C^1-EXTENSION OF SUBHARMONIC FUNCTIONS

M.S. MELNIKOV, P.V. PARAMONOV

ABSTRACT. For a Jordan Dini-Lyapunov type domain D in $\mathbb{R}^2$ we prove the possibility to extend any function subharmonic in D and of the class $C^1(\overline{D})$ to a function subharmonic and of the class C^1 on the whole of $\mathbb{R}^2$ with the uniform estimate of its gradient. We also obtain a localization theorem for C^1-subharmonic extension from closed Jordan domains, and give examples of C^1-smooth Jordan domains which don't have this extension property.

1. INTRODUCTION

In [1, Theorem 5.1] it was proved that any function f subharmonic in the unit ball B of $\mathbb{R}^N$ ($N \geqslant 2$), $f \in C^1(\overline{B})$, can be extended to a function $F \in C^1(\mathbb{R}^N)$ subharmonic on the whole of $\mathbb{R}^N$ with the property $\|\nabla F\|_{\mathbb{R}^N} \leqslant c\,\|\nabla f\|_{\overline{B}}$, where $c \in (0, +\infty)$ depends only on N.

Here as usual, $\|g\|_E$ stands for the uniform norm $\sup_{x\in E}|g(x)|$ of the (complexvalued) function g on the set E.

In this paper we prove (Theorem 2.2) that for $N = 2$ the above formulated result still holds if, instead of the ball (disc) B one takes an arbitrary Dini-Lyapunov type domain D (with c depending on D). We have failed to obtain an analogous theorem for $N \geq 3$. Notice that in [2, Corollary 1] the result on the C^1-extension of subharmonic functions from Lyapunov-Dini domains in R^N was obtained for all $N \geq 2$, but with some additional requirements on functions f under extension.

For an open set Ω in $\mathbb{R}^N$ ($N = 2, 3, \ldots$) and $m \in \mathbb{Z}_+ = \{0, 1, \ldots\}$ we denote by $BC^m(\Omega)$ the class of all (complexvalued) functions f, which have bounded and continuous partial derivatives in Ω up to the order m. The norm in $BC^m(\Omega)$ is defined as usual:

$$\|f\|_{m,\Omega} = \max_{|\beta|\leqslant m} \|\partial^\beta f\|_\Omega\ ,$$

The research of the first author was partially supported by grants BFM 2000-0361 and 2001-SGR-00431.

The research of the second author was partially supported by grants SS – 2040.2003.1 (Russia) and SAB 98 – 0049 (Spain).

where the maximum is taken over all N-indices $\beta = (\beta_1, \ldots, \beta_N) \in \mathbb{Z}_+^N$ such that $|\beta| = \beta_1 + \ldots + \beta_N \leqslant m$, and where

$$\partial^\beta f(x) = \frac{\partial^{|\beta|} f(x)}{\partial x_1^{\beta_1} \ldots \partial x_N^{\beta_N}} \ .$$

In particular, $BC^0(\Omega) = BC(\Omega)$ is the space of bounded continuous functions on Ω with the uniform norm.

Let now X be a compact set in $\mathbb{R}^N$, $m \in \mathbb{Z}_+$. Denote by $C^m(X)$ the class of functions f on X, which can be extended (from X) to functions of the class $BC^m(\mathbb{R}^N)$. By then

$$\|f\|_{m,X} = \inf \|F\|_{m,\mathbb{R}^N} \ ,$$

where the infimum is taken over all possible extensions $F \in BC^m(\mathbb{R}^N)$, $F|_X = f$.

Notice that, if the interior X° of X is dense in X (that is $\overline{X^\circ} = X$, where $\overline{E}$ is the closure of E in $\mathbb{R}^N$), then for each $f \in C^m(X)$ the functions $\partial^\beta f(x)$ are uniquely defined on X, $|\beta| \leqslant [m]$. In particular, for $m \geqslant 1$, the vector-gradient $\nabla f(x) = \left\{ \frac{\partial f(x)}{\partial x_1}, \ldots, \frac{\partial f(x)}{\partial x_N} \right\}$ is well-defined and we set

$$|\nabla f(x)| = \max_{1 \leqslant n \leqslant N} \left| \frac{\partial f(x)}{\partial x_n} \right| \ ;$$

for $m \geqslant 2$ the matrix of the second partial derivatives

$$\nabla^2 f(x) = \left(\frac{\partial^2 f}{\partial x_n \partial x_{n'}} \right)_{n,n'=1}^N$$

is defined with

$$|\nabla^2 f(x)| = \max_{1 \leqslant n \leqslant n' \leqslant N} \left| \frac{\partial^2 f(x)}{\partial x_n \partial x_{n'}} \right| \ .$$

If E is open and $f \in BC^m(E)$, or E is compact with $\overline{E^\circ} = E$ and $f \in C^m(E)$, then

$$\|\nabla f\|_E = \max_{1 \leqslant n \leqslant N} \left\| \frac{\partial f}{\partial x_n} \right\|_E \qquad \text{(for } m \geqslant 1),$$

$$\|\nabla^2 f\|_E = \max_{1 \leqslant n \leqslant n' \leqslant N} \left\| \frac{\partial^2 f}{\partial x_n \partial x_{n'}} \right\|_E \qquad \text{(for } m \geqslant 2).$$

Finally, for $m \in \mathbb{Z}_+$ and an open set Ω, we write $C^m(\Omega)$ for the class of functions $\{f \mid f \in BC^m(D)$ for each bounded open D, $\overline{D} \subset \Omega\}$.

Denote by $SH(\Omega)$ the class of all (real valued) subharmonic functions on an open set Ω in $\mathbb{R}^N$. The problem we are interested in can be formulated as follows.

Problem 1.1. *Let X be a compact set in $\mathbb{R}^N$ with $\overline{X^\circ} = X$. What are the conditions on X, necessary and sufficient for the equality*

$$C^1(X) \cap SH(X^\circ) = (C^1(\mathbb{R}^N) \cap SH(\mathbb{R}^N))|_X \tag{1.1}$$

to be satisfied?

In Section 2 we formulate and prove the main result of this paper – Theorem 2.2. As a corollary, we obtain Theorem 2.6 – the localization theorem for C^1-subharmonic extension from closed Jordan domains. In Section 3 we construct (in particular, C^1-smooth) Jordan domains which don't have this C^1-subharmonic extension property.

2. The main result and its proof

In the paper, $B(a,r)$ denotes an open disc in $\mathbb{C}$ with centre a and of radius $r > 0$. By c we denote positive constants (absolute, if there is no other comment), which can be different in different occurrences.

Let $\omega_E(g,\delta) = \sup\{|g(x) - g(y)| \mid x \in E,\, y \in E,\, |x-y| \leqslant \delta\}$ be the modulus of continuity of the (scalar- or vector valued) function g on the set E, $\delta \geqslant 0$.

Definition 2.1. A Jordan domain D in $\mathbb{C}$ will be called *a Dini-Lyapunov type domain*, if some (and then any) conformal mapping k from the domain D onto the unit disc $B = B(0,1)$ satisfies the following properties:

(a) the function k can be extended to a C^1-diffeomorphism of $\overline{D}$ onto $\overline{B}$; in particular, $k' = dk/dz \in C(\overline{D})$ and $k'_0 = \min\limits_{z\in\overline{D}} |k'(z)| > 0$;

(b) the function $\varkappa(t) = \omega_{\overline{D}}(k', t)$, $t \geqslant 0$, satisfies the Dini condition:

$$\int_0^1 \frac{\varkappa(t)}{t}\, dt < +\infty \,.$$

Theorem 2.2. *Let D be a Dini-Lyapunov type domain. Then there is $c = c(D) > 0$ such that for any $f \in C^1(\overline{D}) \cap SH(D)$ there exists $F \in C^1(\mathbb{C}) \cap SH(\mathbb{C})$ with $F|_{\overline{D}} = f$ and*

$$\|\nabla F\|_{\mathbb{C}} \leqslant c\, \|\nabla f\|_{\overline{D}}.$$

In studing the Problem 1.1 we found that the following question (as far as we know) is still open.

Problem 2.3. What are the conditions on a (real valued) function $g \in C^1(\partial B)$, necessary and sufficient for the possibility to extend g to a function of the class $C^1(\overline{B}) \cap SH(B)$?

2.1. Proof of Theorem 2.2

The appearing positive constants $c_1, c_2, \ldots$ either are absolute (by silence) or they depend (finally) only on the domain D (in the last case we write, for instance, $c_1 = c_1(D)$). These constants are fixed till the end of the proof.

Let D, k and $\varkappa$ satisfy the conditions of Definition 2.1. Then D has a smooth boundary ∂D and $\varkappa(\cdot)$ satisfies the "doubling" property ($c_1 = c_1(D)$):

$$\varkappa(2t) \leqslant c_1 \varkappa(t)\,,\ t > 0\,. \tag{2.1}$$

First, we extend the function k *by the Whitney method* in some neighbourhood of $\overline{D}$ as follows. Let $\{Q_j^*, \varphi_j^*\}_{j\in J}$ be a special (Whitney type) partition of unity on the set $\mathbb{C}\setminus\overline{D}$, as constructed in [3, Chapter 6, § 1] (in [3] the index k was used instead of j and the set F instead of our $\overline{D}$), let p_j be some point in ∂D closest to the square Q_j^*. For $z \in \mathbb{C}\setminus\overline{D}$ set

$$k(z) = \sum_{j\in J} \left(k(p_j) + k'(p_j)\,(z-p_j)\right)\,\varphi_j^*(z)\,.$$

Standard arguments (see [3, Chapter 6, § 2.3]) show that there exist a Jordan domain Ω_0 containing $\overline{D}$ and a disc $U_0 = \{w \in \mathbb{C} \mid |w| < r_0\}$, $r_0 \in (1,2)$, such that the following properties hold:

(1) $k \in C^1(\Omega_0)\cap C^2(\Omega_0\setminus\overline{D})$ and, moreover,

$$\left|\frac{\partial k(z)}{\partial z} - k'(z')\right| + \left|\frac{\partial k(z)}{\partial \overline{z}}\right| \leqslant c_2\varkappa(d)\,,$$
$$\left|\nabla^2 k(z)\right| \leqslant \frac{c_2\varkappa(d)}{d}\,,$$

where $z \in \Omega_0\setminus\overline{D}$ and z' is some point in ∂D closest to z, $d = |z - z'|$;

(2) k is bilipschitz map from Ω_0 onto U_0, that is

$$c_3^{-1}|z_1 - z_2| \leqslant |k(z_1) - k(z_2)| \leqslant c_3|z_1 - z_2|$$

for all $z_1, z_2 \in \Omega_0$; here Ω_0, U_0, k, $c_2 \in (1,+\infty)$, $c_3 \in (1,+\infty)$ depend only on D.

Let $g_0 \in C^1(U_0)\cap C^2(U_0\setminus\overline{B})$ be realvalued, $f_0(z) = g_0(k(z))$. We want to find the relation between $\Delta f_0(z)$ and $\Delta g_0(w)$, $w = k(z)$, $z \in \Omega_0\setminus\overline{D}$. Setting $\partial_z = \frac{\partial}{\partial z}$, $\partial_{\overline{z}} = \frac{\partial}{\partial \overline{z}}$, one can easily show that for $z \in \Omega_0\setminus\overline{D}$ the following equalities hold:

$$\frac{1}{4}\Delta f_0(z) = \partial_z\partial_{\overline{z}} f_0(z) =$$
$$= \partial_w\partial_{\overline{w}} g_0(w)|_{k(z)}\,(\partial_z k\partial_{\overline{z}}\overline{k} + \partial_{\overline{z}} k\partial_z\overline{k}) + \partial_w^2 g_0(w)|_{k(z)}\,\partial_{\overline{z}} k\partial_z k +$$
$$+ \partial_{\overline{w}}^2 g_0(w)|_{k(z)}\,\partial_{\overline{z}}\overline{k}\partial_z\overline{k} + \partial_w g_0(w)|_{k(z)}\,\partial_z\partial_{\overline{z}} k + \partial_{\overline{w}} g_0(w)|_{k(z)}\,\partial_z\partial_{\overline{z}}\overline{k}\ .$$

From this, taking into account that g_0 is realvalued, that $\partial_{\overline{z}}\overline{k} = \overline{\partial_z k}$, $\partial_z\overline{k} = \overline{\partial_{\overline{z}} k}$ and the properties (1) hold for k, one finds:

$$\begin{aligned}\Delta f_0(z) =&\Delta g_0(k(z))\,|k'(z')|^2 + |\nabla^2 g_0(k(z))|\,O_1(\varkappa(d)) + \\ &+|\nabla g_0(k(z))|\,O_2(\varkappa(d))/d\ ,\end{aligned} \tag{2.2}$$

where $|O_l(\varkappa(d))| \leqslant c_4\varkappa(d)$, $l = 1$ and 2, $c_4 = c_4(D)$.

The main idea of the proof of Theorem 2.2 is the following. Take $g(w) = f(k^{-1}(w))$, $|w| \leqslant 1$, so that $g \in C^1(\overline{B}) \cap SH(B)$. Using [1, 5.1], extend g to the function $g_1 \in C^1(\mathbb{C}) \cap SH(\mathbb{C})$, satisfying the properties (see [1, Proof of Theorem 5.1]):

$$\|\nabla g_1\|_{\mathbb{C}} \leqslant c_5\|\nabla g\|_{\overline{B}}\ ,\ \ \omega(t) := \omega_{\mathbb{C}}(\nabla g_1, t) \xrightarrow[t\to 0]{} 0\ ; \tag{2.3}$$

in particular, $\omega(t) \leqslant 2c_5\|\nabla g\|_{\overline{B}}$, $t \geqslant 0$.

With the help of the next two lemmas we modify g_1 outside of $\overline{B}$ and find a function $g_0 \in C^1(\mathbb{C}) \cap C^2(\mathbb{C} \setminus \overline{B}) \cap SH(\mathbb{C})$, $g_0 = g$ on $\overline{B}$, and a disc $U = \{|w| < r\}$, $r = r(D) \in (1, r_0)$, such that the conditions (1) and (2) of function k, the equality (2.2) and appropriate estimates on g_0 and its partial derivatives up to the order 2 garantee that $f_0(z) = g_0(k(z))$ satisfies the required properties on the extention of function f, but only in the neighbourhood $\Omega = k^{-1}(U)$ of $\overline{D}$ (notice that Ω will depend only on D). The desired extension F can be found then, using f_0, much more easily.

Lemma 2.4. *Let g_1 and $\omega(\cdot)$ be the above mentioned functions (see (2.3)). One can find $g_2 \in C^1(\mathbb{C}) \cap C^2(\mathbb{C} \setminus \overline{B}) \cap SH(\mathbb{C})$, $g_2 = g_1$ on $\overline{B}$, with the properties:*

$$\|\nabla g_2\|_{\mathbb{C}} \leqslant c_6\|\nabla g\|_{\overline{B}}\ , \tag{2.4}$$

$$|\nabla^2 g_2(w)| \leqslant \frac{c_6\omega(|w|-1)}{|w|-1}\ ,\ \ |w| > 1\ . \tag{2.5}$$

Lemma 2.5. *Let $\sigma(t)$, $t \geqslant 0$, be some continuous nondecreasing function on $[0, +\infty)$ with conditions $\sigma(0) = 0$, $\varliminf_{t\to 0}\frac{\sigma(t)}{t} > 0$ and (Dini's condition) $\int_0^1(\sigma(t)/t)\,dt < +\infty$. Then there exist $r_1 = r_1(\sigma) \in (1, 2)$, $c_7 = c_7(\sigma) > 1$ and a function $g_3 \in C^1(\mathbb{C}) \cap C^2(\mathbb{C} \setminus \overline{B}) \cap SH(\mathbb{C})$ with the properties:*

$$(2.6) \qquad g_3(w) = 0 \ \ \textit{for}\ |w| \leqslant 1\,, \quad \Delta g_3(w) = \frac{\sigma(|w|-1)}{|w|-1} \quad \textit{for}\ |w|>1\,,$$

$$(2.7) \qquad |\nabla g_3(w)| \leqslant \int\limits_1^{|w|} \frac{\sigma(t-1)}{t-1}\,dt\,, \qquad |w|>1,$$

$$(2.8) \qquad |\nabla^2 g_3(w)| \leqslant c_7\,\frac{\sigma(|w|-1)}{|w|-1}\,, \qquad |w| \in (1, r_1).$$

Notice in advance that g_0 will be equal to g_2+g_3, and the function σ in the last lemma will be directly proportional to $\varkappa(t)+t$.

Proof of Lemma 2.4. We first construct some special (Whitney type) partition of unity on $\mathbb{C}\setminus\overline{B}$ as follows. Fix $\psi \in C_0^\infty(B)$ with the following conditions: $0 \leqslant \psi(w) \leqslant 1$, $\psi = 1$ on $B(0,3/4)$, $\psi(w)=\psi(|w|)$ is *radial* and $\iint \psi(w)\,du\,dv = 1$ $(w=u+iv)$. For $\delta>0$ define $\psi^\delta(w) = \frac{1}{\delta^2}\psi\left(\frac{w}{\delta}\right)$. For $s\in\mathbb{Z}$, let $\{w_j^s \mid j \in J_s\}$ be the vertexes of some right polygon centered at the origin, with side-length $\delta_s \in [2^{-s-2}, 2^{-s-1})$ and with the radius of the circumscibed disc equal to $|w_j^s| = 1+2^{-s}$, $j\in J_s$. Let $\rho_s = \frac{3}{4}\,2^{-s}$, $B_j^s = B(w_j^s,\rho_s)$, $j\in J_s$. It is easily seen that $\bigcup\limits_{s\in\mathbb{Z}}\bigcup\limits_{j\in J_s} B(w_j^s, 3\rho_s/4) = \mathbb{C}\setminus\overline{B}$. Therefore, defining

$$\psi_j^s(w) = \frac{\psi((w-w_j^s)/\rho_s)}{\sum\limits_{s'\in\mathbb{Z}}\sum\limits_{j'\in J_s}\psi((w-w_{j'}^{s'})/\rho_{s'})}\,,$$

we find that $\{B_j^s,\psi_j^s\}_{s\in\mathbb{Z},j\in J_s}$ is the partition of unity on $\mathbb{C}\setminus\overline{B}$ with the properties

$$(2.9) \qquad \|\nabla^l\psi_j^s\|_{\mathbb{C}} \leqslant c\rho_s^{-l}, \qquad l=1 \text{ and } 2,$$

since for each $w\in\mathbb{C}\setminus\overline{B}$ the number of indices (s,j) such that $\psi_j^s(w)\neq 0$ does not exceed some positive absolute constant.

Denote by $\Phi(z)$ the standard fundamental solution for the Laplace equation in $\mathbb{R}^2$:

$$\Phi(z) = \frac{1}{2\pi}\log|z|.$$

Let $\varphi\in C_0^\infty(\mathbb{C})$. The Vitushkin operator associated with the Laplace operator Δ and with the function φ is defined as follows:

$$V_\varphi(h) := \Phi*(\varphi\Delta h), \qquad V_\varphi : (C_0^\infty(\mathbb{C}))' \to (C_0^\infty(\mathbb{C}))'\,,$$

where $*$ means the convolution operator. Clearly, $\Delta V_\varphi(h) = \varphi\,\Delta h$ (in the distributional sense).

It is known (see [1, Lemma 3.4]) that if $h \in C^1(B(a,\delta))$ and $\varphi \in C_0^\infty(B(a,\delta))$ then $V_\varphi(h) \in C^1(\mathbb{C})$ and

$$\|\nabla(V_\varphi(h))\|_{\mathbb{C}} \leqslant c \operatorname{osc}(\nabla h, B(a,\delta)) \, \|\nabla\varphi\|_{\mathbb{C}} \, \delta \,, \tag{2.10}$$

where $\operatorname{osc}(e,E) = \sup\{|e(z)-e(w)| \mid z \in E,\, w \in E\}$ and e is a bounded function on E.

Let now $h_j^s = V_{\psi_j^s}(g_1)$, $\tilde h_j^s = \psi^{\rho_s/8} * h_j^s$, $j \in J_s$. The sought for function g_2 is the following:

$$g_2 = g_1 + \sum_{s\in\mathbb{Z}} \sum_{j\in J_s} (\tilde h_j^s - h_j^s).$$

In fact, by (2.10) we have (setting $h = g_1$, $\varphi = \psi_j^s$, $\delta = \rho_s$ and taking (2.9) into account with $l = 1$):

$$\|\nabla h_j^s\|_{\mathbb{C}} \leqslant c\,\omega(\rho_s), \qquad\qquad \|\nabla \tilde h_j^s\|_{\mathbb{C}} \leqslant c\,\omega(\rho_s). \tag{2.11}$$

Since h_j^s is harmonic outside B_j^s (because $\Delta h_j^s = \psi_j^s\,\Delta g_1$), and since $\psi^{\rho_s/8}$ is radial, it easily follows from the mean value theorem for harmonic functions that $h_j^s(w) = \tilde h_j^s(w)$ for $w \notin B\left(w_j^s, \frac{7}{8}\,2^{-s}\right)$, and therefore the function $\tilde h_j^s(w)$ is harmonic outside of the disc $\tilde B_j^s = B(w_j^s, \frac{7}{8}\,2^{-s})$. In particular, $g_1 = g_2$ on $\overline{B}$, and for $w \notin \overline{B}$ one has:

(2.12)
$$g_2(w) - g_1(w) = \sum_{s\in\mathbb{Z}} \sum_{j\in J_s} (\tilde h_j^s(w) - h_j^s(w)) = \sum_{(s,j)\in SJ(w)} \left(\tilde h_j^s(w) - h_j^s(w)\right) ,$$

where $SJ(w) = \left\{(s,j) \;\middle|\; s \in \mathbb{Z},\, j \in J_s,\, w \in \tilde B_j^s\right\}$. Evidently, the number of elements in $SJ(w)$ does not exceed some absolute constant, and that for $(s,j) \in SJ(w)$ one has

$$c^{-1}(|w|-1) \leqslant \rho_s \leqslant c\,(|w|-1) \,, \tag{2.13}$$

where $c > 1$ is some absolute constant. From the last remarks and by (2.11) we have

$$|\nabla g_2(w) - \nabla g_1(w)| \leqslant c\,\omega(|w|-1) \leqslant c_6 \|\nabla g\|_{\overline{B}}, \qquad |w| > 1,$$

and (2.4) is proved.

Since $\Delta h_j^s = \psi_j^s\,\Delta g_1$, for $|w| > 1$ we have

$$\Delta g_2(w) = \sum_{(s,j)\in SJ(w)} \Delta \tilde h_j^s(w) \,,$$

so that $g_2 \in C^1(\mathbb{C}) \cap C^\infty(\mathbb{C}\setminus\overline{B}) \cap SH(\mathbb{C})$, because $\tilde h_j^s \in C^\infty(\mathbb{C}) \cap SH(\mathbb{C})$. Moreover, by (2.11) and from the estimate $\|\nabla\psi^{\rho_s/8}\|_{\mathbb{C}} \leqslant c/\rho_s^3$ we obtain:

$$|\nabla^2 \tilde h_j^s(w)| = |\nabla\psi^{\rho_s/8} * \nabla h_j^s(w)| \leqslant c\,\frac{1}{\rho_s}\,\omega(\rho_s) \,. \tag{2.14}$$

Fix w_0, $|w_0| > 1$, and let us prove (2.5) for $w = w_0$. Put $\delta = (|w_0| - 1)/3$. Let

$$SJ_0(w_0) = \left\{(s,j) \;\middle|\; s \in \mathbb{Z},\, j \in J_s,\, \tilde{B}^s_j \cap B(w_0,\delta) \neq \emptyset\right\} ,$$

so that the number of elements in $SJ_0(w_0)$ does not exceed some absolute constant either. Define

$$\tilde{h}_2(w) = \sum_{(s,j)\in SJ_0(w_0)} \tilde{h}^s_j(w) \,.$$

By (2.14) and (2.13) it follows that

$$|\nabla^2 \tilde{h}_2(w_0)| \leqslant c\, \frac{\omega(\delta)}{\delta} \,.$$

Put $h_2 = g_2 - \tilde{h}_2$. It remains to show that

$$|\nabla^2 h_2(w_0)| \leqslant c\, \frac{\omega(\delta)}{\delta} \,. \tag{2.15}$$

By (2.11) (see also (2.3)), (2.12) and from harmonicity of $\tilde{h}^s_j$ outside $\tilde{B}^s_j$, we find that h_2 is harmonic in $\overline{B(w_0,\delta)}$ and the following estimate holds: $\operatorname{osc}(\nabla h_2, \overline{B(w_0,\delta)}) \leq c\,\omega(\delta)$. The desired estimate (2.15) now follows directly from the mentioned properties of the function h_2 and the Cauchy integral formula in the disc $B(w_0,\delta)$ for the *derivative* of the holomorphic function $\partial_w h_2$.

Lemma 2.4 is proved. □

Proof of Lemma 2.5. Let us seek $g_3(w)$ in the radial-symmetric form: $g_3(w) = h(\rho)$, $\rho = |w|$. Using polar coordinates (ρ,θ) in $\mathbb{R}^2$ and elementary formulas

$$|\nabla g_3(w)| \;=\; |h'(\rho)| \,,\;\; |\nabla^2 g_3(w)| \;\leqslant\; |h''(\rho)| + \frac{1}{\rho}|h'(\rho)| \,,$$

$$\Delta g_3(w) \;=\; h''(\rho) + \frac{1}{\rho} h'(\rho) \,,\;\; |w| = \rho > 1 \,,$$

we see that the conditions (2.6)–(2.8) in terms of h have the form:

$$h''(\rho) + \frac{1}{\rho} h'(\rho) = \frac{\sigma(\rho-1)}{\rho-1} \,,\;\; \rho > 1 \,, \tag{2.16}$$

$$|h'(\rho)| \leqslant \int_1^\rho \frac{\sigma(t-1)}{t-1}\, dt \,, \tag{2.17}$$

$$|h''(\rho)| + \frac{1}{\rho}|h'(\rho)| \;\leqslant\; \frac{c_7\, \sigma(\rho-1)}{\rho-1} \,. \tag{2.18}$$

Set $\mu(t) = 0$ for $t \in (0,1]$, $\mu(t) = \sigma(t-1)/(t-1)$ for $t > 1$, so that $\mu \in L^1_{\mathrm{loc}}(0,+\infty)$. Solving the equation $h''(\rho) + \frac{1}{\rho}h'(\rho) = \mu(\rho)$ on $(0,+\infty)$ by the variation of constants method, one gets:

$$h'(\rho) = \frac{1}{\rho}\int\limits_0^\rho t\mu(t)\,dt \geqslant 0\,,$$

and $h'(\rho)$ is continuous on $(0,+\infty)$ with

$$|h'(\rho)| \leqslant \int\limits_0^\rho \mu(t)\,dt\,,$$

which gives (2.16) and (2.17) (and then (2.6) with (2.7)). The function

$$g_3(w) = h(\rho) = \int\limits_0^\rho \frac{1}{t}\int\limits_0^t \tau\mu(\tau)\,d\tau\,dt$$

is subharmonic, because $\Delta g_3(w) = \mu(|w|)$ in the distributional sense. Finally, for $\rho > 1$, we have

$$h''(\rho) = -\frac{1}{\rho^2}\int\limits_1^\rho t\mu(t)\,dt + \mu(\rho).$$

Since the first summand in the right hand side of the last equality tends to zero as $\rho \to 1$ ($\rho > 1$), but $\varliminf\limits_{\rho\to 1+} \mu(\rho) > 0$, there exists $r_1 = r_1(\sigma) \in (1,2)$ such that $h''(\rho) > 0$ on $(1, r_1)$. But then, for $\rho \in (1, r_1)$, the inequality (2.18) (and hence (2.8)) follows directly from (2.16). Lemma 2.5 is proved. □

Let us continue with the proof of Theorem 2.2. We can suggest that $f \not\equiv \mathrm{const}$ in $\overline{D}$, otherwice there is nothing to do. Starting from f, construct the functions g, g_1, g_2 and g_3 as in Lemmas 2.4 and 2.5 (where the function $\sigma(t)$ will be chosen a little bit later). Put $g_0 = g_2 + g_3$, so that $g_0 = g$ on $\overline{B}$. Let $g_0(k(z)) = f_0(z)$ in Ω_0, and let $\Omega_1 = k^{-1}\{|w| < \min\{r_0, r_1\}\}$. By (2.2) and the properties (2.4) – (2.8), we have for $z \in \Omega_1 \setminus \overline{D}$:

$$\Delta f_0(z) \geqslant \frac{\sigma(|k(z)|-1)}{|k(z)|-1}(k_0')^2 - c_4\varkappa(d)\left(c_6\frac{\omega(|k(z)|-1)}{|k(z)|-1} + c_7\frac{\sigma(|k(z)|-1)}{|k(z)|-1}\right) - \\ - c_4\frac{\varkappa(d)}{d}\left(c_6\|\nabla g_0\|_{\overline{B}} + \int\limits_1^{|k(z)|}\frac{\sigma(t-1)}{t-1}dt\right).$$

Notice that, by (2.1) and (2), there exists $c_1^* = c_1^*(D)$ such that $\varkappa(d) = \varkappa(|z-z'|) \leqslant c_1^* \varkappa(|k(z)|-1)$, $z \in \Omega_1$. Since $\varkappa(\cdot)$ satisfies the Dini condition, the same condition is also satisfied by the function $\sigma(t) = c_8 \|\nabla g\|_{\overline{B}} (\varkappa(t)+t)$, where $c_8 > 0$ ($c_8 = c_8(\overline{D})$) will be found later. We remark that σ satisfies the conditions of Lemma 2.5, and that in defining of r_1, instead of the requirements on σ there appear the corresponding requirements on $\varkappa$, so that $r_1 = r_1(\varkappa) = r_1(D)$. Therefore, by conditions (2) on k and the estimate $\omega(t) \leqslant 2c_5 \|\nabla g_0\|_{\overline{B}}$, we obtain:

$$\Delta f_0(z) \geqslant \frac{\sigma(|k(z)|-1)}{|k(z)|-1}\left(\frac{(k_0')^2}{2} - c_1^* c_4 c_7 \varkappa(|k(z)|-1)\right) + \frac{\|\nabla g_0\|_{\overline{B}}\, \varkappa(|k(z)|-1)}{|k(z)|-1} \times \left(\frac{c_8 (k_0')^2}{2} - (2c_5+1)c_4 c_6 c_1^* c_3 - c_8 c_1^* c_3 c_4 \int_1^{|k(z)|} \left(\frac{\varkappa(t-1)}{t-1}+1\right) dt\right).$$

Clearly, we can choose $r_2 = r_2(D) \in (1,2)$ with the following properties:

$$\varkappa(r_2 - 1) \leqslant \frac{(k_0')^2}{2c_1^* c_4 c_7}$$

$$\int_1^{r_2} \left(\frac{\varkappa(t-1)}{t-1}+1\right) dt \leqslant \frac{(k_0')^2}{4c_1^* c_3 c_4},$$

so that for $c_8 = 4(2c_5+1)c_1^* c_3 c_4 c_6 (k_0')^{-2}$ the function $f_0(z)$ is subharmonic in $\Omega = k^{-1}(U)$, where $U = \{w \mid |w| < \min\{r_0, r_1, r_2\}\}$. In fact, the subharmonicity of f_0 in D is evident, and in $\Omega \setminus \overline{D}$ it follows from the last estimates. Therefore, the subharmonicity of f_0 on (some neighbourhood of) ∂D now follows from the C^1-smoothness of ∂D and from the fact that $f_0 \in C^1(\Omega)$. We give the idea of the proof of the last claim for completeness. By the Gauss-Ostrogradski formula, one can easily show that for each closed disc K in Ω the inequality $\int_{\partial K} \frac{\partial f_0}{\partial \nu}\, d\ell \geqslant 0$ takes place. Here ν is the outer unit normal to ∂K and $d\ell$ is the differential of the *length* on ∂K. The smoothing $\psi^\delta * f_0$ of f_0 with respect to the mollifier ψ^δ (see above) preserves the above mentioned integral property of the function f_0 (in a δ-"smaller" domain, than Ω), from which it is easy to prove that $\psi^\delta * f_0$ is subharmonic. It remains to pass to the limit as $\delta \to 0+$.

We are ready to extend f on all of $\mathbb{C}$. Without loss of generality, we can suggest that $||\nabla f||_{\overline{D}} = 1$, $0 \in D$ and $f(0) = 0$. Fix some Jordan domains G_1 and G_2 such that $\overline{D} \subset G_1 \subset \overline{G_1} \subset G_2 \subset \overline{G_2} \subset \Omega$, and some function $\varphi \in C_0^\infty(G_2)$, $\varphi = 1$ on G_1 ($\varphi = 0$ outside G_2). We also can suggest that G_1, G_2 and φ depend only on D. It follows from the construction of f_0 that $f_0 \in C^1(\overline{G}_2)$ and $||f_0||_{1,\overline{G_2}} \leq c$, where $c > 0$ (here and below) depends only on D. Extend f_0 from the compact set $\overline{G_2}$ on all of $\mathbb{C}$ to some function F_0 of the class $BC^1(\mathbb{C})$ and such that $||\nabla F_0||_{\mathbb{C}} \le c$. Define $F_1 = V_\varphi(F_0)$. By (2.10) we obtain that $F_1 \in C^1(\mathbb{C})$ and $||\nabla F_1||_{\mathbb{C}} \le c$. Since $\Delta F_1 = \varphi \Delta F_0 \ge 0$ (in the distributional sense), we have $F_1 \in SH(\mathbb{C})$. Consider $F_2 = F_0 - F_1$. Since $\Delta F_2 = (1-\varphi)\Delta F_0$, the function F_2 is harmonic in G_1 and $F_2 \in C^1(\overline{G_1})$ with $||F_2||_{1,\overline{G_1}} \le c$. It remains to find an appropriate extension (from $\overline{D}$) of the function F_2.

We can choose some Jordan domain G_0 with *analytic* boundary and with the properties $\overline{D} \subset G_0 \subset \overline{G_0} \subset G_1$. Notice that

$$F_2 \in C^2(\overline{G_0})\,, \ ||F_2||_{2,\overline{G_0}} \le c\,. \tag{2.19}$$

Fix now some open disc $B_R = B(0,R)$, $R > 0$, such that $\overline{G_0} \subset B(0,R/2)$. Let f_3 be the solution of the Dirichlet problem in the domain $G = B_R \setminus \overline{G_0}$ with boundary values equal to zero on ∂B_R and equal to F_2 on ∂G_0. We claim that $f_3 \in C^1(\overline{G})$ and $||f_3||_{1,\overline{G}} \le c$. In fact, it is well known that a conformal mapping of the domain G onto some *appropriate* annulus A can be extended conformally on some neighbourhood of $\overline{G}$. The required claim undoubtedly is true for A: it is important just to recall that (2.19) takes place. The function f_4, which is equal to F_2 on $\overline{G_0}$ and equal to f_3 on $\overline{G}$, satisfies the properties $f_4 \in \mathrm{Lip}_1(\overline{B_R})$ and $||\nabla f_4||_{B_R} \le c$ (notice that $\nabla f_4 \in L_\infty(B_R)$ in the sense of distributions). It is known (can be easily checked using [4, Chapter 5, §27, s. 7]) that the Laplacean (in the sense of distributions) Δf_4 of the function f_4 in B_R is equal to the finite measure $h\ell$, where ℓ is the *length-measure* on ∂G_0 (so that ℓ is positive), and the function $h \in C(\partial G_0)$ has the form

$$h(z) = \frac{\partial f_4}{\partial \nu_+}\Big|_z + \frac{\partial f_4}{\partial \nu_-}\Big|_z\,,\ z \in \partial G_0\,,$$

where ν_+ and ν_- are the outer and inner unit normals to ∂G_0 respectively. Therefore, $||h||_{\partial G_0} \le c_9 = c_9(D)$. Consider the function f_5 in $\overline{B_R}$, which is equal to zero in $\overline{G_0}$ and which is the solution of the Dirichlet problem in G with the boundary values equal to 0 on ∂G_0 and equal to 1 on ∂B_R. It is well known that

$$\frac{\partial f_5}{\partial \nu_+}\Big|_z = h_5(z) \ge c_{10} = c_{10}(D) > 0\ ,\ z \in \partial G_0$$

(one can check this fact using again a conformal mapping from G onto A). The function $f_6 = f_4 + c_9 f_5/c_{10}$ satisfies the equality $\Delta f_6 = (h + (c_9/c_{10})h_5)\ell \ge 0$ in B_R, so that

$$f_6 \in \mathrm{Lip}_1(\overline{B_R}) \cap SH(B_R)\ ,\ ||\nabla f_6||_{B_R} \le c\,.$$

Set $\delta = \operatorname{dist}(\partial D, \partial G_0)/2$. The function $f_7 = \psi^\delta * f_6$, which is well-defined in $B(0, R-\delta)$, coincides (by the mean value theorem) with f on $\overline{D}$, $f_7 \in C^1(\overline{B(0,R/2)}) \cap SH(B(0,R/2))$, and also $||\nabla f_7||_{\overline{B(0,R/2)}} \le c$. It remains to extend the function f_7 from $\overline{B(0,R/2)}$ on all of $\mathbb{C}$ with the help of [1, Theorem 5.1] mentioned at the beginning of this paper.

Theorem 2.2 is proved.

2.2. Extension and localization

With the help of Theorem 2.2 we can easily prove the following localization theorem, which has, from our point of view, its own interest. We restrict our consideration to closed Jordan domains.

Theorem 2.6. *Let D be a Jordan domain in $\mathbb{C}$ and suppose that for any $a \in \partial D$ there exists a disk $B(a, r_a)$, $r_a > 0$, such that for each $f \in C^1(\overline{D}) \cap SH(D)$ one can find $F_a \in C^1(\overline{D_a}) \cap SH(D_a)$, $D_a = D \cup B(a, r_a)$, with $F_a = f$ in $\overline{D}$. Then for any $f \in C^1(\overline{D}) \cap SH(D)$ there is $F \in C^1(\mathbb{C}) \cap SH(\mathbb{C})$ with $F = f$ in $\overline{D}$. If, additionally, each F_a can be chosen with the property $||F_a||_{1,\overline{D_a}} \le c(D,a)||f||_{1,\overline{D}}$ then F also can be found with additional estimate $||\nabla F||_{\mathbb{C}} \le c(D)||f||_{1,\overline{D}}$.*

Proof. First notice that the set $E = \{e \in \partial D \,|\, \exists\, b_e \notin \overline{D}$ with $\overline{B(b_e, |b_e - e|)} \cap \overline{D} = \{e\}\}$ is everywhere dense in ∂D. From the conditions of Theorem 2.6 (and the compactness of ∂D) it easily follows, that there exists a finite subset $E_J = \{e_1, \cdots, e_J\}$ of E and a (say Jordan) neighbourhood Ω of $\overline{D}$ such that each $f \in C^1(\overline{D}) \cap SH(D)$ can be extended (respectively, in addition, with appropriate estimates) to a function $F_J \in C^1(\overline{\Omega_J}) \cap SH(\Omega_J)$, where

$$\Omega_J = \Omega \setminus (\overline{B^1} \cup \cdots \cup \overline{B^J}), \ B^j = B(b_{e_j}, |b_{e_j} - e_j|), \ j = 1, \cdots, J.$$

Using Theorem 2.2 for "rather small" Dini-Lyapunov type domains G_j $(1 \le j \le J)$ such that $G_j \subset \Omega_J$ and $G_j \cup \overline{B^j}$ contains some neighbourhood of e_j, we can extend F_J to a C^1-subharmonic function in some "small" neighbourhood of E_J. So, finally, we can extend f appropriately to some neighbourhood of $\overline{D}$ and it remains to apply (the final step of the proof of) Theorem 2.2. □

3. An example on the lack of extension

The following theorem gives a wide class of examples on the lack of the C^1-subharmonic extension (this class contains some closed Jordan C^1-smooth domains). Unfortunately, this result is not enough to discuss the precision of (the sufficient) conditions in Theorem 2.2.

Theorem 3.1. *Let $d > 0$ and suppose that $h \in C([-d,d])$, $h \geq 0$ on $[-d,d]$, $h(0) = 0$, h is Lipschitz nondecreasing on $[0,d]$ and*

$$\int_0^d \frac{h(x)}{x^2}\,dx \;=\; \infty\,. \tag{3.1}$$

Let D be a Jordan domain in $\mathbb{C}_z$ ($z = x + iy$) such that the graph $\Gamma = \{x+ih(x)\,|\, x \in [-d,d]\}$ of the function h is a subset of ∂D and that near 0 the domain D is "under" Γ. Then there is a function $f \in SH(D)\cap C^1(\overline{D})$, which can not be extended from $\overline{D}$ to a C^1-subharmonic function in some neighbourhood of 0.

Proof. Set $I_n = [d/2^{n+1}, d/2^n]$, $n \in \mathbb{Z}_+$. By [5, Corollary 4.6], for each n there exists a Borel function h_n on I_n such that $h_n(t) \in [0,1]$ for all $t \in I_n$, $\int_{I_n} h_n(t)\,dt \geq d/(c2^n)$ and

$$\left| \int_{I_n} \frac{h_n(t)}{z - t - ih(t)}\,dt \right| \;<\; c \tag{3.2}$$

for all $z \notin \Gamma_n = \{t + ih(t)\,|\, t \in I_n\}$, where $c > 1$ (as before, it can change in different places) depends only on Lip_1-constant of h. Define a positive Borel measure μ_n on Γ_n as $\mu_n(E) = \int_{E_x} h_n(t)\,dt$, where E is a Borel set in Γ_n and E_x is its projection on the x-axis. By (3.2), $|(1/z) * \mu_n| < c$ outside Γ_n, which easily gives that

$$g_n = \Phi * \mu_n \in \mathrm{Lip}_1(\mathbb{C}) \text{ and } ||\nabla g_n||_{\mathbb{C}} < c\,. \tag{3.3}$$

For $n \geq 1$ and $\varepsilon_n > 0$ define $g_n^{\pm}(z) = g_n(z \mp i\varepsilon_n)$, so that $\mu_n^{\pm} = \Delta g_n^{\pm}$ are positive measures ($\pm i\varepsilon_n$-shifts of μ_n) with supports on $\Gamma_n^{\pm}$ (which are $\pm i\varepsilon_n$-shifts of Γ_n respectively). Clearly, we can choose ε_n so small that

$$|g_n^{\pm}(0) - g_n(0)| + |\nabla g_n^{\pm}(z) - \nabla g_n(z)| < 1/2^n \tag{3.4}$$

for all z with $\mathrm{Re}\, z \notin I_{n+1} \cup I_n \cup I_{n-1}$.

Now, for $n \geq 1$ choose $\delta_n < \varepsilon_n$ such that $U_{\delta_n}(\Gamma_n^-) \subset \{z \in D\,|\,\mathrm{Im}\, z > 0\}$, $U_{\delta_n}(\Gamma_n^+)\cap D = \emptyset$, where $U_\delta(E)$ is δ-neighbourhood of E, $\delta > 0$. Let ψ^δ be the function defined above in Section 2. Take $f_n = \psi^{\delta_n} * g_n$ and $f_n^{\pm} = \psi^{\delta_n} * g_n^{\pm}$, so that $f_n \in C^1(\mathbb{C})$ and $f_n^{\pm} \in C^1(\mathbb{C})$, $f_n(z) = g_n(z)$ outside $U_{\delta_n}(\Gamma_n)$ and $f_n^{\pm}(z) = g_n^{\pm}(z)$ outside $U_{\delta_n}(\Gamma_n^{\pm})$ respectively. Moreover, by (3.3) we have

$$||\nabla f_n||_{\mathbb{C}} < c \;\;,\;\; ||\nabla f_n^{\pm}||_{\mathbb{C}} < c \tag{3.5}$$

and, by (3.4),

$$|f_n^{\pm}(0) - f_n(0)| + |\nabla f_n^{\pm}(z) - \nabla f_n(z)| \leq 1/2^n \tag{3.6}$$

whenever $\mathrm{Re}\, z \notin I_{n+1} \cup I_n \cup I_{n-1}$.

Finally, set $f = \sum_{n=1}^{+\infty}(f_n^- - f_n^+)$. We claim that f is as required. In fact, since $\Delta f = \sum_{n=1}^{+\infty} \psi^{\delta_n} * \mu_n^-$ inside D, by (3.5) and (3.6) one has $f \in C^1(\mathbb{C}) \cap SH(D)$.

Suppose, by contradiction, that there exist $r \in (0,d)$ and $F_0 \in SH(B_r) \cap C^1(\overline{B_r})$ (here $B_r = B(0,r)$) such that $F_0 = f$ on $\overline{D} \cap \overline{B_r}$. Without loss of generality we can suppose that $\{z \in B_r \mid \operatorname{Im} z < 0\} \subset D$. Fix $\varphi \in C_0^\infty(B_r)$ such that $\varphi = 1$ in $B(0,r/2)$ and define $F = V_\varphi F_0$ (see Section 2). By (2.10), $F \in C^1(\mathbb{C}) \cap SH(\mathbb{C})$ and $\mu = \Delta F$ is finite positive measure with support in B_r. Choose a natural number N such that $d/2^N < r/2$ and $U_{\delta_n}(\Gamma_n^-) \subset B(0,r/2)$ for all $n \geq N$. Then $\nu = \mu - \sum_{n=N}^{\infty} \psi^{\delta_n} * \mu_n^-$ is also finite positive measure with suport in $\{z \in B_r \mid \operatorname{Im} z \geq 0\}$.

Since $F = \Phi * \mu$, for each $\varepsilon > 0$ one has $(z = x + iy)$:

$$-\frac{\partial F}{\partial y}\Big|_{z=-i\varepsilon} = -\frac{\partial \Phi}{\partial y} * \mu \Big|_{z=-i\varepsilon} = \frac{-y}{2\pi(x^2+y^2)} * \Big(\sum_{n=N}^{+\infty} \psi^{\delta_n} * \mu_n^- + \nu \Big) \Big|_{z=-i\varepsilon} \geq$$

$$\geq \frac{-y}{2\pi(x^2+y^2)} * \Big(\sum_{n=N}^{+\infty} \psi^{\delta_n} * \mu_n^- \Big) \Big|_{z=-i\varepsilon}.$$

Since $f_n^-(z) = g_n^-(z)$ outside $U_{\delta_n}(\Gamma_n^-)$, by (3.4) we have

$$-\frac{\partial F}{\partial y}\Big|_{z=-i\varepsilon} \geq \frac{-y}{2\pi(x^2+y^2)} * \Big(\sum_{n=N}^{+\infty} \mu_n^- \Big) \Big|_{z=-i\varepsilon} \geq$$

$$\geq \frac{-y}{2\pi(x^2+y^2)} * \Big(\sum_{n=N}^{+\infty} \mu_n \Big) \Big|_{z=-i\varepsilon} - 1 .$$

Recall, that $\int_{I_n} h_n(t)\,dt \geq d/(c2^n)$, which gives

$$2\pi - 2\pi \frac{\partial F}{\partial y}\Big|_{z=-i\varepsilon} \geq \sum_{n=N}^{+\infty} \int_{I_n} \frac{(h(t)+\varepsilon)h_n(t)\,dt}{t^2 + (\varepsilon + h(t))^2} \geq$$

$$\geq \frac{1}{c} \int_0^{d/2^{N+1}} \frac{h(t)\,dt}{t^2 + (\varepsilon + h(t))^2}.$$

Since $h(t) \leq ct$ for $t \in (0,d)$, by (3.1) the last integral tends to $+\infty$ as $\varepsilon \to 0$, which gives the desired contradiction. □

Acknowledgment. The second author expresses his deep gratitude to the Centre de Recerca Matemàtica and the Universitat Autònoma de Barcelona for hospitality and financial support.

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Universitat Autònoma de Barcelona, Spain.
Moscow State (M.V. Lomonosov) University, Russia.
E-mail: melnikov@mat.uab.es, petr@paramonov.msk.ru