MARKOV PROCESSES AND FEYNMAN-KAC PROPAGATORS

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ABSTRACT. In this paper, we study the behavior of Feynman-Kac propagators corresponding to free propagators and time-dependent measures. Our main results concern the inheritance of various properties of free propagators by Feynman-Kac propagators. These properties include the $(L^r - L^q)$-boundedness and the boundedness in various spaces of continuous functions.

1. INTRODUCTION

Propagators are two-parametric families of bounded linear operators satisfying the flow condition (forward propagators) or the backward flow condition (backward propagators). The term "propagator" is not the only name used for such a family of operators. Several other names appeared in the mathematical literature (e.g., evolution operators, evolution families, non-autonomous semigroups, etc).

A simple example of a propagator (both forward and backward) is the family of operators $\{S_{t-\tau} : 0 \le \tau \le t < \infty\}$ obtained from a semigroup S_t on a Banach space B. Another example is the free backward propagator Y associated with a transition probability function P on a locally compact second countable topological space E equipped with the Borel σ-algebra $\mathcal{E}$ (see Section 3). Perturbing Y by various multiplicative functionals, we get more examples of backward propagators. For instance, the backward Feynman-Kac propagators Y_V and Y_μ are such perturbations of Y. These backward propagators are associated with a Borel function V on $[0,T] \times E$ and a time-dependent Borel measure $\mu = \{\mu(\tau) : 0 \le \tau \le T\}$, and are defined as follows:

$$Y_V(\tau,t)f(x) = E_{\tau,x} f(X_t) \exp\{-\int_\tau^t V(s,X_s)ds\}, \ \ 0 \le \tau \le t \le T,$$

and

$$Y_\mu(\tau,t)f(x) = E_{\tau,x} f(X_t) \exp\{-A_\mu(t,\tau)\}, \ \ 0 \le \tau \le t \le T.$$

In the formulas above, $X = (X_t, \mathcal{F}_t^\tau, P_{\tau,x})$ is a non-homogeneous Markov process on a probability space $(\Omega, \mathcal{F})$, having $(E, \mathcal{E})$ as its state space and P as its transition probability function. The symbol A_μ in the formula for Y_μ denotes the additive functional of the process X that will be constructed in Section 5. Backward Feynman-Kac propagators are the main objects of our study in this paper. Our preference for the backward case is not restrictive, since the results for backward propagators obtained in this paper can be reformulated for forward propagators using time reversal. This will be explained in Section 11. We refer the reader to [5, 10, 29] for more information on transition probability functions and non-homogeneous Markov processes. One of the cornerstones of the theory of non-homogeneous Markov processes was Kolmogorov's pioneering paper [20].

In Section 4 of the present paper, we define the classes $\mathcal{P}_f^*$ and $\mathcal{P}_m^*$ of functions on $[0,T] \times E$ and time-dependent measures on $\mathcal{E}$. These classes are generalizations of a celebrated Kato class K_n, introduced by Aizenman and Simon in [1, 28]. The definition of K_n is based on a condition used by Kato in [18]. Aizenman and Simon developed the theory of Schrödinger semigroups with Kato class potentials (see [1, 28]). The Feynman-Kac propagators Y_V with $V \in \mathcal{P}_f^*$ and Y_μ with $\mu \in \mathcal{P}_m^*$ are, in a sense, two-parametric relatives of Schrödinger semigroups with Kato class potentials. More information on the Kato class and Schrödinger semigroups can be found in [4, 17, 30, 31]. Various generalizations of the Kato classes of functions and measures were studied in the non-autonomous case (see [11, 12, 13, 14, 21, 22, 24, 25, 26, 27]). The classes $\mathcal{P}_f^*$ and $\mathcal{P}_m^*$ were originally defined in [12, 13] in the case of the Brownian motion on R^n.

One of our main objectives in this paper is to study the similarities in the behavior of free backward propagators and backward Feynman-Kac propagators. We are especially interested in the inheritance of properties of free backward propagators by backward Feynman-Kac propagators. Various results concerning perturbations of semigroups and propagators by functions and measures were obtained in [1, 2, 3, 9, 11, 12, 13, 14, 23, 28, 33]. Some of the methods used in the present paper were borrowed from [8, 28]. In [23], interesting results concerning the inheritance of the Feller property were obtained. The authors of [23] studied perturbations of strongly continuous semigroups by measures in the case of general locally compact spaces. The paper [9] also deserves a special attention. In it, the strong continuity in L^p of semigroups perturbed by signed measures was studied.

An important relation between Y_μ and Y is given by Duhamel's formula,

$$Y_\mu(\tau, t)f(x) = Y(\tau, t)f(x) + \int_\tau^t Y(\tau, s)[\mu(s)Y_\mu(s, t)f](x)ds.$$

For a bounded Borel function f and $\mu \in \mathcal{P}_m^*$, this formula holds for every $x \in E$ (see [15]). If we impose more restrictions, then the function $u(\tau, x) = Y_\mu(\tau, t)f(x)$ is a solution in the sense of measures to the following final value problem:

$$\begin{cases} D_1^+ u(\tau) + A(\tau)u(\tau) - \mu(\tau)u(\tau) = 0, & 0 \le \tau < t \le T, \\ u(t) = f, \end{cases}$$

where D_1^+ stands for the τ-derivative from the right, and the family of operators $A(\tau)$ is defined by

$$A(\tau)h(x) = \lim_{\epsilon \to 0+} \epsilon^{-1}[Y(\tau, \tau + \epsilon)h(x) - h(x)]$$

(see [15]). The previous assertion explains why backward Feynman-Kac propagators are of importance in the theory of partial differential equations.

The structure of the present paper is as follows: Section 2 is devoted to general backward propagators on Banach spaces. Here we prove a theorem concerning the equivalence of separate and joint continuity of backward propagators (Theorem 2.2). In Section 3, we discuss free backward propagators associated with transition probability functions. The classes $\mathcal{P}_f^*$ and $\mathcal{P}_m^*$ are introduced and studied in Section 4. In Section 5, the additive functional A_μ is constructed for a time-dependent measure $\mu \in \mathcal{P}_m^*$. This construction is based on the power type estimates for the functional $A_V(\tau, t) = \int_\tau^t V(s, X_s)ds$, obtained in Section 5, and on an approximation lemma (Lemma 5.1). The latter result shows that for every $\mu \in \mathcal{P}_m^*$ there exists a sequence $g_k \in \mathcal{P}_f^*$ approaching μ in a certain sense. This kind of approximation is weaker then the approximation in the norm of the class $\mathcal{P}_m^*$. Some ideas used in the proof of the existence and uniqueness results for A_μ (Theorem 5.6 and Lemma 5.13) were borrowed from the theory of Dirichlet forms (see [9], theorems 5.1.1 and 5.1.2). Section 6 is devoted to the exponential estimates for the functionals A_V and A_μ. The rest of the paper concerns the inheritance of various properties of backward free propagators by backward Feynman-Kac propagators. In Section 7, we study the boundedness of the propagators Y_V and Y_μ on the space L^r and also the $(L^r - L^q)$-smoothing by Y_V and Y_μ. An important difference between Schrödinger semigroups with Kato class potentials and the backward Feynman-Kac propagators Y_V and Y_μ with $V \in \mathcal{P}_f^*$ and $\mu \in \mathcal{P}_m^*$ is that Schrödinger semigroups with Kato class potentials inherit the L^1-boundedness from the heat semigroup, while there are examples of backward Feynman-Kac propagators for which the L^1-boundedness is not inherited. This was shown in [13]. Section 8 of the present paper is devoted to the behavior of backward propagators on various spaces of continuous functions on E. Here we discuss the inheritance of the strong Feller and the strong BUC-property (see Theorem 8.3), and show that the Feller-Dynkin and the BUC-property is inherited under some additional restrictions on the free backward propagator (see theorems 8.4 and 8.5). In Section 9, we define various subclasses of the classes $\mathcal{P}_f^*$ and $\mathcal{P}_m^*$, and prove that the Feller, the Feller-Dynkin, and the BUC-properties are inherited by backward Feynman-Kac propagators in the case of functions or time-dependent measures from these subclasses. In section 10, we consider free backward propagators associated with fundamental solutions of second order parabolic partial differential equations with time-dependent coefficients, and discuss the properties of the corresponding backward Feynman-Kac propagators. Finally, in Section 11, we translate the results obtained in the previous sections from the language of backward propagators into the language of forward propagators.

2. Backward propagators on Banach spaces

Let B be a Banach space. By $L(B,B)$ will be denoted the space of all bounded linear operators on B. Let $T > 0$ be a given number, and put $D_T = \{(\tau,t) : 0 \le \tau \le t \le T\}$. Put also $D_\infty = \{(\tau,t) : 0 \le \tau \le t < \infty\}$.

Definition 2.1. *A two-parametric family of operators* $\{Q(\tau,t) \in L(B,B) : (\tau,t) \in D_\infty\}$ *is called a backward propagator on* B*, provided that the following conditions hold:*
(1) $Q(\tau,t) = Q(\tau,\lambda)Q(\lambda,t)$ *for* $0 \le \tau \le \lambda \le t < \infty$.
(2) $Q(t,t) = I$ *for* $0 \le t < \infty$.
If the family S *is defined on* D_T *with* $T < 0$*, then it will be assumed in 1 and 2 that* $(\tau,t) \in D_T$.

We will say that a backward propagator Q is strongly continuous if for every $x \in B$, the function $(\tau,t) \to Q(\tau,t)x$ is continuous. For $0 > t \le \infty$, a backward propagator Q will be called uniformly bounded if $||Q(\tau,t)||_{B\to B} \le M$ for all $(\tau,t) \in D_T$. If $T = \infty$, and for every compact subset K of D_∞, $||Q(\tau,t)||_{B\to B} \le M_K$ for all $(\tau,t) \in K$, then Q will be called locally uniformly bounded. A backward propagator Q is called separately strongly continuous if for every fixed t and $x \in B$, the function $\tau \to Q(\tau,t)x$ is continuous on $[0,t]$, and for every fixed τ and $x \in B$, the function $t \to Q(\tau,t)x$ is continuous on $[\tau,T]$ (if $T = \infty$, then we consider the interval $[t,\infty)$ instead of the interval $[t,T]$).

The next theorem shows that the joint continuity and the separate continuity are equivalent if a backward propagator is locally uniformly bounded.

Theorem 2.2. *For a backward propagator* Q *on* B*, the following are equivalent:*

(i) *The strong continuity.*
(ii) *The strong separate continuity and the uniform local boundedness.*

Proof. Using the uniform boundedness principle, we see that (i) implies (ii).

Now let Q be a strongly separately continuous and locally uniformly bounded propagator. Let $(\tau, t) \in D_T$, and suppose t' and τ' are close to t and τ, respectively. We will first assume that $t > \tau$. Then for τ' close to τ, we have $t > \tau'$. Using the local uniform boundedness condition and assuming that $t' \geq t$, we get that for every $x \in B$,

$$\begin{aligned} I &= ||Q(\tau', t')x - Q(\tau, t)x||_B \\ &\leq ||Q(\tau', t')x - Q(\tau', t)x||_B + ||Q(\tau', t)x - Q(\tau, t)x||_B \\ &\leq ||Q(\tau', t)(Q(t, t')x - x)||_B + ||Q(\tau', t)x - Q(\tau, t)x||_B \\ &\leq M||Q(t, t')x - x||_B + ||Q(\tau', t)x - Q(\tau, t)x||_B. \end{aligned}$$

Next the separate continuity condition gives

$$\lim_{t' \to t, \tau' \to \tau} I = 0. \tag{1}$$

If $t' < t$, then

$$\begin{aligned} I &\leq ||Q(\tau', t')x - Q(\tau', t)x||_B + ||Q(\tau', t)x - Q(\tau, t)x||_B \\ &\leq ||Q(\tau', t')(x - Q(t', t)x)||_B + ||Q(\tau', t)x - Q(\tau, t)x||_B \\ &\leq M||Q(t', t)x - x||_B + ||Q(\tau', t)x - Q(\tau, t)x||_B, \end{aligned}$$

and we again get formula (1).

Finally, let $\tau = t < \tau' \leq t'$. Then the separate continuity condition implies that for every $\epsilon > 0$ there exists $\lambda > 0$ such that $\lambda > \tau$ and

$$||Q(\tau, \lambda)x - x||_B \leq \epsilon. \tag{2}$$

If follows from the local uniform boundedness condition and from (2) that

$$\begin{aligned} I &= ||Q(\tau', t')x - x||_B \\ &\leq ||Q(\tau', t')x - Q(\tau', \lambda)x||_B + ||Q(\tau', \lambda)x - Q(\tau, \lambda)x||_B \\ &+ ||Q(\tau, \lambda)x - x||_B \leq ||Q(\tau', t')(x - Q(t', \lambda)x||_B \\ &+ ||Q(\tau', \lambda)x - Q(\tau, \lambda)x||_B + \epsilon \leq M||Q(t', \lambda)x - x||_B \\ &+ ||Q(\tau', \lambda)x - Q(\tau, \lambda)x||_B + \epsilon \leq M||Q(t', \lambda)x - Q(\tau, \lambda)x||_B \\ &+ M||Q(\tau, \lambda)x - x||_B + ||Q(\tau', \lambda)x - Q(\tau, \lambda)x||_B + \epsilon \\ &\leq M||Q(t', \lambda)x - Q(\tau, \lambda)x||_B + ||Q(\tau', \lambda)x - Q(\tau, \lambda)x||_B + (M + 1)\epsilon. \end{aligned} \tag{3}$$

In (3), M depends on t. It follows from (3) and from the separate continuity condition that there exists $\delta > 0$ such that for $\tau \leq \tau' \leq t' < \tau + \delta$, we have $I \leq (2M + 2)\epsilon$. Therefore, (1) holds for $\tau = t < \tau' \leq t'$.

This completes the proof of Theorem 2.2.

3. Transition functions and free backward propagators

In this section, we gather known definitions and facts from the theory of non-homogeneous Markov processes, and define the backward Feynman-Kac propagator Y_V. Let E be a locally compact second countable Hausdorff topological space. Then E is σ-compact and metrisable (see [19]). We will fix a metric $\rho : E \times E \to [0, \infty)$ generating the topology of E, and denote by $\mathcal{E}$ the σ-algebra of Borel subsets of E. The symbol BC will stand for the space of all bounded continuous functions on E equipped with the norm $||f||_C = \sup_{x \in E} |f(x)|$. By C_∞ will be denoted the space of all continuous functions on E vanishing at infinity, and by BUC the space of bounded uniformly continuous functions on E. It is easy to show that C_∞ is a closed subspace of BUC, and BUC is a closed subspace of BC. The symbol $L^\infty_{\mathcal{E}}$ will stand for the space of all bounded Borel functions on E.

Let $P(r, x; s, A)$, where $0 \le r < s < \infty$, $x \in E$, and $A \in \mathcal{E}$, be a transition probability function. This means that the following conditions hold:

(1) For fixed r, s, and A, P is a nonnegative Borel measurable function on E.
(2) For fixed r, s, and x, P is a Borel measure on $\mathcal{E}$.
(3) $P(r, x; s, E) = 1$ for all r, s, and x.
(4) $P(r, x; s, A) = \int_E P(r, x; u, dy) P(u, y; s, A)$ for all $r < u < s$, and A.

Given a transition probability function P, we can define a family of contraction operators on $L^\infty_{\mathcal{E}}$ by

$$\begin{cases} Y(\tau, t) f(x) = \int_E f(y) P(\tau, x; t, dy), & 0 \le \tau < t < \infty \\ Y(t, t) f(x) = f(x), & 0 \le t < \infty \end{cases}$$

for all $x \in E$ and $f \in L^\infty_{\mathcal{E}}$. This family will be called the free backward propagator associated with P.

Let us fix a non-negative Borel measure m on $(E, \mathcal{E}$ (the reference measure). We will write dx instead of $m(dx)$, and will always assume that $0 < m(A) < \infty$ for any compact subset A of E having nonempty interior. By L^r with $1 \le r \le \infty$ will be denoted the usual Lebesgue space with respect to the reference measure m. The space of all Borel functions from L^r will be denoted by $L^r_{\mathcal{E}}$. If P is a transition probability function, then we will say that P possesses density p, if there exists a nonnegative function $p(r, x; s, y)$ such that

$$P(r, x; s, A) = \int_A p(r, x; s, y) dy$$

for all $A \in \mathcal{E}$. In this case, the free backward propagator Y is defined on the space L^∞ by

$$\begin{cases} Y(t,\tau)f(x) = \int_E f(y)p(\tau,x;t,y)dy, & 0 \le \tau < t < \infty, \\ Y(t,t)f(x) = f(x), & 0 \le t < \infty. \end{cases}$$

A rich source of transition probability densities is the theory of second order non-divergence or divergence form parabolic partial differential equations on R^n. If there exists a fundamental solution for such an equation, then it can be used as a transition density. Numerous results concerning the existence of fundamental solutions in the case of equations with time-dependent coefficients can be found in [6, 7, 16, 21] (see also the references in these papers). We will discuss such examples of transition probability functions in Section 10.

Let $\Omega = E^{[0,\infty)}$ denote the path space equipped with the cylindrical σ-algebra $\mathcal{F}$, and let P be a transition probability function. Then there exists a non-terminating non-homogeneous Markov process $(X_t, \mathcal{F}_t^\tau, P_{\tau,x})$, $(\tau,t) \in D_\infty$, on $(\Omega,\mathcal{F})$ with the phase space $(E,\mathcal{E})$. Here $X_t(\omega) = \omega(t)$, $\mathcal{F}_t^\tau = \sigma(X_s : \tau \le s \le t)$, and $P_{\tau,x}$ with $0 \le \tau \le T$ and $x \in E$ is a measure on $\mathcal{F}$ such that

$$P_{\tau,x}(\omega(t_1) \in A_1; \cdots ; \omega(t_{k-1}) \in A_{k-1}; \omega(t_k) \in A_k)$$
$$= \int_{A_1} P(\tau,x;t_1,dx_1) \int_{A_2} P(t_1,x_1;t_2,dx_2) \cdots \int_{A_k} P(t_{k-1},x_{k-1};t_k,dx_k)$$

for all $\tau < t_1 < t_2 < \cdots < t_k \le T$ and $A_i \in \mathcal{E}$ for $1 \le i \le k$. We will also need non-homogeneous Markov processes defined on a general probability space $(\Omega,\mathcal{F})$ (see [5, 10, 29]). Any two such processes with the same transition function P are called stochastically equivalent.

The Markov property of the process X can be formulated as follows: For all $0 \le \tau \le s \le t \le T$, $x \in E$, and $g \in L^\infty_{\mathcal{E}}$,

$$Y(t,s)g(X_s) = E_{\tau,x}(g(X_t)|\mathcal{F}_s^\tau)$$

$P_{\tau,x}$ a.s. In the present paper, we will restrict ourselves to progressively measurable processes. A process X is called $\mathcal{F}_t^\tau$-progressively measurable, or simply progressively measurable, if for every τ and t with $0 \le \tau < t \le T$, the mapping $(s,\omega) \to \omega(s)$ of $[\tau,t] \times \Omega$ into E is $\sigma(\mathcal{B}_{[\tau,t]} \times \mathcal{F}_t^\tau)/\mathcal{E}$-measurable. It is known that every left- or right-continuous process is progressively measurable (see [10, 32] for more information concerning progressive measurability of stochastic processes). We will denote by $\mathcal{M}$ the class of all transition probability functions P such that there exists a progressively measurable process X corresponding to P. If $P \in \mathcal{M}$, then we will always choose a progressively measurable process X to represent P.

A process X associated with P is called stochastically continuous if for all $0 \le \tau < s < t \le T$, $x \in E$, and $\epsilon > 0$, we have

$$\lim_{t-s \to 0+} P_{\tau,x}(\rho(X_s, X_t) > \epsilon) = 0.$$

For $\epsilon > 0$ and $y \in E$, put $G_\epsilon(y) = \{x \in E : \rho(x, y) > \epsilon\}$. Then an equivalent condition for the stochastic continuity of the process X is as follows:

$$\lim_{t-s \to 0+} \int_E P(s, y; t, G_\epsilon(y)) P(\tau, x; s, dy) = 0 \tag{4}$$

where it is assumed that $0 \le \tau < s < t \le T$, $x \in E$, and $\epsilon > 0$. It is known that condition (4) implies $P \in \mathcal{M}$ (this follows from Theorem 4 in Chapter 1.6 in [10]). The uniform stochastic continuity condition for the transition function P,

$$\lim_{t-s \to 0+} \sup_{y \in E} P(s, y; t, G_\epsilon(y)) = 0$$

for all $\epsilon > 0$, is stronger than condition (4). It implies the existence of a process X corresponding to P such that X is right-continuous and possesses left-hand limits (see [10], Theorem 2 on p. 75).

The sample paths of progressively measurable processes are Borel measurable functions. Hence, if X is progressively measurable, then the functional

$$A_V(\tau, t) = \int_\tau^t V(s, X_s) ds$$

is defined for appropriate Borel functions V on $[0, T] \times E$. Moreover, the random variable $A_V(\tau, t)$ is $\mathcal{F}_t^\tau$-measurable.

Definition 3.1. *Let $P \in \mathcal{M}$, and let V be a Borel function on $[0, T] \times E$. We will call the family of linear operators,*

$$Y_V(\tau, t) g(x) = E_{\tau,x} g(X_t) \exp\{-\int_\tau^t V(s, X_s) ds\}, \ \ 0 \le \tau \le t \le T,$$

the backward Feynman-Kac propagator associated with P and V.

The restrictions on V and g under which the backward Feynman-Kac propagator Y_V exists will be given later. For a time-dependent measure μ, the backward propagator Y_μ will be defined in Section 5.

4. Non-autonomous classes of functions and measures

Let us denote by V a Borel function on the set $[0, T] \times E$, where $T > 0$ is a fixed number, and by μ a family $\mu = \{\mu(t) : 0 \le t \le T\}$ of signed measures on $(E, \mathcal{E})$. Recall that in the definition of a signed measure ν it is assumed that at least one of the measures ν^+ and ν^- is finite. Let P be a transition probability function from the class $\mathcal{M}$. For V as above, put

$$N_V(\tau,t,x)=\int_\tau^t Y(\tau,s)V(s)(x)ds,\ \ (\tau,t)\in D_T,\ \ x\in E. \tag{5}$$

The function $N(V)$ is, in a sense, a potential of V. In the case of the family μ as above, we assume that P possesses density p. Then we can define the potential N_μ by

$$N(\mu)(\tau,t,x)=\int_\tau^t Y(\tau,s)\mu(s)(x)ds,\ \ (\tau,t)\in D_T,\ \ x\in E. \tag{6}$$

It is assumed in (5) and (6) that the integrals on the right-hand side make sense.

Definition 4.1. *Let $P\in\mathcal{M}$. Then we say that V belongs to the class $\hat{\mathcal{P}}_f^*$, provided that*

$$\sup_{(\tau,t)\in D_T}\sup_{x\in E}N(|V|)(\tau,t,x)<\infty.$$

Let $V\in\hat{\mathcal{P}}_f^*$. Then we say that V belongs to the class $\mathcal{P}_f^*$, provided that

$$\lim_{t-\tau\to 0+}\sup_{x\in E}N(|V|)(t,\tau,x)=0.$$

Suppose that $P\in\mathcal{M}$ possesses density p. Then we say that μ belongs to the class $\hat{\mathcal{P}}_m^*$, provided that

$$\sup_{\tau:(\tau,t)\in D_T}\sup_{x\in E}N(|\mu|)(\tau,t,x)<\infty.$$

If $\mu\in\hat{\mathcal{P}}_m^*$, then we say that μ belongs to the class $\mathcal{P}_m^*$, provided that

$$\lim_{t-\tau\to 0+}\sup_{x\in E}N(|\mu|)(\tau,t,x)=0.$$

In the case of the heat semigroup, the classes in Definition 4.1 were studied in [12, 13].

Remark 4.2. Note that the function classes $\hat{\mathcal{P}}_f^*$ and $\mathcal{P}_f^*$ are defined for any transition probability function $P\in\mathcal{M}$, while in the case of time-dependent measures we restrict ourselves to transition probability functions possessing densities. In the latter case, the function classes $\hat{\mathcal{P}}_f^*$ and $\mathcal{P}_f^*$ are in one-to-one correspondence, $V(\tau,x)\leftrightarrow d\mu(\tau,x)=V(\tau,x)dx$, with subclasses of the classes $\hat{\mathcal{P}}_m^*$ and $\mathcal{P}_m^*$.

For $V\in\hat{\mathcal{P}}_f^*$ and $\mu\in\hat{\mathcal{P}}_m^*$, we put

$$||V||_f=\sup_{(\tau,t)\in D_T}\sup_{x\in E}N(|V|)(\tau,t,x),$$

and

$$||\mu||_m = \sup_{(\tau,t)\in D_T} \sup_{x\in E} N(|\mu|)(\tau,t,x).$$

It is clear that

$$||V||_f = \sup_{\tau:0\le\tau\le T} \sup_{x\in E} N(|\mu|)(\tau,T,x)$$

and

$$||\mu||_m = \sup_{\tau:0\le\tau\le T} \sup_{x\in E} N(|\mu|)(\tau,T,x).$$

Remark 4.3. Let l denote the Lebesgue measure on the σ-algebra $\mathcal{B}_{[0,T]}$ of all Borel subsets of $[0,T]$, and $l_{[\tau,T]}$ denote the restriction of l to $[\tau,T]$. For every $\tau\in[0,T]$ and $x\in E$, define a measure $\xi_{\tau,x}$ on the sigma algebra $\sigma(\mathcal{B}_{[\tau,T]}\times\mathcal{E})$ as follows: For $U\in\sigma(\mathcal{B}_{[\tau,T]}\times\mathcal{E})$,

$$\xi_{\tau,x}(U)=\int_U P(\tau,x;u,dy)du.$$

Then for $V\in\hat{\mathcal{P}}_f^*$, the condition $||V||_f=0$ means that for all $\tau\in[0,T)$ and $x\in E$, we have $V(u,y)=0$ $\xi_{\tau,x}$-a.e. on $[\tau,T]\times E$. If P possesses density p such that

$$p(\tau,x;u,y)>0 \tag{7}$$

for all τ, x, u, and y, then we have another equivalent condition: $V(u,y)=0$ $l\times m$-a.e. on $[0,T]\times E$. If the density p exists and $\mu\in\hat{\mathcal{P}}_m^*$, then the condition $||\mu||_m=0$ means that

$$\int_\tau^T\int_E p(\tau,x;u,y)d|\mu(u)|(y)du=0$$

for all τ and x. If p satisfies (7), then we get the following equivalent condition: $\mu(u)=0$ for l-a.a. $u\in[0,T]$.

If we take into account the identifications described in Remark 4.3, then the spaces $(\hat{\mathcal{P}}_f^*,||\cdot||_f)$ and $(\hat{\mathcal{P}}_m^*,||\cdot||_m)$ become normed spaces . The next result shows that more is true.

Lemma 4.4. *Let $P\in\mathcal{M}$. Then $(\hat{\mathcal{P}}_f^*,||\cdot||_f)$ is a Banach space, and $(\mathcal{P}_f^*,||\cdot||_f)$ is its closed subspace. Moreover if P possesses a strictly positive density p, then $(\hat{\mathcal{P}}_m^*,||\cdot||_m)$ is a Banach space, and $\mathcal{P}_m^*$ is its closed subspace.*

Proof. We will prove that if p is strictly positive, then the space $\hat{\mathcal{P}}_m^*$ is complete, and $\mathcal{P}_m^*$ is a closed subspace of $\hat{\mathcal{P}}_m^*$. The rest of the proof of Lemma 4.4 is similar.

Let $\mu_k\in\hat{\mathcal{P}}_m^*$, $k\ge1$, be a sequence such that

$$\sum_{k=1}^\infty||\mu_k||_m=\sum_k\sup_{\tau:0\le\tau\le T}\sup_{x\in E}\int_\tau^T du\int_E p(\tau,x;u,y)d|\mu_k(u,y)|<\infty. \tag{8}$$

Then for every $x \in E$, we have

$$\int_0^T du \int_E p(0,x;u,y) d \sum_{k=1}^{\infty} |\mu_k(u,y)| < \infty,$$

Hence, there exists a Borel set $U_x \in [0,T]$ such that $l(U_x) = T$ and

$$\int_E p(0,x;u,y) d \sum_{k=1}^{\infty} |\mu_k(u,y)| < \infty \tag{9}$$

for all $u \in U_x$. Fix $x \in E$. Then (9) implies that for every $j \geq 1$ and $u \in U_x$,

$$\sum_{k=1}^{\infty} |\mu_k(u)|(A_{j,u}) < \infty$$

where $A_{j,u} = \{y \in E : p(0,x;u,y) > j^{-1}\}$. Hence, $\sum \mu_k(u)$ is a finite signed Borel measure on every set $A_{j,u}$ for all $u \in U_x$. Since the strict positivity of p implies $\cup_{j=1}^{\infty} A_{j,u} = E$ for all $u \in U_x$, the measure $\mu(u) = \sum \mu_k(u)$ is a signed Borel measure on E for all $u \in U_x$, and hence l-a.e. on $[0,T]$. It follows from (8) that $\mu \in \hat{\mathcal{P}}_m^*$, and it is not difficult to prove using (8) that the series $\sum_{k=1}^{\infty} \mu_k$ converges to μ in the space $\hat{\mathcal{P}}_m^*$. This proves the completeness of $\hat{\mathcal{P}}_m^*$.

Now let $\nu_k \in \mathcal{P}_m^*$, $k \geq 1$, be such that $\nu_k \to \nu$ in $\hat{\mathcal{P}}_m^*$. We have

$$\int_\tau^t Y(\tau,u)|\mu(u)|(x)du \leq \int_\tau^t Y(\tau,u)|\mu(u) - \mu_k(u)|(x)du + \int_\tau^t Y(\tau,u)|\mu_k(u)|(x)du. \tag{10}$$

It follows from (10) that $\mu \in \mathcal{P}_m^*$. Hence, the class $\mathcal{P}_m^*$ is a closed subspace of the space $\hat{\mathcal{P}}_m^*$.

This completes the proof of Lemma 4.4.

The next result provides a description of the classes $\mathcal{P}_f^*$ and $\mathcal{P}_m^*$ in terms of the potential operator N.

Lemma 4.5. (a) *Let $P \in \mathcal{M}$ and $V \in \hat{\mathcal{P}}_f^*$. Then $V \in \mathcal{P}_f^*$ if and only if*

$$\lim_{t'-t \to 0+} \sup_{\tau: 0 \leq \tau \leq t} \sup_{x \in E} [N(|V|)(\tau,t',x) - N(|V|)(\tau,t,x)] = 0. \tag{11}$$

(b) *Suppose that $P \in \mathcal{M}$ possesses density p, and let $\mu \in \hat{\mathcal{P}}_m^*$. Then $\mu \in \mathcal{P}_m^*$ if and only if (11) holds with μ instead of V.*

Proof. Part (a). Let $V \in \mathcal{P}_f^*$. Then we have

$$\begin{aligned} N(|V|)(\tau,t',x) - N(|V|)(\tau,t,x) &= \int_t^{t'} Y(\tau,u)|V(u)|(x)du \\ &= Y(\tau,t)N(|V|)(t,t')(x). \end{aligned}$$

It follows that

$$\sup_{x\in E}[N(|V|)(\tau,t',x)-N(|V|)(\tau,t,x)]\le\sup_{x\in E}N(|V|)(t,t',x).$$

It is clear that the previous estimate implies (11).

Now assume that (11) holds. Then we have

$$\lim_{t-\tau\to 0}\sup_{x\in E}N(|V|)(\tau,t,x)=\lim_{t-\tau\to 0}\sup_{x\in E}[N(|V|)(\tau,t,x)-N(|V|)(\tau,\tau,x)]=0.$$

This implies $V\in\mathcal{P}_f^*$.

The proof of part (b) is similar.

Remark 4.6. It is easy to see from the proof of Lemma 4.5 that

$$\lim_{t'-t\to 0+}\sup_{\tau:0\le\tau\le t}\sup_{x\in E}|N(V)(\tau,t',x)-N(V)(\tau,t,x)|=0.$$

In the sequel, we will use the following notation:

$$M(V)(\tau,t)=\sup_{r:\tau\le r\le t}\sup_{x\in E}|N(V)(r,t,x)|$$

and

$$M(\mu)(\tau,t)=\sup_{r:\tau\le r\le t}\sup_{x\in E}|N(\mu)(r,t,x)|.$$

5. Construction of A_μ

Our main objective in this section is to define and study the additive functional A_μ where μ is a time-dependent measure from the class $\mathcal{P}_m^*$. The functional A_μ will replace the functional,

$$A_V(\tau,t)=\int_\tau^t V(s,X_s)ds, \tag{12}$$

in the definition of the Feynman-Kac propagator Y_μ for $\mu\in\mathcal{P}_m^*$. A natural idea is to try to approximate the time-dependent measure μ by a sequence of functions $g_k\in\mathcal{P}_f^*$ such that the corresponding sequence of functionals A_{g_k} converges in an appropriate sense. Then the functional A_μ can be defined as the limit of the sequence A_{g_k}. Since the approximation of time-dependent measures by functions in the norm topology of the space $\hat{\mathcal{P}}_m^*$ is not always possible (see Lemma 4.4), we should look for weaker conditions. The second part of the next assertion will be helpful in the construction of A_μ.

Lemma 5.1. *(a) Let $P\in\mathcal{M}$ and $V\in\mathcal{P}_f^*$. For $k\ge 1$, $0\le\tau\le T$, and $x\in E$, put*

$$g_k(\tau,x)=kN(V)(\tau,\min(\tau+\frac{1}{k},T),x). \tag{13}$$

Then the following conditions hold:

$$g_k\in\mathcal{P}_f^* \tag{14}$$

for all $k \geq 1$;

$$\lim_{k\to\infty} \sup_{(\tau,t)\in D_T} \sup_{x\in E} |N(V-g_k)(\tau,t,x)| = 0; \tag{15}$$

and

$$\lim_{t-\tau\to 0+} \sup_{k\geq 1} \sup_{x\in E} N(|g_k|)(\tau,t,x) = 0. \tag{16}$$

(*b*) *Suppose that $P \in \mathcal{M}$ possesses density p, and let $\mu \in \mathcal{P}_m^*$. For $k \geq 1$, $0 \leq \tau \leq T$, and $x \in E$, put*

$$g_k(\tau,x) = kN(V)(\tau, \min(\tau + \frac{1}{k}, T), x). \tag{17}$$

Then conditions (14)-(16) in part (*a*) *of Lemma 5.1 hold with μ instead of V.*

Proof. (*a*) We have

$$\begin{aligned} N(g_k)(\tau,t,x) &= k\int_\tau^t Y(\tau,s)ds \int_s^{\min(s+\frac{1}{k},T)} Y(s,u)V(u)(x)du \\ &= k\int_\tau^t ds \int_s^{\min(s+\frac{1}{k},T)} Y(\tau,u)V(u)(x)du \\ &= k\int_\tau^{\min(t+\frac{1}{k},T)} Y(\tau,u)V(u)(x)du \int_\tau^t \chi_{C_k(u)}(s)ds, \end{aligned} \tag{18}$$

where $\chi_{C_k(u)}$ is the characteristic function of the set $C_k(u) = \{s : s \leq u \leq \min(s + \frac{1}{k}, T)\}$. It follows from (18) that

$$\begin{aligned} N(g_k)(\tau,t,x) &= k\int_\tau^{\min(\tau+\frac{1}{k},T)} Y(\tau,u)V(u)(x)du \int_\tau^t \chi_{C_k(u)}(s)ds \\ &+ \int_{\min(\tau+\frac{1}{k},t)}^t Y(\tau,u)V(u)(x)du \\ &+ k\int_t^{\min(t+\frac{1}{k},T)} Y(\tau,u)V(u)(x)du \int_\tau^t \chi_{C_k(u)}(s)ds. \end{aligned} \tag{19}$$

Using (19), we obtain

$$\begin{aligned} |N(V-g_k)(\tau,t,x)| &\leq \int_\tau^{\min(\tau+\frac{1}{k},t)} Y(\tau,u)|V(u)|(x)du \\ &+ k\int_\tau^{\min(\tau+\frac{1}{k},t)} Y(\tau,u)|V(u)|(x)du \int_\tau^t \chi_{C_k(u)}(s)ds \\ &+ k\int_t^{\min(t+\frac{1}{k},T)} Y(\tau,u)|V(u)|(x)du \int_\tau^t \chi_{C_k(u)}(s)ds. \end{aligned}$$

Since the Lebesgue measure of the set $C_k(u)$ does not exceed $\frac{1}{k}$, we get

$$\begin{aligned}|N(V-g_k)(\tau,t,x)| &\leq 2N(|V|)(\tau,\min(\tau+\frac{1}{k},t),x) \\ &+ Y(\tau,t)\int_t^{\min(t+\frac{1}{k},T)} Y(t,u)|V(u)|(x)du \\ &= 2N(|V|)(\tau,\min(\tau+\frac{1}{k},t),x) \\ &+ Y(\tau,t)N(|V|)(t,\min(t+\frac{1}{k},T)(x).\end{aligned}$$

Therefore,

$$\begin{aligned}\sup_{x\in E}|N(V-g_k)(\tau,t,x)| &\leq 2\sup_{x\in E}N(|V|)(\tau,\min(\tau+\frac{1}{k},t),x) \\ &+ \sup_{x\in E}N(|V|)(t,\min(t+\frac{1}{k},T),x). \qquad (20)\end{aligned}$$

Now it is clear that (20) and the definition of the class $\mathcal{P}_f^*$ imply condition (15) in Lemma 5.1.

Since

$$\begin{aligned}N(|g_k|)(\tau,t,x) &\leq k\int_\tau^t Y(\tau,s)ds\int_s^{\min(s+\frac{1}{k},T)} Y(s,u)|V(u)|(x)du \\ &= k\int_\tau^t ds\int_s^{\min(s+\frac{1}{k},T)} Y(\tau,u)|V(u)|(x)du \\ &= k\int_\tau^{\min(t+\frac{1}{k},T)} Y(\tau,u)|V(u)|(x)du\int_{\tau,t}\chi_{C_k(u)}(s)ds,\end{aligned}$$

we get

$$N(|g_k|)(\tau,t,x)\leq k\min(t-\tau,\frac{1}{k})N(|V|)(\tau,\min(t+\frac{1}{k},T),x). \qquad (21)$$

It is clear that (14) follows from (21).

It remains to show that (16) holds. Let $\epsilon>0$. Then (21) and Lemma 4.5 imply that there exist $\delta_1>0$ and $k_0>1$ such that

$$\sup_{x\in E}N(|g_k|)(\tau,t,x)<\epsilon \qquad (22)$$

for all $t-\tau<\delta_1$ and $k\geq k_0$. Moreover, since $g_k\in\mathcal{P}_f^*$, there exists $\delta_2>0$ such that $\delta_2<\delta_1$ and (22) holds for all $t-\tau<\delta_2$ and $k\leq k_0$. Hence, (22) holds for all $k\geq 1$ and $t-\tau<\delta_2$, and we get (16).

This completes the proof of part (a) of Lemma 5.1. The proof of part (b) is similar.

Remark 5.2. Suppose that the conditions in part (b) of Lemma 5.1 hold. Then it follows from (15) that

$$\lim_{k\to\infty} M(g_k)(\tau,t) = M(\mu)(\tau,t).$$

Moreover, (21) implies

$$\limsup_{k\to\infty} N(|g_k|)(\tau,t,x) \le N(|\mu|)(\tau,t,x)$$

and

$$\limsup_{k\to\infty} M(|g_k|)(\tau,t) \le M(|\mu|)(\tau,t).$$

The next definition will be useful in the sequel.

Definition 5.3. *Let $P \in \mathcal{M}$, $V \in \mathcal{P}_f^*$, and $V_k \in \mathcal{P}_f^*$ for all $k \ge 1$. Then we will say that the sequence V_k ζ-approaches V provided that*

$$\lim_{k\to\infty} \sup_{(\tau,t)\in D_T} \sup_{x\in E} |N(V - V_k)(\tau,t,x)| \to 0,$$

and

$$\lim_{t-\tau\to 0+} \sup_{k\ge 1} \sup_{x\in E} N(|V_k|)(\tau,t,x) = 0.$$

If $P \in \mathcal{M}$ possesses density p, then we will say that a sequence $\mu_k \in \mathcal{P}_m^*$, $k \ge 1$, ζ-approaches $\mu \in \mathcal{P}_f^*$ provided that

$$\lim_{k\to\infty} \sup_{(\tau,t)\in D_T} \sup_{x\in E} |N(\mu - \mu_k)(\tau,t,x)| \to 0,$$

and

$$\lim_{t-\tau\to 0+} \sup_{k\ge 1} \sup_{x\in E} N(|\mu_k|)(\tau,t,x) = 0.$$

Remark 5.4. It follows from (13), (17), and the definition of the class $\hat{\mathcal{P}}_f^*$ that the functions g_k in Lemma 5.1 are bounded. Hence, for any $V \in \mathcal{P}_f^*$ ($\mu \in \mathcal{P}_m^*$) there exists a sequence of bounded functions approaching V (μ) in the ζ-sense.

Lemma 5.5. *If*

$$\lim_{t-\tau\to 0+} \sup_{k\ge 1} \sup_{x\in E} N(|V_k|)(\tau,t,x) = 0,$$

then $\sup_k ||V_k||_f < \infty$. *The same result is true for time-dependent measures.*

Proof. It follows from the assumptions that there exists $\delta > 0$ such that for $t - \tau < \delta$ and $k \ge 1$, we have

$$\sup_{x\in E} \sup_{k\ge 1} N(|V_k|)(\tau,t,x) < 1. \tag{23}$$

For every $(\tau, t) \in D_T$ with $\tau < t$, there exists a partition $\tau = t_0 < \cdots < t_n = t$ such that $\max\{|t_{j+1} - t_j| : 0 \le j \le n-1\} < \delta$ and $n < \delta^{-1}T$. Then we have

$$\begin{aligned} N(|V_k|)(\tau, t, x) &= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} Y(\tau, s)|V_k|(x)ds \\ &= \sum_{j=0}^{n-1} Y(\tau, t_j) \int_{t_j}^{t_{j+1}} Y(t_j, s)|V_k|(x)ds \\ &= \sum_{j=0}^{n-1} Y(\tau, t_j) N(|V_k|)(t_j, t_{j+1})(x). \end{aligned}$$

Therefore, using (23) we get

$$\sup_{k \ge 1} \sup_{x \in E} N(|V_k|)(\tau, t, x) \le \sum_{k \ge 1}^{n-1} N(|N_k|)(t_j, t_{j+1}, x) \le n < \frac{T}{\delta}.$$

This completes the proof of Lemma 5.5 in the case of functions. The case of measures is similar.

It would be interesting to try to define a Hausdorff topology on the class $\mathcal{P}_f^*$ using the conditions in Definition 5.3. This topology should be weaker than the norm topology of $\hat{\mathcal{P}}_f^*$ restricted to $\mathcal{P}_f^*$. Here it is important to answer the following question: Does the equality

$$\sup_{(t,\tau) \in D_T} \sup_{x \in E} |N(V)(t, \tau, x)| = 0$$

imply that $V(\tau, x) = 0$ $l \times m$ a.e. on $[0, T] \times E$?

Let V be a nonnegative function from the class $\mathcal{P}_f^*$. Then the functional A_V defined by formula (12) is a non-decreasing continuous additive functional. More precisely, it has the following properties:

(1) For all $\tau \le t$, the random variable $A_V(\tau, t)$ is $\mathcal{F}_t^\tau$-measurable.
(2) For all τ and $x \in E$, $A_V(\tau, \tau) = 0$ $P_{\tau,x}$-a.s.
(3) For all $\tau \le t$ and $x \in E$, the function $t \to A_V(\tau, t)$, $\tau \le t \le T$, is non-decreasing and continuous $P_{\tau,x}$-a.s.
(4) For all $\tau \le \lambda \le t$, $A_V(\tau, t) = A_V(\tau, \lambda) + A_V(\lambda, t)$ $P_{\tau,x}$-a.s.
(5) For all $\tau \le t$ and $x \in E$, $E_{\tau,x} A_V(\tau, t) = N(V)(\tau, t, x)$.

Our next goal is to define the functional A_μ in the case of a time-dependent measure $\mu \in \mathcal{P}_m^*$. Let g_k be the sequence corresponding to μ by formula (17). We will show below that the limit $A_\mu(\tau, t) = \lim_{k \to \infty} A_{g_k}(\tau, t)$ exists in a certain sense.

Theorem 5.6. *Let $P \in \mathcal{M}$ be a transition probability function possessing density p. Then for every family μ of nonnegative measures from the class $\mathcal{P}_m^*$, there exists a functional $A_\mu(\tau, t)$, $(\tau, t) \in D_T$, for which conditions 1-5 above hold.*

Remark 5.7. It will be shown below that

$$\lim_{k\to\infty} \sup_{\tau:0\le\tau\le T} \sup_{x\in E} E_{\tau,x} \sup_{t:\tau\le t\le T} |A_\mu(\tau,t) - A_{g_k}(\tau,t)|^n = 0, \tag{24}$$

for every $n \ge 1$ (see the proof of Theorem 5.6 below), and that A_μ is unique up to equivalence (see Lemma 5.13).

Proof of Theorem 5.6. The following assertion holds:

Lemma 5.8. *Let $P \in \mathcal{M}$ and $V \in \mathcal{P}_f^*$. Then for every t and τ with $(\tau, t) \in D_T$, $x \in E$, and any integer $n \ge 2$, we have*

$$|E_{\tau,x} A_V(\tau,t)^n| \le n! N(|V|)(\tau,t,x) M(|V|)(\tau,t)^{n-2} M(V)(\tau,t). \tag{25}$$

Proof. Using the Markov property and taking into account that $V \in \mathcal{P}_f^*$, we obtain

$$\begin{aligned} E_{\tau,x} A_V(\tau,t)^2 &= 2E_{\tau,x} \int_\tau^t V(s,X_s)ds \int_s^t V(u,X_u)du \\ &= 2E_{\tau,x} \int_\tau^t V(s,X_s)ds \int_s^t E_{\tau,x}(V(u,X_u)|\mathcal{F}_s^\tau)du \\ &= 2E_{\tau,x} \int_\tau^t V(s,X_s)ds \int_s^t E_{s,z} V(u,X_u)du|_{z=X_s} \\ &\le 2\int_\tau^t dsY(\tau,s)|V(s)|(x)ds \sup_{s:\tau\le s\le t} \sup_{y\in R^n} |\int_s^t Y(s,u)V(u)(y)du|. \end{aligned} \tag{26}$$

Now it is clear that (26) implies (25) with $n = 2$.

Next let $n > 2$ be any positive integer. By induction, we get

$$\begin{aligned} E_{\tau,x} A_V(\tau,t)^n &= n! E_{\tau,x} \int_\tau^t V(t_1,X_{t_1})dt_1 \int_{t_1}^t V(t_2,X_{t_2})dt_2 \cdots \int_{t_{n-1}}^t V(t_n,X_{t_n})dt_n \\ &\le n! \int_\tau^t dsY(\tau,s)|V(s)|(x)ds \sup_{r:\tau\le r\le t} \sup_{y\in E} [\int_r^t Y(r,u)|V(u)|(y)du]^{n-2} \\ &\times \sup_{r:\tau\le r\le t} \sup_{y\in E} |\int_r^t Y(r,u)V(u)(y)du|. \end{aligned}$$

The previous estimate implies (25), and completes the proof of Lemma 5.8.

Corollary 5.9. *Suppose that the conditions in Lemma 5.8 are satisfied. Then for any odd integer $n \ge 3$,*

$$E_{\tau,x}|A_V(\tau,t)|^n \le \sqrt{(n-1)!(n+1)!}N(|V|)(\tau,t,x)M(|V|)(\tau,t)^{n-2}M(V)(\tau,t). \tag{27}$$

Proof. If $n \ge 3$ is odd, then

$$E_{\tau,x}|A_V(\tau,t)|^n \le \{E_{\tau,x}A_V(\tau,t)^{n-1}\}^{\frac{1}{2}}\{E_{\tau,x}A_V(\tau,t)^{n+1}\}^{\frac{1}{2}}.$$

Now it is clear that (27) follows from Lemma 5.8.

Lemma 5.8 and Corollary 5.9 provide pointwise estimates for the expression $E_{\tau,x}|A_V(\tau,t)|^n$. The next lemma shows that stronger uniform estimates hold.

Lemma 5.10. *Let $P \in \mathcal{M}$ and $V \in \mathcal{P}_f^*$. Then for any τ with $0 \le \tau \le T$, any $\delta > 0$ such that $\tau + \delta \le T$, and any even integer $n \ge 2$, the following estimate holds:*

$$\sup_{x\in E} E_{\tau,x} \sup_{t:\tau\le t\le\tau+\delta} A_V(\tau,t)^n \le c_n M(|V|)(\tau,\tau+\delta)^{n-1}M(V)(\tau,\tau+\delta), \tag{28}$$

where

$$c_n = 2^n[(\frac{n}{n-1})^n n! + 1]. \tag{29}$$

Moreover, for any odd integer $n \ge 3$, we have

$$\sup_{x\in E} E_{\tau,x} \sup_{t:\tau\le t\le\tau+\delta} |A_V(\tau,t)|^n \le c_n M(|V|)(\tau,\tau+\delta)^{n-1}M(V)(\tau,\tau+\delta) \tag{30}$$

where $c_n = \sqrt{c_{n-1}c_{n+1}}$.

Remark 5.11. For $n = 1$, we have

$$\sup_{x\in E} E_{\tau,x} \sup_{t:\tau\le t\le\tau+\delta} |A_V(\tau,t)| \le \{c_2 M(|V|)(\tau,\tau+\delta)M(V)(\tau,\tau+\delta)\}^{\frac{1}{2}}. \tag{31}$$

Estimate (31) easily follows from (28) with $n = 2$.

Remark 5.12. It is clear that for $V \ge 0$, Lemma 5.10 follows from Lemma 5.8.

Proof of Lemma 5.10. We will prove estimate (28). Estimate (30) follows from (28) and Hölder's inequality.

Let $n \ge 2$ be an even integer, and let $V \in \mathcal{P}_f^*$. For given τ, x, and t with $\tau \le t \le \tau + \delta$, put

$$M_t = E_{\tau,x}(A_V(\tau,\tau+\delta)|\mathcal{F}_t^\tau).$$

It follows from (25) that M_t is an $\mathcal{F}_t^\tau$-martingale from L^n. Using the Markov property, we see that for every t with $\tau \le t \le \tau+\delta$,

$$\begin{aligned} M_t &= A_V(\tau,t)+\int_t^{\tau+\delta} E_{\tau,x}(V(s,X_s)|\mathcal{F}_t^\tau)ds \\ &= A_V(\tau,t)+\int_t^{\tau+\delta} Y(t,s)V(s)(X_t)ds \end{aligned}$$

$P_{\tau,x}$-a.s. Hence M_t is a modification of the functional $\tilde{M}_t = A_V(\tau,t) + N(V)(t,\tau+\delta,X_t)$.

Fix a partition $\tau = t_0 < t_1 < \cdots < t_k = \tau+\delta$. Using Doob's inequality (see [32]), we get

$$\begin{aligned} E_{\tau,x}\sup_{j:0\le j\le k} A_V(\tau,t_j)^n &\le 2^n E_{\tau,x}\sup_{j:0\le j\le k} M_{t_j}^n \\ &+ 2^n E_{\tau,x}\sup_{j:0\le j\le k}|N(V)(t_j,\tau+\delta,X_{t_j})|^n \\ &\le 2^n(\frac{n}{n-1})^n E_{\tau,x}A_V(\tau,\tau+\delta)^n \\ &+ 2^n[\sup_{z\in E}\sup_{s:\tau\le s\le\tau+\delta}|N(V)(s,\tau+\delta,z)|]^n \\ &\le 2^n(\frac{n}{n-1})^n E_{\tau,x}A_V(\tau,\tau+\delta)^n \\ &+ 2^n M(V)(\tau,\tau+\delta)^n. \qquad (32) \end{aligned}$$

It follows from (25) and (32) that

$$E_{\tau,x}\sup_{j:0\le j\le k} A_V(\tau,t_j)^n \le c_n M(|V|)(\tau,\tau+\delta)^{n-1}M(V)(\tau,\tau+\delta) \qquad (33)$$

for all $x\in E$ where c_n is defined by (29). Next we choose a sequence of refinements of the partition $\tau = t_0 < t_1 < \cdots < t_k = \tau+\delta$ on the left-hand side of (33) such that the maximum length of the partition intervals tends to 0, and pass to the limit, using the monotone convergence theorem and the continuity of $A_V(\tau,t)$ with respect to t. This establishes estimate (28), and completes the proof of Lemma 5.10.

Let us continue the proof of Theorem 5.6. For a nonnegative family $\mu\in\mathcal{P}_m^*$, define the sequence g_k by (17). Using (28) and (30), we obtain

$$\begin{aligned} &\sup_{x\in E} E_{\tau,x}\sup_{t:\tau\le t\le\tau+\delta}|A_{g_k-g_j}(\tau,t)|^n \le \\ &\qquad \le c_n M(|g_k-g_j|)(\tau,\tau+\delta)^{n-1}M(g_k-g_j)(\tau,\tau+\delta). \end{aligned}$$

It follows from Lemma 5.1 that

$$\lim_{j,k\to\infty}\sup_{\tau:0\le\tau\le T}\sup_{x\in E}E_{\tau,x}\sup_{t:\tau\le t\le\tau+\delta}|A_{g_k}(\tau,t)-A_{g_j}(\tau,t)|^n = 0.$$

Hence, there exists a functional A_μ such that (24) holds. Using the fact that every functional A_{g_k} satisfies conditions 1-5 in Theorem 5.6, we can prove that the functional A_μ defined by (24), also satisfies these conditions.

This completes the proof of Theorem 5.6.

It is not difficult to see that the functional A_μ does not depend on the choice of a sequence g_k such that g_k ζ-approaches μ. Actually, more is true.

Lemma 5.13. *Let μ be a non-negative family from the class $\mathcal{P}_m^*$. Let A_1 and A_2 be two functionals satisfying conditions 1-5 in Theorem 5.6. Then for every $0 \le \tau \le T$ and $x \in R^n$, the processes $A_1(\tau,t)$ and $A_2(\tau,t)$ are indistinguishable.*

Proof. It is easy to see, using the conditions in Theorem 5.6, that

$$\begin{aligned}
&E_{\tau,x}[A_1(\tau,t) - A_2(\tau,t)]^2 = \\
&\qquad = 2\sum_{i,j=1}^{2}(-1)^{i+j}E_{\tau,x}\int_\tau^t [A_i(\tau,t) - A_i(\tau,s)]dA_j(\tau,s) \\
&\qquad = 2\sum_{i,j=1}^{2}(-1)^{i+j}E_{\tau,x}\int_\tau^t A_i(s,t)dA_j(\tau,s) \qquad (34)
\end{aligned}$$

By condition 1, $A_i(t,s)$ is $\mathcal{F}_t^s$-measurable. Using the Markov property in (34), we get

$$\begin{aligned}
&E_{\tau,x}[A_1(\tau,t) - A_2(\tau,t)]^2 = \\
&\qquad = 2\sum_{i,j=1}^{2}(-1)^{i+j}E_{\tau,x}\int_\tau^t E_{z,s}A_i(s,t)|_{z=X_s}dA_j(\tau,s) \\
&\qquad = 2\sum_{i,j=1}^{2}(-1)^{i+j}E_{\tau,x}\int_\tau^t N_\mu(s,t,X_s)dA_j(\tau,s) = 0. \qquad (35)
\end{aligned}$$

It follows from (35) that for given τ and x, the process A_2 is a modification of A_1. Since both processes are continuous, they are indistinguishable.

This completes the proof of Lemma 5.13.

Remark 5.14. It is clear that under the conditions in Theorem 5.6, we can find a sequence of positive integers k' such that for a given number τ with $0 \le \tau \le T$ and $x \in R^n$,

$$A_{g_{k'}}(\tau,t) \to A_\mu(\tau,t) \qquad (36)$$

$P_{\tau,x}$-a.s. uniformly with respect to $t \in [\tau,T]$.

Definition 5.15. *Let $\mu \in \mathcal{P}_m^*$. Denote by μ^+ and μ^- the positive and the negative variation of the family μ, respectively. Then we define the* *functional* A_μ *as follows:*

$$A_\mu(\tau,t) = A_{\mu^+}(\tau,t) - A_{\mu^-}(\tau,t).$$

Now we are ready to give the definition of the backward Feynman-Kac propagator corresponding to $\mu \in \mathcal{P}_m^*$.

Definition 5.16. *Let $P \in \mathcal{M}$ be a transition probability function possessing density p, and let $\mu \in \mathcal{P}_m^*$. Then the family of operators,*

$$Y_\mu(\tau,t)g(x) = E_{\tau,x}g(X_t)\exp\{-A_\mu(\tau,t)\}, \ \ (\tau,t) \in D_T,$$

is called the backward Feynman-Kac propagator associated with P and μ.

Lemma 5.17. *Let $P \in \mathcal{M}$ be a transition probability function possessing density p, and let $\mu \in \mathcal{P}_m^*$. Then estimates (28) and (30) hold with μ instead of V.*

Proof. We will prove estimate (28) for a time-dependent measure μ. It is clear that estimate (30) for μ follows from (28).

Let $\mu \in \mathcal{P}_m^*$, and let g_k be the sequence constructed for μ in Lemma 5.1. Then $g_k \in \mathcal{P}_f^*$. Applying estimate (28) to the sequence g_k, we see that

$$\sup_{x\in E} E_{\tau,x} \sup_{t:\tau\le t\le\tau+\delta} |A_{g_k}(\tau,t)|^n \le c_n M(|g_k|)(\tau,t)^{n-1} M(g_k)(\tau,t). \tag{37}$$

It follows from Remark 5.2 that

$$\limsup_{k\to\infty} M(|g_k|)(\tau,t) \le M(|\mu|)(\tau,t) \tag{38}$$

and

$$M(g_k)(\tau,t) \to M(\mu)(\tau,t) \tag{39}$$

as $k \to \infty$. Now using (24), (37), (38), and (39) we see that (28) holds for μ.

This completes the proof of Lemma 5.17.

Lemma 5.18. (a) *Let $P \in \mathcal{M}$, $V \in \mathcal{P}_f^*$, and $g \in L^\infty_{\mathcal{E}}$. If t and τ are such that $M(|V|)(\tau,t) < 1$, then*

$$Y_V(\tau,t)g(x) - Y(\tau,t)g(x) = \sum_{k\ge 1}\int_\tau^t dt_1 \int_{t_1}^t dt_2 \cdots$$

$$\int_{t_{k-1}}^t Y(\tau,t_1)V(t_1)Y(t_1,t_2)V(t_2)\cdots Y(t_{k-1},t_k)V(t_k)Y(t_k,t)g(x)dt_k. \tag{40}$$

(b) *If $P \in \mathcal{M}$ possesses density p, $V \in \mathcal{P}_f^*$, $g \in L^\infty$, and $(\tau,t) \in D_T$ is such that $M(|V|)(\tau,t) < 1$, then equality (40) holds.*

(*c*) *If* $P \in \mathcal{M}$ *possesses density* p, $\mu \in \mathcal{P}_m^*$, $g \in L^\infty$, *and* $(\tau, t) \in D_T$ *is such that* $M(|\mu|)(\tau, t) < 1$, *then*

$$Y_\mu(\tau,t)g(x) - Y(\tau,t)g(x) = \sum_{k\geq 1}\int_\tau^t dt_1 \int_{t_1}^t dt_2 \cdots$$
$$\int_{t_{k-1}}^t Y(\tau,t_1)\mu(t_1)Y(t_1,t_2)\mu(t_2)\cdots Y(t_{k-1},t_k)\mu(t_k)Y(t_k,t)g(x)dt_k.$$

Proof. (*a*) Using the formula $Y(\tau,t)g(X_t) = E_{\tau,x}g(X_t)$, Definition 3.1, and expanding the exponential, we obtain

$$\begin{aligned} & Y_V(\tau,t)g(x) - Y(\tau,t)g(x) \\ = & \sum_{k\geq 1}\int_\tau^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{k-1}}^t E_{\tau,x}V(t_1,X_{t_1})V(t_2,X_{t_2})\cdots V(t_k,X_{t_k})g(X_t)dt_k. \end{aligned}$$

Then the Markov property gives

$$\begin{aligned} & Y_V(\tau,t)g(x) - Y(\tau,t)g(x) \\ = & \sum_{k\geq 1}\int_\tau^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{k-1}}^t E_{\tau,x}V(t_1,X_{t_1})V(t_2,X_{t_2})\cdots V(t_k,X_{t_k})E_{\tau,x}(g(X_t)|\mathcal{F}_{t_k}^\tau)dt_k \\ = & \sum_{k\geq 1}\int_\tau^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{k-1}}^t E_{\tau,x}V(t_1,X_{t_1})V(t_2,X_{t_2})\cdots V(t_k,X_{t_k})Y(t_k,t)g(X_{t_k})dt_k. \end{aligned}$$

Repeating the same reasoning several times, we get (40). Condition $M(|V|)(\tau,t) < 1$ implies the convergence of the series in (40) (see the estimates in Section 5). Hence, part (*a*) of Lemma 5.18 holds. Parts (*b*) and (*c*) can be obtained similarly.

Remark 5.19. The series representation for $Y_V - Y$ and $Y_\mu - Y$ is a non-autonomous version of the Dyson series (see [17]).

6. Exponential estimates for non-autonomous functionals

Our main goal in this section is to study the non-autonomous multiplicative functionals $\exp\{-\int_\tau^t V(s,X_s)\}ds$ and $\exp\{-A_\mu(t,\tau)\}$. First we prove Khas'minski's Lemma in the non-autonomous case.

Lemma 6.1. (*a*) *Let* $P \in \mathcal{M}$, $V \in \mathcal{P}_f^*$, *and let* $(\tau,t) \in D_T$ *be such that* $M(|V|)(\tau,t) < 1$. *Then*

$$\sup_{x\in E} E_{\tau,x}\exp\{\int_\tau^t |V(s,X_s)|ds\} \leq \frac{1}{1 - M(|V|)(\tau,t)}. \tag{41}$$

(*b*) *If $P \in \mathcal{M}$ possesses density p, $\mu \in \mathcal{P}_m^*$, and $(\tau, t) \in D_T$ is such that $M(|\mu|)(\tau, t) < 1$, then*

$$\sup_{x \in E} E_{\tau,x} \exp\{A_{|\mu|}(\tau, t)\} \le \frac{1}{1 - M(|\mu|)(\tau, t)}. \tag{42}$$

Proof. Using estimate (25), we get

$$E_{\tau,x} \frac{A_{|V|}(\tau, t)^n}{n!} \le M(|V|)(\tau, t). \tag{43}$$

It is clear that (43) implies (41). Estimate (42) follows from (41) using the same ideas as in the proof of Lemma 5.17.

The next assertions contain more exponential estimates:

Lemma 6.2. (*a*) *Let $P \in \mathcal{M}$, $V \in \mathcal{P}_f^*$, and let $(\tau, t) \in D_T$ be such that $M(|V|)(t, \tau) < 1$. Then*

$$\begin{aligned} E_{\tau,x} \exp\{|A_V(\tau, t)|\} &\le 1 + \{2N(|V|)(\tau, t, x)M(V)(\tau, t)\}^{\frac{1}{2}} \\ &+ \frac{2\sqrt{3}}{3} \frac{N(|V|)(\tau, t, x)M(V)(\tau, t)}{1 - M(|V|)(\tau, t)}. \end{aligned} \tag{44}$$

(*b*) *If $P \in \mathcal{M}$ possesses density p, $\mu \in \mathcal{P}_m^*$, and $(\tau, t) \in D_T$ is such that $M(|\mu|)(\tau, t) < 1$, then estimate (44) holds with μ instead of V.*

Proof. Estimate (44) follows from Lemma 5.8, Corollary 5.9, and the fact that

$$\sqrt{(m-1)!(m+1)!} \le \frac{2\sqrt{3}}{3}(m!)$$

for all $m \ge 3$. The proof of part (*b*) is similar. Here we reason as in the proof of Lemma 5.17.

Lemma 6.3. (*a*) *Let $P \in \mathcal{M}$, $q \ge 1$, $1 < r < \infty$, $\frac{1}{r} + \frac{1}{r'} = 1$, and let V and W be functions from the class $\mathcal{P}_f^*$. Suppose that $(\tau, t) \in D_T$ is such that $M(rq|W|)(\tau, t) < 1$ and $M(r'q|V - W|)(\tau, t) < 1$. Then*

$$\begin{aligned} &E_{\tau,x}|\exp\{A_V(\tau, t)\} - \exp\{A_W(\tau, t)\}|^q \\ \le\ & \frac{1}{(1 - M(rq|W|)(\tau, t))^{\frac{1}{r}}} \{[2N(r'q|V - W|)(\tau, t, x)M(r'q(V - W))(\tau, t)]^{\frac{1}{2}} \\ +\ & \frac{2\sqrt{3}}{3} \frac{N(r'q|V - W|)(\tau, t, x)M(r'q(V - W))(\tau, t)}{1 - M(r'q|V - W|)(\tau, t)}\}^{\frac{1}{r'}}. \end{aligned} \tag{45}$$

(*b*) *Suppose that $P \in \mathcal{M}$ possesses the density p, and let $q \ge 1$, $1 < r < \infty$, $\frac{1}{r} + \frac{1}{r'} = 1$, and $\mu, \nu \in \mathcal{P}_m^*$. Let $(\tau, t) \in D_T$ be such that $M(rq|\nu|)(\tau, t) < 1$*

and $M(r'q|\mu-\nu|)(\tau,t)<1$. *Then*

$$\begin{aligned}
&E_{\tau,x}|\exp\{A_\mu(\tau,t)\}-\exp\{A_\nu(\tau,t)\}|^q\\
\leq\ & \frac{1}{(1-M(rq|\nu|)(\tau,t))^{\frac{1}{r}}}\{[2N(r'q|\mu-\nu|)(\tau,t,x)M(r'q(\mu-\nu))(\tau,t)]^{\frac{1}{2}}\\
+\ & \frac{2\sqrt{3}}{3}\frac{N(r'q|\mu-\nu|)(\tau,t,x)M(r'q(\mu-\nu))(\tau,t)}{1-M(r'q|\mu-\nu|)(\tau,t)}\}^{\frac{1}{r'}}. \qquad (46)
\end{aligned}$$

Proof. Part (a). It is not difficult to prove that

$$|e^a-1|^b\leq e^{|a|b}-1, \qquad (47)$$

for $a\in R$ and $b\geq 1$. Using (47) and Hölder's inequality, we get

$$\begin{aligned}
&E_{\tau,x}|\exp\{A_V(\tau,t)\}-\exp\{A_W(\tau,t)\}|^q\\
\leq\ & \{E_{\tau,x}\exp\{A_{rqW}(\tau,t)\}^{\frac{1}{r}}\{E_{\tau,x}|\exp\{A_{V-W}(\tau,t)\}-1|^{r'q}\}^{\frac{1}{r'}}\\
\leq\ & \{E_{\tau,x}\exp\{A_{rq|W|}(\tau,t)\}^{\frac{1}{r}}\{E_{\tau,x}\exp\{|A_{r'q(V-W)}(\tau,t)|\}-1\}^{\frac{1}{r'}}. \qquad (48)
\end{aligned}$$

Now it is clear that (45) follows from (41),(44) and (48).

Part (b). First we use Lemma 5.1 to find the approximating sequences g_k (for μ) and h_k (for ν). By the assumptions, $M(rq|\nu|)(\tau,t)<1$ and $M(r'q|\mu-\nu|)(\tau,t)<1$. Using remarks 5.2 and 5.14, we see that there exists a sequence k' of positive integers such that

$$M(rq|h_{k'}|)(\tau,t)<1,\quad M(r'q|g_{k'}-h_{k'}|)(\tau,t)<1,$$

$$\lim_{k'\to\infty}A_{g_{k'}}(\tau,t)=A_\mu(\tau,t),\quad \lim_{k'\to\infty}A_{h_{k'}}(\tau,t)=A_\nu(\tau,t),$$

$$\limsup_{k'\to\infty}M(|h_{k'}|)(\tau,t)\leq M(|\mu|)(\tau,t),$$

$$\limsup_{k'\to\infty}M(|g_{k'}-h_{k'}|)(\tau,t)\leq M(|\mu-\nu|)(\tau,t),$$

$$\limsup_{k'\to\infty}N(|g_{k'}-h_{k'}|)(\tau,t,x)\leq N(|\mu-\nu|)(\tau,t,x),$$

and

$$\lim_{k'\to\infty}M(g_{k'}-h_{k'})(\tau,t)=M(\mu-\nu)(\tau,t).$$

It follows from (45) that

$$\begin{aligned}
&E_{\tau,x}|\exp\{A_{g_{k'}}(\tau,t)\}-\exp\{A_{h_{k'}}(\tau,t)\}|^q\\
\leq&\frac{1}{(1-M(rq|h_{k'}|)(\tau,t))^{\frac{1}{r}}}\{[2N(r'q|g_{k'}-h_{k'}|)(\tau,t,x)M(r'q(g_{k'}-h_{k'}))(\tau,t)]^{\frac{1}{2}}\\
+&\frac{2\sqrt{3}}{3}\frac{N(r'q|g_{k'}-h_{k'}|)(\tau,t,x)M(r'q(g_{k'}-h_{k'}))(\tau,t)}{1-M(r'q|g_{k'}-h_{k'}|)(\tau,t)}. \qquad (49)
\end{aligned}$$

Now using Fatou's Lemma in (49), we see that estimate (46) holds.

This completes the proof of Lemma 6.3.

It follows from formula (29) that $c_n \leq c2^n n!$. However, this inequality does not allow us to get an exponential estimate for the functional A_V using (28) and (30). We will obtain such an estimate by modifying the proof of Lemma 5.10.

Lemma 6.4. (a) *Let* $P \in \mathcal{M}$ *and* $V \in \mathcal{P}_f^*$. *Then for every* τ *with* $0 \leq \tau \leq T$ *and every* $\delta > 0$ *such that* $\tau + \delta \leq T$ *and* $M(|V|)(\tau, \tau+\delta) < 1$, *the following estimate holds:*

$$\sup_{x \in E} E_{\tau,x} \exp\{\sup_{t:\tau \leq t \leq \tau+\delta} |A_V(\tau,t)|\} \leq \exp\{M(V)(\tau,\tau+\delta)\} \times$$
$$\times(1 + c\{M(|V|)(\tau,\tau+\delta)M(V)(\tau,\tau+\delta)\}^{\frac{1}{2}} +$$
$$+c\frac{M(|V|)(\tau,\tau+\delta)M(V)(\tau,\tau+\delta)}{1 - M(|V|)(\tau,\tau+\delta)}). \tag{50}$$

(b) *Suppose that* $P \in \mathcal{M}$ *possesses density* p, *and let* $\mu \in \mathcal{P}_m^*$. *Then for every* τ *with* $0 \leq \tau \leq T$ *and every* $\delta > 0$ *such that* $\tau + \delta \leq T$ *and* $M(|\mu|)(\tau, \tau+\delta) < 1$, *the following estimate holds:*

$$\sup_{x \in E} E_{\tau,x} \exp\{\sup_{t:\tau \leq t \leq \tau+\delta} |A_\mu(\tau,t)|\} \leq \exp\{M(\mu)(\tau,\tau+\delta)\} \times$$
$$\times(1 + c\{M(|\mu|)(\tau,\tau+\delta)M(\mu)(\tau,\tau+\delta)\}^{\frac{1}{2}} +$$
$$+c\frac{M(|\mu|)(\tau,\tau+\delta)M(\mu)(\tau,\tau+\delta)}{1 - M(|\mu|)(\tau,\tau+\delta)}).$$

Proof. Using the same notation as in the proof of Lemma 5.10, and applying Doob's inequality, we get for every $n \geq 2$,

$$E_{\tau,x} \sup_{j:0 \leq j \leq k} |M_{t_j}|^n \leq (\frac{n}{n-1})^n E_{\tau,x}|A_V(\tau,\tau+\delta)|^n$$
$$\leq (\frac{n}{n-1})^n n! M(|V|)(\tau,\tau+\delta)^{n-1} M(V)(\tau,\tau+\delta).$$

Dividing the previous inequality by $n!$, adding up the resulting inequalities, and using (28) and the equality $M_t = A_V(\tau,t) + N(V)(t, t+\delta, X_t)$, we get

$$\begin{aligned}
E_{\tau,x}\exp\{\sup_{j:0\le j\le k}|M_{t_j}|\} \le \\
&\le 1+E_{\tau,x}\sup_{j:0\le j\le k}|M_{t_j}| \\
&\quad +c\frac{M(|V|)(\tau,\tau+\delta)M(V)(\tau,\tau+\delta)}{1-M(|V|)(\tau,\tau+\delta)}) \\
&\le 1+E_{\tau,x}\sup_{j:0\le j\le k}|A_V(\tau,t_j)| \\
&\quad +E_{\tau,x}\sup_{j:0\le j\le k}|N(V)(t_j,\tau+\delta,X_{t_j})| \\
&\quad +c\frac{M(|V|)(\tau,\tau+\delta)M(V)(\tau,\tau+\delta)}{1-M(|V|)(\tau,\tau+\delta)}) \\
&\le 1+\{E_{\tau,x}\sup_{j:0\le j\le k}|A_V(\tau,t_j)|^2\}^{\frac{1}{2}} \\
&\quad +M(V)(\tau,\tau+\delta)+c\frac{M(|V|)(\tau,\tau+\delta)M(V)(\tau,\tau+\delta)}{1-M(|V|)(\tau,\tau+\delta)}) \\
&\le 1+\{c_2M(|V|)(\tau,\tau+\delta)M(V)(\tau,\tau+\delta)\}^{\frac{1}{2}} \\
&\quad +M(V)(\tau,\tau+\delta)+c\frac{M(|V|)(\tau,\tau+\delta)M(V)(\tau,\tau+\delta)}{1-M(|V|)(\tau,\tau+\delta)}).
\end{aligned} \tag{51}$$

We also have

$$\begin{aligned}
E_{\tau,x}\exp\{\sup_{j:0\le j\le k}|M_{t_j}|\} \ge \\
&\ge E_{\tau,x}\exp\{\sup_{j:0\le j\le k}|A_V(\tau,t_j)|-\sup_{j:0\le j\le k}|N(V)(t_j,\tau+\delta,X_{t_j})|\} \\
&\ge \exp\{-M(V)(\tau,\tau+\delta)\}E_{\tau,x}\exp\{\sup_{j:0\le j\le k}|A_V(\tau,t_j)|\}.
\end{aligned} \tag{52}$$

It follows from (51) and (52) that

$$\begin{aligned}
E_{\tau,x}\exp\{\sup_{j:0\le j\le k}|A_V(\tau,t_j)|\} \le \\
&\le \exp\{M(V)(\tau,\tau+\delta)\} \\
&\quad \times[1+\{c_2M(|V|)(\tau,\tau+\delta)M(V)(\tau,\tau+\delta)\}^{\frac{1}{2}} \\
&\quad +M(V)(\tau,\tau+\delta)+c\frac{M(|V|)(\tau,\tau+\delta)M(V)(\tau,\tau+\delta)}{1-M(|V|)(\tau,\tau+\delta)}].
\end{aligned}$$

Therefore,

$$E_{\tau,x}\exp\{\sup_{j:0\le j\le k}|A_V(\tau,t_j)|\}\le\exp\{M(V)(\tau,\tau+\delta)\}\times$$
$$\times(1+c\{M(|V|)(\tau,\tau+\delta)M(V)(\tau,\tau+\delta)\}^{\frac{1}{2}}+$$
$$+c\frac{M(|V|)(\tau,\tau+\delta)M(V)(\tau,\tau+\delta)}{1-M(|V|)(\tau,\tau+\delta)}). \tag{53}$$

Finally, we can consider a sequence of refinements of the partition $\tau = t_0 < t_1 < \cdots < t_k = \tau+\delta$ on the left-hand side of (53) such that the maximum length of the partition intervals tends to 0, and pass to the limit, using the monotone convergence theorem and the continuity of the functional $A_V(\tau,t)$ with respect to t. This shows that estimate (50) holds. The proof of part (b) of Lemma 6.4(a) is similar. Here we use the ideas in the proof of part (b) of Lemma 6.3.

7. $(L^r - L^q)$-smoothing by backward Feynman-Kac propagators

In this section we study the behavior of backward Feynman-Kac propagators on the scale of Lebesgue spaces L^r.

Theorem 7.1. (a) *Let $P\in\mathcal{M}$. Then for any $V\in\mathcal{P}_f^*$, Y_V is a backward propagator on $L^\infty_{\mathcal{E}}$.*
(b) *Suppose that $P\in\mathcal{M}$ possesses density p, and let $V\in\mathcal{P}_f^*$. Then Y_V is a backward propagator on L^∞.*
(c) *Suppose that $P\in\mathcal{M}$ possesses density p, and let $\mu\in\mathcal{P}_m^*$. Then Y_μ is a backward propagator on L^∞.*

Proof. (a) Let $g\in L^\infty_{\mathcal{E}}$. Then the function $Y_V(\tau,t)g$ is Borel measurable. Using Lemma 6.1, we get

$$\sup_{x\in E}|Y_V(\tau,t)g(x)|\le\|g\|_\infty\sup_{x\in E}E_{\tau,x}\exp\{A_{|V|}(\tau,t)\}\le\|g\|_\infty\frac{1}{1-M(|V|)(\tau,t)}$$

for all τ,t with $M(|V|)(\tau-t)<1$. It follows that $Y_V(\tau,t)\in L(L^\infty_{\mathcal{E}},L^\infty_{\mathcal{E}})$ for all τ and t such that $t-\tau$ is small. Using the Markov property, we see that Y_V is a backward propagator on $L^\infty_{\mathcal{E}}$.
(b) The existence of the density allows us to define Y_V on L^∞. The rest of the proof is the same as in part (a).
(c) The proof is the same as in part (a).

Remark 7.2. *The following estimate holds in part (b) of Theorem 7.1:*

$$\|Y_V(\tau,t)\|_{\infty\to\infty}\le\exp\{A([\frac{t-\tau}{\delta}]+1)\}$$

where $\delta>0$ is any number such that $\rho(\delta)=\sup\{M(|V|)(\eta,\lambda):\lambda-\eta<\delta\}<1$, and $A=\ln\frac{1}{1-\rho(\delta)}$. Similar estimates hold in parts (a) and (c).

Our next result explains why the ζ-approximation is useful in the theory of Feynman-Kac propagators.

Theorem 7.3. *Let $P \in \mathcal{M}$, and let $V \in \mathcal{P}_f^*$ and $V_k \in \mathcal{P}_f^*$ be such that V_k ζ-approaches V. Then*

$$\lim_{k\to\infty} \sup_{(\tau,t)\in D_T} ||Y_V(\tau,t) - Y_{V_k}(\tau,t)||_{L^\infty_{\mathcal{E}}\to L^\infty_{\mathcal{E}}} = 0.$$

Suppose that $P \in \mathcal{M}$ possesses density p, and let $\mu \in \mathcal{P}_m^*$ and $\mu_k \in \mathcal{P}_m^*$ be such that μ_k ζ-approaches μ. Then

$$\lim_{k\to\infty} \sup_{(\tau,t)\in D_T} ||Y_\mu(\tau,t) - Y_{\mu_k}(\tau,t)||_{\infty\to\infty} = 0.$$

Proof. We will proof the second part of Theorem 7.3. The proof of the first part is similar.

Let μ and μ_k be such as in the formulation of Theorem 7.3, and let $f \in L^\infty$. Then by Lemma 6.3(b) with $q = 1$ and $r = 2$, there exists $\delta_0 > 0$ such that

$$|Y_\mu(\tau,t)f(x) - Y_{\mu_k}(\tau,t)f(x)| \le \alpha||f||_\infty(\{M(\mu-\mu_k)(\tau,t)\}^{\frac{1}{2}} + M(\mu-\mu_k)(\tau,t)) \tag{54}$$

for all τ and t with $t - \tau < \delta_0$ and all $x \in E$. In (54), the constant α does not depend on x, t, τ, and k. It follows from (54) that

$$\lim_{k\to\infty} \sup_{(\tau,t)\in D_T : t-\tau<\delta_0} ||Y_\mu(\tau,t) - Y_{\mu_k}(\tau,t)||_{L^\infty\to L^\infty_{\mathcal{E}}} = 0. \tag{55}$$

In order to get rid of the restriction $t - \tau < \delta_0$ in (55), we can reason as follows: Consider a partition $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$ of the interval $[0,T]$ such that $t_{j+1} - t_j < \delta_0$ for all j with $0 \le j \le n-1$. Then estimate (54) holds for t and τ inside of each of the intervals $[t_j, t_{j+1}]$. Using the previous assertion and the properties of backward propagators, we can finish the proof of the second part of Theorem 7.3. We will illustrate how this can be done by considering a special case where $t_j \le \tau \le t_{j+1} \le t \le t_{j+2}$ with $0 \le j \le n-2$. The general case is similar. Using the uniform boundedness of the propagators Y_{μ_k} on the space L^∞ (this follows from Remark 7.2), we obtain

$$\begin{aligned}
&|||Y_\mu(\tau,t) - Y_{\mu_k}(\tau,t)||_{\infty\to\infty} \\
&\le \ ||Y_\mu(\tau,t_{j+1})Y_\mu(t_{j+1},t) - Y_{\mu_k}(\tau,t_{j+1})Y_{\mu_k}(t_{j+1},t)||_{\infty\to\infty} \\
&\le \ ||Y_\mu(\tau,t_{j+1})Y_\mu(t_{j+1},t) - Y_\mu(\tau,t_{j+1})Y_{\mu_k}(t_{j+1},t)||_{\infty\to\infty} \\
&+ \ ||Y_\mu(\tau,t_{j+1})Y_{\mu_k}(t_{j+1},t) - Y_{\mu_k}(\tau,t_{j+1})Y_{\mu_k}(t_{j+1},t)||_{\infty\to\infty} \\
&\le \ \alpha||Y_\mu(t_{j+1},t) - Y_{\mu_k}(t_{j+1},t)||_{\infty\to\infty} \\
&+ \ \alpha||Y_\mu(\tau,t_{j+1}) - Y_{\mu_k}(\tau,t_{j+1})||_{\infty\to\infty}.
\end{aligned} \tag{56}$$

Since $t - t_{j+1} < \delta_0$ and $t_{j+1} - \tau < \delta_0$, we can apply (55) to (56). It follows that (55) can be extended to include all $t_j \leq \tau \leq t_{j+1} \leq t \leq t_{j+2}$ with $0 \leq j \leq n-2$.

This completes the proof of Theorem 7.3.

Corollary 7.4. *Let $P \in \mathcal{M}$ and $V \in \mathcal{P}_f^*$. Define g_k by (13). Then*

$$\lim_{k\to\infty} \sup_{(\tau,t)\in D_T} ||Y_V(\tau,t) - Y_{g_k}(\tau,t)||_{L_{\mathcal{E}}^\infty \to L_{\mathcal{E}}^\infty} = 0.$$

Suppose that $P \in \mathcal{M}$ possesses density p and let $\mu \in \mathcal{P}_m^*$. Define g_k by (17). Then

$$\lim_{k\to\infty} \sup_{(\tau,t)\in D_T} ||Y_\mu(\tau,t) - Y_{g_k}(\tau,t)||_{\infty\to\infty} = 0.$$

Corollary 7.4 follows from Theorem 7.3 and from the fact that the sequence g_k defined by (13) ζ-approaches V, and the sequence g_k defined by (17) ζ-approaches μ (see Lemma 5.1).

The next lemma will be important in the sequel.

Lemma 7.5. (a) *Let $P \in \mathcal{M}$. Then for any $V \in \mathcal{P}_f^*$, we have*

$$\lim_{t-\tau\to 0+} ||Y_V(\tau,t) - Y(\tau,t)||_{L_{\mathcal{E}}^\infty \to L_{\mathcal{E}}^\infty} = 0. \tag{57}$$

(b) *Suppose that $P \in \mathcal{M}$ possesses density p. Then for any if $V \in \mathcal{P}_f^*$, we have*

$$\lim_{t-\tau\to 0+} ||Y_V(\tau,t) - Y(\tau,t)||_{\infty\to\infty} = 0. \tag{58}$$

(c) *Suppose that $P \in \mathcal{M}$ possesses density p. Then for any $\mu \in \mathcal{P}_m^*$, we have*

$$\lim_{t-\tau\to 0} ||Y_\mu(\tau,t) - Y(\tau,t)||_{\infty\to\infty} = 0. \tag{59}$$

Proof. It follows from Lemma 6.1(a) and the definition of the class $\mathcal{P}_f^*$ that

$$\begin{aligned}\limsup_{t-\tau\to 0+} ||Y_V(\tau,t) - Y(\tau,t)||_{\infty\to\infty} &\leq \limsup_{t-\tau\to 0+} \sup_{x\in E} (E_{\tau,x} \exp\{A_{|V|}(\tau,t)\} - 1) \\ &\leq \limsup_{t-\tau\to 0+} \sup_{x\in E} \frac{M(|V|)(\tau,t)}{1 - M(|V|)(\tau,t)} = 0.\end{aligned}$$

This gives equality (58). The proof of (57) and (59) is similar. In the proof of (59), we use Lemma 6.1(b).

The next result provides sufficient conditions for the existence of backward Feynman-Kac propagators on the space L^p.

Theorem 7.6. *Let $1 < s < \infty$ and $1 \leq r < s$. Then the following are true:*
(a) *Let $P \in \mathcal{M}$ and $V \in \mathcal{P}_f^*$. Suppose that the free backward propagator Y*

satisfies $Y(\tau,t) \in L(L^r_{\mathcal{E}}, L^r_{\mathcal{E}})$ *for all* $(\tau,t) \in D_T$*. Then* Y_V *is a backward propagator on* $L^s_{\mathcal{E}}$*. If, in addition,* Y *is uniformly bounded on* $L^r_{\mathcal{E}}$ *and strongly continuous on* $L^s_{\mathcal{E}}$*, then* Y_V *is a strongly continuous backward propagator on* $L^s_{\mathcal{E}}$*.*
(b) *If* $P \in \mathcal{M}$ *possesses density* p*, and if* $Y(\tau,t) \in L(L^r, L^r)$ *for all* $(\tau,t) \in D_T$*, then* Y_V *is a backward propagator on* L^s*. If, in addition,* Y *is uniformly bounded on* L^r *and strongly continuous on* L^s*, then* Y_V *is a strongly continuous backward propagator on* L^s*.*
(c) *Suppose that* $P \in \mathcal{M}$ *possesses density* p *and let* $\mu \in \mathcal{P}^*_m$*. If* $Y(\tau,t) \in L(L^r, L^r)$ *for all* $0 \le \tau < t \le T$*, then* Y_μ *is a backward propagator on* L^s*. If in addition,* Y *is uniformly bounded on* L^r *and strongly continuous on* L^s*, then* Y_μ *is a strongly continuous backward propagator on* L^s*.*

Remark 7.7. We do not know whether Theorem 7.6 holds for $r = p$. In the case of the heat semigroup, Theorem 7.6 may fail for $p = 1$. This was established in [13].

Proof of Theorem 7.6. Part (b) Assume that the conditions in part (b) of Theorem 7.6 hold, and let $g \in L^s$. Using Hölder's inequality, we get

$$\begin{aligned} |Y_V(\tau,t)g(x)| &\le \{E_{\tau,x}|g(X_t)|^{\frac{s}{r}}\}^{\frac{r}{s}}\{E_{\tau,x}\exp\{\frac{s}{s-r}A_{|V|}(\tau,t)\}^{\frac{s-r}{s}} \\ &= \{Y(\tau,t)|g|^{\frac{s}{r}}(x)\}^{\frac{r}{s}}\{Y_{\frac{s}{s-r}|V|}(\tau,t)1(x)\}^{\frac{s-r}{s}} \\ &\le ||Y_{\frac{s}{s-r}|V|}(\tau,t)||_{\infty\to\infty}^{\frac{s-r}{s}}\{Y(\tau,t)|g|^{\frac{s}{r}}(x)\}^{\frac{r}{s}} \end{aligned} \tag{60}$$

for all t and τ with $(\tau,t) \in D_T$. It follows from Remark 7.2 that

$$||Y_{\frac{s}{s-r}|V|}\tau,t)||_{\infty\to\infty} \le c \tag{61}$$

where $c \ge 1$ depends on s, r, and V. Next (60) and (61) give

$$|Y_V(\tau,t)g(x)| \le c\{Y(\tau,t)|g|^{\frac{s}{r}}(x)\}^{\frac{r}{s}}. \tag{62}$$

and (61) implies that

$$||Y_V(\tau,t)||_{s\to s} \le c||Y(\tau,t)||^{\frac{r}{s}}_{r\to r}. \tag{63}$$

Therefore, Y_V is a propagator on L^s. The proofs of the corresponding assertions in parts (a) and (c) are similar.

Remark 7.8. Inequality (63) gives a norm estimate in Theorem 7.6.

Let us return to the proof of part (b) of Theorem 7.6.

Lemma 7.9. *Let* $1 < s < \infty$ *and* $1 \le r < s$*. Then the following are true:*
(a) *Let* $P \in \mathcal{M}$ *and* $V \in \mathcal{P}^*_f$*. Suppose that the free backward propagator* Y *is uniformly bounded on* $L^r_{\mathcal{E}}$*. Then we have*

$$\lim_{t-\tau\to 0+} ||Y_V(\tau,t) - Y(\tau,t)||_{L^s_{\mathcal{E}}\to L^s_{\mathcal{E}}} = 0. \tag{64}$$

(b) *If* $P \in \mathcal{M}$ *possesses density* p, $V \in \mathcal{P}_f^*$, *and* Y *is uniformly bounded on* L^r, *then*

$$\lim_{t-\tau\to 0+} ||Y_V(\tau,t) - Y(\tau,t)||_{s\to s} = 0. \tag{65}$$

(c) *If* $P \in \mathcal{M}$ *possesses density* p, $\mu \in \mathcal{P}_m^*$, *and* Y *is uniformly bounded on* L^r, *then*

$$\lim_{t-\tau\to 0+} ||Y_\mu(\tau,t) - Y(\tau,t)||_{s\to s} = 0. \tag{66}$$

Remark 7.10. Lemmas 7.3 and 7.9 were obtained in [12, 13] in the case of the heat semigroup. A similar result, concerning time-independent perturbations of semigroups on L^1, was obtained earlier in [23], Lemma 4.2.

Proof of Lemma 7.9. We begin with the proof of part (b) of Lemma 7.9. It follows from Theorem 7.6(b) that Y and Y_V are backward propagators on L^s. Let $g \in L^s$. Then, using Hölder's inequality and inequality (47), we obtain

$$\begin{aligned} &|Y_V(\tau,t)g(x) - Y(\tau,t)g(x)| \le \\ &\quad \le \{E_{\tau,x}|g(X_{\tau,x}^t)|^{\frac{s}{r}}\}^{\frac{r}{s}}\{E_{\tau,x}|\exp\{A_V(\tau,t)\}-1|^{\frac{s}{s-r}}\}^{\frac{s-r}{s}} \\ &\quad \le \{Y(\tau,t)|g|^{\frac{s}{r}}(x)\}^{\frac{r}{s}}\{E_{\tau,x}\exp\{\frac{s}{s-r}A_{|V|}(\tau,t)\}-1\}^{\frac{s-r}{s}}. \end{aligned} \tag{67}$$

It follows from (67), Lemma 6.1(a), the definition of the class $\mathcal{P}_f^*$, and the uniform boundedness of Y on L^r that

$$\begin{aligned} &\limsup_{t-\tau\to 0+} ||Y_V(\tau,t) - Y(\tau,t)||_{s\to s} \le \\ &\quad \le a(s,r,V)\limsup_{t-\tau\to 0+}\sup_{x\in R^n}\{E_{\tau,x}\exp\{\frac{s}{s-r}A_{|V|}(\tau,t)\}-1\}^{\frac{s-r}{s}} \\ &\quad \le c(s,r,V)\limsup_{t-\tau\to 0+}[\frac{\frac{s}{s-r}M(|V|)(\tau,t)}{1-\frac{s}{s-r}M(|V|)(\tau,t)}]^{\frac{s-r}{s}} = 0. \end{aligned}$$

This gives equality (65). The proof of equality (64) is similar. In the proof of equality (66), we use Lemma 6.1(b) instead of Lemma 6.1(a).

Let us continue the proof of part (b) of Theorem 7.6. Suppose that Y is locally uniformly bounded on L^r and strongly continuous on L^s. We have already shown that Y_V is a backward propagator on L^s. moreover, Y_s is uniformly bounded (see estimate (63)). Therefore, in order to prove the strong continuity of Y_V it is sufficient to show the separate strong continuity (see Theorem 2.2).

Let $(\tau,t) \in D_T$, and suppose that $t' \ge t$ and $g \in L^s$. Then

$$\begin{aligned} ||Y_V(\tau,t')g - Y_V(\tau,t)g||_s &= ||Y_V(\tau,t)(Y_V(t,t')g-g)||_s \le M||Y_V(t,t')g-g||_s \\ &\le M||g||_s||Y_V(t,t') - Y(t,t')||_s + M||Y(t,t')g-g||_s. \end{aligned}$$

Next it follows from Lemma 7.9 and from the strong continuity of Y that

$$\lim_{t'\to t+}||Y_V(\tau,t')g - Y_V(\tau,t)g||_s = 0. \tag{68}$$

Similarly, we get

$$\lim_{t'\to t-}||Y_V(\tau,t')g - Y_V(\tau,t)g||_s = 0. \tag{69}$$

Now assume that $\tau' \le \tau$. Then

$$\begin{aligned} ||Y_V(\tau',t)g - Y_V(\tau,t)g||_s &= ||(Y_V(\tau,\tau') - I)Y_V(\tau,t)g||_s \\ &\le ||Y_V(\tau',\tau) - Y(\tau',\tau)||_{s\to s}||Y_V(\tau,t)g||_s \\ &+ ||Y(\tau',\tau)Y_V(\tau,t)g - Y_V(\tau,t)g||_s. \end{aligned}$$

It follows from (63), Lemma 7.9, and from the strong continuity of Y that

$$\lim_{\tau'\to\tau-}||Y_V(\tau',t)g - Y_V(\tau,t)g||_s = 0. \tag{70}$$

Finally, let $\tau < \tau' < t$, and let λ be such that $\tau' < \lambda < t$. Then

$$\begin{aligned} ||Y_V(\tau',t)g - Y_V(\tau,t)g||_s &= ||(Y_V(\tau',\lambda) - Y_V(\tau,\lambda))Y_V(\lambda,t)g||_s \\ &\le ||(Y(\tau',\lambda) - Y(\tau,\lambda))Y_V(\lambda,t)g||_s \\ &+ ||Y_V(\tau',\lambda) - Y(\tau',\lambda)||_{s\to s}||Y_V(\lambda,t)g||_s \\ &+ ||Y_V(\tau,\lambda) - Y(\tau,\lambda)||_{s\to s}||Y_V(\lambda,t)g||_s \\ &\le ||(Y(\tau',\lambda) - Y(\tau,\lambda))Y_V(\lambda,t)g||_s \\ &+ M||Y_V(\tau',\lambda) - Y(\tau',\lambda)||_{s\to s}||g||_s \\ &+ M||Y_V(\tau,\lambda) - Y(\tau,\lambda)||_{s\to s}||g||_s \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{71}$$

For every $\epsilon > 0$, fix λ such that $\tau < \lambda < t$ and $I_2 + I_3 \le \frac{\epsilon}{2}$ for all τ' with $\tau < \tau' < \lambda$. This can be done using Lemma 7.9. Then the strong continuity of Y implies the existence of $\delta > 0$ such that $I_1 \le \frac{\epsilon}{2}$ for all τ' with $\tau < \tau' \le \tau + \delta < \lambda$. Hence, (71) gives

$$\lim_{\tau'\to\tau+}||Y_V(\tau',t)g - Y_V(\tau,t)g||_s = 0, \tag{72}$$

and it follows from (68), (69), (70), and (72) that Y_V is separately strongly continuous.

This completes the proof of Theorem 7.6.

The next result is a smoothing theorem for backward Feynman-Kac propagators.

Theorem 7.11. *Let $1 < s < q \le \infty$ and $1 \le r < s$. Then the following are true:*
(a) Let $P \in \mathcal{M}$ and $V \in \mathcal{P}_f^*$. Suppose that $Y(\tau,t) \in L(L^r_{\mathcal{E}}, L^{\frac{rq}{s}}_{\mathcal{E}})$ for all $0 \le \tau < t \le T$. Then $Y_V(\tau,t) \in L(L^s_{\mathcal{E}}, L^q_{\mathcal{E}})$ for all $0 \le \tau < t \le T$.

(b) If $P \in \mathcal{M}$ possesses density p, $V \in \mathcal{P}_f^*$, and $Y(\tau,t) \in L(L^r, L^{\frac{rq}{s}})$ for all $0 \le \tau < t \le T$, then $Y_V(\tau,t) \in L(L^s, L^q)$.
(c) If $P \in \mathcal{M}$ possesses density p, $\mu \in \mathcal{P}_m^*$, and $Y(\tau,t) \in L(L^r, L^{\frac{rq}{s}})$ for all $0 \le \tau < t \le T$, then $Y_\mu(\tau,t) \in L(L^s, L^q)$.

Proof. We will prove part (b) in the case $q \neq \infty$. The proof in the case $q = \infty$ is similar.

Let $g \in L^s$. Using estimate (62), we get

$$||Y_V(\tau,t)g||_q \le c\{\int_{R^n} \{Y(\tau,t)|g|^{\frac{s}{r}}(x)\}^{\frac{rq}{s}} dx\}^{\frac{1}{q}}.$$

It follows from the assumptions in Theorem 7.11 that

$$||Y_V(\tau,t)g||_q \le c||Y(\tau,t)||^{\frac{r}{s}}_{r\to\frac{rq}{s}} ||g||_s. \tag{73}$$

Now it is clear that part (b) of Theorem 7.11 follows from (73). The proof of parts (a) and (c) is similar.

8. Feller, Feller-Dynkin, and BUC-property

In this section, we study the behavior of backward propagators on the spaces of continuous functions.

Definition 8.1. *A backward BC-propagator is called a backward Feller propagator. A backward C_∞-propagator is called a backward Feller-Dynkin propagator. If a backward $L^\infty_{\mathcal{E}}$-propagator Q is such that $Q(\tau,t) \in L(L^\infty_{\mathcal{E}}, BC)$ for all $0 \le \tau < t \le T$, then it is said that Q satisfies the strong Feller condition. If a backward $L^\infty_{\mathcal{E}}$-propagator Q is such that $Q(\tau,t) \in L(L^\infty_{\mathcal{E}}, BUC)$ for all $0 \le \tau < t \le T$, then it is said that Q satisfies the strong BUC-condition.*

Remark 8.2. If Q is a backward L^∞-propagator, then we may replace the space $L^\infty_{\mathcal{E}}$ by the space L^∞ in the definition of the strong Feller and the strong BUC-condition.

Theorem 8.3. *Let $P \in \mathcal{M}$ and $V \in \mathcal{P}_f^*$. Then the following assertions hold:*
(a) If Y satisfies the strong Feller condition, then Y_V also satisfies the same condition.
(b) If Y satisfies the strong BUC-condition, then Y_V also satisfies the same condition.

We do not know whether Theorem 8.3 holds for backward Feller, Feller-Dynkin, or BUC propagators. However, this is true under additional restrictions.

Theorem 8.4. *Let $P \in \mathcal{M}$, $V \in \mathcal{P}_f^*$, and suppose that Y satisfies the strong Feller condition. Then the following assertions hold:*
(a) *If Y is a backward Feller-Dynkin propagator, then Y_V also has the same property.*
(b) *If Y is a strongly continuous backward Feller-Dynkin propagator, then Y_V also has the same property.*

If Y satisfies the strong BUC-condition, then Theorem 8.3(b) implies that Y_V is a backward BUC-propagator. Moreover, the following assertion holds:

Theorem 8.5. *Let $P \in \mathcal{M}$, $V \in \mathcal{P}_f^*$, and suppose that Y satisfies the strong BUC-condition. Then the following assertion holds: If Y is a strongly continuous backward BUC-propagator, then Y_V also has the same property.*

The next theorem provides sufficient conditions for the continuity of the function $(\tau, x) \to Y_V(\tau, t)g(x)$ on the set $[0, t) \times E$.

Theorem 8.6. *Let $P \in \mathcal{M}$. Then the following assertions hold:*
(a) *Suppose that Y is a backward strong Feller propagator. Suppose also that Y is continuous on BC in the topology of uniform convergence on compact subsets of E. Let $V \in \mathcal{P}_f^*$. Then for every $t \in (0, T]$ and any $g \in L_{\mathcal{E}}^{\infty}$, the function $(\tau, x) \to Y_V(\tau, t)g(x)$ is continuous on the set $[0, t) \times R^n$.*
(b) *Suppose that Y is a strongly continuous backward BUC-propagator possessing the strong BUC-property. Let $V \in \mathcal{P}_f^*$. Then for every $t \in (0, T]$ and any $g \in L_{\mathcal{E}}^{\infty}$, the function $(\tau, x) \to Y_{\mu}(\tau, t)g(x)$ is continuous on the set $[0, t) \times R^n$.*

Remark 8.7. If $P \in \mathcal{M}$ possesses density p, then Theorem 8.6 holds for all $g \in L^{\infty}$. Moreover, theorems 8.3-8.6 hold for a time-dependent measure μ from the class $\mathcal{P}_m^*$.

Proof of Theorem 8.3. We will prove part (a) of Theorem 8.3. The proof of Part (b) is similar.

Lemma 8.8. (a) *Let $P \in \mathcal{M}$ and $V \in \mathcal{P}_f^*$. Then for all $x, x' \in E$, $0 \le \tau < t \le T$, $g \in L_{\mathcal{E}}^{\infty}$, and $\lambda > 0$ with $\tau + \lambda < t$, we have*

$$\begin{aligned} &|Y_V(\tau,t)g(x') - Y_V(\tau,t)g(x)| \le \\ &\quad 2||Y_V(\tau,\tau+\lambda) - Y(\tau,\tau+\lambda)||_{\infty\to\infty}||Y_V(\tau+\lambda,t)g||_{\infty} \\ &\quad +|Y(\tau,\tau+\lambda)Y_V(\tau+\lambda,t)g(x') - Y(\tau,\tau+\lambda)Y_V(\tau+\lambda,t)g(x)|. \end{aligned} \tag{74}$$

(b) *Suppose that $P \in \mathcal{M}$ possesses density p. Then (74) holds for all $g \in L^{\infty}$.*
(c) *Suppose that $P \in \mathcal{M}$ possesses density p. If $\mu \in \mathcal{P}_m^*$, then for every*

$x, x' \in E$, $0 \le \tau < t \le T$, $g \in L^\infty$, *and* $\lambda > 0$ *with* $\tau + \lambda < t$, *we have*

$$\begin{aligned} &|Y_\mu(\tau,t)g(x') - Y_\mu(\tau,t)g(x)| \le \\ &\quad 2||Y_\mu(\tau,\tau+\lambda) - Y(\tau,\tau+\lambda)||_{\infty\to\infty}||Y_\mu(\tau+\lambda,t)g||_\infty \\ &\quad +|Y(\tau,\tau+\lambda)Y_\mu(\tau+\lambda,t)g(x') - Y(\tau,\tau+\lambda)Y_\mu(\tau+\lambda,t)g(x)|. \end{aligned}$$

Proof of Lemma 8.8. We will prove only part (a) of Lemma 8.8. The proof of parts (b) and (c) is similar. We have

$$\begin{aligned} & |Y_V(\tau,t)g(x') - Y_V(\tau,t)g(x)| \\ = \; & |Y_V(\tau,\tau+\lambda)Y_V(\tau+\lambda,t)g(x') - Y_V(\tau,\tau+\lambda)Y_V(\tau+\lambda,t)g(x)| \\ \le \; & |Y_V(\tau,\tau+\lambda)Y_V(\tau+\lambda,t)g(x') - Y(\tau,\tau+\lambda)Y_V(\tau+\lambda,t)g(x')| \\ + \; & |Y(\tau,\tau+\lambda)Y_V(\tau+\lambda,t)g(x') - Y(\tau,\tau+\lambda)Y_V(\tau+\lambda,t)g(x)| \\ + \; & |Y_V(\tau,\tau+\lambda)Y_V(\tau+\lambda,t)g(x) - Y(\tau,\tau+\lambda)Y_V(\tau+\lambda,t)g(x)| \\ \le \; & 2||Y_V(\tau,\tau+\lambda) - Y(\tau,\tau+\lambda)||_{\infty\to\infty}||Y_V(\tau+\lambda,t)g||_\infty \\ + \; & |Y(\tau,\tau+\lambda)Y_V(\tau+\lambda,t)g(x') - Y(\tau,\tau+\lambda)Y_V(\tau+\lambda,t)g(x)|. \end{aligned}$$

This completes the proof of Lemma 8.8.

Let us go back to the proof of Theorem 8.3(a). Suppose that the conditions in Theorem 8.3(a) hold, and let $g \in L^\infty_{\mathcal{E}}$. Since Y_V is a uniformly bounded backward $L^\infty_{\mathcal{E}}$-propagator (see Remark 7.2), we have

$$||Y_V(\tau,t)||_{\infty\to\infty} < M \tag{75}$$

for all $(\tau,t) \in D_T$. It follows from (75) and Lemma 7.9 that for every $\epsilon > 0$ there exists $\lambda > 0$ such that $\tau + \lambda < t$ and

$$2||Y_V(\tau,\tau+\lambda) - Y(\tau,\tau+\lambda)||_{\infty\to\infty}||Y_V(\tau+\lambda,t)g||_\infty < \frac{\epsilon}{2}. \tag{76}$$

Moreover, for λ as above and any fixed $x \in E$ there exists $\delta > 0$ such that

$$|Y(\tau,\tau+\lambda)Y_V(\tau+\lambda,t)g(x') - Y(\tau,\tau+\lambda)Y_V(\tau+\lambda,t)g(x)| < \frac{\epsilon}{2} \tag{77}$$

for all x' such that $\rho(x',x) < \delta$. This follows from (75) and the assumption that Y is a backward strong Feller propagator. Now it is easy to see that Theorem 8.3(a) can be obtained from (76), (77), and Lemma 8.8.

This completes the proof of Theorem 8.3.

Proof of Theorem 8.4. Part (a). Let $g \in C_\infty$. Then $Y_V(\tau,t)g \in BC$ for all $(\tau,t) \in D_T$, by Theorem 8.3(a).

For every $\epsilon > 0$ there exists a compact set K_ϵ such that $|g(x)| < \epsilon$ for all $x \in E \backslash K_\epsilon$. Moreover, Urysohn's Lemma implies that there exists a continuous function g_ϵ on E with compact support such that $g_\epsilon(x) = 1$ for

$x \in K_\epsilon$. Then we have

$$
\begin{aligned}
|Y_V(\tau,t)g(x)| &\leq |Y_V(\tau,t)gg_\epsilon(x)| + |Y_V(\tau,t)g(1-g_\epsilon(x)| \\
&\leq c_{\tau,t}[\{Y(\tau,t)|gg_\epsilon|^2(x)\}^{\frac{1}{2}} + \epsilon].
\end{aligned}
$$

It is clear that this implies part (a) of Theorem 8.4.

(b) Let Y be a strongly continuous Feller-Dynkin propagator. By part (a) of Theorem 8.4, Y_V is a Feller-Dynkin propagator. Reasoning as in the proof of the strong continuity of Y_V in the space L^s in Theorem 7.6, and using the C-norm instead of the L^s-norm, we see that Y_V is strongly continuous on C_∞.

Proof of Theorem 8.5. It is clear that Y_V is a backward BUC-propagator (see Theorem 8.3). Now we can obtain the strong continuity of Y_V on the space BUC, reasoning as in the proof of the strong continuity of Y_V on L^s in Theorem 7.6, and using the C-norm instead of the L^s-norm.

Proof of Theorem 8.6. We will first prove part (b) of Theorem 8.6. Suppose that Y is a strongly continuous backward BUC-propagator, possessing the strong BUC-property, and let t and g be such as in the formulation of Theorem 8.6. Fix τ with $0 \leq \tau < t$ and $x \in E$, and let τ' be close to τ, and $x' \in E$ be close to x. Then we have

$$
\begin{aligned}
|Y_V(\tau',t)g(x') - Y_V(\tau,t)g(x)| &\leq ||Y_V(\tau',t)g - Y_V(\tau,t)g||_\infty \\
&+ |Y_V(\tau,t)g(x') - Y_V(\tau,t)g(x)| \\
&= I_1 + I_2. \qquad (78)
\end{aligned}
$$

By Theorem 8.3(b), Y_V satisfies the strong BUC-condition. Hence,

$$
\lim_{x' \to x} I_2 = 0. \qquad (79)
$$

If τ' is close to τ in the expression I_1, then, for some $\delta > 0$, we have $t - \delta > \max(\tau', \tau)$. Since Y_V is a backward BUC propagator, satisfying the strong BUC-condition, we get

$$
I_1 = ||Y_V(\tau', t-\delta)Y_V(t-\delta,t)g - Y_V(\tau,t-\delta)Y_V(t-\delta,t)g||_\infty \to 0 \qquad (80)
$$

if $\tau' \to \tau$. Now it is clear that part (b) of Theorem 8.6 follows from (78), (79), and (80).

Next we will prove part (a) of Theorem 8.6. Suppose that Y is a backward strong Feller propagator. Suppose also that Y is continuous on BC in the topology of uniform convergence on compact subsets of E. Using Theorem 8.3(a), we see that Y_V is a backward strong Feller propagator. Let t and g be such as in the formulation of Theorem 8.6(a), and fix τ with $0 \leq \tau < t$ and $x \in E$. Suppose that τ' is close to τ, and $x' \in U(x)$ where $U(x)$ is a relatively compact neighborhood of x. Then we have

$$\begin{aligned}|Y_V(\tau',t)g(x')-Y_V(\tau,t)g(x)| &\leq |Y_V(\tau',t)g(x')-Y_V(\tau,t)g(x')| \\ &+ |Y_V(\tau,t)g(x')-Y_V(\tau,t)g(x)| \\ &= J_1+J_2. \end{aligned} \tag{81}$$

Since Y_V is a backward strong Feller propagator,

$$\lim_{x'\to x} J_2 = 0. \tag{82}$$

For δ such as in the proof of part (b) above, we have the following:

$$\sup_{x'\in U(x)} J_1 \leq \sup_{x'\in U(x)} |Y_V(\tau',t-\delta)Y_V(t-\delta,t)g(x')-Y_V(\tau,t-\delta)Y_V(t-\delta,t)g(x')|. \tag{83}$$

Using the fact that Y_V is a backward strong Feller propagator again, we obtain $Y_V(t-\delta,t)g\in BC$. Moreover, for every $h\in BC$ and $\tau'<\tau$, we have

$$\begin{aligned}\sup_{x'\in U(x)} |Y_V(\tau',t-\delta)h(x')-Y_V(\tau,t-\delta)h(x')| &\leq \\ &\leq \sup_{x'\in U(x)} |(Y_V(\tau',\tau)-I)Y_V(\tau,t-\delta)h(x')| \\ &\leq M||Y_V(\tau',\tau)-Y(\tau',\tau)||_{BC\to BC}||h||_C \\ &+ \sup_{x'\in U(x)} |(Y(\tau',\tau)-I)Y_V(\tau,t-\delta)h(x')|.\end{aligned}$$

Now Lemma 7.5 and the continuity assumption for Y imply

$$\lim_{\tau'\to\tau-} \sup_{x'\in U(x)} |Y_V(\tau',t-\delta)h(x')-Y_V(\tau,t-\delta)h(x')| = 0. \tag{84}$$

Next let $\tau<\tau'<t-\delta$, and let λ be such that $\tau'<\lambda<t-\delta$. Then

$$\begin{aligned}&\sup_{x'\in U(x)} |Y_V(\tau',t-\delta)h(x')-Y_V(\tau,t-\delta)h(x')| \\ \leq\ &\sup_{x'\in U(x)} |(Y_V(\tau',\lambda)-Y_V(\tau,\lambda))Y_V(\lambda,t-\delta)h(x')| \\ \leq\ &\sup_{x'\in U(x)} |(Y(\tau',\lambda)-Y(\tau,\lambda))Y_V(\lambda,t-\delta)h(x')| \\ +\ &M||Y_V(\tau',\lambda)-Y(\tau',\lambda)||_{BC\to BC}||h||_C \\ +\ &M||Y_V(\tau,\lambda)-Y(\tau,\lambda)||_{BC\to BC}||h||_C \\ =\ &C_1+C_2+C_3.\end{aligned} \tag{85}$$

It follows from Lemma 7.5 that for every $\epsilon>0$ there exists λ such that $C_2+C_3<\frac{\epsilon}{2}$. Then the continuity assumption for Y implies that there exists $\eta > 0$ such that $\tau \le \tau' \le \tau + \eta \le \lambda$ and $C_1 \le \frac{\epsilon}{2}$. Therefore, (85) gives

$$\lim_{\tau' \to \tau+} \sup_{x' \in U(x)} |Y_V(\tau', t-\delta)h(x') - Y_V(\tau, t-\delta)h(x')| = 0. \tag{86}$$

Since (83), (84), and (86) hold,

$$\lim_{\tau' \to \tau} \sup_{x' \in U(x)} J_1 = 0. \tag{87}$$

Now it is clear that (81), (82), and (87) imply Theorem 8.6(a).

Remark 8.9. Analyzing the proof of Theorem 8.6(a), we see that this theorem holds under the following weaker conditions:

(1) Y is a backward strong Feller propagator;
(2) $Y(\tau,t)h$ is continuous on D_T in the topology of uniform convergence on compact subsets of E for all functions of the form $h = Y_V(r,s)g$ with $g \in L^\infty$ and $(r,s) \in D_T$.

It is not difficult to see that Theorem 8.6(b) follows from the version of Theorem 8.6(a) described in Remark 8.9.

9. Subclasses of the classes $\mathcal{P}_f^*$ and $\mathcal{P}_m^*$

In this section, we define certain subclasses of the classes $\mathcal{P}_f^*$ and $\mathcal{P}_m^*$, and show that backward Feynman-Kac propagators associated with functions and time-dependent measures from these new classes inherit the Feller, the Feller-Dynkin, and the BUC-property from free backward propagators.

Definition 9.1. *Assuming that $P \in \mathcal{M}$, we define the function classes $\mathcal{P}_{f,c}^*$ and $\mathcal{P}_{f,u}^*$ as follows:*

$$V \in \mathcal{P}_{f,c}^* \iff V \in \mathcal{P}_f^* \text{ and } N(V)(\tau,t,\cdot) \in BC \text{ for all } (\tau,t) \in D_T,$$

$$V \in \mathcal{P}_{f,u}^* \iff V \in \mathcal{P}_f^* \text{ and } N(V)(\tau,t,\cdot) \in BUC \text{ for all } (\tau,t) \in D_T.$$

Definition 9.2. *We define the class $\mathcal{D}_{f,c}^*$ as follows: A function $V \in \mathcal{P}_f^*$ belongs to this class if there exists exists a sequence $V_k \in \mathcal{P}_f^*$ such that $V_k(t,\cdot) \in BC$ for all $k \ge 1$ and $0 \le t \le T$, and V_k ζ-approaches V. The class $\mathcal{D}_{f,u}^*$ is defined similarly. Here we require the condition $V_k(t,\cdot) \in BUC$ for all $k \ge 1$ and $0 \le t \le T$, and V_k ζ-approaches V.*

Remark 9.3. If P possesses density p, then the classes of time-dependent measures $\mathcal{P}_{m,c}^*$, $\mathcal{P}_{m,u}^*$, $\mathcal{D}_{m,c}^*$, and $\mathcal{D}_{m,u}^*$ can be defined similarly.

Lemma 9.4. *The following assertions hold:*
1. $\mathcal{P}_{f,c}^* \subset \mathcal{D}_{f,c}^*$ *and* $\mathcal{P}_{m,c}^* \subset \mathcal{D}_{m,c}^*$.
2. $\mathcal{P}_{f,u}^* \subset \mathcal{D}_{f,u}^*$ *and* $\mathcal{P}_{m,u}^* \subset \mathcal{D}_{m,u}^*$.

3. *If* $V \in \mathcal{P}_f^*$, *and there exists a sequence* $V_k \in \mathcal{P}_{f,c}^*$ *such that* V_k ζ-*approaches* V, *then* $V \in \mathcal{P}_{f,c}^*$. *Similarly, if* $\mu \in \mathcal{P}_m^*$, *and there exists a sequence* $V_k \in \mathcal{P}_{f,c}^*$ *such that* V_k ζ-*approaches* V, *then* $\mu \in \mathcal{P}_{m,c}^*$.
4. *If* $V \in \mathcal{P}_f^*$, *and there exists a sequence* $V_k \in \mathcal{P}_{f,u}^*$ *such that* V_k ζ-*approaches* V, *then* $V \in \mathcal{P}_{f,u}^*$. *Similarly, if* $\mu \in \mathcal{P}_m^*$, *and there exists a sequence* $V_k \in \mathcal{P}_{f,u}^*$ *such that* V_k ζ-*approaches* V, *then* $\mu \in \mathcal{P}_{m,u}^*$.

Proof. Part 1. Let $V \in \mathcal{P}_{f,c}^*$. Since for the sequence g_k defined by (13), we have $g_k(t,\cdot) \in BC$ for all $k \geq 1$ and $0 \leq t \leq T$, and since g_k ζ-approaches V (see Lemma 5.1), we get $V \in \mathcal{D}_{f,c}^*$. The proof for the measures, and that of part 2 of Lemma 9.4 is similar.

Part 3. Let $V \in \mathcal{P}_f^*$, and assume that there exists a sequence $V_k \in \mathcal{P}_{f,c}^*$ such that V_k ζ-approaches V. Using Definition 9.2 and Lemma 5.1, we see that $V \in \mathcal{P}_{f,c}^*$. The proof for the measures and that of part 4 of Lemma 9.4 is similar.

Theorem 9.5. *Let* $P \in \mathcal{M}$ *and* $V \in \mathcal{D}_{f,c}^*$. *Then the following assertions hold:*
(a) *If* Y *is a backward Feller propagator, then* Y_V *has the same property.*
(b) *If* Y *is a backward Feller-Dynkin propagator, then* Y_V *has the same property. If, in addition,* Y *is strongly continuous on* C_∞, *then* Y_V *is also strongly continuous on* C_∞.

Theorem 9.6. *Let* $P \in \mathcal{M}$ *and* $V \in \mathcal{D}_{f,u}^*$. *Then if* Y *is a backward* BUC-*propagator, then* Y_V *has the same property. If, in addition,* Y *is strongly continuous on* BUC, *then* Y_V *is also strongly continuous on* BUC.

Remark 9.7. Theorem 9.5 (Theorem 9.6) holds for a time-dependent measure $\mu \in \mathcal{D}_{m,c}$ ($\mu \in \mathcal{D}_{m,u}$), provided that $P \in \mathcal{M}$ possesses density p.

Proof of theorems 9.5 and 9.6. We start with the proof of part (b) of Theorem 9.5. Let $V \in \mathcal{D}_{f,c}^*$, $g \in C_\infty$, and let $V_k \in \mathcal{P}_f^*$ be a sequence of functions such that $V_k(t,\cdot) \in BC$ for all $k \geq 1$ and $0 \leq t \leq T$, and V_k ζ-approaches V. Then using Lemma 5.1(a) and estimate (45) with $q = 1$ and $r = 2$, we get

$$\begin{aligned} & ||Y_V(\tau,t)g - Y_{V_k}(\tau,t)g||_C \\ \leq\ & ||g||_C \frac{1}{(1-M(2|V|)(\tau,t))^{\frac{1}{2}}}\{[2M(2|V-V_k|)(\tau,t)M(2(V-V_k))(\tau,t)]^{\frac{1}{2}} \\ +\ & \frac{2\sqrt{3}}{3}\frac{M(2|V-V_k|)(\tau,t)M(2(V-V_k))(\tau,t)}{1-M(2|V-V_k|)(\tau,t)}\}^{\frac{1}{2}} \end{aligned} \tag{88}$$

for all $k \geq k_0$ and $t - \tau < \delta$ where $\delta > 0$ is small and does not depend on k. It follows from (88) and Lemma 5.1 that for $t - \tau < \delta$, we have

$$\lim_{k\to\infty} ||Y_V(\tau, t)g - Y_{V_k}(\tau, t)g||_C = 0. \tag{89}$$

Hence, it is sufficient to prove Theorem 9.5(b) for all functions $V \in \mathcal{P}_f^*$ such that $V(t, \cdot) \in BC$ for all $0 \leq t \leq T$. Indeed, if this is true, then Y_{V_k} is a backward Feller-Dynkin propagator for all $k \geq 1$. It follows from (89) and from the fact that C_∞ is closed in BC that $Y_V(\tau, t)g \in C_\infty$ for all $g \in C_\infty$ and $t - \tau < \delta$. Next the properties of backward propagators show that Y_V is a backward Feller-Dynkin propagator. The previous reasoning also implies that the Feller-Dynkin property is stable under the approximation in the ζ-sense. Hence, Theorem 9.5(b) holds for all $V \in \mathcal{D}_{f,c}^*$.

Our final goal is to prove Theorem 9.5(b) for a function $V \in \mathcal{P}_f^*$ for which $V(t, \cdot) \in BC$ for all $0 \leq t \leq T$. Let $g \in C_\infty$, and assume that Y is a backward Feller-Dynkin propagator. Then using formula (40), we get

$$Y_V(\tau,t)g(x) - Y(\tau,t)g(x) = \sum_{j\geq 1}\int_\tau^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{j-1}}^t Y(\tau,t_1)V(t_1)Y(t_1,t_2)V(t_2) \cdots Y(t_{j-1},t_j)V(t_j)Y(t_j,t)g(x)dt_j. \tag{90}$$

for all $k \geq k_0$ and $t - \tau < \delta$. The family Y consists of contractions on $L_{\mathcal{E}}^\infty$ which map the space C_∞ into itself. Moreover, the definition of the class $\mathcal{P}_{f,c}^*$ shows that $V(t_k, \cdot) \in BC$ for every fixed t. The integrands in (90) are Borel functions of the variables $t_1, \cdots, t_j$, and belong to the space C_∞ in the variable x. It follows that the integrals in (90) also belong to the space C_∞ (use the dominated convergence theorem). Since the series in (90) converges in C, and C_∞ is a closed subspace of BC, the function on the right-hand side of (90) is in C_∞. By assumption, we have $Y(\tau, t)g \in C_\infty$ for all $k \geq k_0$ and $t - \tau < \delta$. Using the backward propagator properties, we see that Y_V is a backward Feller-Dynkin propagator. If in addition, Y is strongly continuous on C_∞, then we can prove the strong continuity of Y_V on C_∞, using the same methods as in the proof of part (b) of theorem 7.6 with $s = \infty$.

This completes the proof of part (b) of Theorem 9.5. The proof of part (a) and that of Theorem 9.6 is similar.

10. Sufficient conditions and examples

A well-known source of examples of transition probability densities is the theory of second order parabolic partial differential equations. Let us consider the equation

$$\frac{\partial u}{\partial \tau} + Lu = 0. \tag{91}$$

In (91), L stands for a differential operator given by

$$L = \sum_{i,j=1}^{n} a_{ij}(\tau, x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(\tau, x) \frac{\partial}{\partial x_i} \tag{92}$$

(non-divergence form), or by

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} [a_{ij}(\tau, x) \frac{\partial}{\partial x_j}] + \sum_{i=1}^{n} b_i(\tau, x) \frac{\partial u}{\partial x_i} \tag{93}$$

(divergence form). Let us also consider the final value problem,

$$\begin{cases} \frac{\partial u}{\partial \tau} + Lu = 0, & 0 \le \tau < t \le T, \\ u(t) = f, \end{cases} \tag{94}$$

for equation (91). Solutions to problem (94) with L in the divergence form are understood in the weak sense. If there exists a fundamental solution p for equation (91), then p can be used as a transition probability density. The following results are known:

Non-divergence form. Let L be as in formula (92), and assume that

(1) The functions a_{ij} and b_i are bounded and measurable on $[0, T] \times R^n$;
(2) There exists a constant $\gamma > 0$ such that for all $(\tau, x) \in [0, T] \times R^n$ and any collection of real numbers $\lambda_1, \cdots, \lambda_n$,

$$\sum_{i,j=1}^{n} a_{ij}(\tau, x) \lambda_i \lambda_j \ge \gamma \sum_{i=1}^{n} \lambda_i^2;$$

(3) There exists a constant δ with $0 < \delta \le 1$ such that

$$\sum_{i,j=1}^{n} |a_{ij}(\tau_1, x_1) - a_{ij}(\tau_2, x_2)| + \sum_{i=1}^{n} |b_i(\tau_1, x_1) - b_i(\tau_2, x_2)| \le$$
$$\le C(|x_1 - x_2|^\delta + |\tau_1 - \tau_2|^\delta)$$

for all $(\tau_1, x_1), (\tau_2, x_2) \in [0, T] \times R^n$.

Then there exists a unique fundamental solution $p(\tau, x; t, y)$ of equation (91). The function p satisfies the following conditions: it is jointly continuous, strictly positive,

$$p(\tau, x; t, y) \le M(t - \tau)^{-\frac{n}{2}} \exp\{-\frac{\alpha |x - y|^2}{t - \tau}\}, \tag{95}$$

and

$$|\frac{\partial p}{\partial x_i}(\tau, x; t, y)| \le M(t - \tau)^{-\frac{n+1}{2}} \exp\{-\frac{\alpha |x - y|^2}{t - \tau}\}. \tag{96}$$

For $f \in C_0^\infty$ and $t > 0$, the function

$$u(\tau, x) = \int_{R^n} f(y) p(\tau, x; t, y) dy$$

is in $C_b^{1,2}([0,t] \times R^n)$ and satisfies (94) (see e.g., [6, 7, 16]). It follows from the upper Gaussian estimate (95) that the process X corresponding to p is a continuous process. It is not difficult to prove, using estimates (95) and (96) that the backward free propagator Y associated with the density p is $(L^r - L^q)$-smoothing for all $1 \le r \le q \le \infty$, and possesses the strong Feller and the strong BUC-property. It follows that Y is a backward BC- and BUC-propagator. Moreover, Y is a backward C_∞-propagator (use estimate (95)).

We will need the following simple assertion concerning general transition probability densities:

Lemma 10.1. *For every function $f \in BUC$ and $\epsilon > 0$, we have*

$$||f - Y(\tau,t)f||_C \le \sup_{x,y \in E: \rho(x,y) \le \epsilon} |f(x) - f(y)| + 2||f||_C \sup_{x \in E} \int_{y: \rho(x,y) > \epsilon} p(\tau, x; t, y) dy.$$

We leave the proof of Lemma 10.1 as an exercise for the reader.

Lemma 10.2. *Let p be a fundamental function for equation (91) in non-divergence form satisfying estimates (95) and (96). Then the free backward propagator Y associated with p is strongly continuous on C_∞ and BUC.*

Proof. Since Y is a backward BC-propagator, the strong BUC-continuity implies the strong C_∞-continuity. Moreover, it follows from estimate (95) that p is uniformly stochastically continuous (see the definition in the introduction).

Let $f \in BUC$. Then using Lemma 10.1, estimate (95), and the uniform stochastic continuity of a scaled Gaussian kernel, we get that

$$\lim_{t - \tau \to 0+} ||f - Y(\tau,t)f||_C = 0. \tag{97}$$

By Theorem 2.2, the strong continuity of Y on BUC follows from the separate continuity. Equality (97) means the strong continuity of Y on BUC on the diagonal $t = \tau$. Next we fix t and τ with $\tau < t$, and assume that $t' > t$. Then

$$||Y(\tau,t')f - Y(\tau,t)f||_C = ||Y(\tau,t)(Y(t,t')f - f)||_C \le M||Y(t,t')f - f||_C.$$

Similar reasoning applies in the case where $t' \le t$. Combining these two cases and using (97), we see that

$$\lim_{t' \to t} ||Y(\tau,t')f - Y(\tau,t)f||_C = 0. \tag{98}$$

Let $\tau > \tau'$. Then

$$||Y(\tau,t)f - Y(\tau',t)f||_C = ||Y(\tau',\tau)Y(\tau,t)f - Y(\tau,t)f||_C.$$

Since t and τ are fixed, Y is a BUC-propagator, and (97) holds, we get

$$\lim_{\tau'\to\tau-} ||Y(\tau',t)f - Y(\tau,t)f||_C = 0. \tag{99}$$

Now let $\tau < \tau'$. Then

$$||Y(\tau,t)f - Y(\tau',t)f||_C = ||Y(\tau,\tau')Y(\tau',t)f - Y(\tau',t)f||_C. \tag{100}$$

Applying Lemma 10.1 to (100), we get

$$\begin{aligned} ||Y(\tau,t)f - Y(\tau',t)f||_C \le &\sup_{x,y\in R^n:|x-y|\le\epsilon} |Y(\tau',t)f(x) - Y(\tau',t)f(y)| \\ &+2||Y(\tau',t)f||_C \sup_{x\in R^n} \int_{y:\rho(x,y)>\epsilon} p(\tau,x;\tau',y)dy. \end{aligned} \tag{101}$$

The first term on the right-hand side of (101) tends to 0 as $\epsilon \to 0$ uniformly with respect to τ' near τ, by the gradient estimate (96) (note that τ' and τ are separated from t). For every fixed $\epsilon > 0$, the second term tends to 0 as $\tau' \to \tau$, by the upper Gaussian estimate (95) and the uniform stochastic continuity of a scaled Gaussian density. This gives

$$\lim_{\tau'\to\tau+} ||Y(\tau',t)f - Y(\tau,t)f||_C = 0. \tag{102}$$

Now (97), (98), (99), and (102) imply the separate continuity of Y on BUC. It follows from Theorem 2.2 that Y is jointly continuous on BUC.

This completes the proof of Lemma 10.2.

Remark 10.3. Since the backward free propagator in the example above is $(L^r - L^q)$-smoothing for all $1 \le r \le q \le \infty$, possesses the strong Feller and the strong BUC-property, and is a strongly continuous propagator on C_∞ and BUC, the perturbed backward propagator Y_μ with $\mu \in \mathcal{P}_m^*$ satisfies $Y_\mu(\tau,t) \in L(L^r, L^q)$ for all $(\tau,t) \in D_T$ and $1 < r \le q \le \infty$, possesses the strong Feller and the strong BUC-property, and is a strongly continuous propagator on C_∞ and BUC (see theorems 7.1, 7.6, 7.11, 8.4, and 8.5).

Divergence form. Let L be as in (93), and assume that the following conditions hold:

1. The functions a_{ij} and b_i are bounded and measurable on $[0,T]\times R^n$;
2. There exists a constant $\gamma > 0$ such that for all $(\tau,x) \in [0,T] \times R^n$ and any collection of real numbers $\lambda_1, \cdots, \lambda_n$,

$$\sum_{i,j=1}^n a_{ij}(\tau,x)\lambda_i\lambda_j \ge \gamma \sum_{i=1}^n \lambda_i^2.$$

Then there exists a unique fundamental solution $p(\tau, x; t, y)$ of equation (91). The function p satisfies the Gaussian estimate (92) (more information on the fundamental solutions in the divergence case can be found in [21]).

Remark 10.4. As in the non-divergence case, the backward free propagator Y in the divergence case is such that $Y(\tau, t) \in L(L^r, L^q)$ for all $(\tau, t) \in D_T$ and $1 \leq r \leq q \leq \infty$ (this follows from the Gaussian estimate). The strong Feller property also holds for Y (this follows from the Gaussian estimate and the continuity of p). Moreover, Y is strongly continuous on C_∞ (this fact can be obtained from the strong Feller property, the Gaussian estimate, Lemma 10.1, and the ideas in the proof of part (b) of Theorem 7.6). However, the validity of the strong BUC-property for Y is not clear. Hence, by the results obtained in the present paper, the perturbed backward propagator Y_μ with $\mu \in \mathcal{P}_m^*$ satisfies $Y_\mu \in L(L^r, L^q)$ for all $(\tau, t) \in D_T$ and $1 < r \leq q \leq \infty$, possesses the strong Feller property, and is a strongly continuous backward propagator on C_∞.

11. Backward transition functions and forward propagators

In this section we discuss forward Feynman-Kac propagators. There is a simple connection between the forward and the backward cases which is based on the idea of time reversion. We will start with a quick overview of all the necessary notions in the case of forward propagators.

A propagator S on a Banach space B is a two-parametric family $S(t, \tau) \in L(B, B)$, $(t, \tau) \in D_T$, such that $S(t, \lambda)S(\lambda, \tau) = S(t, \tau)$ for all $\tau \leq \lambda \leq t$, and $S(t, t) = I$ for all $0 \leq t \leq T$.

Suppose that $\tilde{P}(\tau, A; t, y)$ satisfies the following conditions:

(1) For fixed τ, A, and t, $\tilde{P}$ is a nonnegative Borel function on E.
(2) For fixed τ, t, and y, $\tilde{P}$ is a Borel measure on $\mathcal{E}$.
(3) The normality condition, $\tilde{P}(\tau, E; t, y) = 1$ holds for all τ, t, and y.
(4) The Chapman-Kolmogorov equation,

$$\tilde{P}(\tau, A; t, y) = \int_E \tilde{P}(\tau, A; \lambda, x)\tilde{P}(\lambda, dx; t, y)$$

holds for all $\tau < \lambda < t$, A, and y. Then $\tilde{P}$ is called a backward transition probability function.

The free propagator associated with $\tilde{P}$ is defined on $L^\infty_{\mathcal{E}}$ by

$$\begin{cases} U(t, \tau)g(y) = \int_E g(x)\tilde{P}(\tau, dx; t, y) \\ U(t, t)f = f, \end{cases}$$

for all τ, t, and $f \in L^{\infty}_{\mathcal{E}}$. If $\tilde{P}$ possesses density $\tilde{p}$, then

$$\begin{cases} U(t,\tau)g(y) = \int_E g(x)\tilde{p}(\tau,x;t,y)dx \\ U(t,t)f = f, \end{cases}$$

for all $x \in E$, $0 \leq \tau < t < \infty$, and $f \in L^{\infty}$.

We define time reversal η by $\eta(t) = T - t$ where $t \in [0,T]$. For a function V on $[0,T] \times E$ and a time-dependent measure μ, we put $\eta(V)(t,x) = (\eta(t), x)$ and $\eta(\mu)(t) = \mu(\eta(t))$. An important connection between the forward and backward cases is as follows: If $\tilde{P}$ is a backward transition probability function, then

$$P(\tau,x;t,A) = \tilde{P}(\eta(t),A;\eta(\tau),x) \tag{103}$$

is a transition probability density. If $(X, \mathcal{F}^{\tau}_t, P_{\tau,x})$ is a non-homogeneous progressively measurable Markov process on $(\Omega, \mathcal{F})$ with transition probability function P, then we can define a progressively measurable backward Markov process $\tilde{X}$ on $(\Omega, \mathcal{F})$ by $\tilde{X}_t = X_{\eta(t)}$.

Suppose that a backward transition probability function $\tilde{P}$ is given. If V is a Borel function on $[0,T] \times E$, then we will say that V belongs to the class $\mathcal{P}_f$ provided that $\eta(V) \in \mathcal{P}^*_f$ (we should take into account (103)). Similarly, if $\tilde{P}$ possesses density $\tilde{p}$ and μ is a time-dependent measure, we will say that μ belongs to the class $\mathcal{P}_m$ provided that $\eta(\mu) \in \mathcal{P}^*_m$. The potentials of V and μ are defined by

$$\tilde{N}(V)(t,\tau,x) = \int_{\tau}^{t} U(t,s)V(s)(x)ds$$

and

$$\tilde{N}(\mu)(t,\tau,x) = \int_{\tau}^{t} U(t,s)\mu(s)(x)ds,$$

respectively. It $\tilde{P}$ possesses density $\tilde{p}$, then the functional C_{μ} corresponding to a time-dependent measure $\mu \in \mathcal{P}_m$ is given by

$$C_{\mu}(t,\tau) = A_{\eta(\mu)}(T-t, T-\tau). \tag{104}$$

Here we should take into account the correspondence between $\tilde{P}$ and P given by (103), and use Theorem 5.6. Since $\eta(\mu) \in \mathcal{P}^*_m$, we only need the progressive measurability of the process X (or equivalently, the progressive measurability of the process $\tilde{X}$) in order the right-hand side of (104) to be defined.

Let $\tilde{P}$ be a backward transition probability function, and suppose that the process $\tilde{X}$ is progressively measurable. Then for any $V \in \mathcal{P}_f$, the

Feynman-Kac propagator U_V is defined on the space $L^\infty_{\mathcal{E}}$ by

$$U_V(t,\tau)g(y) = E_{\eta(t),y} g(X_{\eta(\tau)}) \exp\{-\int_\tau^t V(s, X_{\eta(s)})ds\}.$$

Similarly, if $\tilde{P}$ possesses density $\tilde{p}$ and if the process $\hat{X}$ is progressively measurable, then the Feynman-Kac propagator U_μ is defined on the space $L^\infty_{\mathcal{E}}$ by

$$U_\mu(t,\tau)g(y) = E_{\eta(t),y} g(X_{\eta(\tau)}) \exp\{-C_\mu(t,\tau)\}.$$

If $\tilde{P}$ is a backward transition probability function, the process $\tilde{X}$ is progressively measurable, and $V \in \mathcal{P}_f$, then Duhamel's formula takes the following form:

$$U_V(t,\tau)g(y) = U(t,\tau)g(y) + \int_\tau^t U(t,s)[V(s)U_V(s,\tau)g](y)ds$$

for all $y \in E$ and $g \in L^\infty$. If $\tilde{P}$ possesses density $\tilde{p}$, $\tilde{X}$ is progressively measurable, and $\mu \in \mathcal{P}_m$, then Duhamel's formula holds for μ.

Now it is clear that all the results for backward Feynman-Kac propagators obtained in the present paper can be reformulated in the case of forward propagators using time reversal techniques.

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