CCC FORCING AND SPLITTING REALS

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Abstract. Prikry asked if it is relatively consistent with the usual axioms of ZFC that every nontrivial ccc forcing adds either a Cohen or a random real. Both Cohen and random reals have the property that they neither contain nor are disjoint from an infinite set of integers in the ground model, i.e. they are splitting reals. In this note I show that that it is relatively consistent with ZFC that every non atomic weakly distributive ccc forcing adds a splitting real. This holds, for instance, under the Proper Forcing Axiom and is proved using the P-ideal dichotomy first formulated by Abraham and Todorcevic [AT] and later extended by Todorcevic [T]. In the process, I show that under the P-ideal dichotomy every weakly distributive ccc complete Boolean algebra carries an exhaustive submeasure, a result which has some interest in its own right. Using a previous theorem of Shelah [Sh1] it follows that a modified Prikry conjecture holds in the context of Souslin forcing notions, i.e. every non atomic ccc Souslin forcing either adds a Cohen real or its regular open algebra is a Maharam algebra.

1. Introduction

Given two forcing notions $\mathcal{P}$ and $\mathcal{Q}$, let us write $\mathcal{P} \leq \mathcal{Q}$ iff forcing with $\mathcal{Q}$ introduces a $\mathcal{P}$-generic over V. It is fairly easy to see that this is equivalent to saying that there is $p \in \mathcal{P}$ and an embedding of the complete Boolean algebra $RO(\mathcal{P} \restriction p)$ into $RO(\mathcal{Q})$. Let Σ be a given class of posets. We say that $\Sigma_0 \subseteq \Sigma$ is a *basis* for Σ if for every $\mathcal{Q} \in \Sigma$ there is $\mathcal{P} \in \Sigma_0$ such that $\mathcal{P} \leq \mathcal{Q}$. We are interested in finding a basis for the class of ccc forcing notions. Clearly, both Cohen forcing $\mathcal{C}$ and random real forcing $\mathcal{R}$ have to be in any such basis. Prikry asked if it is consistent that $\{\mathcal{C}, \mathcal{R}\}$ form a basis for the class of all ccc posets. This is equivalent to saying that every nontrivial ccc poset adds a Cohen or a random real. Another version of this problem is to identify a basis for the class of appropriately definable ccc posets. Under ZFC definable is interpreted to mean the class of Souslin posets, i.e. posets $\mathcal{P}$ such that the domain of $\mathcal{P}$ is an analytic set of reals and both the order and the incompatibility relation of $\mathcal{P}$ are analytic. Under

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suitable large cardinal or determinacy assumptions we can replace Souslin by some higher order definability. One common property of both Cohen and random reals is that they are both splitting reals, i.e. they neither contain nor are disjoint from an infinite set of integers in the ground model. Thus, one test question going in the direction of Prikry's conjecture is whether it is relatively consistent that every nontrivial ccc forcing adds a splitting real.

Another related problem is a well-known question of von Neumann [Mau] who asked if there is an (ω, ω)-weakly distributive complete Boolean algebra which is not a measure algebra. Recall that a poset $\mathcal{P}$ is *weakly distributive* iff every real (i.e. element of ω^ω) in $V^{\mathcal{P}}$ is dominated by a ground model real, i.e. $\mathcal{P}$ is ω^ω-bounding. Maharam [Mah] formulated the notion of an exhaustive submeasure and found an algebraic characterization for a complete Boolean algebra to carry one; such algebras are now known as Maharam algebras. Any Maharam algebra is both ccc and weakly distributive and it is not known if it is necessarily a measure algebra. The question whether every Maharam algebra is, in fact, a measure algebra is equivalent to the well known Control Measure Problem.

In this note we show that it is relatively consistent with ZFC that every non atomic ccc poset adds a splitting real. We use the P-ideal dichotomy formulated by Abraham and Todorcevic [AT]. In fact, we show that if the P-ideal dichotomy holds for ideals on a set of size κ then every weakly distributive ccc complete Boolean algebra of size at most κ carries an exhaustive submeasure, i.e. is a Maharam algebra. We then show in ZFC that any non atomic Maharam algebra adds a splitting real. The full P-ideal dichotomy is a consequence of the Proper Forcing Axiom and was also proved relatively consistent with GCH, assuming the existence of a supercompact cardinal, by Todorcevic [T]. While for the consistency that every ccc forcing adds a splitting real we do not need to assume any large cardinal axioms, for the proof that every weakly distributive ccc Boolean algebra is a Maharam measure we need the full P-ideal dichotomy which requires large cardinal assumptions. The main motivation for our proof comes from the PhD thesis of Quickert [Qu1] who showed that under the P-ideal dichotomy no weakly distributive ccc forcing can have the Sacks property.

One consequence of our result and a previous result of Shelah ([Sh1], see also [Ve1]) is that a modified version of Prikry's conjecture holds in the context of Souslin ccc forcing: i.e. every Souslin ccc forcing either adds a Cohen real or is a Maharam algebra. Indeed, Shelah showed that any Souslin ccc forcing which is not ω^ω-bounding adds a Cohen real. It is not known if it is consistent that this holds without any definability restriction on the forcing. However, Błaszyck and Shelah [BłSh], using a previous result of Shelah [Sh2], showed that it is relatively consistent with ZFC that every ccc σ-centered forcing notion adds a Cohen real.

2. P-ideal dichotomy and exhaustive submeasures

Recall that a submeasure on a Boolean algebra $\mathcal{B}$ is a function $\mu : \mathcal{B} \to [0,1]$ such that $\mu(a) = 0$ iff $a = \mathbf{0}$, $a \le b$ implies $\mu(a) \le \mu(b)$, and $\mu(a \vee b) \le \mu(a) + \mu(b)$, for every $a, b \in \mathcal{B}$. We say that μ is *exhaustive* if for every sequence $\{a_n : n < \omega\}$ of disjoint elements of $\mathcal{B}$ we have $\lim_{n\to\infty} \mu(a_n) = 0$, and that it is *uniformly exhaustive* if for every $\epsilon > 0$ there is an integer n such that any family of pairwise disjoint elements of $\mathcal{B}$ each of μ-submeasure at least ϵ has size at most n. Finally μ is called *continuous* if for every sequence $\{a_n : n < \omega\}$ of elements of $\mathcal{B}$ if $\limsup_n a_n = \liminf_n a_n = a$, for some a, then $\lim_n \mu(a_n) = \mu(a)$. It is well known that if a weakly distributive ccc Boolean algebra $\mathcal{B}$ admits an exhaustive submeasure then it admits a continuous submeasure. We call such algebras *Maharam algebras.*

Our goal in this section is to show that it is relatively consistent, modulo a supercompact cardinal, to assume that every weakly distributive ccc complete Boolean algebra carries an exhaustive submeasure. In fact, we will deduce this from the P-ideal dichotomy. Recall that an ideal $\mathcal{I}$ of subsets of a set X is called a *P-ideal* if for every sequence $\{A_n : n < \omega\}$ of elements of $\mathcal{I}$ there is $A \in \mathcal{I}$ such that $A_n \subseteq_* A$, for all n. Here $\subseteq_*$ denotes inclusion modulo finite sets. We say that a set Y is *orthogonal* to a family $\mathcal{A}$ provided the intersection of Y with any element of $\mathcal{A}$ is finite.

Let $(*)_\kappa$ be the following principle:

$(*)_\kappa$ *Let X be a set of size at most κ and let $\mathcal{I}$ be a P-ideal of countable subsets of X. Then one of the following two alternatives holds:*

(a) *there is an uncountable subset Y of X such that $[Y]^{\le\omega} \subseteq \mathcal{I}$*
(b) *we can write $X = \bigcup X_n$, where each X_n is orthogonal to $\mathcal{I}$.*

The P-deal dichotomy is the statement that $(*)_\kappa$ holds, for all κ. This principle, which follows from the Proper Forcing Axiom, was first studied by Abraham and Todorcevic [AT] who proved that a limited version of the principle is relatively consistent with CH and used it to show some consequences of Martin's Axiom are consistent with CH. Later, Todorcevic [T] proved that the full version of the P-ideal dichotomy is relatively consistent with GCH assuming the existence of a supercompact cardinal is consistent.

Theorem 1. *Assume $(*)_\kappa$ holds. Let $\mathcal{B}$ be a ccc weakly distributive complete Boolean algebra of size at most κ. Then $\mathcal{B}$ is a Maharam algebra.*

Proof: The motivation for this proof comes from the work of Quickert [Qu2] and Solecki [So1]. We will break the proof into several lemmas. Let $\mathcal{B}$

be a complete ccc weakly distributive Boolean algebra. Following Quickert let us define the following ideal $\mathcal{I}$ on $[\mathcal{B}\setminus\{\mathbf{0}\}]^{\leq\omega}$:

$$X \in \mathcal{I} \text{ if and only if } \Vdash_{\mathcal{B}} A \cap \dot{G} \text{ is finite}$$

Thus, $X \in \mathcal{I}$ iff there is a maximal antichain $\mathcal{A}$ such that every member of $\mathcal{A}$ is compatible with only finitely many members of X. The following two lemmas are from [Qu2]. We reproduce the proofs for completeness.

Lemma 1. *$\mathcal{I}$ is a P-ideal.*

Proof: Suppose $X_n \in \mathcal{I}$, for all n. We may assume all the X_n are infinite and fix, for each n, an enumeration $X_n = \{p_{n,k} : k < \omega\}$. Define a name $\dot{f}$ for a function in ω^ω by letting $\dot{f}(n)$ be the name for the least integer k such that $p_{n,l} \notin \dot{G}$, for all $l \geq k$. By the definition of $\mathcal{I}$ this is well defined. Now, by weak distributivity and the ccc of $\mathcal{B}$ we can fix a function $g \in \omega^\omega$ such that $\Vdash \dot{f} \leq_* g$. Let

$$X = \bigcup_n X_n \setminus \{p_{n,i} : i < g(n)\}.$$

It follows that $\Vdash X \cap \dot{G}$ is finite, i.e. $X \in \mathcal{I}$ and clearly we have $X_n \subseteq_* X$, for all n. □

Lemma 2. *There is no uncountable X such that $[X]^{\leq\omega} \subseteq \mathcal{I}$.*

Proof: Assume otherwise and let X be a counterexample. Since $\mathcal{B}$ satisfies the ccc there is a condition $b \in \mathcal{B}$ such that $b \Vdash$ "$X \cap \dot{G}$ is uncountable". Then, again by the ccc, there is a countable $A \subseteq X$ such that $b \Vdash$ "$A \cap \dot{G}$ is infinite ". Therefore, $A \notin \mathcal{I}$, a contradiction. □

Quickert [Qu2] used these two lemmas in her proof that under the P-ideal dichotomy no non atomic ccc forcing has the Sacks property. Moreover, she showed that if the P-ideal dichotomy holds then every weakly distributive ccc forcing satisfies the σ-finite chain condition.

Definition 1. *A set $U \subseteq \mathcal{B}$ is* large *if it is downward closed and $A \subseteq_* U$, for every $A \in \mathcal{I}$; i.e. if $\mathcal{B}\setminus U$ is orthogonal to $\mathcal{I}$.*

Note that our definition of largeness is not the same as the one in [Qu2]. In fact, a set U is large in our sense iff the set $\{-b : b \in U\}$ is large in the sense of Quickert. While we could have, of course, used her definition, we find this one slightly more convenient.

Now, by $(*)_\kappa$ and Lemma 2, we can write $\mathcal{B}\setminus\{\mathbf{0}\} = \bigcup_n X_n$, where X_n is orthogonal to $\mathcal{I}$, for all n. By replacing X_n by its upward closure we may assume that if $a \in X_n$ and $a \leq b$ then $b \in X_n$. Moreover, by replacing X_n

by $\bigcup_{i\leq n} X_i$, we may assume that $X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots$. Let $U_n = \mathcal{B} \setminus X_n$. Then, by definition, each U_n is large, we have $U_0 \supseteq U_1 \supseteq U_2 \supseteq \ldots$ and $\bigcap\{U_n : n < \omega\} = \{\mathbf{0}\}$.

Our first goal is to improve this sequence in order to have the additional property that $U_{n+1} \vee U_{n+1} \subseteq U_n$, for every n, where $U \vee V = \{u \vee v : u \in U \text{ and } v \in V\}$. For this, it is clearly sufficient to show that for every large U there is a large V such that $V \vee V \subseteq U$.

Remark 1. Note that if we assume that $\mathcal{B}$ does not add splitting reals we immediately have that for every n there is k such that $U_k \vee U_k \subseteq U_n$. To see this, assume otherwise and fix for each $k \geq n$, $b_k^0, b_k^1 \in U_k$ such that $b_k = b_k^0 \vee b_k^1 \notin U_n$. Since, each b_k is not in U_n we can show that there is an infinite $I \subseteq \mathbb{N}$ such that $c = \bigwedge\{b_k : k \in I\} \neq \mathbf{0}$. Then we define a name τ for an element of 2^ω by letting $||\tau(k) = 1|| = b_k^1$. Again, by the assumption that no splitting reals are added below c we find an infinite $J \subseteq I$ and $\epsilon \in \{0, 1\}$ such that $d = \bigwedge\{b_k^\epsilon : k \in J\} \neq \mathbf{0}$. Let $l \geq n$ be such that $d \in X_l$. Since, X_l is upward closed it follows that $b_k^\epsilon \in X_l$, for every $k \in J$. But if $k \geq l$ we have $b_k^\epsilon \in U_l$, a contradiction.

The following argument was motivated by a similar argument from [So1].

Definition 2. *Let V be large and let Z_V be the set of all $x \in V$ such that the set $V(x) = \{a \in V : x \vee a \in V\}$ is large.*

Claim 1. *If V is large then so is Z_V.*

PROOF: Otherwise there would be an infinite set $X \in \mathcal{I}$ which is disjoint from Z_V. Fix an enumeration $X = \{x_n : n < \omega\}$, and for each n, a set $A_n \in \mathcal{I}$ such that for every $a \in A_n$, $x_n \vee a \notin V$. Since $\mathcal{I}$ is a P-ideal we can find $A \in \mathcal{I}$ such that $A_n \subseteq_* A$, for all n. Now, choose a $1-1$ sequence $\{a_n : n < \omega\}$ of elements of A such that $x_n \vee a_n \notin V$. Since both X and A belong to $\mathcal{I}$ it follows that the set $\{x_n \vee a_n : n < \omega\}$ is also in $\mathcal{I}$. But then there must be an n such that $x_n \vee a_n \in V$, a contradiction. □

Lemma 3. *For every large V there is a large W such that $W \vee W \subseteq V$.*

PROOF: Assume otherwise and fix a large V for which this fails. Let $V_0 = V \cap U_0$. Since Z_{V_0} is large by our assumption there are x_0 and y_0 in Z_{V_0} such that $x_0 \vee y_0 \notin V$. Let $V_1 = V_0(x_0) \cap V_0(y_0) \cap U_1$. Since Z_{V_1} is large we can again pick x_1 and y_1 in Z_{V_1} such that $x_1 \vee y_1 \notin V$. Let $V_2 = V_1(x_1) \cap V_1(y_1) \cap U_2$. Then V_2 is large and then so is Z_{V_2}. We pick x_2 and y_2 in Z_{V_2} such that $x_2 \vee y_2 \notin V$. We proceed by recursion. Given V_n which is large we can pick x_n and y_n in Z_{V_n} such that $x_n \vee y_n \notin V$. We then let $V_{n+1} = V_n(x_n) \cap V_n(y_n) \cap U_{n+1}$. Notice that for every n and k we have $x_n \vee x_{n+1} \vee x_{n+2} \ldots \vee x_{n+k} \in U_n$.

Claim 2. *$\{x_n : n < \omega\}$ and $\{y_n : n < \omega\}$ belong to $\mathcal{I}$.*

PROOF: Since the statement is symmetric let us assume, towards contradiction, that $\{x_n : n < \omega\}$ is not in $\mathcal{I}$. Fix a condition $b \in \mathcal{B} \setminus \{\mathbf{0}\}$ such that $b \Vdash$ "$\{n : x_n \in \dot{G}\}$ is infinite". Since $\mathcal{B}$ is weakly distributive we can pick a strictly increasing function f in ω^ω and a nonzero $c \leq b$ such that

$$c \Vdash \bigvee\{x_i : f(n) < i \leq f(n+1)\} \in \dot{G}$$

for every n. Let $z_n = \bigvee\{x_i : f(n) < i \leq f(n+1)\}$. Then, by our construction, we have that $z_n \in U_{f(n)}$ and on the other hand $c \leq z_n$, for all n. Since the U_n are downward closed it follows that $c \in \bigcap\{U_l : l < \omega\} = \{\mathbf{0}\}$, a contradiction. □

Now, since both $\{x_n : n < \omega\}$ and $\{y_n : n < \omega\}$ are in $\mathcal{I}$ it follows that $\{x_n \vee y_n : n < \omega\}$ is in $\mathcal{I}$, as well. But this set is disjoint from V and V was supposed to be large, a contradiction. This finishes the proof of Lemma 3. □

Now, using Lemma 3 we can improve the original decreasing sequence $U_0 \supseteq U_1 \supseteq \ldots$ to assume that $U_{n+1} \vee U_{n+1} \subseteq U_n$, for all n. Let us define the function $\varphi : \mathcal{B} \to [0,1]$ by:

$$\varphi(a) = \inf\{2^{-n} : a \in U_n\}$$

Now we define a submeasure $\mu : \mathcal{B} \to [0,1]$ as follow

$$\mu(b) = \inf\{\sum_{i=1}^{l} \varphi(a_i) : b \leq \bigvee_{i=1}^{l} a_i\} \cup \{1\}$$

Lemma 4. *μ is a positive exhaustive submeasure on $\mathcal{B}$.*

PROOF: It is clear that if $a \leq b$ then $\mu(a) \leq \mu(b)$ and that $\mu(a \vee b) \leq \mu(a \vee b)$, for every $a, b \in \mathcal{B}$. We need to show that μ is positive on every nonzero element of $\mathcal{B}$ and that it is exhaustive. The following fact is immediate.

Fact 1. *Suppose $n_1 < n_2 < \ldots < n_k$ and $a_i \in U_{n_i+1}$, for $i = 1, \ldots k$. Then $\bigvee_{i=1}^k a_i \in U_{n_1}$.* □

From this it follows that if $a \notin U_n$ then $\mu(a) \geq 2^{-n}$, therefore μ is positive. Finally, to see that μ is exhaustive suppose $\{x_n : n < \omega\}$ is a disjoint sequence of elements of $\mathcal{B}$. Given $\epsilon > 0$ fix an integer k such that $2^{-k} \leq \epsilon$. Since $\{x_n : n < \omega\} \in \mathcal{I}$ and U_k is large, there is n such that $x_l \in U_k$ for all $l \geq n$. This means that

$$\mu(x_l) \leq \varphi(x_l) \leq 2^{-k} \leq \epsilon$$

for all $l \geq n$. Thus μ is exhaustive. □

Corollary 1. *Let $\mathcal{P}$ be a non atomic Souslin ccc forcing notion. Then either there is $p \in \mathcal{P}$ such that forcing with $\mathcal{P}$ below p adds a Cohen real or else the regular open algebra $RO(\mathcal{P})$ is a Maharam algebra.*

Proof: In [Sh1] Shelah showed that every ccc Souslin forcing which adds an unbounded real adds a Cohen real. Suppose now $\mathcal{P}$ is a weakly distributive ccc Souslin forcing. In the proof of the consistency of the P-ideal dichotomy one shows that if alternative (b) of $(*)$ does not hold then one can add an uncountable set X witnessing (a). In the way we applied that in the proof of Theorem 1 this would give an uncountable antichain in $\mathcal{P}$. However, for Souslin forcing notions, the ccc is an absolute property and therefore we must have alternative (b) in the ground model. □

3. Splitting Reals

A common feature of both Cohen and random reals is that they are splitting reals, i.e. they do not contain nor are disjoint from an infinite set of integers in the ground model. The goal of this section is to prove that under the P-ideal dichotomy every non atomic ccc weakly distributive forcing adds a splitting real. By Theorem 1 it suffices to show that every non atomic Maharam algebra adds a splitting real. We start with the following simple lemma.

Lemma 5. *Let $\mathcal{B}$ be a ccc complete Boolean algebra which does not add splitting reals below any condition. Let X be an infinite subset of $\mathcal{B}$. Then there is an infinite subset Y of X such that either $\bigwedge Y \neq \mathbf{0}$ or $\Vdash_{\mathcal{B}} Y \cap \dot{G}$ is finite.*

Proof: Fix an enumeration $X = \{b_n : n < \omega\}$ and let τ be the name for an element of 2^ω defined by $||\tau(n) = 1|| = b_n$. Since τ is forced not to be a splitting real there is an infinite $I_0 \subseteq \mathbb{N}$ and a nonzero c_0 such that $c \Vdash$ "$\tau \restriction I_0$" is constant. We recursively build an antichain $\{c_\xi : \xi < \delta\}$ and a decreasing mod finite sequence $I_0 \supseteq_* \ldots \supseteq_* I_\xi \supseteq_* \ldots$ such that $c_\xi \Vdash_{\mathcal{B}}$ "$\tau \restriction I_\xi$ is almost constant", for all ξ. At a countable limit stage λ we first diagonalize to find an infinite J such that $J \subseteq_* I_\xi$, for all $\xi < \lambda$. If $\{c_\xi : \xi < \lambda\}$ is not already a maximal antichain, by using the fact that $\tau \restriction J$ is forced not to be a splitting real, we find c_λ incompatible with all the c_ξ, for $\xi < \lambda$, and an infinite $I_\lambda \subset J$ such that $c_\lambda \Vdash_{\mathcal{B}}$ "$\tau \restriction J$ is constant". Since $\mathcal{B}$ is ccc the construction must stop after countably many steps. At this stage we get an infinite I such that $\Vdash$ "$\tau \restriction I$ is almost constant". Let $Y = \{b_n : n \in I\}$. If there is $c \in \mathcal{B} \setminus \{\mathbf{0}\}$ and an integer n such that $c \Vdash_{\mathcal{B}} \tau \restriction (I \setminus n) \equiv 1$, then it follows that $c \leq \bigwedge Y$. Otherwise $\Vdash_{\mathcal{B}}$ "$Y \cap \dot{G}$ is finite ". □

Theorem 2. *Let $\mathcal{B}$ be a non atomic Maharam algebra. Then forcing with $\mathcal{B}$ adds a splitting real.*

Proof: Let μ be an exhaustive submeasure on $\mathcal{B}$. If μ is uniformly exhaustive, by a theorem of Kalton and Roberts [KR] $\mathcal{B}$ is a measure algebra and therefore it adjoins a random real. Assume now $\mathcal{B}$ is not uniformly exhaustive and fix an $\epsilon > 0$ which witnesses this. We can now fix, for each n, a family $A_n = \{a_{n,1} \ldots, a_{n,n}\}$ of pairwise disjoint sets of μ-submeasure $\geq \epsilon$. Note that by Lemma 5 if X is an infinite set of members of $\mathcal{B}$ each of μ-submeasure $\geq \epsilon$ then there is an infinite subset Y of X such that $\bigwedge Y \neq \mathbf{0}$. Fix a family $\{f_\xi : \xi < \omega_1\}$ of functions in $\prod_n \{1, \ldots, n\}$ such that for $\xi \neq \eta$ there is l such that $f_\xi(k) \neq f_\eta(k)$, for all $k \geq l$. We build a tower of infinite subsets of $\mathbb{N}$, $I_0 \supseteq_* I_1 \supseteq_* \ldots \supseteq_* I_\xi \supseteq_* \ldots$, for $\xi < \omega_1$, such that

$$b_\xi = \bigwedge \{a_{n,f_\xi(n)} : n \in I_\xi\} \neq \mathbf{0},$$

for each ξ. At a stage α we do the following. If α is a limit ordinal we first find an infinite set J such that $J \subseteq_* I_\xi$, for all $\xi < \alpha$; if $\alpha = \beta + 1$ let $J = I_\beta$. Now, look at the family $\{a_{n,f_\alpha(n)} : n \in J\}$. By Lemma 5 we can find an infinite $I_\alpha \subseteq_* J$ such that $b_\alpha = \bigwedge \{a_{n,f_\alpha(n)} : n \in I_\alpha\} \neq \mathbf{0}$. Notice that if $\xi \neq \eta$ then b_ξ and b_η are incompatible. Therefore $\{b_\xi : \xi < \omega_1\}$ is an uncountable antichain in $\mathcal{B}$, a contradiction. □

For the following corollary we only need a limited version of the P-ideal dichotomy since every complete ccc Boolean algebra contains a complete subalgebra of size $2^{\aleph_0}$. Therefore for the consistency result no large cardinal assumptions are necessary.

Corollary 2. *Assume $(*)_{2^{\aleph_0}}$. Then every non atomic weakly distributive ccc forcing adds a splitting real.* □

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