BOUNDED ELEMENTS IN A PRO-C*-ALGEBRA AND APPLICATIONS

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ABSTRACT. A pro-C*-algebra is a projective limit of C*-algebras in the category of Hausdorff topological *-algebras (and their continuous morphisms). In this paper, we investigate the relationship between bounded and spectrally bounded elements of a pro-C*-algebra and we give sufficient conditions for a pro-C*-algebra to be trivial. Then, let $\mathcal{A}$ be a pro-C*-algebra and denote by $b(\mathcal{A}) = \{a \in \mathcal{A} : \|a\|_\infty = \sup\{p(a), p \in \mathrm{S}\} < +\infty\}\}$ the C*-algebra consisting of bounded elements of $\mathcal{A}$. We prove that if $\mathcal{A}$ is a pro-C*-algebra such that its topology is metrizable and its elements are all spectrally bounded, then $\mathcal{A}$ is necessarily a C*-algebra. On the other hand, we check the preservation of the exactness of short sequences by the covariant functor $b \to b(\mathcal{A})$ from the category of pro-C*-algebras (and continuous *-homomorphims) to the category of C*-algebras (and *-homomorphisms). As an application of this result, we show that if $b(\mathcal{A})$ is simple, then $\mathcal{A}$ is necessarily a C*-algebra. Finally, we establish a characterization of the connected component of the identity in the group of unitary elements in a pro-C*-algebra.

1. Introduction and Notations

Pro-C*-algebras have been carefully studied by E. J. Dubuc and H. Porta in [?] and by N.C. Phillips in [?] because of their mathematical interest and their physical applications. Autors have used different terminology such as, e.g., LCM^*-algebra in [?], locally C*-algebra ([?] and [?]) and σ-algebra in the merizable case (see [?] and [?]). We shall present an equivalent definition: A pro-C*-algebra $\mathcal{A}$ is a complete topological algebra with some involution such that its topology is determined by a point-separating set S of C*-seminorms. A seminorm p on $\mathcal{A}$ is a C*-seminorm if for every $a \in \mathcal{A}$,

2000 *Mathematics Subject Classification.* 18A05, 46H05, 46J05, 46K05.
Finacially supported by CRM, Universitat Autònoma de Barcelona, Spain.

we have

$$p(a^*a) = p(a)^2 \tag{1}$$

Due to Z. Sebestyen [11], every C*-seminorm p on $\mathcal{A}$ satisfies

$$p(ab) \leq p(a)p(b) \ \textit{for all} \ a, b \in \mathcal{A} \tag{2}$$

and so,

$$p(a^*) = p(a) \tag{3}$$

It follows from (2) that a pro-C*-algebra is a complete locally m-convex algebra (its topology is induced by a separating directed set of submultiplicative seminorms [?]) and by (3), it is a complete locally m-convex *-algebra. It is then a fact that a pro-C*-algebra $\mathcal{A}$ is a projective limit of C*-algebras $(\mathcal{A}_p, p \in \mathrm{S})$ in the category of Hausdorff topological *-algebras (and their continuous *-homomorphisms $\pi_p : \mathcal{A} \to \mathcal{A}_p$) (see [?] and [?]. Such an algebra $\mathcal{A}$ is a topological Q-algebra if the group $G_{\mathcal{A}}$ of all invertible elements in $\mathcal{A}$ is open [?]. A pro-C*-algebra which is a C*-algebra is said to be trivial. We denote by $b(\mathcal{A}) = \{a \in \mathcal{A} : \ \|a\|_\infty = \sup\{p(a), p \in \mathrm{S}\} < +\infty\}\}$ the C*-algebra consisting of all bounded elements of $\mathcal{A}$ ([?], [?]). We briefly mention some examples of non trivial pro-C*-algebras.

(1) The infinite product of countable family of full matrix algebras $\prod_{n\geq 1} \mathbb{M}_n(\mathbb{C})$ with the cartesian product topology.

(2) The algebra $C(\sigma)$ of continuous functions on a Kelley space σ [?].

(3) The algebra $C_c([0,1])$ of continuous functions on $[0,1]$ with the topology of uniform convergence on the countable compact subsets of $[0,1]$.

(4) The tangent algebra constructed by W. Arveson and considered as the solution of the universal problem on derivations (see [?]).

Notice that the first and the second examples are σ-algebras.

The purpose of this paper is to investigate the relationship between spectrally bounded elements and bounded elements of a pro-C*-algebra and establish some structural results for pro-C*-algebras. The section 1 contains the basic notions about pro-C*-algebras. In the section 2, we will show in Theorem 2.1 that there is at least one spectrally unbounded element in a non trivial σ-algebras. On the other hand, we establish in Proposition 3.4 the preservation of exactness for short sequences by the covariant functor $b \to b(\mathcal{A})$ which acts on the category of pro-C*-algebras (and continuous *-homomorphims) and values in the category of C*-algebras (and

*-homomorphims) and we show that if $0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{B} \to 0$ is a short sequence that is exact, where $\mathcal{A}$ and $\mathcal{B}$ are pro-C*-algebras and $\mathcal{I}$ is a closed two-sided ideal of a pro-C*-algebra $\mathcal{A}$, then $b(\mathcal{I})$ is a closed two-sided ideal of $b(\mathcal{A})$ and the sequence $0 \to b(\mathcal{I}) \to b(\mathcal{A}) \to b(\mathcal{A}\backslash\mathcal{I}) \to 0$ is also exact. We preface to show this result Proposition 3.2 which states that $b(\mathcal{A})$ does not depend of the choice of the familly $(p \in \mathrm{S})$ and if $\phi : \mathcal{A} \to \mathcal{B}$ is an injective *-homomorphism with a closed range, then $b(\mathcal{A}) \cong b(\mathcal{B}) \cap \phi(\mathcal{A})$. We will deduce in Corollary 3.1 that for every continuous C*-norm p on a pro-C*-algebra $\mathcal{A}$, every algebra A_p as above is *-isomorphic to $b(\mathcal{A})/b(\ker(p)))$. In Theorem 2.2, it is shown that the C*-algebra $b(\mathcal{A})$ of a non trivial pro-C*-algebra, is not simple. In the section 3, Lemmas 3.1 and 3.2 will be used to characterize in Theorem 3.1 the connected component of the identity in the group of unitary elements in a pro-C*-algebra.

2. Preliminaries

Let $(\mathcal{A}, p \in \mathrm{S})$ be a pro-C*-algebra. For each $a \in \mathcal{A}$, the spectrum $sp_{\mathcal{A}}(a)$ of a is the set of $\lambda \in \mathbb{C}$ such that $\lambda - a$ is non invertible in $\mathcal{A}$. The real number $\rho_{\mathcal{A}}(a) = sup\{|\lambda| : \ \lambda.1 \in sp_{\mathcal{A}}(a)\}$ is called the spectral radius of a. With the above notation, an element $a \in \mathcal{A}$ is invertible in $\mathcal{A}$ if, and only if, $\pi_p(a)$ is invertible in $\mathcal{A}_p$ for each $p \in \mathrm{S}$, and so

$$sp_{\mathcal{A}}(a) = \cup_{p \in p} sp_{\mathcal{A}_p}(\pi_p(a)) \quad for\ all\ a \in \ \mathcal{A}. \tag{4}$$

We say that an element a of $\mathcal{A}$ is spectrally bounded if its spectrum is bounded. Now, we recall a collection of elementary properties of a pro-C*-algebra indicating certain references in the literature for each of them.

Proposition 2.1. [**?**] *Let $\mathcal{A}$ be a pro-C*-algebra. Then:*

(1) *Every closed star subalgebra of $\mathcal{A}$ is a pro-C*-algebra.*
(2) *Every closed two-sided ideal is a star subalgebra of $\mathcal{A}$.*

We denote by $X_{\mathcal{A}}$ the set of all continuous characters on $\mathcal{A}$ endowed with the Gelfand topology (if $a \in \mathcal{A}$ and $\chi \in X_{\mathcal{A}}$, we let $\hat{a}(a) = \chi(a)$, the Gelfand topology is the weakest one for which the form map $\hat{a}$ is countinuous on $X_{\mathcal{A}}$). Notice that X_A is a Kelley space and it is not empty whenever $\mathcal{A}$ is commutative. Note that commutative proc-C*-algebras are well discussed in Section 3 of [**?**] and in Section 2 of [**?**]. In the following theorem, we presente a characterization of commutative pro-C*-algebras.

Theorem 2.1. [?] *A commutative unital topological algebra is a pro-C^*-algebra if and only if, it is of the form $C(X)$ where X is a Kelley space and $C(X)$ is the algebra of all continuous fonctions on X. Moreover, a unital pro-C^*-algebra is *-isomorphic to $C(X_{\mathcal{A}})$.*

Like in the category of C*-algebras, the continuous and the holomorphic functional calculus has been well established.

Theorem 2.2. [?] *Let $\mathcal{A}$ be a unital pro-C^*-algebra. For each normal element $a \in \mathcal{A}$, there exists a unique continuous *-homomorphism $f \to f(a)$ from $C(sp_{\mathcal{A}}(a))$ to $\mathcal{A}$, sending the identity function to a.*

Refering to [?] and [?], we also recall the following result.

Theorem 2.3. *Let $\mathcal{A}$ be a unital pro-C^*-algebra. For every element $a \in \mathcal{A}$, there exists a unique continuous *-homomorphism $f \to f(a)$ from the algebra $\mathcal{O}(U)$ of holomorphic function on an open set U containing $sp_{\mathcal{A}}(a)$, to $\mathcal{A}$ sending the identity function to a. This homomorphism satisafies $sp_{\mathcal{A}}(f(a)) = f(sp_{\mathcal{A}}(a))$.*

In general, a *-homomorphism between pro-C*-algebras is not necessarily continuous. In the case of σ-algebras, we have:

Theorem 2.4. [?] *Let $\mathcal{A}$ be a σ-algebra. Then for each pro-C^*-algebra $\mathcal{B}$, every *-homomorphim $\phi : \mathcal{A} \to \mathcal{B}$ is continuous.*

Theorem 2.5. [?] *Let $\phi : \mathcal{A} \to \mathcal{B}$ be a *-homomorphim of pro-C^*-algebra. Then $\phi(b(\mathcal{A})) \subseteq \mathcal{B}$ and hence, $\phi : b(\mathcal{A}) \to b(\mathcal{B})$ is a continuous *-homomorphism.*

With the above notation, recall that $b(\mathcal{A})$ is dense in $\mathcal{A}$ and for every p$\in$ S, $\mathcal{A} \setminus \ker(p)$ is complete and hence, the mappings $\pi_p = \mathcal{A} \to \mathcal{A}_p$ is surjective (for more details, see for example [?] and [?]).

3. Bounded and Spectrally Bounded Elements

We start with some statements that we need to prove the main results. The following lemma is proved in [?] in the case of a countable projective limit of Banach spaces and it holds for the general case.

Lemma 3.1. *Let* $\mathcal{X} = \lim_{\leftarrow}(\mathcal{X}_p, p \in \mathcal{S})$ *be a projective limit of Banach spaces with the canonical morphisms* $\pi_p : \mathcal{X} \to \mathcal{X}_p$*, and assume that* $\mathcal{Y}$ *is a closed subset of* $\mathcal{X}$*. Then,* $\mathcal{Y} = \lim_{\leftarrow}((\overline{\pi_p(\mathcal{Y})}, p \in \mathcal{S})$ *is a projective limit of closed sets* $(\overline{\pi_p(\mathcal{Y})}, \pi_p)$*, where each* $\pi_p : \mathcal{Y} \to \overline{\pi_p(\mathcal{Y})}$ *is the associate canonical morphism.*

Proof. It is clear that $\lim_{\leftarrow}(\overline{\pi_p(\mathcal{Y}_p)},\ \pi_p, p \in \mathcal{S}) = \mathcal{X} \cap (\cap\{\pi_p^{-1}(\mathcal{Y}_p, p \in \mathcal{S}\})$. Note that the family of sets $\mathrm{B} = \{\pi_p^{-1}(V_p) \cap \mathcal{X},\ p \in \mathrm{S}\}$ forms a base for the topology of $\mathcal{X}$ where V_p is an open set of $\mathcal{X}_p$. Let $\mathcal{Y}_p = \overline{\pi_p(\mathcal{Y})}$. It is easy to see that $\mathcal{Y} \subseteq \cap\{\pi_p^{-1}(\mathcal{Y}_p),\ p \in \mathcal{S}\} \cap \mathcal{X}$. To prove the other inclusion, let $x \notin \mathcal{Y}$. Since $\mathcal{X} \setminus \mathcal{Y}$ is open, there exists a neighbourhood $V_p \subseteq \mathcal{X}_p$ of $\pi_p(x)$ such that $\pi_p(V_p) \cap \mathcal{Y} = \emptyset$, and thus, $V_p \cap \pi_p(\mathcal{Y}) = \emptyset$. Then $\pi_p(x) \notin \overline{\pi_p(\mathcal{Y})}$. It follows that $x \notin \cap\{\pi_p^{-1}(\mathcal{Y}_p)\}$. This complete the proof.

As a first consequence, we obtain the following result that is true in the C*-algebras case .

Proposition 3.1. *Let* $(\mathcal{A}, p \in \mathrm{S})$ *be a pro-C*-algebra and* $\mathcal{B}$ *be a closed star subalgebra. Then each element* $a \in B$ *satisfies*

$$sp_{\mathcal{B}}(a) = sp_{\mathcal{A}}(a).$$

Proof. Let $a \in \mathcal{B}$. By Lemma 3.1, we have $\mathcal{B} = \lim_{\leftarrow}(\overline{\pi_p(\mathcal{B})},\ \pi_p)$. Set $\overline{\pi_p(\mathcal{B})} = \mathcal{B}_p$. It is clear that $\mathcal{B}_p$ is a closed star subalgebra of the C*-algebra $\mathcal{A}_p$ and thus, $sp_{\mathcal{B}_p}(\pi(a)) = sp_{\mathcal{A}_p}(\pi_p(a))$. By (4), the result follows.

Remark 3.1. Let $\mathcal{A}$ be a pro-C*-algebra with the identity. It is easy to check that if $a \in b(\mathcal{A})$, then the spectrum $sp_{\mathcal{A}}(a)$ is bounded. The following remark show that the converse is not true. Consider the infinite matrix

$$N = \begin{pmatrix} 0 & 1 & 0 & . & . & . & . \\ 0 & 0 & 2 & 0 & . & . & . \\ . & . & . & . & . & . & \\ . & . & . & 0 & n & 0 & . \\ . & . & . & . & . & . & . \end{pmatrix}$$

Then N is a quasinilpotent element of the pro-C*-algebra $\Pi_{n\geq 1}\mathbb{M}_n(\mathbb{C})$ and

$$\|N\|_\infty \geq \sqrt{\rho(\pi_n(N)\pi_n(N)^*)} \geq (n-1) \quad for \quad all \quad n \geq 1$$

where π_n is the canonical homomorphism from $\Pi_{n\geq 1}\mathbb{M}_n(\mathbb{M})$ to $\mathbb{M}_n(\mathbb{M})$. The algebra $\Pi_{n\geq 1}\mathbb{M}_n(\mathbb{M})$ is a non trivial σ-algebra having many spectrally unbounded elements.

In fact, every spectrally bounded normal element of a pro-C*-algebra $\mathcal{A}$ is in $b(\mathcal{A})$ [**?**] and hence, these two properties are equivalent in the commutative case.

Theorem 3.1. *Let $\mathcal{A}$ be a σ-algebra with the identity. Then, every element a of $\mathcal{A}$ is spectrally bounded, if and only if, $\mathcal{A}$ is topologically *-isomorphic to a C^*-algebra.*

Proof. Assume that $sp_{\mathcal{A}}(a)$ is bounded for all $a \in \mathcal{A}$. Then, all hermitian elements of $\mathcal{A}$ are in $b(\mathcal{A})$ and thus, $\mathcal{A} = b(\mathcal{A})$. Applying Theorem 2.4, the identity map $id_{\mathcal{A}} : \mathcal{A} \to b(\mathcal{A})$ is continuous. Therefore $b(\mathcal{A})$ is *-isomorphic to $\mathcal{A}$. This completes the result.

The following result is a consequence of Theorem 3.1, and it is proved in the general case in [**?**].

Corollary 3.1. *A σ-algebra that is a Q-algebra is trivial.*

Proof. We have only to prove that in a pro-C*-algebra $\mathcal{A}$ which is a Q-algebra, all elements are spectrally bounded. If not, there exists an element $a \in \mathcal{A}$ with $sp_{\mathcal{A}}(a)$ unbounded, and then there exists a sequence $(\lambda_n) \subseteq sp_{\mathcal{A}}(a)$ such that $|\lambda_n| \to +\infty$. Therefore $a/\lambda_n \to 1$. Since $G_{\mathcal{A}}$ is open, $G_{\mathcal{A}} \bigcap \{a/\lambda_n,\ n \in \mathbb{N}\} \neq \{0\}$. It follows that some $1 - \lambda_n$ is invertible. This is a contradiction.

Notice that in the example 3, every element of the algebra is spectrally bounded. By Theorem 3.1, such an algebra is not topologically *-isomorphic to a σ-algebra and by Proposition 1.14 in [**?**]), it is not be a Q-algebra. Examples 1, 2 and 4 are non trivial σ-algebras having at least one spectrally unbounded element.

Remark 3.2. In general, we define the algebra $b(\mathcal{A})$ of a complete locally m-convex algebra $\mathcal{A}$, for each family Γ of submultiplicative seminorms which determines the topology [**?**]. We can write $b_\Gamma(\mathcal{A})$ insteed of $b(\mathcal{A})$. Notice that such a family of seminorms is not unique and if an element a is spectrally unbounded in $\mathcal{A}$, then there exists a collection of seminorms defining the same toplogy for that $a \in b_\Gamma(\mathcal{A})$ [**?**]. By the following proposition, we deduce that $b(\mathcal{A})$ does not depend on the choice of the family of C*-seminorms which defines the topology of $\mathcal{A}$ whenever $\mathcal{A}$ is a pro-C*-algebra.

Proposition 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two pro-C^*-algebras. If $\phi : \mathcal{A} \to \mathcal{B}$ is an injective *-homomorphism from $\mathcal{A}$ into $\mathcal{B}$ with a closed image, then $b(\mathcal{A}) \cong b(\mathcal{B}) \cap \phi(\mathcal{A})$. In particular, if $\mathcal{A}$ is a closed star subalgebra of $\mathcal{B}$, then $b(\mathcal{A}) = b(\mathcal{B}) \cap \mathcal{A}$.*

Proof. Note that $\phi(\mathcal{A})$ is a closed star subalgebra of $\mathcal{B}$ and then it is a pro-C*-algebra. By [**?**], we have $\phi(b(\mathcal{A})) \subseteq b(\phi(\mathcal{A})) \subseteq b(\mathcal{B})$. Since ϕ is injective, we have $sp_{\mathcal{A}}(a) = sp_{\phi(\mathcal{A})}(a),\ a \in \mathcal{A}$. Then, $b(\mathcal{A})$ is algebrcally (and so, topologically) *-isomorphic to $b(\phi(\mathcal{A}))$. It remains to show that $b(\phi(\mathcal{A})) = b(\mathcal{B}) \cap \phi(\mathcal{A})$. Indeed, let $a \in \mathcal{A}$ and assume that $\phi(a) \in b(\mathcal{B})$. Applying Proposition 3.1, we obtain that $sp_{\phi(\mathcal{A})}(\phi(a)) = sp_{\mathcal{B}}(a)$ is bounded. Therefore, the result follows. The second statement holds whenever ϕ is chosen as the identity map.

The action $b \to b(\mathcal{A})$ is a covariant functor from the category of pro-C*-algebras to the category of C*-algebras. Now, we prove the preservation of exactness of some short sequences by this functor. Here, sequences are consider, have the form $0 \to \mathcal{J} \to \mathcal{A} \to \mathcal{B} \to 0$ where $\mathcal{J}$ is a closed two-sided ideal of $\mathcal{A}$.

Proposition 3.3. *Let $\mathcal{J}$ be a closed two-sided ideal of a pro-C*-algebra $\mathcal{A}$, then $b(\mathcal{J})$ is a closed two-sided ideal of $b(\mathcal{A})$. Moreover, if a sequence of the form*

$$0 \to \mathcal{J} \xrightarrow{\alpha} \mathcal{A} \xrightarrow{\beta} \mathcal{B} \to 0$$

*is exact, where α is the identity map and β a surjective *-homomorphism , then the sequence*

$$0 \longrightarrow b(\mathcal{J}) \xrightarrow{\alpha} b(\mathcal{A}) \xrightarrow{\beta} b(\mathcal{B}) \longrightarrow 0$$

is also exact.

Proof. By Proposition 3.2, $b(\mathcal{J}) = b(\mathcal{A}) \cap \mathcal{J}$. In particular, it is a two-sided ideal of $b(\mathcal{A})$. The C*-algebra $b(\mathcal{J})$ embeds into the C*-algebra $b(\mathcal{A})$, and thus it is closed there. It is easily seen that $im(\alpha) \subseteq \ker(\beta)$. To show the other inclusion, let a an element of $\mathcal{A}$ such that $\beta(a) = 0$. Then $a \in \mathcal{J}$, and therefore $a \in B(\mathcal{J}) = b(\mathcal{A}) \cap \mathcal{J}$. It remains to prove that $\beta : b(\mathcal{A} \to b(\mathcal{B}))$ is surjective. Indeed, since $b(\mathcal{A})$ is dense in $\mathcal{A}$, we have:

$$\mathcal{B} = \beta(\mathcal{A}) = \beta(\overline{b(\mathcal{A})}).$$

Note that $\beta(b(\mathcal{A}))$ is closed in $\mathcal{B}$. Then $\overline{\beta(b(\mathcal{A}))} = \beta(b(\mathcal{A}))$ and thus,

$$\beta(\overline{b(\mathcal{A})}) \subseteq \beta(b(\mathcal{A})).$$

Therefore, $\mathcal{B} \subseteq \beta(b(\mathcal{A}))$. Since $\beta(b(\mathcal{A}))$ is a C*-algebra, we obtain $b(\beta(b(\mathcal{A}))) = \beta(b(\mathcal{A}))$. Applying Theorem 2.5 to the identity map $\mathcal{B} \to \beta(b(\mathcal{A}))$, the result follows.

Corollary 3.2. *If p is a continuous C*-seminorm on a pro-C*-algebra $\mathcal{A}$, then*

$$b(\mathcal{A})/b(\ker(p)) \cong \mathcal{A}_p.$$

Proof. Using Remark 2.1 and applying Proposition 3.3 for $\mathcal{J} = \ker(p)$ and $B = \mathcal{A}_p$, we obtain the result.

Remark 3.3. Note that the quotient of a σ-algebra by a closed two-sided ideal is complete and thus, it is a σ-algebra (Corollary 5.4 of [**?**]). In general, the quotient of a complete topological algebra by a closed two-ideal is not necessarily complete. The only example indicated by various authors is well presented in [**?**]. It is a projective limit of Banach spaces having a closed subspace such that the quotient is not complete. By giving this space the zero multiplication, it will be a complete locally m-convexe algebra that is not a pro-C*-algebra. A closed two-sided ideal $\mathcal{I}$ of a non trivial pro-C*-algebra $\mathcal{A}$ such that $\mathcal{A}/\mathcal{I}$ is not complete, has yet to be found.

Corollary 3.3. *Let $\mathcal{J}$ be a closed two-sided ideal of a σ-algebra $\mathcal{A}$, then the sequence*

$$0 \longrightarrow b(\mathcal{J}) \xrightarrow{\alpha} b(\mathcal{A}) \xrightarrow{\beta} b(\mathcal{A}/\mathcal{J}) \longrightarrow 0$$

is exact where α is the identity map and $\beta : \mathcal{A} \to \mathcal{A}_p$ is the canonical projection

Theorem 3.2. *Let $\mathcal{A}$ be a pro-C*-algebra. If $b(\mathcal{A})$ is simple, then $\mathcal{A}$ is a C*-algebra.*

Proof. Let p be a continuous non-zero C*-seminorm on $\mathcal{A}$. As it was shown in Proposition 3.3, $\mathcal{L}$ is a closed two-sided ideal of $b(\mathcal{A})$. Since p is non-zero and $b(\mathcal{A})$ is dense in $\mathcal{A}$, $\mathcal{L} \neq b(\mathcal{A})$. Thus, $\mathcal{L} = 0$, because $b(\mathcal{A})$ is simple. By ([**?**], 1.10) and ([**?**], 3.1), $\mathcal{L}$ is dense in $\ker(p)$, and hence p is a C*-seminorm. The algebra $\mathcal{A}/\ker(p)$ is also complete with respect to p (see 1.12 in [**?**]). Therefore, $(\mathcal{A},\ p)$ is a C*-algebra. By the uniqueness of the C*-norm on an *-algebra if it exsits, the pro-C*-algebra $\mathcal{A}$ is a C*-algebra.

Problem. Let $(\mathcal{A}, \|\ \|)$ be a C*-algebra which is not simple. Is there a non trivial pro-C*-algebra $\mathcal{B}$ such that $\mathcal{A}$ is the C*-algebra of bounded elements of $\mathcal{B}$?

Remark 3.4. Let $\mathcal{A}$ be a simple C*-algebra. Assume that it has a structure of a complete locally m-convexe algebra induced by a family $(p \in \mathrm{S})$ of seminorms such that $p(a) \leq \|a\|$, then $(\mathcal{A}, p \in \mathrm{S})$ is a pro-C*-algebra (see [**?**]) and it is a solution of the problem. In particular, every unital commutative C*-algebra is a solution of the problem. Indeed, it is well known that a unital commutative C*-algebra is *-isomorphic to an algebra of continuous fonctions on a Hausdorff compact set K. Let $C(K)$ be such an algebra and let $C_c(K)$ be the algebra $C(K)$ with the topology of the uniform convergence on countable compact sets of K. By Theorem 3.1 in [**?**], $C_c(K)$ is a pro-C*-algebra. Since the identity map $C(K) \to C_c(K)$

is continuous, $C(K) \subseteq b(C_c(K))$ [?] and then, $C(K) \cong b(C_c(K))$.

4. Unitary Elements in a Pro-C*-algebra

Let $(\mathcal{A},\ p \in \mathrm{S})$ be a pro-C*-algebra with an identity $1_{\mathcal{A}}$ and $\mathcal{A}_{sa}$ be the real space of hermitian elements of $\mathcal{A}$. We denote by $\mathcal{U}(\mathcal{A})$ the group of unitary elements of $\mathcal{A}$, and $\mathcal{U}_0(\mathcal{A})$ be the connected component of the identity in it. Let $a \in \mathcal{A}_{sa}$. By Theorem 2.2, $exp(ia) \in \mathcal{A}$. Moreover, $exp(ia) \in \mathcal{U}(\mathcal{A})$. Set $expi\mathcal{A}_{sa}$ be the set of those elements. Then the subgroup $\mathcal{E}(\mathcal{A}) \subset \mathcal{U}(\mathcal{A})$ generated by the set $expi\mathcal{A}_{sa}$ is non empty.

Remark 4.1 Let u be an element of $\mathcal{U}(\mathcal{A})$ such that $\|1-u\|_\infty < 1$. Then u is the form $expia$, where $a \in \mathcal{A}_{su}$. Indeed, note that $sp_{\mathcal{A}}(u) \subseteq sp_{b(\mathcal{A}})(u) \subseteq \{z \in \mathbb{C} : |z| = 1 and |z-1| < 1\}$. So, there exists a continuous fonction $f(z) = logz = i.arg(z)$ on $sp_{\mathcal{A}}(u)$ such that $expf = id_{sp(u)}$. By Theorem 2.2, $f(u)$ is the form ia with $a \in \mathcal{A}_{sa}$ and thus $u = expf(u) = expia$.

Theorem 4.1. *Let $\mathcal{A}$ be a pro-C*-algebra. Then the subgroup $\mathcal{E}(\mathcal{A})$ is a path-connected component set of $\mathcal{U}(\mathcal{A})$ included in $\mathcal{U}_0(\mathcal{A})$.*

Proof. By Theorem 2.2, $\mathcal{E}(\mathcal{A})$ is not empty. Let $u = exp(ia) \in expi\mathcal{A}_{sa}$. Put $u(t) = exp(ita)$, for all $t \in \mathbb{R}$. We note that $\mathcal{U}(\mathcal{A}) = \mathcal{U}(b(\mathcal{A}))$ as sets and thus $\{u(t),\ t \in \mathbb{R}\} \subseteq \mathcal{U}(b(\mathcal{A}))$. By using the same argument of Theorem 7.3 in [?], $\{u(t),\ t \in \mathbb{R}\}$ is a continuous one-parameter subgroup of $\mathcal{U}(b(\mathcal{A}))$. It follows that $\{u(t),\ t \in \mathbb{R}\} \subseteq \mathcal{E}(b(\mathcal{A}))$ which equal to $\mathcal{U}_0(b(\mathcal{A}))$ and so, the connectedness of the subgroup $\{u(t),\ t \in \mathbb{R}\}$ in $\mathcal{U}(\mathcal{A})$. Therefore $\mathcal{E}(\mathcal{A}) \subset \mathcal{U}(\mathcal{A})$ is a path-connected component set included in $\mathcal{U}_0(\mathcal{A})$.

Remark 4.2 Let $(\mathcal{A}, \|\ \|)$ be a C*-algebra. By Remark 4.1 and by using the same way in the proof of Theorem 7.3 in [?], we obtain that $\mathcal{E}(\mathcal{A})$ is open and closed in $\mathcal{U}(\mathcal{A})$ and so, it is equal to $\mathcal{U}_0(\mathcal{A})$.

Notice that $\mathcal{U}_0(b(\mathcal{A})) \subseteq \mathcal{U}_0(\mathcal{A})$ and the inclusion maybe strict: Since $b(\mathcal{A})$ is a C*-algebra, $\mathcal{E}(b(\mathcal{A})) = \mathcal{U}_0(b(\mathcal{A}))$. One has $\mathcal{E}(b(\mathcal{A})) \subseteq \mathcal{E}(\mathcal{A}) \subseteq \mathcal{U}_0(\mathcal{A})$, so if $\mathcal{U}_0(b(\mathcal{A})) = \mathcal{U}_0(\mathcal{A})$, then $\mathcal{E}(\mathcal{A}) = \mathcal{U}_0(\mathcal{A})$. But this is possible only if $\mathcal{E}(\mathcal{A})$ is closed, while $\mathcal{E}(\mathcal{A})$ need not to be closed (see the example 3.7 in [?]). This also shows that the topologies of $\mathcal{U}_0(\mathcal{A})$ and $\mathcal{U}_0(b(\mathcal{A}))$ may be different.

Lemma 4.1. *If $\theta : \mathcal{A} \longrightarrow \mathcal{B}$ is a *-homomorphism of pro-C^*-algebra that is onto, then $\theta(\mathcal{E}(\mathcal{A})) = \mathcal{E}(\mathcal{B})$.*

Proof. Let $b \in \mathcal{B}$ be a hermitian element. Since θ is a surjective *-homomorphism, there exists a hermitian element $a \in \mathcal{A}$ such that $\theta(a) = b$. Thus, $\theta(expia) = expib$, and since $\mathcal{E}(\mathcal{B})$ is generated by elements of $expi\mathcal{A}_{sa}$, the statement follows.

Remark 4.1. It is easy to check that if $(\mathcal{A},\ p \in \mathrm{S})$ is a pro-C*-algebra, then

$$\mathcal{U}(\mathcal{A}) \cong \lim_{\longleftarrow}(\mathcal{U}(\mathcal{A}_p),\ \pi_p,\ p \in \mathrm{S}).$$

Now we shall use Lemma 3.1 and Lemma 4.1 to establish the following theorem:

Theorem 4.2. *Let $\mathcal{A} \cong \lim_{\longleftarrow}(\mathcal{A}_p, \pi_p, p \in \mathrm{S})$ be a pro-C^*-algebra. Then*

$$\mathcal{U}_0(\mathcal{A}) \cong \lim_{\longleftarrow}(\mathcal{U}_0(\mathcal{A}_p), \pi_p,\ p \in \mathrm{S}).$$

Proof. We note that $\mathcal{U}_0(\mathcal{A})$ and $\mathcal{U}_0(\mathcal{A}_p)$ are closed in $\mathcal{U}(\mathcal{A})$ and $\mathcal{U}(\mathcal{A}_p)$ respectively. By Lemma 3.1, we have

$$\mathcal{U}_0(\mathcal{A}) \cong \lim_{\longleftarrow}(\overline{\pi_p(\mathcal{U}_0(\mathcal{A}_p))}, \pi_p,\ p \in p).$$

Since $\mathcal{A}_p$ is a C*-algebra, $\mathcal{E}(\mathcal{A}_p) = \mathcal{U}_0(\mathcal{A}_p)$ and by Lemma 4.1,

$$\pi_p(\mathcal{E}(\mathcal{A})) = \mathcal{E}(\mathcal{A}_p) = \mathcal{U}_0(\mathcal{A}_p)\ for\ all\ p \in \mathrm{S}. \tag{5}$$

On the other hand, since $\mathcal{U}_0(\mathcal{A}_p)$ is closed in $\mathcal{U}(\mathcal{A}_p)$, we have

$$\overline{\pi_p(\mathcal{E}(\mathcal{A}))} \subseteq \overline{\pi_p(\mathcal{U}_0(\mathcal{A}))} \subseteq \mathcal{U}_0(\mathcal{A}_p). \tag{6}$$

Therefore, all sets in equations (5) and (6) are equal and thus, for each $p \in \mathrm{S}$, we have $\overline{\pi_p(\mathcal{U}_0(\mathcal{A}))} = \mathcal{U}_0\ (\mathcal{A}_p)$.

Corollary 2. *The set $\mathcal{E}(\mathcal{A})$ is dense in $\mathcal{U}_0(\mathcal{A})$.*

Proof. It follows from Lemma 3.1 that $\overline{\mathcal{E}(\mathcal{A})}$ is the projective limit of the projective system $(\overline{\pi_p((\overline{\mathcal{E}(\mathcal{A})})}, \pi_p, p \in \mathcal{S})$. Moreover,

$$\overline{\pi_p(\overline{\mathcal{E}(\mathcal{A})})} \subseteq \overline{\pi_p(\mathcal{E}(\mathcal{A})}). \tag{7}$$

By equations (5), (6) and (7), we obtain $\overline{\pi_p((\overline{\mathcal{E}(\mathcal{A})})} = \mathcal{U}_0(\mathcal{A}_p)$. Applying Thoerem 4.1, we deduce that $\overline{\mathcal{E}(\mathcal{A})} = \mathcal{U}_0(\mathcal{A})$.

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