BIFURCATION OF LIMIT CYCLES FROM A 4–DIMENSIONAL CENTER IN CONTROL SYSTEMS

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ABSTRACT. We study the bifurcation of limit cycles from the periodic orbits of a 4–dimensional center in a class of piecewise linear differential systems, which appears in a natural way in control theory. Our main result shows that three is an upper bound for the number of limit cycles, up to first order expansion of the displacement function with respect to the small parameter. Moreover, this upper bound is reached. For proving this result we use the averaging method in a form where the differentiability of the system is not needed.

1. INTRODUCTION

Piecewise linear differential systems appear in a natural way in control theory. These systems can present the complicated dynamical phenomena as the general nonlinear differential systems. Some of the main ingredients in the qualitative description of the dynamical behaviour of differential systems are the number and distribution of the limit cycles.

The purpose of this paper is to study the existence of limit cycles of the 4–dimensional control system

$$x' = A_0 x + \varepsilon F(x), \tag{1}$$

for $|\varepsilon| \neq 0$ a sufficiently small real parameter, where

$$A_0 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and $F : \mathbb{R}^4 \to \mathbb{R}^4$ is given by

$$F(x) = Ax + \varphi(k^T x)b,$$

with $A \in \mathcal{M}_4(\mathbb{R})$, $k, b \in \mathbb{R}^4 \setminus \{0\}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ the piecewise linear function

$$\varphi(x) = \begin{cases} -1, & \text{for } x \in (-\infty, -1), \\ x, & \text{for } x \in [-1, 1], \\ 1, & \text{for } x \in (1, \infty). \end{cases} \tag{2}$$

The independent variable is denoted by t, vectors of $\mathbb{R}^4$ are column vectors, and k^T denotes the transposed vector.

For $\varepsilon = 0$ system (1) becomes

$$x_1' = -x_2, \ x_2' = x_1, \ x_3' = -x_4, \ x_4' = x_3. \tag{3}$$

We notice that the origin is a *global isochronous center* for (3), i.e. all its orbits are periodic with the same period. A *limit cycle* is an isolated periodic orbit in the set of periodic orbits. The *Poincaré mapping* (or, equivalently, the *displacement map*) is a suitable tool for studying limit cycles of autonomous systems (detailed explanations can be found in [3, 5]). We remind that a limit cycle of some system corresponds to an isolated zero of the displacement function. Our main result is the following.

Theorem 1. *Three is the upper bound for the number of limit cycles of system (1) which bifurcate from the periodic orbits of system (3), up to first order expansion of the displacement function of (1) with respect to the small parameter ε . Moreover, there are systems (1) having exactly three limit cycles.*

The proof of Theorem 1 is based on the first order averaging method. We will present this method in Section 2, in the form recently obtained in [1]. The advantage of this result is that the smoothness assumptions for the vector field of the differential system are minimal. In particular, it can be applied to piecewise linear differential systems, which are not C^2 (not even C^1), as it was required in its classical version, see for instance, Theorem 11.5 of Verhulst [13].

The proof of Theorem 1 will be the subject of Section 3. The first step in the study of system (1), Lemma 1, is to reduce the number of parameters by a linear change of variables. The next objective is to transform the system into one which is in the standard form for averaging. This is accomplished in Lemma 2 by a change of variables which is related to the first integrals of system (3). A difficult problem will be the estimation of the number of isolated zeros of some 3–dimensional function with 3 variables, or, equivalently, of the number of equilibrium points of the averaged system. Proposition 4 gives the answer to this question and it will be used in the proof of Theorem 1, after noticing the relation between the averaging method and the displacement function. In Section 4 we will give the proof of Proposition 4.

Some of the above ideas can be found in the literature, and sometimes they were treated in a general setting.

The authors of [2] study canonical forms for n–dimensional continuous piecewise linear differential systems. Using their terminology, system (1) belongs to the class of symmetrical 4-dimensional continuous piecewise–linear circuits with three zones. As a brief explanation of this terminology, it is easy to see that the phase space of (1) can be divided into three regions, such that this system restricted to each region is linear. The change of the variable x to $-x$ leave the system invariant, so there is a symmetry. We notice also that electric circuits are described using piecewise linear systems.

Due to the specific form of our problem, the results of [2] can not be applied.

Reference [6] can be seen for a theoretical discussion about suitable transformations of high dimensional differential systems which are small perturbations of a center, into the standard form for averaging. The general idea is to relate this change of variables to the first integrals of the center.

The averaging method has been used in [8] to show that, given an arbitrary positive integer n, there exist planar piecewise linear systems with at least n limit cycles. This is a new argument in support of the idea that the piecewise linear differential systems can present almost all the complex dynamics one can see in the nonlinear differential systems.

As far as we know, high–dimensional piecewise linear systems have not been studied much. There exists a systematic study of the planar ones, for example in [4, 9, 10, 12]. In [8] the authors studied also limit cycles of a 3–dimensional control system using the harmonic balance method.

Finally, we would like to add some comments related to our approach to the problem of counting the limit cycles of piecewise linear differential systems. We chosen here to study bifurcation with respect to a small parameter from the periodic orbits of a center, up to first order expansion of the displacement mapping. For some values of the coefficients, this is sufficient in order to find the exact number of limit cycles. But, in some other cases, the first order expansion of the displacement mapping can have non isolated zeros. Then, a higher order expansion is needed. The study can be done by using second, third,... order averaging method. A key point could be the relation between the averaging method and the displacement map due to the fact that the displacement map of a piecewise linear differential system is analytic in a neighborhood of a limit cycle. Also, we would like to notice that our approach open the possibility to study piecewise linear differential systems with more than three zones, or of arbitrary even dimension.

2. First Order Averaging Method

The aim of this section is to present the first order averaging method as it was obtained in [1]. Differentiability of the vector field is not needed. The specific conditions for the existence of a simple isolated zero of the averaged function are given in terms of the Brouwer degree. In fact, the Brouwer degree theory is the key point in the proof of this theorem. We remind here that continuity of some finite dimensional function is a sufficient condition for the existence of its Brouwer degree (see [11] for precise definitions).

Theorem 2. *We consider the following differential system*

$$x'(t) = \varepsilon H(t, x) + \varepsilon^2 R(t, x, \varepsilon), \tag{4}$$

where $H : \mathbb{R} \times D \to \mathbb{R}^n$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ are continuous functions, T–periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. We define $h : D \to \mathbb{R}^n$ as

$$h(z) = \int_0^T H(s, z) ds, \tag{5}$$

and assume that:
(i) H and R are locally Lipschitz with respect to x;
(ii) for $a \in D$ with $h(a) = 0$, there exists a neighborhood V of a such that $h(z) \neq 0$ for all $z \in \overline{V} \setminus \{a\}$ and $d_B(h, V, 0) \neq 0$.
Then, for $|\varepsilon| > 0$ sufficiently small, there exists an isolated T–periodic solution $\varphi(\cdot, \varepsilon)$ of system (4) such that $\varphi(\cdot, \varepsilon) \to a$ as $\varepsilon \to 0$.

Here we will need some facts from the proof of Theorem 2. Hypothesis (i) assures the existence and uniqueness of the solution of each initial value problem on the interval $[0, T]$. Hence, for each $z \in D$, it is possible to denote by $x(\cdot, z, \varepsilon)$ the solution of (4) with the initial value $x(0, z, \varepsilon) = z$. We consider also the function $\zeta : D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ defined by

$$\begin{aligned} \zeta(z, \varepsilon) &= \int_0^T \big[\varepsilon H(t, x(t, z, \varepsilon)) + \\ &\quad \varepsilon^2 R(t, x(t, z, \varepsilon), \varepsilon) \big] \, dt. \end{aligned} \tag{6}$$

From the proof of Theorem 2 we extract the following facts.

Remark 1. *For every $z \in D$ the following relation holds*

$$x(T, z, \varepsilon) - x(0, z, \varepsilon) = \zeta(z, \varepsilon).$$

The function ζ can be written in the form

$$\zeta(z, \varepsilon) = \varepsilon h(z) + \varepsilon^2 O(1),$$

where h is given by (5) and the symbol $O(1)$ denotes a bounded function on every compact subset of $D \times (-\varepsilon_f, \varepsilon_f)$. Moreover, for $|\varepsilon|$ sufficiently small, $z = \varphi(0, \varepsilon)$ is an isolated zero of $\zeta(\cdot, \varepsilon)$.

For concrete systems there is the possibility that the function ζ is not globally differentiable, but the function h is. In fact, only differentiability in some neighborhood of a fixed isolated zero of h could be enough. When this is the case, one can use the following remark in order to verify the hypothesis (ii) of Theorem 2.

Remark 2. *Let $h : D \to \mathbb{R}^n$ be a C^1 function, with $h(a) = 0$, where D is an open subset of $\mathbb{R}^n$ and $a \in D$. Whenever a is a simple zero of h (i.e. $J_h(a) \neq 0$), there exists a neighborhood V of a such that $h(z) \neq 0$ for all $z \in \overline{V} \setminus \{a\}$. Then $d_B(h, V, 0) \in \{-1, 1\}$.*

3. Proof of Theorem 1

The proof of Theorem 1 is based on the first order averaging method presented in the previous section. In order to apply this result, we will prove first that it is possible to reduce the number of parameters and after we will change the variables in order to transform the system into one which is in the standard form for averaging. The computation of the averaged function (5) will be also a special task. Another difficult problem will be the estimation of the number of its isolated zeros. We will use here the solution of this problem (written in Proposition 4), but we will give the proof in the next section. The relation between the averaging method and the displacement function will be also discussed.

In the next lemma we will prove that, by a linear change of variables, it is possible to reduce the number of parameters. We denote by e_i, $i = 1, ..., 4$ the vectors which form the canonical base of $\mathbb{R}^4$.

Lemma 1. *By a linear change of variables, and eventually a permutation of the variables, system (1) can be transformed into the system*

$$x' = A_0 x + \varepsilon \overline{A} x + \varepsilon \varphi(x_1) \overline{b}, \tag{7}$$

where $\overline{A} \in \mathcal{M}_4(\mathbb{R})$ is an arbitrary matrix and $\overline{b} = e_1$ or $\overline{b} = e_3$.

Proof. Since the linear change of variables $x = Jy$, with J an invertible matrix, transforms system (1) into

$$y' = J^{-1} A_0 J y + \varepsilon J^{-1} A J y + \varepsilon \varphi(k^T J y) J^{-1} b,$$

we have to find J such that

$$J^{-1}A_0 J = A_0, \tag{8}$$

$$J^{-1}b = \bar{b}, \tag{9}$$

$$k^T J = e_1^T. \tag{10}$$

We denote by z_{ij}, $i,j = 1,2,3,4$ the elements of the matrix J. It is easy to check that (8) is equivalent to

$$J = \begin{pmatrix} z_{11} & z_{12} & z_{13} & z_{14} \\ -z_{12} & z_{11} & -z_{14} & z_{13} \\ z_{31} & z_{32} & z_{33} & z_{34} \\ -z_{32} & z_{31} & -z_{34} & z_{33} \end{pmatrix}. \tag{11}$$

From (9) we have, when $\bar{b} = e_1$,

$$z_{11} = b_1, \ z_{12} = -b_2, \ z_{31} = b_3, \ z_{32} = -b_4, \tag{12}$$

or, when $\bar{b} = e_3$,

$$z_{13} = b_1, \ z_{14} = -b_2, \ z_{33} = b_3, \ z_{34} = -b_4. \tag{13}$$

We replace J given by (11) in (10) and obtain

$$\begin{aligned} k_1 z_{11} + k_2 z_{12} + k_3 z_{13} + k_4 z_{14} &= 1, \\ -k_1 z_{12} + k_2 z_{11} - k_3 z_{14} + k_4 z_{13} &= 0, \\ k_1 z_{31} + k_2 z_{32} + k_3 z_{33} + k_4 z_{34} &= 0, \\ -k_1 z_{32} + k_2 z_{31} - k_3 z_{34} + k_4 z_{33} &= 0. \end{aligned} \tag{14}$$

These relations, together with (12) or (13) form a linear algebraic system of 8 equations with 8 unknowns (z_{ij}, $i = 1,3$, $j = 1,2,3,4$). Straightforward calculations show that, when $k_3^2 + k_4^2 \neq 0$, one can find a unique solution of the system (12) and (14), and when $k_3^2 + k_4^2 = 0$ one can find a unique solution of the system (13) and (14). This solution provides a matrix J whose determinant is $(b_3^2 + b_4^2)/(k_3^2 + k_4^2)$, and $(b_3^2 + b_4^2)/(k_1^2 + k_2^2)$, respectively. We remind that we work in the hypotheses that neither k, nor b is the null vector. Thus, when $k_3^2 + k_4^2 = 0$ we must have $k_1^2 + k_2^2 \neq 0$, so at least one of the above expressions is well defined. The matrix J is invertible only if $b_3^2 + b_4^2 \neq 0$. But if this is not the case, we must have that $b_1^2 + b_2^2 \neq 0$. By permuting x_1 with x_3 and, respectively, x_2 with x_4 we are again in the situation that we already have studied. $\square$

An equivalent system, which is in the standard form for averaging, will be found in the next lemma by a proper change of the variables.

Lemma 2. *Changing the variables (x_1, x_2, x_3, x_4) to (θ, r, ρ, s) by*

$$x_1 = r\cos\theta, \qquad x_2 = r\sin\theta,$$
$$x_3 = \rho\cos(\theta + s), \qquad x_4 = \rho\sin(\theta + s),$$

system (7) is transformed into a system of the form

$$\frac{dr}{d\theta} = \varepsilon H_1(\theta, r, \rho, s) + \varepsilon^2 O(1),$$

$$\frac{d\rho}{d\theta} = \varepsilon H_2(\theta, r, \rho, s) + \varepsilon^2 O(1), \qquad (15)$$

$$\frac{ds}{d\theta} = \varepsilon H_3(\theta, r, \rho, s) + \varepsilon^2 O(1),$$

where,

$$H_1 = \cos\theta F_1 + \sin\theta F_2,$$
$$H_2 = \cos(\theta + s)F_3 + \sin(\theta + s)F_4,$$
$$H_3 = \frac{1}{r}\cos\theta F_2 - \frac{1}{r}\sin\theta F_1 -$$
$$\frac{1}{\rho}\cos(\theta + s)F_4 + \frac{1}{\rho}\sin(\theta + s)F_3,$$

and for every $i = 1, ..., 4$,

$$F_i = a_{i1}r\cos\theta + a_{i2}r\sin\theta + a_{i3}\rho\cos(\theta + s) +$$
$$a_{i4}\rho\sin(\theta + s) + \varphi(r\cos\theta)b_i.$$

We take ε_f sufficiently small, n arbitrarily large, and $D_n = (1/n, n) \times (1/n, n) \times \mathbb{R}$. Then, the vector field of system (15) is well defined and continuous on $\mathbb{R} \times D_n \times (-\varepsilon_f, \varepsilon_f)$. Also, it is 2π–periodic with respect to θ and locally Lipschitz with respect to (r, ρ, s).

Proof. System (7) in the variables (θ, r, ρ, s) becomes

$$\theta' = 1 + \varepsilon\frac{1}{r}(\cos\theta F_2 - \sin\theta F_1),$$
$$r' = \varepsilon H_1(\theta, r, \rho, s),$$
$$\rho' = \varepsilon H_2(\theta, r, \rho, s),$$
$$s' = \varepsilon H_3(\theta, r, \rho, s).$$

We notice that for $|\varepsilon|$ sufficiently small, $\theta'(t) > 0$ for each t when $(\theta, r, \rho, s) \in \mathbb{R} \times D_n$. Now we eliminate the variable t in the above system by considering θ as the new independent variable. It is easy to see that, indeed, the right hand side of the new system is well defined and continuous on $\mathbb{R} \times D_n \times (-\varepsilon_f, \varepsilon_f)$, it is 2π–periodic with respect to the independent variable θ and locally Lipschitz with respect to (r, ρ, s). Form (15) is obtained

after an expansion with respect to the small parameter ε. $\square$

We will apply Theorem 2 to system (15). Our next step is to find the corresponding function (5). We will denote it by $h : D_n \to \mathbb{R}^3$, $h = (h_1, h_2, h_3)^T$. For each $i = 1, ..., 3$, the component h_i is defined by the formula

$$h_i(r, \rho, s) = \int_0^{2\pi} H_i(\theta, r, \rho, s) d\theta,$$

where H_i are like in Lemma 2.

In order to calculate the exact expression of h, we use the following formulas

$$\int_0^{2\pi} \cos^2 \theta d\theta = \pi,$$

$$\int_0^{2\pi} \sin^2 \theta d\theta = \pi,$$

$$\int_0^{2\pi} \cos \theta \sin \theta d\theta = 0,$$

$$\int_0^{2\pi} \cos \theta \cos(\theta + s) d\theta = \pi \cos s,$$

$$\int_0^{2\pi} \cos \theta \sin(\theta + s) d\theta = \pi \sin s,$$

$$\int_0^{2\pi} \sin \theta \cos(\theta + s) d\theta = -\pi \sin s,$$

$$\int_0^{2\pi} \sin \theta \sin(\theta + s) d\theta = \pi \cos s,$$

$$\int_0^{2\pi} \cos^2(\theta + s) d\theta = \pi,$$

$$\int_0^{2\pi} \sin^2(\theta + s) d\theta = \pi,$$

$$\int_0^{2\pi} \cos(\theta + s) \sin(\theta + s) d\theta = 0.$$

For each $r > 0$ we denote

$$I_1(r) = \int_0^{2\pi} \varphi(r \cos \theta) \cos \theta d\theta, \qquad (16)$$

$$I_2(r) = \int_0^{2\pi} \varphi(r \cos \theta) \sin \theta d\theta, \qquad (17)$$

where φ is the piecewise–linear function given by (2).

We claim that the integrals I_1 and I_2 given by (16) and (17), respectively, have the following expressions

$$I_1(r) = \begin{cases} r\pi, & \text{for } 0 < r \le 1, \\ 2\dfrac{\sqrt{r^2 - 1}}{r} + r\pi - \\ \quad 2r \arctan \sqrt{r^2 - 1}, & \text{for } r > 1. \end{cases} \qquad (18)$$

and
$$I_2(r) = 0, \quad \text{for all } r > 0. \tag{19}$$
Now we shall prove the claim. Whenever $0 < r \leq 1$ we have that $|r \cos \theta| \leq 1$ and $|r \sin \theta| \leq 1$ for all $\theta \in [0, 2\pi)$. Then $\varphi(r \cos \theta) = r \cos \theta$ for every θ. Thus
$$I_1(r) = r \int_0^{2\pi} \cos^2 \theta d\theta = r\pi,$$
$$I_2(r) = r \int_0^{2\pi} \sin \theta \cos \theta d\theta = 0.$$

We fix now $r > 1$ and consider $\theta_c \in (0, \pi/2)$ such that $\cos \theta_c = 1/r$. Then we can write
$$I_1(r) = \int_0^{\theta_c} \cos \theta d\theta + r \int_{\theta_c}^{\pi - \theta_c} \cos^2 \theta d\theta -$$
$$\int_{\pi - \theta_c}^{\pi + \theta_c} \cos \theta d\theta + r \int_{\pi + \theta_c}^{2\pi - \theta_c} \cos^2 \theta d\theta +$$
$$\int_{2\pi - \theta_c}^{2\pi} \cos \theta d\theta,$$

and
$$I_2(r) = \int_0^{\theta_c} \sin \theta d\theta + r \int_{\theta_c}^{\pi - \theta_c} \sin \theta \cos \theta d\theta -$$
$$\int_{\pi - \theta_c}^{\pi + \theta_c} \sin \theta d\theta + r \int_{\pi + \theta_c}^{2\pi - \theta_c} \sin \theta \cos \theta d\theta +$$
$$\int_{2\pi - \theta_c}^{2\pi} \sin \theta d\theta.$$

Straightforward computations lead to the following expressions
$$I_1(r) = 2 \sin \theta_c + r\pi - 2r\theta_c, \text{ and } I_2(r) = 0.$$
Since $\theta_c \in (0, \pi/2)$ with $\cos \theta_c = 1/r$, we have $\sin \theta_c = \sqrt{r^2 - 1}/r$ and $\theta_c = \arctan \sqrt{r^2 - 1}$. This ends the proof of the claim.

We obtain the following expressions for the components of h,
$$\begin{aligned}
h_1 &= c_1 r + (c_2 \cos s + c_3 \sin s)\rho + b_1 I_1(r), \\
h_2 &= (c_5 \cos s + c_6 \sin s)r + c_7 \rho + b_3 \cos s I_1(r), \\
h_3 &= c_4 + (c_2 \sin s - c_3 \cos s)\frac{\rho}{r} + \\
&\quad (c_5 \sin s - c_6 \cos s)\frac{r}{\rho} + b_3 \sin s \frac{I_1(r)}{\rho},
\end{aligned} \tag{20}$$

where we made also the notations

$$
\begin{aligned}
c_1 &= (a_{11} + a_{22})\pi, \\
c_2 &= (a_{13} + a_{24})\pi, \\
c_3 &= (a_{14} - a_{23})\pi, \\
c_4 &= (a_{21} - a_{12} - a_{43} + a_{34})\pi, \\
c_5 &= (a_{31} + a_{42})\pi, \\
c_6 &= (a_{41} - a_{32})\pi, \\
c_7 &= (a_{33} + a_{44})\pi.
\end{aligned}
$$

Now we come back to system (7). To each constant $(r^*, \rho^*, s^*) \in D_n$ we associate a periodic orbit of system (3) (i.e. (7) with $\varepsilon = 0$), since

$$
r = \sqrt{x_1^2 + x_2^2}, \ \rho = \sqrt{x_3^2 + x_4^2},
$$
$$
s = \arctan \frac{x_4}{x_3} - \arctan \frac{x_2}{x_1},
$$

are first integrals of this system. One can prove that any isolated 2π–periodic solution of (15) with $|\varepsilon| \neq 0$ sufficiently small, corresponds to a limit cycle of (7).

Lemma 2 states that the hypotheses of Theorem 2 are fulfilled for system (15), where the function h is given by (20). Using also Remark 2 we conclude that, for $|\varepsilon|$ sufficiently small, and for each simple zero $(r^*, \rho^*, s^*) \in D_n$ of h, there exists an isolated 2π–periodic solution $\varphi(\cdot, \varepsilon)$ of system (15) such that $\varphi(\cdot, \varepsilon) \to (r^*, \rho^*, s^*)$ as $\varepsilon \to 0$. In short, we have proved the next result.

Proposition 3. *From each periodic orbit of system (3), which corresponds to a simple zero of h in D_n, it bifurcates a branch of limit cycles of system (7).*

We continue by proving

Lemma 3. *The displacement function of system (7) for the transversal section $x_2 = 0$, written in the coordinates (2), has the form*

$$
\varepsilon h(r, \rho, s) + \varepsilon^2 O(1).
$$

Proof. In order to see the validity of this result, we focus on the function ζ given by (6), and Remark 1, where the notations correspond to system (15) written in the form (4). This function ζ is the displacement function of system (7) associated to the transversal section $x_2 = 0$, since it applies to every initial vector $(r(0), \rho(0), s(0))$, the vector

$$
(r(2\pi) - r(0), \rho(2\pi) - \rho(0), s(2\pi) - s(0)), \tag{21}
$$

and the points of the transversal section are also characterized by $\theta = 0$ (mod 2π). We recall that r, ρ, s are the first integrals of system (7) when $\varepsilon = 0$. $\square$

The proof of the following result will be the subject of the next section. We notice that, in order to find the zeros of h in D_n (defined in Lemma 2), it is sufficient to look for them in $(0, \infty) \times (0, \infty) \times [0, 2\pi)$. This is due to the fact that n can be chosen arbitrarily large, and h, as well as the transformation (2) are 2π–periodic with respect to the variable s.

Proposition 4. *Let $h : (0, \infty) \times (0, \infty) \times \mathbb{R} \to \mathbb{R}^3$ be the function whose components are given by (20), where c_i, $i = 1, ..., 7$ are arbitrary real parameters, $b_1, b_3 \in \{0, 1\}$ with $b_1 b_3 = 0$ and $b_1^2 + b_3^2 \neq 0$. The function I_1 is given by (18). Then,*
(i) h is of class C^1;
(ii) the maximum number of isolated zeros of h in $(0, \infty) \times (0, \infty) \times [0, 2\pi)$ is three.
Moreover, for the following values of the coefficients,

$$c_1 = -2, \; c_2 = -2, \; c_3 = 0, \; c_4 = 21/10, \; c_5 = 0,$$

$$c_6 = 1, \; c_7 = -1, \; b_1 = 1, \; b_2 = b_3 = b_4 = 0,$$

h has exactly three simple zeros.

The conclusion of Theorem 1 follows by Lemmas 1 and 3 and Propositions 3 and 4.

4. PROOF OF PROPOSITION 4

Function h is a composition of some functions which are elementary, and function I_1. A direct study of I_1 shows that it is of class C^1 on $(0, \infty)$. Thus, conclusion (i) is valid.

We will divide the proof of conclusion (ii) of Proposition 4 into several lemmas. Two auxiliary results will be given at the beginning, followed by a discussion with respect to different values of the coefficients.

The following notations will be needed

$$
\begin{align}
d(s) &= b_3 \cos s(c_2 \cos s + c_3 \sin s) - b_1 c_7, \tag{22} \\
k_1(s) &= (-c_1 b_3 + c_5 b_1) \cos s + c_6 b_1 \sin s, \tag{23} \\
k_2(s) &= -c_2 c_5 \cos^2 s - (c_2 c_6 + c_3 c_5) \cos s \sin s - \\
&\quad c_3 c_6 \sin^2 s + c_1 c_7, \tag{24}
\end{align}
$$

$$\begin{aligned} f(s) \quad = \quad & c_4 d(s) k_1(s) + (c_2 \sin s - c_3 \cos s) k_1^2(s) + \\ & (c_5 \sin s - c_6 \cos s) d^2(s) + \\ & b_3 \sin s d(s) k_2(s) \end{aligned} \qquad (25)$$

In the study of the zeros for the function h it will be necessary to study the zeros of f.

Lemma 4. *The function $f : [0, 2\pi) \to \mathbb{R}$ given by formula (25) can have at most six isolated zeros, and they appear in pairs of the form s^* together with $\xi^* = s^* + \pi$ (mod 2π). Moreover, when $b_1 = 0$ and $b_3 = 1$ two of them are $s^* = \pi/2$ and $\xi^* = 3\pi/2$.*

Proof. We have to study the solvability of

$$f(s) = 0. \qquad (26)$$

With the notations $\cos s = x$ and $\sin s = \sqrt{1 - x^2}$, and using the algebraic manipulator *Mathematica*, we obtain the following equivalent form for (26):

$$C_1 x + C_3 x^3 + (D_0 + D_2 x^2)\sqrt{1 - x^2} = 0, \qquad (27)$$

where

$$\begin{aligned} C_1 \quad = \quad & b_1^2(2c_2 c_5 c_6 - c_3 c_6^2 - c_4 c_5 c_7 - c_6 c_7^2) + \\ & b_3^2(c_1 c_3 c_7 - c_3^2 c_6), \\ C_3 \quad = \quad & b_1^2(c_3 c_6^2 - c_3 c_5^2 - 2c_2 c_5 c_6) + \\ & b_3^2(c_3^2 c_6 - c_1^2 c_3 - c_1 c_2 c_4 - c_2^2 c_6 - c_1 c_3 c_7), \\ D_0 \quad = \quad & b_1^2(c_2 c_6^2 - c_4 c_6 c_7 + c_5 c_7^2), \\ D_2 \quad = \quad & b_1^2(c_2 c_5^2 - 2c_3 c_5 c_6 - c_2 c_6^2) + \\ & b_3^2(c_1^2 c_2 + c_1 c_2 c_7 - c_1 c_3 c_4 - 2c_2 c_3 c_6). \end{aligned}$$

When we consider the case $\sin s = -\sqrt{1 - x^2}$, equation (26) becomes

$$C_1 x + C_3 x^3 - (D_0 + D_2 x^2)\sqrt{1 - x^2} = 0. \qquad (28)$$

It follows that we have to find $x \in [-1, 1]$ solution of (27) or of (28). This is equivalent to

$$\left(C_1 x + C_3 x^3\right)^2 - (D_0 + D_2 x^2)^2(1 - x^2) = 0, \qquad (29)$$

which is the polynomial equation

$$\begin{aligned} -D_0^2 + (C_1^2 + D_0^2 - 2D_0 D_2)x^2 + & \qquad (30) \\ (2C_1 C_3 + 2D_0 D_2 - D_2^2)x^4 + (C_3^2 + D_2^2)x^6 \quad = \quad & 0. \end{aligned}$$

This equation can have at most six roots in the interval $[-1, 1]$. Then (26) has at most six solutions $s \in [0, 2\pi)$. Since $f(s + \pi) = -f(s)$ for all $s \in [0, 2\pi)$, it is clear that, if s^* is a zero of f then $s^* + \pi$ is also a zero.

If $b_1 = 0$ and $b_3 = 1$ then $D_0 = 0$ and equation (30) has the solution $x = 0$. Since $\cos s = x$, we have that $s^* = \pi/2$ and $\xi^* = 3\pi/2$ are solutions of (26) in this case. $\square$

The following result will be also needed later on.

Lemma 5. *We consider the equation*

$$I_1(r) = cr, \ r > 0, \tag{31}$$

with I_1 given by (18) and c a real parameter. In the study of its solvability the following situations arise.
(a) If $0 < c < \pi$ then (31) has a unique solution $r^* > 1$.
(b) If $c = \pi$ then (31) has the interval $(0, 1]$ as the set of solutions.
(c) If $c \leq 0$ or $c > \pi$ then (31) has no solution.

Proof. It is easy to see that the result is valid for $r \in (0, 1]$, since in this case $I_1(r) = \pi r$.

For $r > 1$ equation (31) becomes

$$2\frac{\sqrt{r^2 - 1}}{r} + r\pi - 2r \arctan \sqrt{r^2 - 1} = cr.$$

Changing the variable $u = \sqrt{r^2 - 1}$, we obtain the equivalent equation

$$\arctan u = \frac{\pi - c}{2} + \frac{u}{u^2 + 1}, \ u > 0.$$

We study the graphic representation on the interval $(0, \infty)$ for the left and the right hand side, respectively. Then we can see that they intersect if and only if $0 < (\pi - c)/2 < \pi/2$, i.e. $0 < c < \pi$. In this case the intersection point is unique. This completes the proof of the lemma. $\square$

The following three lemmas study the zeros of h. It will be convenient to denote the nonlinear algebraic system which provide the zeros of h, as

$$h_1(r, \rho, s) = 0, \tag{32}$$
$$h_2(r, \rho, s) = 0, \tag{33}$$
$$h_3(r, \rho, s) = 0. \tag{34}$$

Lemma 6. *The function h can have at most three isolated zeros $(r^*, \rho^*, s^*) \in (0, \infty) \times (0, \infty) \times [0, 2\pi)$ with $d(s^*) \neq 0$. When $b_1 = 0$, $b_3 = 1$, $c_1 \neq 0$ then h has at most two such zeros, and when $b_1 = 0$, $b_3 = 1$, $c_1 = 0$ then h has no such zeros.*

Proof. In this case, system $(32)-(33)$ is equivalent to

$$\rho \;=\; \frac{k_1(s)}{d(s)} r, \tag{35}$$

$$I_1(r) \;=\; \frac{k_2(s)}{d(s)} r, \tag{36}$$

where d, k_1 and k_2 are given in (22)–(24).

If we replace (35) and (36) in h_3 we obtain an expression independent of r and ρ. More exactly, we obtain the expression of function f given by (25). Then, equation (34) is equivalent to $f(s) = 0$. Now we fix s^*, a zero of f. We want to study the solvability of (36) with respect to $r > 0$. Using Lemma 5, there exists an isolated solution $r > 0$ of (36) if and only if

$$0 < \frac{k_2(s^*)}{d(s^*)} < \pi, \tag{37}$$

and in this case it is unique. Also, for a fixed r^*, corresponding to some s^*, whenever

$$\frac{k_1(s^*)}{d(s^*)} > 0, \tag{38}$$

we can uniquely find $\rho^* > 0$ from (35). We will prove that condition (38) is satisfied only for at most half of the zeros of f. In order to do this, we notice that $k_1(s + \pi) = -k_1(s)$, $d(s + \pi) = d(s)$, and we remind that the zeros of f appear in pairs, s^* together with $s^* + \pi$. Condition (38) is satisfied either for s^*, or for $s^* + \pi$, unless $k_1(s^*) = 0$. Hence, function h can have at most three isolated zeros $(r^*, \rho^*, s^*) \in (0, \infty) \times (0, \infty) \times [0, 2\pi)$ with $d(s^*) \neq 0$.

If $b_1 = 0$ and $b_3 = 1$ then $k_1(s) = -c_1 \cos s$. When $c_1 \neq 0$, by Lemma 4, two of the zeros of f are $s^* = \pi/2$ and $s^* = 3\pi/2$, which cannot provide any solution since (38) is not valid. When $c_1 = 0$, k_1 is identically zero, so (38) cannot be satisfied for any s. $\square$

Lemma 7. *When $b_1 = 1$, $b_3 = 0$, $c_7 = 0$, the function h can have at most two isolated zeros.*

Proof. First we notice that the function d is identically zero, so we cannot apply Lemma 6. In our conditions, one can prove that system (32)–(34) is equivalent to

$$\frac{I_1(r)}{r} \;=\; -c_1 - q(s)\frac{\rho}{r}, \tag{39}$$

$$k_1(s) \;=\; 0, \tag{40}$$

$$p_1(s)\left(\frac{\rho}{r}\right)^2 \;+\; c_4\frac{\rho}{r} + p_2(s) = 0, \tag{41}$$

where $q(s) = c_2 \cos s + c_3 \sin s$, $p_1(s) = c_2 \sin s - c_3 \cos s$ and $p_2(s) = c_5 \sin s - c_6 \cos s$. For a fixed s, (41) provides at most two isolated values of ρ/r and, for fixed s and fixed ρ/r, (39) gives at most one isolated value for r (by Lemma 5). It is easy now to see that our system has no isolated solution when k_1 is identically zero. If k_1 is not identically zero, then we denote by s^* and $\xi^* = s^* + \pi$, the zeros of k_1. If we replace in (41) s^* and ξ^*, we obtain

$$p_1(s^*) \left(\frac{\rho}{r}\right)^2 + c_4 \frac{\rho}{r} + p_2(s^*) = 0,$$

and,

$$p_1(s^*) \left(\frac{\rho}{r}\right)^2 - c_4 \frac{\rho}{r} + p_2(s^*) = 0,$$

respectively. These two equations can have together at most two positive solutions ρ/r. We conclude that h can have at most two isolated zeros. $\square$

Lemma 8. *When $b_1 = 0$ and $b_3 = 1$, the function h can have at most two isolated zeros $(r^*, \rho^*, s^*) \in (0, \infty) \times (0, \infty) \times [0, 2\pi)$ with $d(s^*) = 0$. Moreover, either $c_1 = 0$, or h has at most one isolated zero.*

Proof. Whenever $b_1 = 0$ system (32)–(34) is equivalent to

$$q(s)\frac{\rho}{r} = -c_1, \tag{42}$$

$$\cos s \frac{I_1(r)}{r} = -\tilde{q}(s) - c_7 \frac{\rho}{r}, \tag{43}$$

$$h_3(r, \rho, s) = 0, \tag{44}$$

where $q(s) = c_2 \cos s + c_3 \sin s$ and $\tilde{q}(s) = c_5 \cos s + c_6 \sin s$. In this case,

$$d(s) = \cos s(c_2 \cos s + c_3 \sin s).$$

We study the cases when $d(s^*) = 0$, i.e. either $c_2 \cos s + c_3 \sin s = 0$ and equation (42) is degenerate, or $\cos s = 0$ and equation (43) is degenerate.

If (42) is degenerate, but (43) is not, then we can write $I_1(r)/r$ as a function of s and ρ/r. Discussion is similar now to the one done in the proof of Lemma 7. The conclusion will be that h can have at most two isolated solutions (r^*, ρ^*, s^*) with $d(s^*) = 0$, and they can be found knowing that

$$c_2 \cos s^* + c_3 \sin s^* = 0,$$

$$\cos s^* p_1(s^*) \left(\frac{\rho}{r}\right)^2 + p_3(s^*)\frac{\rho}{r} - c_6 = 0,$$

$$\frac{I_1(r)}{r} = -\frac{\tilde{q}(s^*)}{\cos s^*} - \frac{c_7}{\cos s^*}\frac{\rho}{r},$$

where $p_1(s) = c_2 \sin s - c_3 \cos s$ and $p_3(s) = c_4 \cos s - c_7 \sin s$. Therefore, necessary conditions for the existence of isolated solutions are

$$c_2^2 + c_3^2 \neq 0 \quad \text{and} \quad c_1 = 0.$$

It remains to study if $s^* = \pi/2$ and $\xi^* = 3\pi/2$ provide zeros of h. When we replace these values in (42)-(43) we obtain

$$c_3 \frac{\rho}{r} = -c_1, \ c_7 \frac{\rho}{r} = -c_6,$$

and

$$c_3 \frac{\rho}{r} = c_1, \ c_7 \frac{\rho}{r} = c_6,$$

respectively. From each system we can obtain at most one isolated value of ρ/r, but it is easy to see that only one of them is positive. Thus, h can have at most one zero with $s^* = \pi/2$ or $3\pi/2$, and $c_1 \neq 0$ is a necessary condition for the existence of such zero. $\square$

In order to conclude that h can have at most three zeros in D_n, three different cases will be considered.

If $b_1 = 1$, $b_3 = 0$, $c_7 \neq 0$, then $d(s) = c_7 \neq 0$ for all $s \in [0, 2\pi)$. The conclusion follows by Lemma 6.

If $b_1 = 1$, $b_3 = 0$, $c_7 = 0$, then we apply Lemma 7.

If $b_1 = 0$, $b_3 = 1$, then the conclusion holds by Lemmas 6 and 8.

For the concrete values of the coefficients given in the hypotheses of the Proposition 4, the components of function h are

$$
\begin{aligned}
h_1 &= 2r - 2\rho \cos s + I_1(r), \\
h_2 &= r \sin s - \rho, \\
h_3 &= 21/10 - 2\sin s \frac{\rho}{r} - \cos s \frac{r}{\rho},
\end{aligned}
$$

and the auxiliary functions given in the beginning of this section become in this case

$$
\begin{aligned}
d(s) &= 1, \\
k_1(s) &= \sin s, \\
k_2(s) &= 2 + 2\cos s \sin s, \\
f(s) &= -10 \cos s + 21 \sin s - 20 \sin^3 s.
\end{aligned}
$$

With the notation $x = \cos s$, equation $f(s) = 0$ becomes $\sin s = 10x/(1 + 20x^2)$, or equivalently

$$1 - 61x^2 + 360x^4 - 400x^6 = 0.$$

This polynomial equation has six solutions $x_{1,2} = \pm 1/\sqrt{5}$, $x_{3,4,5,6} = \pm\sqrt{(7 \pm 2\sqrt{11})/20}$, and all are in the interval $(-1, 1)$.

We obtain that f has the following zeros,

$$s_1 = \arccos \frac{1}{\sqrt{5}}, \ s_{2,3} = \arccos \sqrt{\frac{7 \pm 2\sqrt{11}}{20}},$$

$$\xi_{1,2,3} = s_{1,2,3} + \pi.$$

One can prove that condition (37) is satisfied for every zero of f, but condition (38) is satisfied only for the zeros ξ_1, ξ_2 and ξ_3. Then, h has exactly three zeros. We have to prove now that $J_h(r^*, \rho^*, s^*) \neq 0$ for each (r^*, ρ^*, s^*) zero of h. We notice that $\rho^* = r^* \sin s^*$, and $r^* > 1$ is the unique solution of $I_1(r) = (2 + 2 \sin s^* \cos s^*)r$. One can find the following simplified form of the Jacobian determinant of h,

$$\frac{4}{5r^{*2}}(\sqrt{r^{*2} - 1} - r^{*2}) \cos s^* (15 \cos s^* - 16 \sin s^*).$$

It is easy to see that this expression is different from zero. $\square$

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