically closed field. If I is a prime ideal containing no linear form, then $\operatorname{reg} R/I \leq e - c$.

If this conjecture holds, then Theorem 4.1 would give a much better bound for $|e_i|$ in terms of e for all domains. We give here a partial result:

Corollary 4.4. *Assume that K is an algebraically closed field of characteristic 0 and* $\operatorname{Proj}(R/I)$ *is a reduced and pure-dimensional smooth subscheme in $\mathbb{P}^{n-1}$. Then for all $i \geq 1$ we have*

$$|e_i| < (i+2)^i e^{c+i}.$$

Proof. By Bertini's theorems (see [FOV, Corollary 3.4.6 and Corollary 3.4.14]) we may assume that all $\operatorname{Proj}(T_i)$ are reduced and pure-dimensional smooth subschemes. By Mumford's bound: $\operatorname{reg} T_{i+1} \leq (i+2)(e-2)+1$ (see [BM, Theorem 3.12(ii)]). Moreover, in this case $h^1_{T_{i+1}}(-1) = 0$ for all $i \geq 1$ and $\operatorname{adeg} I = e$. Hence, by Theorem 4.1 and Remark 4.2, we get

$$|e_i| \leq e^c((i+2)(e-2)+2)^i < (i+2)^i e^{c+i}.$$

□

Remark 4.5. Let S be a reduced ring of dimension at least two.

i) It is known that for any K-algebra S, $e_1 \leq e(e-1)/2$ (see [Bl, Remark 3.10]). Hence in the statement (i) of Proposition 4.3 only the following inequality is new: $e_1 \geq -(\operatorname{adeg} S)^c(\frac{e(e-1)}{2} + \operatorname{adeg} S)$.

ii) If the Eisenbud-Goto conjecture holds, then following the proof of Proposition 1.6 one can show that $\operatorname{reg} S \leq \operatorname{adeg} S$ in any dimension. From that it would imply $|e_i| \leq \frac{3}{2}(\operatorname{adeg} S)^{c+i}$. This also indicates that bounds in Theorem 0.2 and Proposition 4.3 are far from being sharp.

iii) There is a bound on $|e_i|$ in terms of the so-called homological degree which also holds for any standard graded algebra over an artinian ring, see [RVV, Theorem 4.3]. However the homological degree is very big.

Now we estimate $|e_i|$ by mean of the defining degrees. Recall that homogeneous elements $y_1, ..., y_m$ of S form a *filter regular sequence* if $[(y_1, ..., y_{i-1}) : y_i]_t = (y_1, ..., y_{i-1})_t$ for all $t \gg 0$ and $i = 0, ..., m$. On the other words, y_i is a filter regular element on $S/(y_1, ..., y_{i-1})S$.

Theorem 4.6. *Let I be an arbitrary homogeneous ideal. Assume that $d \geq 2$ and $x_{c+1}, ..., x_n$ is a filter regular sequence on S. Let $S_d = S$, $S_{d-1} = S/x_nS_d$, ... Set $\pi = \delta_1 \cdots \delta_c$. Then*

(i) $|e_1| \leq \pi \cdot (\operatorname{reg} S_2 + 1) \leq \pi \cdot \operatorname{reg} I$.

(ii) *For $i \geq 2$, $|e_i| \leq \frac{3}{2}\pi \cdot (\operatorname{reg} S_{i+1} + 1)^i \leq \frac{3}{2}\pi \cdot (\operatorname{reg} I)^i$.*

Proof. The proof is similar to that of Theorem 4.1 after noticing that $\operatorname{reg} S_i \leq \operatorname{reg} S_{i+1}$ and $\delta_1(S_i) \leq \delta_1(S_{i+1}), ..., \delta_c(S_i) \leq \delta_c(S_{i+1})$ for all $i \geq 2$. □

Combining it with Theorem 2.1 we immediately get

Proposition 4.7. *Let $d \geq 1$. Then*

(i) $|e_1| \leq \pi(\sigma + \pi)^2$.

(ii) *For all $i \geq 2$, we have $|e_i| \leq \frac{3}{2}\pi(\sigma + \pi)^{i2^i}$.*

In particular $|e_i| < (2\Delta^c)^{1+i2^i}$ for all $i \geq 1$.

Sometimes Theorem 4.6 also gives much better bounds for $|e_i|$. For example, using [BEL] one immediately gets that $|e_i| \leq c^i\Delta^{c+i}$, provided that $\operatorname{Proj}(R/I)$ is a reduced and pure-dimensional smooth subscheme. Another case is

Corollary 4.8. *Let I be an ideal generated by monomials of degree at most Δ in n variables. Then for all $i \geq 1$ we have*

$$|e_i| \leq \frac{3}{2} \min\{(\operatorname{adeg} I)^{c+i},\ n^i\Delta^{c+i}\}.$$

Proof. By [HT, Theorem 1.1], $\operatorname{reg} I \leq \operatorname{adeg} I$ and by [HT, Theorem 1.2], $\operatorname{reg} I \leq n\Delta$. Hence the statement follows from theorems 4.1 and 4.6. □

The following example shows that the bounds in Theorem 4.6 and Corollary 4.8 are rather good.

Example 4.9. Let $n > c + 1$ and

$$I = (x_1, ..., x_c) \cap (x_1^r, ..., x_c^r, x_{c+1}^{r-1}, ..., x_{n-1}^{r-1}).$$

Using the exact sequence

$$0 \to R/I \to R/P \oplus R/J \to R/(P + J) \to 0,$$

where $P = (x_1, ..., x_c)$ and $J = (x_1^r, ..., x_c^r, x_{c+1}^{r-1}, ..., x_{n-1}^{r-1})$, one can check that

$$\operatorname{reg} R/I = n(r - 1) + 1 - d,$$

and

$$P_{R/I}(t) = \binom{t+d-1}{d-1} + [r^c(r-1)^{d-1} - (r-1)^{d-1}].$$

Hence $|e_{d-1}| = (r^c - 1)(r-1)^{d-1}$, while by Corollary 4.8 $|e_{d-1}| \le 3r^{c+d-1}n^{d-1}/2$, and by Theorem 4.6 $|e_{d-1}| \le 3r^c[(n-1)r-2n+c+3]^{d-1}/2$.

5. Finiteness of Hilbert functions

In this section we prove Theorem 0.1. We need some further preliminary results.

Lemma 5.1. *Let I be an arbitrary homogeneous ideal. Let*

$$b = \min\{\delta_1 \cdots \delta_c,\ (\operatorname{adeg} S)^c\}.$$

For all $t \ge 0$ we have

$$H_S(t) \le (b-1)\binom{t+d-2}{d-1} + \binom{t+d-1}{d-1}.$$

Proof. We may assume that $x_{c+1}, ..., x_n$ are chosen generically. In particular, $x_{c+1}, ..., x_n$ form a s.o.p. of S. Set $B = \ell(S/(x_{c+1}, ..., x_n)S)$. By [RVV, Proposition 2.4] for all $t \ge 0$ we have

$$H_S(t) \le (B-1)\binom{t+d-2}{d-1} + \binom{t+d-1}{d-1}.$$

As shown in the proof of Theorem 3.4, $B \le b$. Hence the lemma is proven. □

Lemma 5.2. *Assume that K is an algebraically closed field, I is an intersection of prime ideals and I contains no linear form. Then $c \le d(\operatorname{adeg} I - 1)$.*

Proof. By the assumption

$$I = \cap_{i=1}^s \mathfrak{p}_i,$$

where $\mathfrak{p}_i$ are prime ideals of height at least c. Since $s \le \operatorname{adeg} I$, the statement is derived from the following inequality

$$c \le \operatorname{adeg} I - s + (s-1)d.$$

We prove this inequality by induction on s. The case $s = 1$ is well known. Let $s > 1$. Put $J = \cap_{i=1}^{s-1}\mathfrak{p}_i$. Let a and b be the maximal number of independent linear forms contained in J and $\mathfrak{p}_s$, respectively. By induction we have $c - a \le \operatorname{adeg} J - (s-1) + (s-2)d$ and $c - b \le e(R/\mathfrak{p}_s) - 1$. Since $\operatorname{adeg} I = \operatorname{adeg} J + e(R/\mathfrak{p}_s)$, we get

$$2c \le \operatorname{adeg} I - s + (s-2)d + a + b.$$

If $a+b>n$ it would imply that there is a linear form in $J\cap\mathfrak{p}_s=I$, a contradiction. Hence $a+b\leq n=d+c$. The above inequality then yields $c\leq \operatorname{adeg} I - s+(s-1)d$. □

As mentioned in the introduction, Theorem 0.1 is an immediate consequence of Theorem 0.2, Lemma 5.2 and [RTV2, Theorem 2.3]. We give here a direct proof without the use of [RTV2].

Proof of Theorem 0.1. Without the loss of generality we may assume from the beginning that I contains no linear form. Note that $e\leq \operatorname{adeg} S$. Therefore, by Proposition 4.3 and Lemma 5.2, there are only finitely many Hilbert polynomials associated to reduced algebras such that $\operatorname{adeg} S\leq a$ and $\dim S\leq d$. By Lemmas 5.1 and 5.2, there are only finitely many choices for the initial values of Hilbert functions, while Theorem 0.2 says that for $t\geq (\frac{e(e-1)}{2}+\operatorname{adeg} S)^{2^{d-2}}$ each Hilbert function agrees with the corresponding Hilbert polynomial. This implies the finiteness of the number of Hilbert functions. □

Example 1.8 shows that without the assumption S being a reduced ring Theorem 0.1 does not hold.

Applying Proposition 4.7 and Theorem 2.1, as in the proof of Theorem 0.1, we get a similar finiteness result in terms of the defining degrees.

Corollary 5.3. *Given two numbers δ and n, there exist only finitely many Hilbert functions associated to homogeneous ideals generated by forms of degrees at most δ in at most n variables.*

6. Castelnuovo-Mumford regularity of initial ideals

In this last section we apply results in the previous sections to study the Castelnuovo-Mumford regularity of an initial ideal in I of I with respect to any given term order and coordinates. We even consider a much bigger class: the class of all ideals J having the same Hilbert function as I. Then one can easily bound $\operatorname{reg} J$ in terms of some data of I. This approach was initiated in [CM] and developed further in [HH]. Let us recall some notations. The Hilbert polynomial can be uniquely written in the form

$$P_{R/I}(t)=\binom{c_1+t}{t}+\binom{c_2+t-1}{t-1}+\cdots+\binom{c_s+t-s+1}{t-s+1},$$

where $c_1\geq c_2\geq\cdots\geq c_s\geq 0$ are integers (see, e.g., [V, Section B6]). For $0\leq i\leq d-1$ set

$$B_i=\sharp\{j;\ c_j\geq (d-1)-i\}.$$

Thus in the above notations, $s = B_{d-1}$ (for convenience, we set $B_{-1} = 0$). The following result easily follows from Gotzmann's regularity theorem:

Lemma 6.1. [HH, Lemma 5] *Let I, J be homogeneous ideals having the same Hilbert function. Then*

$$\operatorname{reg} J \leq \max\{\operatorname{reg} I,\ B_{d-1}\}.$$

Since we already know bounds for $\operatorname{reg}(I)$ (see Theorems 0.2 and 2.1), we have only to estimate B_{d-1}. For this purpose we need some relations between the invariants B_i just defined and the Hilbert coefficients which were given in [Bl, Proposition 3.9] (see also [CM, Lemma 1.5]).

Lemma 6.2. *For all $0 \leq j \leq d-1$ we have*

$$B_i = (-1)^i e_i + \binom{B_{i-1}+1}{2} - \binom{B_{i-2}+1}{3} + \cdots + (-1)^{i+1}\binom{B_0+1}{i+1}.$$

Note that $B_{d-1} \geq \cdots \geq B_0 = e$. In order to estimate B_j, we need the following combinatorial result.

Lemma 6.3. *Assume that*

$$|e_i| \leq M^{\alpha+i\beta 2^i} \quad \textit{for all } i \geq 0,$$

where $M \geq 2$ and $\alpha, \beta \geq 1$. Then for all $0 \leq j \leq d-1$ we have

$$B_j < M^{(\alpha+j\beta)2^j}.$$

Proof. We have $B_0 = e = e_0 \leq M^\alpha$ by the assumption. By Lemma 6.2 the following holds

$$B_1 = -e_1 + \binom{B_0+1}{2} \leq |e_1| + \frac{e(e+1)}{2} < M^{\alpha+2\beta} + M^{2\alpha} \leq M^{2(\alpha+\beta)},$$

(since $M \geq 2$).

Let $j \geq 2$. Assume that

$$B_{j-l} < M^{(\alpha+(j-l)\beta)2^{j-l}} \tag{10}$$

for all $l \geq 1$. Lemma 6.2 yields:

$$\begin{aligned} B_j &= (-1)^j e_j + \binom{B_{j-1}+1}{2} - \binom{B_{j-2}+1}{3} + \cdots + (-1)^{j-1}\binom{B_0+1}{j-1} \\ &\leq |e_j| + \binom{B_{j-1}+1}{2} + \binom{B_{j-3}+1}{4} + \cdots \end{aligned} \tag{11}$$

By (10), for all $l \geq 1$ we have

$$\binom{B_{j-l}+1}{l+1} \leq \frac{(B_{j-l}+1)^{l+1}}{(l+1)!} \leq \frac{(B_{j-l}+1)^{2^l}}{(l+1)!} \leq \frac{M^{(\alpha+j\beta)2^j}}{(l+1)!}. \tag{12}$$

From (11), (12) and the assumption $|e_j| \le M^{\alpha+j\beta 2^j}$ it follows that

$$\begin{array}{rl} B_j & \le M^{\alpha+j\beta 2^j} + M^{(\alpha+j\beta)2^j}\{\frac{1}{2!} + \frac{1}{4!}\ \cdots\} \\ & < M^{\alpha+j\beta 2^j} + \frac{2}{3}M^{(\alpha+j\beta)2^j} \\ & \le M^{(\alpha+j\beta)2^j}. \end{array}$$

□

By Macaulay's theorem $H_{R/\operatorname{in} I}(t) = H_{R/I}(t)$ for all $t \in \mathbb{Z}$. Hence, Theorem 0.3 stated in the introduction is a special case of the following result.

Theorem 6.4. *Let K be an arbitrary field. Let J be an arbitrary homogeneous ideal of $R = K[x_1, ..., x_n]$ such that $H_{R/J}(t) = H_{R/I}(t)$ for all t. Then*

(i) $\operatorname{reg}(J) < (2\Delta^c)^{d2^{d-1}}$.

(ii) *Moreover, if R/I is a reduced algebra, we also have*

$$\operatorname{reg}(J) < (\operatorname{adeg}(I))^{(n-1)2^{d-1}}.$$

Proof. Replacing K by $K(u)$, where u is a new indeterminate, one can assume that K is an infinite field as usual.

(i) By Proposition 4.7, $|e_i| \le (2\Delta^c)^{1+i2^i}$ for all $i \ge 0$. Applying Lemma 6.3 to $M = 2\Delta^c$, $\alpha = 1$, $\beta = 1$ and $j = d-1$, we get $B_{d-1} < (2\Delta^c)^{d2^{d-1}}$. Then (i) follows from Lemma 6.1 and Theorem 2.1.

(ii) For short, set $a = \operatorname{adeg} I$. Note that $a \ge e$ and

$$\frac{e(e-1)}{2} + a \le a^2. \tag{13}$$

Hence, by Proposition 4.3(i)

$$|e_1| \le a^{c+2}.$$

Let $i \ge 2$. Since $\Delta \ge 2$, $a \ge 2$. By Proposition 4.3(ii) and (13), we have

$$\begin{array}{rl} |e_i| & \le \frac{3}{2}a^c(\frac{a(a+1)}{2})^{i2^{i-1}} = a^c(\frac{a(a+1)}{2})^{i2^{i-1}-4}\cdot[\frac{3}{2}(\frac{a(a+1)}{2})^4] \\ & \le a^{c+2(i2^{i-1}-4)}a^8 = a^{c+i2^i}. \end{array}$$

Thus, applying Lemma 6.3 to $M = a$, $\alpha = c$, $\beta = 1$ and $j = d-1$, we get $B_{d-1} < a^{(n-1)2^{d-1}}$. By Lemma 6.1 and Theorem 0.2 this implies $\operatorname{reg} J < a^{(n-1)2^{d-1}}$. □

Note that if R/I is a Cohen-Macaulay ring of dimension $d \ge 2$ (but not necessarily reduced), then one can get a little bit better bound (see [HH, Theorem 9]):

$$\operatorname{reg} J \le e^{2^{d-1}}/2^{2^{d-2}}.$$

Example 6.5. Let I^{lex} denote the lex-segment ideal associated to the Hilbert function $H_{R/I}(t)$. This is the ideal generated by all first $H_I(m)$ monomials of degrees m with respect to the lexicographic order, when m runs through all positive integers. It has the same Hilbert function as I. If R/I is a Cohen-Macaulay ring of dimension $d \geq 2$, then from [CM, Theorem 2.5] it follows that $\operatorname{reg}(I^{lex}) = B_{d-1}$.

i) Let I be an ideal generated by a regular sequence consisting of forms of degrees $\delta_1 \geq \cdots \geq \delta_c$ such that $c \geq 2$ and $\delta_2 \geq 35$ ($d \geq 2$). It was shown in [HH, Example 13] that

$$\operatorname{reg}(I^{lex}) \geq 9\frac{\Delta^{c2^{d-1}}}{9^{2^{d-2}}}.$$

ii) Let S be a Veronesian embedding $K[y_1, ..., y_d]^{(p)}$, i.e. S_1 is generated by all monomials of degree p in the variables $y_1, ..., y_d$, where $d \geq 3$. This is a Cohen-Macaulay domain and $P_S(t) = \binom{pt+d-1}{d-1}$. Hence $\operatorname{adeg} S = e = p^{d-1}$ and $e_1 = dp^{d-2}(p-1)$. Let $p \geq 35$. Then $e_1 < e^2/36$ and $e \geq 35^2$. Let $S = K[x_1, ..., x_q]/I$, where $q = \binom{p+d-1}{d-1}$. By [HH, Proposition 12] we get

$$\operatorname{reg}(I^{lex}) \geq 9\frac{e^{2^{d-1}}}{9^{2^{d-2}}}.$$

This shows that the bound in the second part of Theorem 6.4 is very closed to be sharp too.

Since $\operatorname{reg}(\operatorname{in} I) \geq \operatorname{reg} I$, the ideals of Mayr and Meyer again show that the bound $(2\Delta^c)^{d2^{d-1}}$ of Theorem 0.3 is rather good (see Remark 2.6). We do not know whether one can construct a reduced algebra R/I such that there is a term order with $\operatorname{reg}(\operatorname{in} I)$ close to $(\operatorname{adeg}(I))^{(n-1)2^{d-1}}$.

Finally we would like to make the following remark: In the proof of Theorem 0.3 we use very rough estimation for $\operatorname{reg} I$ and $|e_i|$. It could suspect that if $\operatorname{reg} I$ and $|e_i|$ are small, then one could get a bound for $\operatorname{reg}(\operatorname{in} I)$, which would be a single exponent of d. But this is not the case as shown by [HH, Section 4].

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