FEYNMAN-KAC PROPAGATORS AND VISCOSITY SOLUTIONS

ARCHIL GULISASHVILI AND JAN A. VAN CASTEREN

Abstract. We study viscosity solutions to the following partial differential equation on the set $[0,t] \times E$:

$$\frac{\partial u}{\partial \tau}(\tau,x) + [A(\tau)u(\tau)](x) - V(\tau,x)u(\tau,x) = 0,$$

where E is a locally compact second countable Hausdorff topological space and V satisfies a Kato type condition. The family of operators $A(\tau)$ in the equation above is defined by

$$A(\tau)h(x) = \lim_{\epsilon\to 0+} \epsilon^{-1}[Y(\tau+\epsilon,\tau)h(x) - h(x)],$$

where Y is the free backward propagator associated with the given transition probability density p. It is shown in the paper that under some restrictions on p, V, $\tau_0 \in [0,t)$, and $x_0 \in E$, the backward Feynman-Kac propagator Y_V associated with p and V generates a viscosity solution to the equation above at the point (τ_0, x_0). Similar result holds in the case where the function V is replaced by a time-dependent family μ of Borel measures on E.

1. Introduction

Our main objective in the present paper is to show that backward Feynman-Kac propagators associated with time-dependent measures generate viscosity solutions to terminal value problems. We restrict our attention to backward propagators. However, all our results in this paper can be reformulated for forward propagators and initial value problems. We refer the reader to [1, 2] for the information on viscosity solutions, and to [6-10] for the properties of Feynman-Kac propagators.

Let E be a locally compact second countable Hausdorff topological space. Then E is σ-compact and metrisable (see [11]). We will fix a metric $\rho : E \times E \to R^+$ generating the topology of E. By $\mathcal{E}$ will be denoted the σ-algebra of Borel subsets of E, and the symbol $B(x,r)$ will stand for

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the open ball of radius r centered at $x \in E$. The space of bounded continuous functions on E will be denoted by BC, and the symbol Z will stand for the set of integers. It is known that the family of Radon measures on E coincides with the family

of Borel measures that are finite on compact sets (see [5], Theorem 7.8). It will be assumed throughout the paper that a non-negative Radon measure m with full support is given. This measure is called the reference measure. We will write dx instead of $dm(x)$. Let $p(\tau, x; t, y)$ with $0 \le \tau < t < \infty$ and x, $y \in E$ be a transition probability density, and denote by $Y(t, \tau)$ the corresponding free backward propagator on L^∞, defined by

$$Y(t,\tau)f(x) = \int_E f(y)p(\tau, x; t, y)dy$$

for all $0 \le \tau < t < \infty$. If ν is a Borel measure on E, we put

$$Y(t,\tau)f(x) = \int_E p(\tau, x; t, y)d\nu(y).$$

We will denote by X_t the Markov process associated with the density p, and by $\mathcal{F}_t = \sigma(X_s : 0 \le s \le t)$ the filtration generated by the process X_s. It will be assumed throughout the paper that X_t is a progressively measurable process. Examples of such processes are left- or right continuous processes, or more generally, $\mathcal{F}_t$-predictable or $\mathcal{F}_t$-well measurable processes (see [14] for the definitions and more information concerning progressive measurability).

The following classes of time-dependent measures were introduced and studied in [9, 10] (see also [6-8]):

Definition 1. *Let $\mu = \{\mu(t) : 0 \le t \le T\}$ be a time-dependent family of Radon measures on E. Then μ belongs to the class $\hat{\mathcal{P}}^*$, provided that*

$$\sup_{(t,\tau):0\le\tau\le t\le T} \sup_{x\in E} \int_\tau^t Y(s,\tau)|\mu(s)|(x)ds < \infty.$$

If $\mu \in \hat{\mathcal{P}}^*$, then μ belongs to the class $\mathcal{P}^*$, provided that

$$\lim_{t-\tau\to 0+} \sup_{x\in E} \int_\tau^t Y(s,\tau)|\mu(s)|(x)ds = 0.$$

If $d\mu(\tau) = V(\tau)dx$, then we will write $V \in \hat{\mathcal{P}}^*$ and $V \in \mathcal{P}^*$ instead of $\mu \in \hat{\mathcal{P}}^*$ and $\mu \in \mathcal{P}^*$. The class $\mathcal{P}^*$ is a generalization of the Kato class of measures (see [6-10] and the references therein).

Definition 2 (see [9, 10]). *Let $V \in \mathcal{P}^*$. Then the backward Feynman-Kac propagator Y_V is defined by*

$$Y_V(t,\tau)f(x) = E_{\tau,x} f(X_t) \exp\{-\int_\tau^t V(s,X_s)ds\}$$

for $0 \le \tau \le t \le T$ and $f \in L^\infty$.

It was shown in [9, 10] that if $\mu \in \mathcal{P}^*$, then there exists an additive functional $A_\mu(t,\tau)$ such that

$$\lim_{k\to\infty} \sup_{\tau:0\le\tau\le T} \sup_{x\in E} E_{\tau,x} \sup_{t:\tau\le t\le T} |A_\mu(t,\tau) - \int_\tau^t g_k(s,X_s)ds|^2 = 0,$$

where

$$g_k(s,x) = kN(\mu)(\min(s+\frac{1}{k},T),s,x) \tag{1}$$

and

$$N(\mu)(t,\tau,x) = \int_\tau^t Y(s,\tau)\mu(s)(x)ds$$

(see [10], Lemma 3).

Definition 3 (see [9, 10]). *Let $\mu \in \mathcal{P}^*$. Then the backward Feynman-Kac propagator Y_μ is defined by*

$$Y_\mu(t,\tau)f(x) = E_{\tau,x} f(X_t) \exp\{-A_\mu(t,\tau)\}$$

for $0 \le \tau \le t \le T$ and $f \in L^\infty$.

2. Duhamel's formula and informal computations

Our first result in the present paper is Duhamel's formula for backward Feynman-Kac propagators. We will show that for $\mu \in \mathcal{P}^*$, Duhamel's formula holds pointwise.

Theorem 1. *Let $f \in L^\infty$ and $\mu \in \mathcal{P}^*$. Then the function $u(\tau,x) = Y_\mu(t,\tau)f(x)$ satisfies the following Volterra type integral equation:*

$$u(\tau,x) = Y(t,\tau)f(x) - \int_\tau^t Y(s,\tau)[\mu(s)u(s)](x)ds \tag{2}$$

for all $x \in E$ and $0 \le \tau \le t \le T$.

Proof. We will first prove Theorem 1 for a family μ of the form $d\mu(\tau) = V(\tau)dx$. Under the assumptions in Theorem 1, we have

$$\int_\tau^t Y(s,\tau)[V(s)Y_V(t,s)f](x)ds =$$

$$\int_\tau^t E_{\tau,x} V(s, X_s) E_{s,X_s} f(X_t) \exp\{-\int_s^t V(\lambda, X_\lambda) d\lambda\} ds.$$

Using the Markov property, we obtain

$$\begin{aligned} &\int_\tau^t Y(s,\tau)[V(s) Y_V(t,s) f](x) ds \\ &= \int_\tau^t E_{\tau,x} V(s, X_s) E_{\tau,x}(f(X_t) \exp\{-\int_s^t V(\lambda, X_\lambda) d\lambda\} | \mathcal{F}_s) ds \\ &= \int_\tau^t E_{\tau,x} f(X_t) V(s, X_s) \exp\{-\int_s^t V(\lambda, X_\lambda) d\lambda\} ds \\ &= \int_\tau^t E_{\tau,x} f(X_t) \frac{\partial}{\partial s} \exp\{-\int_s^t V(\lambda, X_\lambda) d\lambda\} ds \\ &= E_{\tau,x} f(X_t) - E_{\tau,x} f(X_t) \exp\{-\int_\tau^t V(\lambda, X_\lambda) d\lambda\} \\ &= Y(t,\tau) f(x) - Y_V(t,\tau) f(x). \end{aligned}$$

It follows that

$$Y_V(t,\tau) f(x) = Y(t,\tau) f(x) - \int_\tau^t Y(s,\tau)[V(s) Y_V(t,s) f](x) ds. \qquad (3)$$

This gives Theorem 1 in the case $d\mu(\tau) = V(\tau) dx$.

Now let $\mu \in \mathcal{P}^*$, and denote by g_k the sequence of functions defined by (1). It was shown in [10], Theorem 4, that

$$\lim_{k\to\infty} \sup_{0\le\tau\le t\le T} \sup_{x\in E} |Y_\mu(t,\tau) f(x) - Y_{g_k}(t,\tau) f(x)| = 0 \qquad (4)$$

for all $f \in L^\infty$. The functions g_k belong to the class $\mathcal{P}^*$ (see [10], Lemma 3). It follows from (3) that

$$Y_{g_k}(t,\tau) f(x) = Y(t,\tau) f(x) - \int_\tau^t Y(s,\tau)[g_k(s) Y_{g_k}(t,s) f](x) ds. \qquad (5)$$

Using the properties of backward propagators, we get

$$\int_\tau^t Y(s,\tau)[g_k(s)Y_{g_k}(t,s)f](x)ds =$$

$$\int_\tau^t Y(s,\tau)k\int_s^{\min(s+\frac{1}{k},T)} Y(\lambda,s)[\mu(\lambda)Y_{g_k}(t,s)f](x)d\lambda ds$$

$$\int_\tau^t k\int_s^{\min(s+\frac{1}{k},T)} Y(\lambda,\tau)[\mu(\lambda)Y_{g_k}(t,s)f](x)d\lambda ds$$

$$= \int_\tau^{\min(t+\frac{1}{k},T)} d\lambda k\int_{\max(\lambda-\frac{1}{k},\tau)}^{\lambda} Y(\lambda,\tau)[\mu(\lambda)Y_{g_k}(t,s)f](x)ds$$

$$= \int_\tau^{\min(t+\frac{1}{k},T)} d\lambda Y(\lambda,\tau)[\mu(\lambda)k\int_{\max(\lambda-\frac{1}{k},\tau)}^{\lambda} Y_{g_k}(t,s)fds](x). \tag{6}$$

Passing to the limit as $k\to\infty$ in (6) and using (4) and Definition 1, we get

$$\lim_{k\to\infty}\int_\tau^t Y(s,\tau)[g_k(s)Y_{g_k}(t,s)f](x)ds = \int_\tau^t Y(\lambda,\tau)[\mu(\lambda)Y_\mu(t,\lambda)f](x)d\lambda. \tag{7}$$

Now we see that (2) follows from (4), (5), and (7).

This completes the proof of Theorem 1.

Our next goal is to provide a motivation for the use of viscosity solutions in the present paper. We will reason informally, assuming that all functions appearing in the reasoning are differentiable as many times as needed.

Let t and τ be such that $0\le\tau<t\le T$, and put $u(z,x)=Y_\mu(t,z)f(x)$ where $\tau\le z<t$. Applying the operator $Y(z,\tau)$ to equation (2), we get

$$Y(z,\tau)u(z)(x) = Y(z,\tau)Y(t,z)f(x) - \int_z^t Y(z,\tau)Y(s,z)[\mu(s)u(s)](x)ds.$$

$$= Y(t,\tau)f(x) - \int_\tau^z Y(s,\tau)[\mu(s)u(s)](x)ds.$$

Next differentiating the previous equation with respect to z, we obtain

$$\frac{\partial}{\partial z}Y(z,\tau)u(z)(x) - Y(z,\tau)[\mu(z)u(z)](x) = 0.$$

We can also compute the right derivative of $Y(z,\tau)u(z)(x)$ at τ, which gives

$$\lim_{\epsilon\to0+}\frac{Y(\tau+\epsilon,\tau)u(\tau+\epsilon)(x)-u(\tau,x)}{\epsilon} - \lim_{\epsilon\to0+}\frac{1}{\epsilon}\int_\tau^{\tau+\epsilon} Y(s,\tau)[\mu(s)u(s)](x)ds = 0. \tag{8}$$

Next we get from (8) that

$$\lim_{\epsilon\to0+}\frac{Y(\tau+\epsilon,\tau)u(\tau+\epsilon)(x)-u(\tau,x)}{\epsilon} - \mu(\tau,x)u(\tau,x) = 0$$

The previous equality should be understood as an equality for Radon measures. It follows that

$$\lim_{\epsilon\to 0+}\frac{Y(\tau+\epsilon,\tau)[u(\tau+\epsilon)-u(\tau)](x)}{\epsilon}+\lim_{\epsilon\to 0+}\frac{(Y(\tau+\epsilon,\tau)-I)u(\tau)(x)}{\epsilon} -\mu(\tau,x)u(\tau,x)=0. \quad (9)$$

The first term on the left-hand side of (9) is equal to the right derivative $D_1^+u(\tau,x)$. Moreover, assuming that

$$\frac{\partial Y(s,\tau)g}{\partial s}=A(s)Y(s,\tau)g,$$

where $A(s)$ is a family of operators, we derive from (9) the following equality:

$$D_1^+u(\tau,x)+A(\tau)u(\tau)(x)-\mu(\tau,x)u(\tau,x)=0. \quad (10)$$

Finally, we see that the backward Feynman-Kac propagator Y_μ generates a solution to the final value problem

$$\begin{cases} \frac{\partial}{\partial\tau}u(\tau,x)+[A(\tau)u(\tau)](x)-\mu(\tau,x)u(\tau,x)=0 \\ u(t,x)=f(x). \end{cases} \quad (11)$$

The previous informal reasoning was based on the differentiability assumption for the functions appearing in the proof. Next we will explain how to make the arguments used above rigorous. Here the idea of a pointwise solution in the viscosity sense to problem (11) will be helpful.

3. Viscosity solutions to terminal value problems

This section is concerned with putting the ideas leading to equalities (8) and (10) on the solid ground. Here the idea of a viscosity solution, will be helpful. However, in the new setting, the equalities in (8) and (10) become inequalities.

Theorem 2. *Let $\mu\in\mathcal{P}^*$, $f\in L^\infty$, and fix t such that $0<t\le T$. Then the following assertions hold:*
(*a*) *Suppose that ψ is a bounded continuous function on $[0,T]\times E$, and let $(\tau_0,x_0)\in[0,t)\times E$ and $\delta>0$ with $\tau_0+\delta<t$ be such that*

$$Y_\mu(t,\tau_0)f(x_0)-\psi(\tau_0,x_0)=\min_{(\tau,x)\in[\tau_0,\tau_0+\delta]\times E}(Y_\mu(t,\tau)f(x)-\psi(\tau,x)).$$

Then for every $0<\epsilon<\delta$ we have

$$\frac{Y(\tau_0+\epsilon,\tau_0)\psi(\tau_0+\epsilon)(x)-\psi(\tau_0,x_0)}{\epsilon}-\frac{1}{\epsilon}\int_{\tau_0}^{\tau_0+\epsilon}Y(s,\tau_0)[\mu(s)Y_\mu(t,s)f](x_0)ds\le 0. \quad (12)$$

(b) Suppose that ψ is a bounded continuous function on $[0,T]\times E$, and let $(\tau_0,x_0)\in[0,t)\times E$ and $\delta>0$ with $\tau_0+\delta<t$ be such that

$$Y_\mu(t,\tau_0)f(x_0)-\psi(\tau_0,x_0)=\max_{(\tau,x)\in[\tau_0,\tau_0+\delta]\times E}(Y_\mu(t,\tau)f(x)-\psi(\tau,x))$$

for some $(\tau_0,x_0)\in[0,t)\times E$. Then for every $0<\epsilon<\delta$ we have

$$\frac{Y(\tau_0+\epsilon,\tau_0)\psi(\tau_0+\epsilon)(x)-\psi(\tau_0,x_0)}{\epsilon}-\frac{1}{\epsilon}\int_{\tau_0}^{\tau_0+\epsilon}Y(s,\tau_0)[\mu(s)Y_\mu(t,s)f](x_0)ds\geq 0. \tag{13}$$

Proof of Theorem 2. We will prove part (a) of Theorem 2. The proof of part (b) is similar. Let ψ be any bounded Borel function on $[0,T]\times E$, M be any real number, and put

$$G(\tau,x)=Y_\mu(t,\tau)f(x)-\psi(\tau,x)-M. \tag{14}$$

We will need the following lemma:

Lemma 1. *Let $\mu\in\mathcal{P}^*$, and let $\epsilon>0$ be such that $\tau+\epsilon<t$. Then we have the following equality for G defined by (14):*

$$\begin{aligned}G(\tau,x)-Y(\tau+\epsilon,\tau)G(\tau+\epsilon)(x)\\=Y(\tau+\epsilon,\tau)\psi(\tau+\epsilon)(x)-\psi(\tau,x)-\int_\tau^{\tau+\epsilon}Y(s,\tau)[\mu(s)Y_\mu(t,s)f](x)ds.\end{aligned} \tag{15}$$

Proof of Lemma 1. We have

$$\begin{aligned}Y_\mu(t,\tau)f(x)-Y(\tau+\epsilon,\tau)Y_\mu(t,\tau+\epsilon)f(x)\\=(Y_\mu(\tau+\epsilon,\tau)-Y(\tau+\epsilon,\tau))Y_\mu(t,\tau+\epsilon)f(x).\end{aligned} \tag{16}$$

Using formula (2) in (16), we get

$$\begin{aligned}Y_\mu(t,\tau)f(x)-Y(\tau+\epsilon,\tau)Y_\mu(t,\tau+\epsilon)f(x)\\=-\int_\tau^{\tau+\epsilon}Y(s,\tau)[\mu(s)Y_\mu(\tau+\epsilon,s)Y_\mu(t,\tau+\epsilon)f](x)ds\\=-\int_\tau^{\tau+\epsilon}Y(s,\tau)[\mu(s)Y_\mu(t,s)f](x)ds.\end{aligned} \tag{17}$$

Now it is clear that (15) follows from (17).

This completes the proof of Lemma 1.

Let us go back to the proof of Theorem 2. For the function ψ such as in the formulation of Theorem 2 and for

$$M=\min_{(\tau,x)\in[\tau_0,\tau_0+\delta]\times E}(Y_\mu(t,\tau)f(x)-\psi(\tau,x)),$$

define the function G by (14). Using Lemma 1 with $\tau=\tau_0$, $x=x_0$, and $\epsilon>0$ such that $\tau_0+\epsilon<t$, we obtain

$$G(\tau_0,x_0)-Y(\tau_0+\epsilon,\tau_0)G(\tau_0+\epsilon)(x_0)=$$
$$Y(\tau_0+\epsilon,\tau_0)\psi(\tau_0+\epsilon)(x_0)-\psi(\tau_0,x_0)-\int_{\tau_0}^{\tau_0+\epsilon}Y(s,\tau_0)[\mu(s)Y_\mu(t,s)f](x_0)ds \quad (18)$$

Dividing (18) by ϵ, and using the facts that $G(\tau_0+\epsilon,y)\geq 0$ for all $\epsilon<\delta$ and $y\in E$, and $G(\tau_0,x_0)=0$, we get estimate (12).

This completes the proof of Theorem 2.

The next theorem is a local version of Theorem 2.

Theorem 3. *Let $\mu\in\mathcal{P}^*$, $f\in L^\infty$, and fix t such that $0<t\leq T$. Then the following assertions hold:*
(a) Suppose that ψ is a bounded continuous function on $[0,T]\times E$, (τ_0,x_0) is a point in $[0,t)\times E$, and assume that there exists $\delta>0$ with $\tau_0+\delta<t$ and a relatively compact neighborhood Q of x_0 in E such that

$$Y_\mu(t,\tau_0)f(x_0)-\psi(\tau_0,x_0)=\min_{(\tau,x)\in[\tau_0,\tau_0+\delta]\times\bar{Q}}(Y_\mu(t,\tau)f(x)-\psi(\tau,x))$$

where $\bar{Q}$ denotes the closure of Q in E. Then for every $0<\epsilon<\delta$ we have

$$\frac{Y(\tau_0+\epsilon,\tau_0)\psi(\tau_0+\epsilon)(x)-\psi(\tau_0,x_0)}{\epsilon}-\frac{1}{\epsilon}\int_{\tau_0}^{\tau_0+\epsilon}Y(s,\tau_0)[\mu(s)Y_\mu(t,s)f](x_0)ds$$
$$\leq\frac{\alpha}{\epsilon}\int_{E\setminus\bar{Q}}p(\tau_0,x_0;\tau_0+\epsilon,y)dy \quad (19)$$

where $\alpha>0$ and $\delta>0$ do not depend on ϵ.

(b) Suppose that ψ is a bounded continuous function on $[0,T]\times E$, (τ_0,x_0) is a point in $[0,t)\times E$, and suppose that there exists a neighborhood Q of (τ_0,x_0) in $[0,t)\times E$ such that

$$Y_\mu(t,\tau_0)f(x_0)-\psi(\tau_0,x_0)=\max_{(\tau,x)\in\bar{Q}}(Y_\mu(t,\tau)f(x)-\psi(\tau,x)).$$

Then for every small $\epsilon>0$ we have

$$\frac{Y(\tau_0+\epsilon,\tau_0)\psi(\tau_0+\epsilon)(x)-\psi(\tau_0,x_0)}{\epsilon}-\frac{1}{\epsilon}\int_{\tau_0}^{\tau_0+\epsilon}Y(s,\tau_0)[\mu(s)Y_\mu(t,s)f](x_0)ds$$
$$\geq-\frac{\alpha}{\epsilon}\int_{E\setminus\bar{Q}}p(\tau_0,x_0;\tau_0+\epsilon,y)dy \quad (20)$$

where $\alpha>0$ and $\delta>0$ do not depend on ϵ.

Proof. We will prove part (a) of Theorem 3. The proof of part (b) is similar. Suppose that the conditions in part (a) are satisfied. Define G by (14) with ψ as in the formulation of Theorem 3 and

$$M = \min_{(\tau,x)\in[\tau_0,\tau_0+\delta]\times\bar{Q}} (Y_\mu(t,\tau)f(x) - \psi(\tau,x)).$$

Using equality (15) with $\tau = \tau_0$, $x = x_0$, and $\epsilon > 0$ such that $\epsilon < \delta$, and taking into account that $G(\tau_0, x_0) = 0$, $G(\tau, x) \geq 0$ for $(\tau, x) \in [\tau_0, \tau_0 + \delta] \times \bar{Q}$, and $|G(\tau, x)| \leq \alpha$, we get

$$\begin{aligned}
&\frac{Y(\tau_0+\epsilon,\tau_0)\psi(\tau_0+\epsilon)(x_0) - \psi(\tau_0,x_0)}{\epsilon} - \frac{1}{\epsilon}\int_{\tau_0}^{\tau_0+\epsilon} Y(s,\tau_0)[\mu(s)Y_\mu(t,s)f](x_0)ds \\
&\quad = -Y(\tau_0+\epsilon,\tau_0)\chi_{\bar{Q}}G(\tau_0+\epsilon)(x_0) - Y(\tau_0+\epsilon,\tau_0)\chi_{E\setminus\bar{Q}}G(\tau_0+\epsilon)(x_0) \\
&\quad \leq \frac{\alpha}{\epsilon}\int_{E\setminus\bar{Q}} p(\tau_0, x_0; \tau_0+\epsilon, y)dy.
\end{aligned}$$

Hence, estimate (19) holds.

This completes the proof of Theorem 3.

Next we will proceed with making simplifications in inequalities (12), (13), (19), and (20). The following theorem concerning the continuity conditions for the backward Feynman-Kac propagator, will be useful in the sequel.

Theorem 4 ([10], Theorem 10)**.** *Suppose that the free backward propagator Y is such that $Y(t,\tau) : L^\infty \to BC$ for all t and τ with $0 \leq \tau < t < \infty$, and for every function $g \in BC$, the BC-valued function $(t,\tau) \to Y(t,\tau)g$ is continuous in the topology of uniform convergence on compact subsets of E. Let $\mu \in \mathcal{P}^*$. Then for every fixed t with $0 < t \leq T$, and every $f \in L^\infty$, the function $Y_\mu(t,\tau)f(x)$ is continuous on $[0,t) \times E$.*

The next lemma concerns the first term on the left-hand side of estimates (12), (13), (19), and (20). For $h \in BC$ and $(\tau, x) \in [0,T) \times E$, we put

$$A(\tau)h(x) = \lim_{\epsilon\to 0+} \frac{Y(\tau+\epsilon,\tau)h(x) - h(x)}{\epsilon}, \tag{21}$$

provided that the limit in (21) exists and is finite. We will say that a bounded continuous function ψ on $[0,T] \times E$ is differentiable from the right at $\tau_0 \in [0,T)$ uniformly with respect to $y \in E$, if there exists a function $D_1^+(\tau_0,\cdot) \in BC$ such that

$$\lim_{\epsilon\to 0+} \sup_{y\in E} \left|\frac{\psi(\tau_0+\epsilon,y) - \psi(\tau_0,y)}{\epsilon} - D_1^+\psi(\tau_0,y)\right| = 0. \tag{22}$$

Lemma 2. *Suppose that the free backward propagator Y satisfies the conditions in Theorem 4, and let ψ be a bounded continuous function on $[0,T] \times$*

R^n. Let $\tau_0 \in [0,t)$ and $x_0 \in E$ be such that ψ is differentiable from the right at $\tau_0 \in [0,T)$ uniformly with respect to $y \in E$, and $A(\tau_0)\psi(\tau_0)(x_0)$ exists and is finite. Then

$$\lim_{\epsilon\to 0+} \frac{Y(\tau_0+\epsilon,\tau_0)\psi(\tau_0+\epsilon)(x_0)-\psi(\tau_0,x_0)}{\epsilon} = D_1^+\psi(\tau_0,x_0)+[A(\tau_0)\psi(\tau_0)](x_0).$$

Proof. We have

$$\begin{aligned} &\frac{Y(\tau_0+\epsilon,\tau_0)\psi(\tau_0+\epsilon)(x_0)-\psi(\tau_0,x_0)}{\epsilon} \\ &= Y(\tau_0+\epsilon,\tau_0)\{\frac{\psi(\tau_0+\epsilon)-\psi(\tau_0)}{\epsilon} - D_1^+\psi(\tau_0)\}(x_0) \\ &+[Y(\tau_0+\epsilon,\tau_0)D_1^+\psi(\tau_0)](x_0)+[\frac{Y(\tau_0+\epsilon,\tau_0)-I}{\epsilon}\psi(\tau_0)](x_0) \\ &= I_1+I_2+I_3. \end{aligned} \tag{23}$$

Since Y is a family of contraction operators on L^∞,

$$|I_1| \le \sup_{y\in E} |\frac{\psi(\tau_0+\epsilon,y)-\psi(\tau_0,y)}{\epsilon} - D_1^+\psi(\tau_0,y)|.$$

It follows from (22) that

$$\lim_{\epsilon\to 0} I_1 = 0. \tag{24}$$

Since $D_1^+(\tau_0,\cdot) \in BC$, Theorem 4 gives

$$\lim_{\epsilon\to 0} I_2 = D_1^+\psi(\tau_0,x_0). \tag{25}$$

Finally, (21) implies

$$\lim_{\epsilon\to 0} I_3 = [A(\tau_0)\psi(\tau_0)](x_0). \tag{26}$$

Now it is clear that Lemma 2 follows from (23)-(26).

Next we turn our attention to the second term on the right-hand side of estimates (12), (13), (19), and (20). For $\mu \in \mathcal{P}^*$, consider its Radon-Nikodym-Lebesgue decomposition $d\mu(s) = V(s)dm + d\lambda(s)$ where $\lambda(s)$ is the singular part of $\mu(s)$ with respect to m. It is clear that $V \in \mathcal{P}^*$ and $\lambda \in \mathcal{P}^*$. Let $x_0 \in E$ and $\tau_0 \in [0,t)$ be given, and suppose that C_k with $-\infty < k < \infty$ is a strictly increasing sequence of Borel sets of positive measure m such that $x_0 \in C_k$ for all k, $\mathrm{diam}(C_k)+m(C_k) \to 0$ as $k \to -\infty$, and $\cup_{k=0}^{\infty} C_k = E$. For every integer j, put

$$\gamma_j(s) = \sup_{y\in E\setminus C_j} p(\tau_0,x_0;s,y),$$

and define the majorant p^* of p with respect to the family $\{C_k\}$ as follows:

$$p^*(\tau_0,x_0;s,z) = \gamma_j(s)$$

where j is the unique integer such that $z \in C_{j+1} \backslash C_j$. Let us also recall that the function $Y_\mu(t,s)f(x)$ is bounded on $[0,t] \times E$. Moreover it is continuous on $[0,t) \times E$, by Theorem 3. The following conditions will be used in the sequel:

$$\sup_{s:\tau_0 \le s \le \tau_0+\delta} \frac{1}{m(C_k)} \int_{C_k} |V(s,y) - V(\tau_0,x_0)| dy \to 0 \tag{27}$$

as $k \to -\infty$, where $\delta > 0$ is a number such that $\tau_0 + \delta < t$;

$$\sup_{k:k \ge j} \sup_{s:\tau_0 \le s \le \tau_0+\delta} \frac{1}{m(C_k)} \int_{C_k} |V(s,y)| dy \le M_{1,j} \tag{28}$$

for all $j \in Z$;

$$\sup_{s:\tau_0 \le s \le \tau_0+\delta} \frac{|\lambda(s)|(C_k))}{m(C_k)} \to 0 \tag{29}$$

as $k \to -\infty$;

$$\sup_{k:k \ge j} \sup_{s:\tau_0 \le s \le \tau_0+\delta} \frac{|\lambda(s)|(C_k))}{m(C_k)} \le M_{2,j} \tag{30}$$

for all $j \in Z$;

$$\sup_{s:\tau_0 \le s \le \tau_0+\delta} \int_{C_k} p^*(\tau_0, x_0; s, z) dz \le M_{3,k} \tag{31}$$

for all $k \in Z$, and

$$\lim_{s \to \tau_0+} \int_{E \backslash C_k} p^*(\tau_0, x_0; s, z) dz = 0 \tag{32}$$

for all $k \in Z$.

Remark 1. Condition (27) means that x_0 is a Lebesgue point of the function $V(\tau, \cdot)$ uniformly with respect to τ near τ_0. Condition (28) resembles a uniform local integrability condition for V. Similarly, condition (29) is a differentiability condition for the singular part λ of μ, while (30) is a uniform local integrability condition for λ. Conditions (31) and (32) concern the majorant p^* of the transition density p. They are based on the integrability condition for the majorant of an approximation of the identity (see Theorem 1.25 in [13]). Condition (31) concerns the local integrability of p^*, while (32) is a stochastic continuity condition for p^*.

Lemma 3. *Let $\mu \in \mathcal{P}^*$, and assume that the free backward propagator Y satisfies the conditions in Theorem 3. Let $\tau_0 \in [0,t)$, $x_0 \in E$, and $\{C_k\}$ be such that conditions (27)-(32) hold. Then*

$$\lim_{\epsilon \to 0+} \frac{1}{\epsilon} \int_{\tau_0}^{\tau_0+\epsilon} Y(s,\tau_0)[\mu(s) Y_\mu(t,s) f)](x_0) ds = V(\tau_0,x_0) Y_\mu(t,\tau_0) f(x_0). \tag{33}$$

Proof. Put $D(s,y)=V(s,y)Y_\mu(t,s)f(y)$ and $d\nu(s)=Y_\mu(t,s)f(y)d\lambda(s)$. Since the function $Y_\mu(t,s)f(y)$ is continuous on $[0,t)\times E$ and bounded on $[0,t]\times E$, it follows from (27) that

$$\sup_{s:\tau_0\le s\le\tau_0+\delta}\frac{1}{m(C_k)}\int_{C_k}|D(s,y)-D(\tau_0,x_0)|dy\rightarrow 0 \tag{34}$$

as $k\rightarrow-\infty$. In (34), $\delta>0$ is a number such that $\tau_0+\delta<t$. Moreover, (28)-(30) imply that

$$\sup_{k:k\ge j}\sup_{s:\tau_0\le s\le\tau_0+\delta}\frac{1}{m(C_k)}\int_{C_k}|D(s,y)|dy\le M_{4,j} \tag{35}$$

for all $j\in Z$;

$$\sup_{s:\tau_0\le s\le\tau_0+\delta}\frac{|\nu(s)|(C_k)}{m(C_k)}\rightarrow 0 \tag{36}$$

as $k\rightarrow-\infty$; and

$$\sup_{k:k\ge j}\sup_{s:\tau_0\le s\le\tau_0+\delta}\frac{|\nu(s)|(C_k)}{m(C_k)}\le M_{5,j} \tag{37}$$

for all $j\in Z$.

Our next goal is to show that

$$\lim_{\epsilon\rightarrow 0+}\frac{1}{\epsilon}\int_{\tau_0}^{\tau_0+\epsilon}Y(s,\tau_0)[|D(s)-D(\tau_0,x_0)|](x_0)ds=0 \tag{38}$$

and

$$\lim_{\epsilon\rightarrow 0+}\frac{1}{\epsilon}\int_{\tau_0}^{\tau_0+\epsilon}Y(s,\tau_0)|\nu(s)|(x_0)ds=0. \tag{39}$$

Put

$$T(s,y)=|D(s,y)-D(\tau_0,x_0)|.$$

Then we have

$$Y(s,\tau_0)T(s)(x_0)=\int_E T(s,y)p(\tau_0,x_0;s,y)dy \tag{40}$$

$$\le\int_E T(s,y)p^*(\tau_0,x_0;s,y)dy=\int_0^\infty d\lambda\int_{\{y:p^*(\tau_0,x_0;s,y)\ge\lambda\}}T(s,y)dy.$$

Since $\gamma_k(s)$ is a non-increasing sequence, it follows from (40) that there exists $\delta>0$ such that

$$Y(s,\tau_0)T(s)(x_0)\le\sum_{k\in Z}(\gamma_k(s)-\gamma_{k+1}(s))\int_{C_k}T(s,y)dy. \tag{41}$$

For any $j \in Z$, (41) gives

$$\begin{aligned} & Y(s,\tau_0)T(s)(x_0) \le \\ & \sup_{s:\tau_0 \le s \le \tau_0+\delta} \sum_{k=-\infty}^{j} (\gamma_k(s)-\gamma_{k+1}(s))m(C_k)\frac{1}{m(C_k)}\int_{C_k} T(s,y)dy + \\ & + \sum_{k=j+1}^{\infty} (\gamma_k(s)-\gamma_{k+1}(s))m(C_k)\frac{1}{m(C_k)}\int_{C_k} T(s,y)dy = \\ & = J_1(j)+J_2(j,s). \end{aligned} \tag{42}$$

We have

$$J_1(j) \le \{\sup_{s:\tau_0 \le s \le \tau_0+\delta} \sup_{k:-\infty<k\le j} \frac{1}{m(C_k)}\int_{C_k} |D(s,y)-D(\tau_0,x_0)|dy\} \times \sup_{s:\tau_0 \le s \le \tau_0+\delta} \int_{C_j} p^*(\tau_0,x_0;s,y)dy. \tag{43}$$

Moreover, for every $j \in Z$,

$$\begin{aligned} J_2(j,s) &\le (M_{4,j+1}+|D(\tau_0,x_0)|)\sum_{k=j+1}^{\infty}(\gamma_k(s)-\gamma_{k+1}(s))m(C_k) \\ &\le (M_{4,j+1}+|D(\tau_0,x_0)|)[\gamma_{j+1}(s)m(C_{j+1})+\sum_{k=j+2}^{\infty}\gamma_k(s)m(C_k\backslash C_{k-1})] \\ &\le (M_{4,j+1}+|D(\tau_0,x_0)|)[\frac{m(C_{j+1})}{m(C_{j+1}\backslash C_j)}+1]\int_{E\backslash C_j} p^*(\tau_0,x_0;s,y)dy \\ &\le (M_{4,j+1}+|D(\tau_0,x_0)|)[\frac{m(C_{j+1})}{m(C_{j+1}\backslash C_j)}+1]\int_{E\backslash C_j} p^*(\tau_0,x_0;s,y)dy. \end{aligned} \tag{44}$$

It follows from (42), (43), and (44) that

$$\begin{aligned} & Y(s,\tau_0)[|D(s)-D(\tau_0,x_0)|](x_0) \\ & \le \{\sup_{s:\tau_0 \le s \le \tau_0+\delta} \sup_{k:-\infty<k\le j} \frac{1}{m(C_k)}\int_{C_k} |D(s,y)-D(\tau_0,x_0)|dy\} \times \\ & \times \sup_{s:\tau_0 \le s \le \tau_0+\delta} \int_{C_j} p^*(\tau_0,x_0;s,y)dy + \\ & +(M_{4,j+1}+|D(\tau_0,x_0)|)[\frac{m(C_{j+1})}{m(C_{j+1}\backslash C_j)}+1]\int_{E\backslash C_j} p^*(\tau_0,x_0;s,y)dy \end{aligned}$$

for all $j \in Z$. Now it is not difficult to prove that conditions (34)-(35) imply

$$\lim_{s\to\tau_0+} Y(s,\tau_0)[|D(s)-D(\tau_0,x_0)|](x_0) = 0.$$

This gives equality (38). The proof of equality (39) is similar. Here we use (36) and (37) instead of (34) and (35). Now it is clear that (38) and (39) combined imply (33).

This completes the proof of Lemma 3.

Now we are ready to formulate our main results. The first of them concerns viscosity solutions in the case of global maxima or minima.

Theorem 5. *Let $\mu \in \mathcal{P}^*$, $f \in L^\infty$, $0 < t \le T$, and suppose that the transition density p is such that the corresponding free backward propagator Y satisfies the conditions in Theorem 3. Then the following are true:*
(a) *Let $(\tau_0, x_0) \in [0,t) \times E$, $\delta > 0$ with $\tau_0 + \delta < t$, and ψ be such that:*
(1) *ψ is a bounded continuous function on $[0,T] \times E$;*
(2) *ψ is differentiable from the right at τ_0 uniformly with respect to $y \in E$;*
(3) *$A(\tau_0)\psi(\tau_0)(x_0)$ exists and is finite;*
(4) *There exists a sequence of sets C_k such that conditions (27)-(30) hold;*
(5) *Equality*
$$Y_\mu(t,\tau_0)f(x_0) - \psi(\tau_0,x_0) = \min_{(\tau,x)\in[\tau_0,\tau_0+\delta]\times E}(Y_\mu(t,\tau)f(x) - \psi(\tau,x))$$
holds.

Then
$$D_1^+\psi(\tau_0,x_0) + [A(\tau_0)\psi(\tau_0)](x_0) - V(\tau_0,x_0)Y_\mu(t,\tau_0)f(x_0) \le 0.$$
(b) *Suppose that conditions 1-4 in part (a) are satisfied. Suppose also that*
$$Y_\mu(t,\tau_0)f(x_0) - \psi(\tau_0,x_0) = \max_{(\tau,x)\in[\tau_0,\tau_0+\delta]\times E}(Y_\mu(t,\tau)f(x) - \psi(\tau,x)).$$
Then
$$D_1^+\psi(\tau_0,x_0) + [A(\tau_0)\psi(\tau_0)](x_0) - V(\tau_0,x_0)Y_\mu(t,\tau_0)f(x_0) \ge 0.$$

It is clear that Theorem 5 follows from Theorem 2, Lemma 2, and Lemma 3.

Our next result concerns viscosity solutions in the case of local maxima and minima.

Theorem 6. *Let $\mu \in \mathcal{P}^*$, $f \in L^\infty$, $0 < t \le T$, and suppose that the transition density p is such that the corresponding free backward propagator Y satisfies the conditions in Theorem 3. Then the following are true:*
(a) *Let $(\tau_0, x_0) \in [0,t) \times E$, $\delta > 0$ with $\tau_0 + \delta < t$, and ψ be such that:*
(1) *ψ is a bounded continuous function on $[0,T] \times E$;*
(2) *ψ is differentiable from the right at τ_0 uniformly with respect to $y \in E$;*

(3) *$A(\tau_0)\psi(\tau_0)(x_0)$ exists and is finite;*
(4) *There exists a sequence of sets C_k such that conditions (27)-(30) hold;*
(5) *Equality*

$$\lim_{\epsilon\to 0+}\frac{1}{\epsilon}\int_{E\setminus\bar{Q}} p(\tau_0,x_0;\tau_0+\epsilon,y)dy=0 \tag{45}$$

holds for every relatively compact neighborhood Q of x_0;
(6) *There exists a relatively compact neighborhood Q of (τ_0,x_0) in $[0,t)\times E$ such that*

$$Y_\mu(t,\tau_0)f(x_0)-\psi(\tau_0,x_0)=\min_{(\tau,x)\in[\tau_0,\tau_0+\delta]\bar{Q}}(Y_\mu(t,\tau)f(x)-\psi(\tau,x)).$$

Then

$$D_1^+\psi(\tau_0,x_0)+[A(\tau_0)\psi(\tau_0)](x_0)-V(\tau_0,x_0)Y_\mu(t,\tau_0)f(x_0)\le 0.$$

(b) *Suppose that conditions 1-5 in part* (a) *are satisfied. Suppose also that there exists a relatively compact neighborhood Q of (τ_0,x_0) in $[0,t)\times E$ such that*

$$Y_\mu(t,\tau_0)f(x_0)-\psi(\tau_0,x_0)=\max_{(\tau,x)\in[\tau_0,\tau_0+\delta]\bar{Q}}(Y_\mu(t,\tau)f(x)-\psi(\tau,x)).$$

Then

$$D_1^+\psi(\tau_0,x_0)+[A(\tau_0)\psi(\tau_0)](x_0)-V(\tau_0,x_0)Y_\mu(t,\tau_0)f(x_0)\ge 0.$$

Theorem 6 follows from Theorem 3, Lemma 2, and Lemma 3.

4. Examples

Let E be n-dimensional Euclidean space R^n equipped with its standard norm $|\cdot|$. We will assume that the reference measure m coincides with the Lebesgue measure on R^n. It is well-known that the theory of second order parabolic partial differential equations with time-dependent coefficients is a rich source of transition probability densities on R^n. If such an equation possesses a fundamental function p, then p can be used as a transition density. In many special cases, the fundamental function p satisfies the upper Gaussian estimate,

$$p(\tau,x;t,y)\le\alpha_1 p_0(\alpha_2(t-\tau),x-y) \tag{46}$$

where α_1 and α_2 are positive constants. In estimate (46), p_0 stands for the Gaussian density given by

$$p_(s,z)=(2\pi s)^{\frac{n}{2}}\exp\{-\frac{|z|^2}{2s}\}.$$

For the examples of fundamental functions of second order divergence or non-divergence form parabolic partial differential equations with time-dependent coefficients for which estimate (46) holds, see [4, 12] and the references therein.

Define the radial majorant of the transition density p by

$$p^*(\tau,x;t,y)=\sup_{z:|z-x|\geq|y-x|}p(\tau,x;t,z).$$

It is clear that if estimate (46) holds for p, then

$$p^*(\tau,x;t,y)\leq\alpha_1 p_0(\alpha_2(t-\tau),x-y),$$

and hence, conditions (31) and (32) hold for p^*. It is not difficult to prove that condition (45) also holds. We will assume that the sets C_k in the formulation of Theorem 5 and Theorem 6 are given by $C_k=B(x_0,r_k)$ where $r_k\to 0+$ as $k\to-\infty$ and $r_k\to\infty$ as $k\to\infty$. The next assertion follows from Theorem 4.

Corollary 1. *Let p be a transition probability density on R^n such that estimate (46) holds for p. Let $\mu\in\mathcal{P}^*$, $f\in L^\infty$, $0<t\leq T$, and suppose that Y satisfies the conditions in Theorem 3. Then the following are true:*
(a) Let $(\tau_0,x_0)\in[0,t)\times E$, $\delta>0$ with $\tau_0+\delta<t$, and ψ be such that:

1. *ψ is a bounded continuous function on $[0,T]\times E$;*
2. *ψ is differentiable from the right at τ_0 uniformly with respect to $y\in E$;*
3. *$A(\tau_0)\psi(\tau_0)(x_0)$ exists and is finite;*
4. *Conditions (27)-(30) hold with $C_k=B(x_0,r_k)$ where r_k are such that $r_k\to 0+$ as $k\to-\infty$ and $r_k\to\infty$ as $k\to\infty$;*
5. *There exists a relatively compact neighborhood Q of x_0 in E such that*

$$Y_\mu(t,\tau_0)f(x_0)-\psi(\tau_0,x_0)=\min_{(\tau,x)\in[\tau_0,\tau_0+\delta]\bar{Q}}(Y_\mu(t,\tau)f(x)-\psi(\tau,x)).$$

Then

$$D_1^+\psi(\tau,x)+[A(\tau)\psi(\tau)](x)-V(\tau_0,x_0)Y_\mu(t,\tau_0)f(x_0)\leq 0.$$

(b) Suppose that conditions 1-4 in part (a) are satisfied. Suppose also that there exists a relatively compact neighborhood Q of x_0 in E such that

$$Y_\mu(t,\tau_0)f(x_0)-\psi(\tau_0,x_0)=\max_{(\tau,x)\in[\tau_0,\tau_0+\delta]\times\bar{Q}}(Y_\mu(t,\tau)f(x)-\psi(\tau,x)).$$

Then

$$D_1^+\psi(\tau,x)+[A(\tau)\psi(\tau)](x)-V(\tau_0,x_0)Y_\mu(t,\tau_0)f(x_0)\geq 0$$

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Department of Mathematics, Ohio Univ., Athens, Ohio 45701, USA.
E-mail: guli@math.ohiou.edu

Dep. of Mathematics and Computer Science, Univ. of Antwerp (UA), Middelheimlaan 1, 2020 Antwerp, Belgium.
E-mail: jan.vancasteren@ua.ac.be