THE STRUCTURE OF A SUBCLASS OF AMENABLE BANACH ALGEBRAS

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Abstract. We give sufficients conditions that allow contractible (resp. reflexive amenable) Banach algebras to be a finite-dimensional and semisimple algebras. Moreover, we show that any contractible (resp. reflexive amenable) Banach algebra in which every maximal left ideal has a Banach space complement, is indeed a direct sum of finitely many full matrix algebras. Finally, we characterize hermitian *-algebras that are contractible.

1. Introduction and notations.

The purpose of this paper is to establish the structure of some class of amenable Banach algebras. Let $\mathcal{A}$ be a Banach algebra over the complex field $\mathbb{C}$. A Banach $\mathcal{A}$-bimodule $\mathcal{X}$ is a Banach space $\mathcal{X}$, such that $\mathcal{X}$ is an $\mathcal{A}$-bimodule algebrically and the operation

$$\mathcal{A} \times \mathcal{X} \times \mathcal{A} \to \mathcal{X}, \quad (a, x, b) \mapsto axb$$

is jointly continuous, where $\mathcal{A}\times\mathcal{X}\times\mathcal{A}$ carries the cartesian product topology. For each Banach $\mathcal{A}$-bimodule $\mathcal{X}$, the dual $\mathcal{X}^*$ is naturally a Banach $\mathcal{A}$-bimodule with the module actions defined by

$$(a.T)(x) = T(xa), \quad (T.a)(x) = T(ax), \quad for\ all\ a \in \mathcal{A},\ T \in \mathcal{X}^*\ and\ x \in \mathcal{X},$$

where $T(x)$ denotes the evaluation of T at x.

A derivation from $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule $\mathcal{X}$ is a linear operator $d : \mathcal{A} \to \mathcal{X}$ which satisfies $D(ab) = d(a)b + ad(b)$, $\forall a, b \in \mathcal{A}$. Recall that for any $x \in \mathcal{X}$, the mapping $\delta_x : \mathcal{A} \to \mathcal{X}$ defined by $\delta_x(a) = ax - xa,\ a \in \mathcal{A}$, is a continuous derivation, called an inner derivation. A Banach algebra $\mathcal{A}$ is

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said to be contractible if for every Banach $\mathcal{A}$-bimodule $\mathcal{X}$, each continuous derivation from $\mathcal{A}$ into $\mathcal{X}$ is inner. We say that amenable whenever every continuous derivation from $\mathcal{A}$ into $\mathcal{X}^*$ is inner for each Banach $\mathcal{A}$-bimodule $\mathcal{X}$. Obviously, every contractible Banach algebra is amenable. The algebra $\mathbb{M}_n(\mathbb{C})$ of all $n \times n$ complex matrices, is contractible algebra and every direct sum of a certain finite family of full matrix algebras is contractible or amenable.

The facus here is to setlle the following raised in [**?**] and [**?**] and [**?**] :

QUESTION 1. Is every contractible Banach algebra semisimple ?.

QUESTION 2. Is every contractible Banach algebra finite-dimensional?.

QUESTION 3. Is every reflexive amenable Banach algebra finite-dimensional and semisimple?

Recall that a Banach algebra is called a reflexive Banach algebra if it is reflexive as a Banach space. In this paper, we shall presente two situations in which a contractible Banach algebra is finite-dimensional. First, we shall give a partial answer to the questions above where we assume that each maximal left ideal is complemented as a Banach space. This result improves (Proposition IV.4.3, [**?**],) for contractible Banach algebras and (Corollary 2.3, [**?**]) for reflexive amenable Banach algebras where the autors suppose only that all of whose primitive ideals have finite-codimensional. In the seconte time, we will show that a hermitian Banach *-algebra is contractible if, and only if, it is a finite-dimensional semisimple algebra. Finally, we establish some equivalent statements to questions 2 and 3.

2. Preliminaries.

In this second section, we recall some facts about the structure of contractible and amenable Banach algebras. First, consider on $\mathcal{A}^{**}$ the Banach $\mathcal{A}$-bimodule structure defined by $aT = \eta(a)T$ and $Ta = T\eta(a)$ with $\eta : \mathcal{A} \to \mathcal{A}^{**}$ the injective linear map. Notice that if a Banach algebra $\mathcal{A}$ has a bounded approximate identity, then its bidual $\mathcal{A}^{**}$ has an identity. It is a fact that a contractible Banach algebra has an identity and an amenable Banach algebra admits bounded right, left, bilateral approximate identities. Of course, a reflexive amenable Banach algebra must to be unital. Let $\mathcal{A}$

be a Banach algebra over the complex $\mathbb{C}$. We denotes the identity element in a Banach algebra by $1_{\mathcal{A}}$ and we write $\mathcal{A}\hat{\otimes}\mathcal{A}$ for the completed projective tensorial product (see [?]).

For a unital Banach algebra $\mathcal{A}$, a diagonal of $\mathcal{A}$ is an element $d \in \mathcal{A}\hat{\otimes}\mathcal{A}$ such that

$$ad = da, \ \forall a \in \mathcal{A} \quad and \quad \pi(d)) = 1_{\mathcal{A}}$$

where $\pi : \mathcal{A}\hat{\otimes}\mathcal{A} \to \mathcal{A}$ is the canonical Banach $\mathcal{A}$-module morphism. For a such Banach algebra $\mathcal{A}$, a virtual diagonal of $\mathcal{A}$ is an element $d \in \mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that

$$ad = da, \ \forall a \in \mathcal{A} \quad and \quad \pi^{**}(d)) = 1_{\mathcal{A}}$$

where $\pi^{**} : \mathcal{A}\hat{\otimes}(\mathcal{A} \to \mathcal{A})^{**}$ is the bidual Banach $\mathcal{A}$-module morphism of π. In the following theorems, we presente characterization of contractible (resp. amenable) Banach algebra.

Theorem 2.1. [?] *Let $\mathcal{A}$ be a Banach algebra. The following are equivalent:*

(1) *$\mathcal{A}$ is contractible.*
(2) *$\mathcal{A}$ has a diagonal.*

Theorem 2.2. [?] *Let $\mathcal{A}$ be a Banach algebra. The following are equivalent:*

(1) *$\mathcal{A}$ is amenable .*
(2) *$\mathcal{A}$ has a virtual diagonal.*

Next, the following properties state:

Proposition 2.1. *Let $\mathcal{A}$ be a $\langle$ contractible, amenable $\rangle$ Banach algebra. Then:*

If $\theta : \mathcal{A} \longrightarrow \mathcal{B}$ is a continuous homomorphism from $\mathcal{A}$ into another Banach algebra $\mathcal{B}$ with dense range, then $\mathcal{B}$ is $\langle$ contractible, amenable $\rangle$.

In particular, if $\mathcal{I}$ is a closed two-sided ideal of a $\langle$contractible, amenable$\rangle$ Banach algebra, then $\mathcal{A}/\mathcal{I}$ is a $\langle$ contractible, amenable $\rangle$ too.

Proposition 2.2. *Let $\mathcal{A}$ be a contractible or reflexive amenable Banach algebra and assume that $\mathcal{I}$ is a a closed $\langle$ left, two-sided$\rangle$ ideal of $\mathcal{A}$ which has a Banach space complement. Then there exists a closed $\langle$ left, two-sided$\rangle$ ideal $\mathcal{J}$ of $\mathcal{A}$ such that*

$$\mathcal{A} = \mathcal{I} + \mathcal{J}.$$

Notice that for each closed two-sided ideal $\mathcal{I}$ of a reflexive Banach algebra, $\mathcal{I}$ and the quotient $\mathcal{A}/\mathcal{I}$ are reflexive Banach algebra too.

3. Main results.

Theorem 3.1. *Let $\mathcal{A}$ be a contractible or reflexive amenable Banach algebra. Assume that each maximal left ideal of $\mathcal{A}$ is complemented as a Banach space in $\mathcal{A}$. Then, there are $n_1, n_2, \ldots, n_k \in \mathbb{N}$ such that*

$$\mathcal{A} = \mathbb{M}_{n_1}(\mathbb{C}) \oplus \mathbb{M}_{n_2}(\mathbb{C}) \oplus \cdots \oplus \mathbb{M}_{n_k}(\mathbb{C})$$

Proof. By Preliminaries, the algebra $\mathcal{A}$ has an identity $1_{\mathcal{A}}$. Let $(\mathcal{M}_i)_{i\in I}$ be a family of all maximal left ideals. Since $\mathcal{M}_i$ is complemented as Banach space for each i, there exists a left ideal $\mathcal{J}_i$ such that $\mathcal{A} = \mathcal{M}_i \oplus \mathcal{J}_i$. Notice that

$$Rad(\mathcal{A}) = \bigcap_i \mathcal{M}_i$$

is the Jacobson radical of $\mathcal{A}$ and

$$\bigoplus_i \mathcal{J}_i \subseteq Soc(\mathcal{A}) \quad (*)$$

where $Soc(\mathcal{A})$ is the socle of the algebra $\mathcal{A}$, i.e, it is the sum of all minimal left ideals of $\mathcal{A}$ and it coinsides with the sum of all minimal right ideals of $\mathcal{A}$. Recall that every minimal left ideal of $\mathcal{A}$ is of the form $\mathcal{A}e$ where e is a minimal idempotent, i.e, $e^2 = e \neq 0$ and $e\mathcal{A}e = \mathbb{C}e$. On the other hand, for a finite family of minimal idempotents $(e_k)_{k\in K}$, we have

$$\mathcal{A} = \bigoplus_{k\in K} \mathcal{A}e_k \bigoplus \bigcap_{k\in K} \mathcal{A}(1_{\mathcal{A}} - e_k) \quad (**).$$

It follows from (*) and (**) that $Soc(\mathcal{A})$ is dense in $\mathcal{A}/Rad(\mathcal{A})$. This shows that $\mathcal{A}/Rad(\mathcal{A})$ is finite-dimensional. Therefore

$$\mathcal{A} = Rad(\mathcal{A}) \bigoplus Soc(\mathcal{A}).$$

If $Rad(\mathcal{A}) \neq \{0\}$, this would mean that $Rad(\mathcal{A})$ has an identity which i is impossible. So, $\mathcal{A} = Soc(\mathcal{A})$ and then it is a finite direct sum of certain full matrix algebras.

Corollary 3.1. *Let $\mathcal{A}$ be a contractible or reflexive amenable Banach algebra such that every irreducible representation of $\mathcal{A}$ is finite-dimensional. Then $\mathcal{A}$ is finite-dimensional and semisimple.*

Proof. It is easy to check that every primitive ideal of a Banach algebra is finite-codimensional if, and only if, every its maximal left ideal is finite-codimensional. So, the corollary follows.

Now, we give some equivalents statements to the questions 2 and 3 .

Theorem 3.2. *The following statments are equivalent:*

(1) *Every $\langle$ contractible, reflexive amenable $\rangle$ Banach algebra is finite-dimensional and semisimple.*
(2) *Every $\langle$ contractible, reflexive amenable $\rangle$ primitive Banach algebra is finite-dimensional.*
(3) *Every $\langle$ contractible, reflexive amenable $\rangle$ simple contractible Banach algebra is finite-dimentional.*

Proof. It is clear that $1 \Rightarrow 2$ and $2 \Rightarrow 3$. For $3 \Rightarrow 1$, let $\mathcal{A}$ be a contractible Banach algebra. Let $\mathcal{P}$ be a primitive ideal of $\mathcal{A}$. Then the algebra $\mathcal{A}/\mathcal{P}$ is a $\langle$ contractible, reflexive amenable $\rangle$ Banach algebra. Put $\mathcal{B} = \mathcal{A}/\mathcal{P}$ and consider some maximal two-sided ideal $\mathcal{M}$ of $\mathcal{B}$. Since $\mathcal{B}/\mathcal{M}$ is a $\langle$ contractible, reflexive amenable $\rangle$ simple Banach algebra, it is finite-dimensional. There exists then a closed two-sided ideal $\mathcal{J}$ such that $\mathcal{A} = \mathcal{M} \oplus \mathcal{J}$. Recall that in a simple algebra, every nonzero ideal is essential, i.e, it has a nonzero intersection with every nonzero ideal of the algebra. It follow that $\mathcal{M} = 0$ and so, $\mathcal{B}$ is finite dimensional. Using Corollary 3.1, $\mathcal{A}$ must be a finite-dimensinal and semisimple algebra. This complete the proof.

Note that to show that every $\langle$ contractible, reflexive amenable $\rangle$ simple contractible Banach algebra is finite-dimentional, one can not use Thoerem 3.1, because a such algebra does not contain a maximal left ideal complemented as a Banach space. Indeed, if not, the algebra has to have a non trivial minimal left ideal. Since $Soc(\mathcal{A}) = 0$, this is a contradiction.

Finally, assume that $\mathcal{A}$ is a unital Banach *-algebra which admits at least one state τ. Then there exists a *-representation π_τ of $\mathcal{A}$ on a Hilbert space H_τ, with cyclic vector ζ of norm 1 in H_τ such that $\tau(a) =< \pi_\tau(a)\zeta, \zeta >$, for all a $\in \mathcal{A}$.

Theorem 3.3. *A hermitian Banach *-algebra $\mathcal{A}$ is contractible if, and only if, there are $n_1, n_2, \ldots, n_k \in \mathbb{N}$ such that*

$$\mathcal{A} \cong \mathbb{M}_{n_1}(\mathbb{C}) \oplus \mathbb{M}_{n_2} \oplus \mathbb{M}_{n_2}(\mathbb{C}) \oplus \mathbb{M}_{n_k}(\mathbb{C}).$$

Proof. Let $\mathrm{T}(\mathcal{A})$ be the set of all states of $\mathcal{A}$ and let $R^*(\mathcal{A})$ be the *-radical of $\mathcal{A}$, i.e, the intersection of the kernels of all *-representations of $\mathcal{A}$ on Hilbert spaces. Since $\mathcal{A}$ is hermitian and has an identity, $\mathrm{T}(\mathcal{A}) \neq \emptyset$ and so $R^*(\mathcal{A}) \neq \mathcal{A}$. Put

$$\pi = \bigoplus_{\tau \in \mathrm{T}(\mathcal{A})} \pi_\tau \quad \text{and} \quad H = \bigoplus_{\tau \in \mathrm{T}(\mathcal{A})} H_\tau.$$

Then π is a *-representation of $\mathcal{A}$ on H. Consider

$$\|\pi(a)\| = \sup_{\mathrm{T}(\mathcal{A})} \{\pi_\tau(a)\}.$$

Then $\| \ \|$ is a C*-norm on $\pi(A)$. Let $\mathcal{B}$ denote the cloture of $(\pi(\mathcal{A}), \| \ \|)$. Moreover, $\pi : \mathcal{A} \to \mathcal{B}$ is continuous mapping into C*-algebra $\mathcal{B}$ such that $\ker(\pi(\mathcal{A})) = R^*(\mathcal{A})$. As $\mathcal{A}$ is contractible, $\mathcal{B}$ is also contractible. Using [**?**], the algebra $\mathcal{B}$ has to be finite dimensional. Notice that $\mathcal{A}/R^*(\mathcal{A})$ is isometric with the *-subalgebra $\pi(\mathcal{A})$ of $\mathcal{B}$. Thus, it follows that $\mathcal{A}/R^*(\mathcal{A})$ finite-dimensional. Since $R^*(\mathcal{A})$ is a finite-codimensional closed two-sided *-ideal, there exists a closed two-sided ideal $\mathcal{K}$ such that

$$\mathcal{A} = R^*(\mathcal{A}) \oplus \mathcal{K}.$$

Next, note that $\|\pi(a)\|^2 = sup\{\tau(a^*a), \quad \tau \in \mathrm{T}(\mathcal{A})\} \leq |a^*a|_\sigma$ where $|a|_\sigma$ is the spectral radius of $a \in \mathcal{A}$. By Ptak [**?**], we obtain $\|\pi(a)\|^2 \geq |a|_\sigma^2$. So, if $a \in R^*(\mathcal{A})$, then $|a|_\sigma = 0$. Therefore every element of $R^*(\mathcal{A})$ is quasinilpotent. Notice that in general $Rad(\mathcal{A}) \subseteq R^*(\mathcal{A})$. Since $R^*(\mathcal{A})$ is a closed two-sided *-ideal, we have $R^*(\mathcal{A}) = Rad(\mathcal{A})$ and so, $\mathcal{A}$ is finite-dimensinal and semisimple.

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