POSITIVE QUATERNION-KÄHLER 16-MANIFOLDS WITH $b_2 = 0$

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ABSTRACT. It is known that if the second Betti number of a $4n$-dimensional positive quaterion-Kähler manifold M satisfies $b_2(M) \geq 1$ then M is homothetic to the complex Grassmannian $\mathbb{G}r_2(\mathbb{C}^{n+2})$. In 16 dimensions, it is also known that if $b_4(M) = 1$ then M is homothetic to the quaternionic projective space $\mathbb{HP}^4$.

Here, we prove that the fourth Betti number of a positive quaternion-Kähler 16-manifold is greater than or equal to 3, if $b_2(M) = 0$. We also explore the consequences of other restrictions suggested by the vanishing of certain indices of twisted Dirac operators on the Wolf spaces.

1. INTRODUCTION

A quaternion-Kähler $4n$-dimensional manifold is an irreducible Riemannian manifold whose holonomy group is contained in $Sp(n)Sp(1)$ for $n \geq 2$, and self-dual Einstein when $n = 1$. We shall call a quaternion-Kähler manifold *positive* if its metric is complete and of positive scalar curvature. The only known examples are the Wolf spaces [15]: each compact centerless Lie group G is the isometry group of a quaternion-Kähler symmetric space given as the conjugacy class of a copy of $Sp(1)$ in G, determined by a highest root of G. More precisely, they are

$$\mathbb{HP}^n = \frac{Sp(n+1)}{Sp(n) \times Sp(1)}, \ \mathbb{G}r_2(\mathbb{C}^{n+2}) = \frac{SU(n+2)}{S(U(n) \times U(2))},$$

$$\mathbb{G}r_4(\mathbb{R}^{n+4}) = \frac{SO(n+4)}{SO(n) \times SO(4)},$$

$$\frac{G_2}{SO(4)}, \quad \frac{F_4}{Sp(3)Sp(1)}, \quad \frac{E_6}{SU(6)Sp(1)}, \quad \frac{E_7}{Spin(12)Sp(1)} \quad \text{and} \quad \frac{E_8}{E_7 Sp(1)}.$$

Positive quaternion-Kähler manifolds are conjectured to be symmetric spaces, which has been proved in dimensions 4, 8 and 12 in [6], [10] and [5]

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respectively. Here, we address this problem in dimension 16 under the topological restriction $b_2(M) = 0$.

Theorem 1. *Let M be a 16-dimensional positive quaternion-Kähler manifold. If $b_2(M) = 0$ and $M \not\cong \mathbb{HP}^4$, then $b_4(M) \geq 3$.*

Within the proof of Theorem 1, various indices of Dirac operators with coefficients in quaternionic vector bundles become relevant. Although not all the values of the indices are known, by using only their integrality we can prove Theorem 1. Therefore, we compute their value on the Wolf spaces and find that they vanish. Assuming such vanishings in 16 dimensions enables us to prove the aforementioned conjecture.

The note is organized as follows. In Section 2 we give some preliminaries in quaternion-Kähler geometry. In Section 3 we prove Theorem 1. In Section 4 we assume the vanishing of certain indices which, if true in general, allow further progress in the classification in 16 dimensions. We include an Appendix where we prove the vanishing of the relevant indices on each one of the Wolf spaces.

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2. Preliminaries on quaternion-Kähler manifolds

The group $Sp(4)Sp(1)$ is the quotient of $Sp(4) \times Sp(1)$ by its center $\mathbb{Z}_2 = \{\pm 1\}$, where $Sp(4)$ acts on $\mathbb{R}^{16} \cong \mathbb{H}^4$ by matrix multiplication on the left and $q \in Sp(1)$ acts by multiplication with q^{-1} on the right. The quaternionic structure given by $Sp(4)Sp(1)$ shows up in the expression of the complexified tangent bundle of M as

$$TM_c = E \otimes H, \qquad (1)$$

where E and H are locally defined vector bundles whose fibres $\mathbb{C}^8$ and $\mathbb{C}^2$ are the standard representations of $Sp(4)$ and $Sp(1) = SU(2)$ respectively. In this dimension, all quaternion-Kähler manifolds are spin [11] so that there is a Dirac operator $\partial\!\!\!/$ and twisted versions of it with coefficients in the vector bundles

$$R^{p,q} = \textstyle\bigwedge_0^p E \otimes S^q H,$$

where $\bigwedge_0^p E$ is the primitive part (with respect to an invariant 2-form) of the p-th exterior power of E and $S^q H$ is the q-th symmetric power of H. In fact, $p + q$ must be even to ensure that these operator are globally defined so that their indices are given by the Atiyah-Singer theorem

$$i^{p,q}(M) := \mathrm{ind}(\partial\!\!\!/ \otimes R^{p,q}) = \langle \widehat{A}(M) \cdot \mathrm{ch}(\textstyle\bigwedge_0^p E) \cdot \mathrm{ch}(S^q H), [M] \rangle,$$

where ch denotes the Chern character of a vector bundle [11, 9]. We quote the following theorem in a version particular to 16 dimensions.

Theorem 2. [11, 9] *Let M be a 16-dimensional positive quaternion-Kähler manifold. If $p+q$ is even,*

$$i^{p,q}(M) := \begin{cases} 0 & \text{if } p+q < 4, \\ (-1)^p (b_{2p-2} + b_{2p}) & \text{if } p+q = 4. \end{cases}$$

Furthermore, the dimension $d = \dim(\mathrm{Isom}(M))$ of the isometry group of M equals $\mathrm{ind}(\partial\!\!\!/ \otimes R^{0,6})$. □

The twistor space Z of a quaternion-Kähler manifold M is the projectivization of H, $Z = \mathbb{P}(H)$, which is globally defined and the projection $\pi \colon Z \longrightarrow M$ has fibre $\mathbb{CP}^1$. In fact, the tiwstor space Z of a positive quaternion-Kähler manifold admits a natural complex structure which is Kähler. The Levi-Civita connection on M determines a horizontal holomorphic distribution $\mathcal{D}$ giving the following short exact sequence

$$0 \longrightarrow \mathcal{D} \longrightarrow T^{1,0}Z \longrightarrow L \longrightarrow 0,$$

which is a complex contact structure and L is the quotient line bundle. In the case of positive scalar curvature, the line bundle L is positive, thus making Z into a Fano manifold that can be embedded into a complex projective space by some power of L. Furthermore

$$\mathcal{D} \cong \pi^* E \otimes L^{1/2} \qquad \text{and} \qquad \pi^* H \cong L^{1/2} \oplus L^{-1/2}.$$

Remarkably, the cohomology of the Dirac operators on M coupled to $S^q H$ is isomorphic to that of the basic Dolbeaut complex on Z coupled to the line bundle $L^{(q-n)/2}$, so that

$$\mathrm{ind}(\partial\!\!\!/ \otimes S^q H) = \chi(Z, \mathcal{O}(L^{(q-n)/2})),$$

where $q-n$ is even [12]. These complexes can be tensored with quaternionic bundles, such as E or its powers to give further identities. For instance [9],

$$\mathrm{ind}(\partial\!\!\!/ \otimes \textstyle\bigwedge^p E \otimes S^q H) = \chi(Z, \mathcal{O}(\bigwedge^p \mathcal{D}^* \otimes L^k)), \tag{2}$$

with $k = (p+q-n)/2$, or equivalently

$$\mathrm{ind}(\partial\!\!\!/ \otimes \textstyle\bigwedge^p E \otimes S^q H) = \chi(Z, \mathcal{O}(\bigwedge^p E \otimes L^{(q-n)/2})), \tag{3}$$

with $p+q-n$ even, where we have dropped π^* from the notation. This is the content of the twistor transform [11].

Lemma 1. (cf. [2]) *Let M be a compact connected quaternion-Kähler $4n$-manifold of positive scalar curvature. The symmetric bilinear form Q on $H^4(M)$ defined by*

$$Q(\alpha,\beta)=\int_M \alpha\wedge\beta\wedge(4u)^{n-2},$$

α, $\beta\in H^4(M)$, is positive definite. □

Let us recall some relevant facts about positive quaternion-Kähler manifolds [14], which explain the hypotheses in Theorem 1. Every positive quaternion-Kähler manifold M is simply connected, and its odd Betti numbers vanish [11]. Therefore, the Euler characteristic of M is positive. It is known [9] that if $b_2(M)\geq 1$, then M is homothetic to the complex Grassmannian $\mathbb{G}r_2(\mathbb{C}^{n+2})$. Therefore, we can assume $b_2(M)=0$. It is also known [8] that $b_4(M)\geq 1$, due to the existence of a non-degenerate closed 4-form which encodes the quaternionic structure of the manifold. In [4], it was proved that if M is 16-dimensional with $b_2(M)=0$ and $b_4(M)=1$, then M is homothetic to the quaternionic projective space $\mathbb{HP}^4$.

3. Proof of Theorem 1

For simplicity, we will denote the Betti numbers $b_i(M)$ by b_i. As mentioned in Section 2, we can assume $b_2=0$ and $b_4>1$. Thus, we shall assume

$$b_2=0,\quad b_4=2$$

and show that this leads to a contradiction.

The following identity was proved in [9]

$$-1+3b_2+3b_4-b_6=2b_8, \tag{4}$$

for a general positive quaternion-Kähler 16-manifold. In our case it implies

$$b_6=1,\quad b_8=2,$$

and the Euler characteristic

$$\chi(M)=10.$$

In view of Theorem 2 and (1), we can study the topology of M by means of the characteristic numbers given by the characteristic classes of both E and H. In practice, this means that we can write down such indices as linear combinations of characteristic numbers.

Let $c_i=c_i(E)$ denote the Chern classes of E and $u=-c_2(H)$. Since $E\cong E^*$ and $H\cong H^*$, the odd Chern classes vanish. In fact, the class $4u$ is integral and corresponds to the aforementioned quaternionic non-degenerate closed 4-form, so let us define the *quaternionic volume* of M by

$$\mathbf{v} = \int_M (4u)^4.$$

We can combine the quaternionic bundles $R^{p,q}$ in the following (virtual) fashion

$$\begin{aligned}
V_1 &= (\textstyle\bigwedge_0^2 E - 3EH + 3S^2H + 3)(S^2H - 3),\\
&= R^{2,2} - 3R^{1,3} + 6R^{0,4} - 3R^{2,0} + 6R^{1,1} - 9R^{0,2} - 3,\\
V_2 &= (S^2H - 3)^2 = R^{0,4} - 5R^{0,2} + 10,\\
V_3 &= (S^2H - 3)^3 = R^{0,6} - 7R^{0,4} + 21R^{0,2} - 35,\\
V_4 &= (E - 4H)(S^2H - 3) = R^{1,3} - 4R^{0,4} - 2R^{1,1} + 4R^{0,2} + 8,\\
V_5 &= (\textstyle\bigwedge_0^2 E - 27)(S^2H - 3)^2\\
&= R^{2,4} - 5R^{2,2} + 10R^{2,0} - 27R^{0,4} + 135R^{0,2} - 270,
\end{aligned}$$

which give the following equations on characteristic numbers

$$\begin{aligned}
4u^4 + 3c_2u^3 + 2c_4u^2 + c_6u = (b_2 + b_4) + 3(b_0 + b_2) + 6b_0 &= 11, && (5)\\
\frac{1}{45}(26u^4 + 17c_2u^3 + 3c_2^2u^2 - c_4u^2) = b_0 &= 1, && (6)\\
\frac{16}{3}(8u^4 + c_2u^3) &= d - 7, && (7)\\
-\frac{1}{90}(224u^4 + 152c_2u^3 + 9c_2^2u^2 + 56c_4u^2 + 3c_4c_2u + 6c_6u) &= \\
-(b_0 + b_2) - 4b_0 &= -5, && (8)\\
-32c_2u^3 = \operatorname{ind}(\partial\!\!\!/ \otimes R^{2,4}) - 5(b_2 + b_4) - 27b_0 &= i^{2,4}(M) - 37 && (9)
\end{aligned}$$

where we have omitted the integral sign $\int_M$. From (9) we see that

$$i^{2,4}(M) = -32c_2u^3 + 37$$

and from (7)

$$c_2u^3 = -8u^4 - \frac{21}{16} + \frac{3}{16}d.$$

By [11, Lemma 7.6]

$$c_2u^3 \geq u^4 > 0,$$

so that

$$79 - 6d < i^{2,4}(M) < 37.$$

Claim 1. *c_2 is not proportional to u.*
If they were, $M \cong \mathbb{HP}^4$ by [4].

Claim 2. *Since $b_8 = 2$, c_2u and u^2 are linearly independent.*

Suppose that they are proportional, $c_2u = \lambda u^2$ for some rational number λ. This implies

$$\begin{aligned} c_2u^3 &= \lambda u^4, \\ c_2^2u^2 &= \lambda c_2 u^3, \\ c_2^3 u &= \lambda c_2^2 u^2, \\ c_4c_2u &= \lambda c_4 u^2. \end{aligned}$$

By substituting in the identities of Proposition 2 and solving the system of equations

$$d = \frac{14141}{105} - \frac{9476}{105}\lambda + \frac{5647}{560}\lambda^2 + \frac{153}{112}\lambda^3$$

where λ is a root of

$$4032 - 2176x - 2224x^2 + 431x^3 + 45x^4 = 0.$$

The last polynomial, however, has no rational roots. Hence, there are no integral solutions for d.

Claim 3. $b_4 = 2$ *cannot occur.*

Since c_2u and u^2 are independent, $b_4 = 2$ implies

$$c_4 = ac_2u + bu^2,$$

for some rational numbers a, b. This implies

$$\begin{aligned} c_4u^2 &= au^4 + bc_2u^3, \\ c_4c_2u &= ac_2u^3 + bc_2^2u^2, \\ c_4c_2^2 &= ac_2^2u^2 + bc_2^3u, \\ c_4^2 &= ac_4u^2 + bc_4c_2u. \end{aligned}$$

By substituting these identities in Proposition 2 and solving the system of equations we get certain equations involving d, $i^{2,4}$ and b. For instance, if $d = 8$,

$$32 \le i^{2,4}(M) \le 36,$$

so that b is a root of

$$\begin{aligned} 10261x^3 + 3315269x^2 + 87765959x - 495250145 &= 0, && \text{if } i^{2,4} = 32, \\ 1241x^3 + 362529x^2 + 2293059x - 20265725 &= 0, && \text{if } i^{2,4} = 33, \\ 5543x^3 + 1549911x^2 + 2343565x - 52897483 &= 0, && \text{if } i^{2,4} = 34, \\ 551x^3 + 148909x^2 - 109823x - 3627637 &= 0, && \text{if } i^{2,4} = 35, \\ 30555x^3 + 8011243x^2 - 16804727x - 150381839 &= 0, && \text{if } i^{2,4} = 36. \end{aligned}$$

None of these polynomials, however, have rational roots. The same happens for all the other possible values of d in Claim 3, and the corresponding finite number of values $79 - 6d < i^{2,4}(M) < 37$. □

4. Further restrictions

From [11, 9] we can see that for the indices $i^{p,q}(M)$, the sum $p+q$ accounts for some kind of level with respect to the given quaternionic dimension n. The vanishings of Theorem 2 can be rephrased as saying that the indices $i^{p,q}(M)$ vanish below level $p+q=n$. In this context, the index $i^{2,n}(M)$ in a given quaternionic dimension n is the analogue of the index $i^{2,4}(M)$ in 16 dimensions.

In Section 3 we saw that the integrality of the index $i^{2,4}$ played an important role in proving Theorem 1. Hence, it makes sense to compute the value of the analogous index on each one of the Wolf spaces. We give the proof of the next proposition in the Appendix.

Proposition 1. *Let M be a $4n$-dimensional Wolf space.*

(1) *If $M \ncong \mathbb{HP}^n$ then*

$$i^{1,n+1}(M) = 0.$$

(2) *If $M \ncong G_2/SO(4)$ then*

$$i^{2,n}(M) = 0.$$

From now on, we will study the consequences of assuming further index vanishings as suggested by the Wolf spaces.

4.1. **Assume** $i^{2,4}(M) = 0$**.** If $i^{2,4}(M) = 0$ in 16 dimensions, we see from (9) that $c_2u^3 = (27+5b_4)/32$ so that

$$d = \frac{128}{3}u^4 + \frac{23}{2} + \frac{5}{6}b_4 = \frac{1}{6}\mathbf{v} + \frac{23}{2} + \frac{5}{6}b_4, \tag{10}$$

which gives

$$d \geq 15,$$

since $b_2 = 0$ and $b_4 \geq 3$.

Corollary 1. *Let M be a 16-dimensional positive quaternion-Kähler manifold with $b_2 = 0$ and such that $i^{2,4}(M) = 0$. Then the dimension d of the isometry group belongs to the following list*

$$d = 15,\ 16,\ 18,\ 20,\ 21,\ 22,\ 24,\ \mathbf{28},\ 36. \tag{11}$$

The list follows from checking the possible Lie groups that have dimension greater than 7 and smaller than 56, with rank smaller than 5 which is the bound given by Bileawski [1] . □

The total deficiency

$$\Delta = 2n + 1 + 2\mathbf{v} - d \geq 0$$

of the twistor space of a $4n$-dimensional positive quaternion-Kähler manifold is non-negative [3], which implies in 16 dimensions

$$d \leq 9 + 2\mathbf{v}.$$

so that the cases with $\mathbf{v} = 1,\ 2$ cannot occur.

Corollary 2. *Let M be a 16-dimensional positive quaternion-Kähler manifold with $b_2 = 0$ and such that $i^{2,4}(M) = 0$. Then*

$$d \leq 9 + 2\mathbf{v} \qquad \textit{and} \qquad b_4 \leq \frac{11}{5}\mathbf{v} - 3.$$

□

For instance, let us consider the case $d = 15$. The only possibilities are either

$$b4 = 3, \quad \mathbf{v} = 6, \qquad \text{or} \qquad b4 = 4, \quad \mathbf{v} = 1.$$

By Corollary 2, $\mathbf{v} = 1$ cannot occur.

Corollary 3. *Let M be a 16-dimensional positive quaternion-Kähler manifold with a 15-dimensional group of isometries, $b_2 = 0$ and such that $i^{2,4}(M) = 0$. Then $b_4 = 3$ and either*

$$b_6 = 0, \quad b_8 = 4, \quad \chi(M) = 12, \qquad \textit{or} \qquad b_6 = 2, \quad b_8 = 3, \quad \chi(M) = 15.$$

□

4.2. **Assume $i^{1,5}(M) = 0$ and $i^{2,4}(M) = 0$.** If we also assume the vanishing of $i^{1,5}$, which is expected from the calculations on the Wolf spaces, we can classify the positive quaternion-Kähler 16-manifolds.

Theorem 3. *Let M be a positive quaternion-Kähler 16-manifold. If*

$$i^{1,5}(M) = 0, \quad \textit{and} \quad i^{2,4}(M) = 0,$$

then M is symmetric, i.e.

$$M \cong \mathbb{G}r_2(\mathbb{C}^6),\ \mathbb{G}r_4(\mathbb{R}^8).$$

Proof. We know that $M \ncong \mathbb{HP}^4$ since $i^{1,5}(\mathbb{HP}^4) = 44 \neq 0$. We can assume $M \ncong \mathbb{G}r_2(\mathbb{C}^6)$, so that $b_2 = 0$ and many our previous considerations apply.

Together with the identities in Section 3, the two extra vanishings give the following

$$\begin{aligned} u^4 &= -\frac{69}{256} - \frac{5}{256}b_4 + \frac{3}{128}d, \\ c_2u^3 &= \frac{27}{32} + \frac{5}{32}b_4 \\ c_2^2u^2 &= \frac{185}{16} - \frac{15}{16}b_4 - \frac{1}{8}d. \end{aligned}$$

Given that the quadratic form Q on $H^4(M)$ of Lemma 1 is positive definite, the discriminant of the quadratic polynomial $Q(u + \lambda c_2, u + \lambda c_2)$ is non-positive so that

$$(c_2u^3)^2 - c_2^2u^2 \cdot u^4 \leq 0,$$

i.e. in terms of b_4 and d

$$\frac{15681}{4096} + \frac{485}{2048}b_4 + \frac{25}{4096}b_4^2 - \frac{39}{128}d + \frac{5}{256}b_4d + \frac{3}{1024}d^2 \leq 0,$$

which together with (11) force

$$b_4 = 3 \quad \text{and} \quad d = 28,\ 36.$$

The reductive Lie algebras of dimension 28 are $\mathfrak{so}(4)$ and $\mathfrak{g}_2 \oplus \mathfrak{g}_2$, and those of dimension 36 are $\mathfrak{sp}(4)$ and $\mathfrak{so}(9)$. In fact, by (10), the possible pairs $(d, \mathbf{v})$ are $(28, 84)$ and $(36, 132)$.

Let K denote the isotropy group at any point. By [11], the curvature tensor R of M splits as $R = tR_0 + R_1$, where R_0 is the curvature tensor of quaternionic projective space, t is the scalar curvature, and R_1 is a section of S^4E. In order to measure the covariant derivative ∇R we must look at ∇R_1 as a section of $S^5E \otimes H$ invariant under K. Consider the map given by the composition of the isotropy representation and projection onto the second factor

$$\mathrm{Lie}(K) \longrightarrow \mathfrak{sp}(4) \oplus \mathfrak{sp}(1) \longrightarrow \mathfrak{sp}(1).$$

There are two cases: either the map is surjective or it is identically zero. By checking the possible groups K in both cases, we see that either they have no invariants in $S^5E \otimes H$ so that $\nabla R_1 = 0$, or the existence of such orbit spaces as submanifolds make M symmetric. □

Appendix

In this appendix we compute the value of the indices $i^{1,n+1}(M)$ and $i^{2,n}(M)$. Let us recall the description of the Wolf spaces [15].

Let G and $\mathfrak{g}$ be a compact simple Lie group and its Lie algebra respectively. Let $\mathfrak{H} \subset \mathfrak{g}_c$ be a Cartan sub-algebra and let us choose an order on it. Let R be the set of roots of $\mathfrak{g}$ and ρ the maximal root with respect to the order on $\mathfrak{H}$. Define the following sub-algebras:

$$\mathfrak{K}_0 = \mathrm{span}(\rho) \oplus \mathfrak{g}_\rho \oplus \mathfrak{g}_{-\rho} \cong \mathfrak{sp}(1), \qquad \mathfrak{K}_1 = \mathfrak{H} \oplus \sum_{\langle \alpha,\rho\rangle=0} \mathfrak{g}_\alpha,$$

where $\langle , \rangle$ is the Killing form on $\mathfrak{g}_c$ and $\mathfrak{g}_\alpha \subset \mathfrak{g}_c$ is the weight space of the root $\alpha \in \mathfrak{H}$. Then $\mathfrak{K}_0 \oplus \mathfrak{K}_1$ is a parabolic sub-algebra of $\mathfrak{g}_c$ and the corresponding real form $\mathfrak{K} = \mathfrak{g} \cap (\mathfrak{K}_0 \oplus \mathfrak{K}_1)$ is the Lie algebra of the centraliser $K = K_1 Sp(1)$ of a copy of $Sp(1)$ in G. Thus $M = G/K$ is a quaternion-Kähler symmetric space. The twistor space is $Z = G/(K_1 U(1))$, where the Lie algebra $\mathfrak{u}(1)$ of $U(1)$ is generated by the maximal root ρ.

$\dim(M)/4$	K_1	G
n	$Sp(n)$	$Sp(n+1)/\mathbb{Z}_2$
n	$U(n)$	$SU(n+2)/\mathbb{Z}_{n+2}$
n	$SO(n) \times SU(2)$	$SO(n+4)[/\mathbb{Z}_2$ for n even$]$
2	$SU(2)$	G_2
7	$Sp(3)$	F_4
10	$SU(6)$	$E_6/\mathbb{Z}_3$
16	$Spin(12)$	E_7
28	E_7	E_8

Table I

By twistor transform (2), the indices

$$i^{p,n+2-p}(M) = \chi(Z, \mathcal{O}(L^{2-p/2} \otimes \textstyle\bigwedge_0^p E)),$$

so that we must compute the dimensions of the cohomology groups

$$H^i(Z, \mathcal{O}(L^{2-p/2} \otimes \textstyle\bigwedge_0^p E)) \cong H^{0,i}(Z, L^{2-p/2} \otimes \textstyle\bigwedge_0^p E) \tag{12}$$

for $p = 1$, 2. By the Bott-Borel-Weil Theorem [7, Theorem 5] at most one of these groups is non-zero. Our task is to prove that, in fact, all of them are zero.

The line bundle L over the twistor space Z is the homogeneous line bundle given by the representation of the torus $U(1)$ generated by the maximal root ρ, i.e. $L = V(\rho)$, where $V(\lambda)$ denotes both the irreducible representation of K with highest weight λ and the associated vector bundle on G/K. The pull-back of the vector bundle E to Z will correspond to an irreducible representation of K_1 with highest weight denoted by γ. Once the weight γ is found for each space, the proof is completed by checking that the highest weight of $L^{2-p/2} \otimes \bigwedge_0^p E$ is orthogonal to a positive root of G. Let $\{e_i\}_{i=1,\dots,N}$ denote the standard basis of $\mathbb{R}^N$.

Type A. Let $n \geq 2$ and

$$\begin{aligned}
\mathfrak{H} &= \operatorname{span}(\{\alpha_j = e_j - e_{j+1},\, j = 1, \ldots, n+1\}) \subset \mathfrak{su}(n+2), \\
\rho &= e_1 - e_{n+2}, \\
\mathfrak{K}_1 &\cong \mathfrak{u}(n), \\
E &= \mathbb{C}^n + \overline{\mathbb{C}^n} \\
\textstyle\bigwedge_0^p E &= \bigoplus_{j=0}^{[p/2]} \left[V(\lambda_{p-j} + s(\lambda_j) + (p-j)\beta) \oplus V(s(\lambda_{p-j}) + \lambda_j - (p-j)\beta)) \right],
\end{aligned}$$

where $\beta = 1/(n+2)\sum_{j=1}^{n+2} e_j - \frac{e_1 + e_{n+2}}{2}$, $\lambda_j = e_2 + \ldots + e_{j+1}$ and $s(x_1, x_2, \ldots, x_{n+2}) = -(x_{n+2}, x_{n+1}, \ldots, x_1)$.

Type B. Let $n = 2m + 1 \geq 3$ and

$$\begin{aligned}
\mathfrak{H} &= \operatorname{span}(\{\alpha_j = e_j - e_{j+1},\, \alpha_{m+2} = e_{m+2},\, j = 1, \ldots, m+1\}), \\
\rho &= e_1 + e_2, \\
\mathfrak{K}_1 &\cong \mathfrak{so}(2m+1) \oplus \mathfrak{su}(2), \\
E &= \mathbb{C}^n \otimes \mathbb{C}^2 \\
\textstyle\bigwedge_0^p E &= \bigoplus_{j=0}^{[p/2]} V((p-2j)(e_1 - e_2) + \lambda_{p-j} + \lambda_j),
\end{aligned}$$

where $\lambda_j = e_3 + \ldots + e_{j+2}$.

Type C. Let $n \geq 2$ and

$$\begin{aligned}
\mathfrak{H} &= \operatorname{span}(\{\alpha_j = e_j - e_{j+1}, \alpha_{n+1} = 2e_{n+1},\, j = 1, \ldots, n\}), \\
\rho &= 2e_1, \\
\mathfrak{K}_1 &\cong \mathfrak{sp}(n), \\
E &= \mathbb{C}^{2n} \\
\textstyle\bigwedge_0^p E &= V(e_2 + \ldots + e_{p+1}).
\end{aligned}$$

In this case $i^{1,n+1} = n(2n+3) \neq 0$.

Type D. Let $n = 2m \geq 4$ and

$$\begin{aligned}
\mathfrak{H} &= \mathrm{span}(\{\alpha_j = e_j - e_{j+1}, \alpha_{m+2} = e_{m+1} + e_{m+2}, j = 1, \ldots, m+1\}),\\
\rho &= e_1 + e_2,\\
\mathfrak{K}_1 &\cong \mathfrak{so}(2m) \oplus \mathfrak{su}(2),\\
E &= \mathbb{C}^n \otimes \mathbb{C}^2\\
\textstyle\bigwedge_0^p E &= \bigoplus_{j=0}^{[p/2]} V((p-2j)(e_1 - e_2) + \lambda_{p-j} + \lambda_j),
\end{aligned}$$

where $\lambda_j = e_3 + \ldots + e_{j+2}$.

Type E_6. For $E_6/(SU(6)Sp(1))$, $n = 10$. Let $\mathfrak{H} \cong \mathbb{R}^6$ be the Cartan sub-algebra spanned by the canonical basis. Since

$$\mathfrak{e}_6 = \mathfrak{sp}(1) \oplus \mathfrak{su}(6) \oplus \textstyle\bigwedge^3 \mathbb{C}^6 \otimes H,$$

the 20-dimensional complex vector space $E = \bigwedge^3 \mathbb{C}^6 = V(e_3)$ and $\bigwedge_0^2 E = V(e_2 + e_4)$.

Type E_7. For $E_7/(Spin(12)Sp(1))$, $n = 16$. Let $\mathfrak{H} \cong \mathbb{R}^7$ be the Cartan sub-algebra spanned by the canonical basis.

$$\mathfrak{e}_7 = \mathfrak{sp}(1) \oplus \mathfrak{so}(12) \oplus \Delta_{12}^+ \otimes H,$$

where Δ_{12}^+ is the positive half-spin representation. Thus, $E = \Delta_{12}^+ = V(e_5)$ and $\bigwedge_0^2 E = V(e_4)$.

Type E_8. For $E_8/(E_7 Sp(1))$, $n = 28$. Let $\mathfrak{H} \cong \mathbb{R}^8$ be the Cartan sub-algebra spanned by the canonical basis. Since

$$\mathfrak{e}_8 = \mathfrak{sp}(1) \oplus \mathfrak{e}_7 \oplus V_{E_7}(e_7) \otimes H,$$

$E = V_{E_7}(e_7)$ denotes the representation of E_7 with highest weight e_7, and $\bigwedge_0^2 E = V(e_6)$.

Type F_4. For $F_4/Sp(3)Sp(1)$, $n = 7$. Let $\mathfrak{H}$ be the Cartan sub-algebra spanned by the basic roots $\{\alpha_1 = (1,-1,0,0)$, $\alpha_2 = (0,1,-1,0)$, $\alpha_3 = (0,0,2,0)$, $\alpha_4 = (-1,-1,-1,1)\}$, and maximal root $\rho = (0,0,0,2)$. Since

$$\mathfrak{f}_4 = \mathfrak{sp}(1) \oplus \mathfrak{sp}(3) \oplus \oplus \textstyle\bigwedge_0^3 \mathbb{C}^6 \otimes H,$$

the 14-dimensional complex representation $E = \bigwedge_0^3 \mathbb{C}^6 = V(1,1,1,0)$ and $\bigwedge_0^2 E = V(2,2,0,0)$.

Type G_2. For $G_2/SO(8)$, $n = 2$. Let $\mathfrak{H}$ be the Cartan sub-algebra spanned by the basic roots $\{\alpha = (1,0), \beta = (-3/2, \sqrt{3}/2)\}$ and $\rho = 3\alpha + 2\beta =$

$(0, \sqrt{3})$. Since

$$\mathfrak{g}_2 = \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \oplus S^3\mathbb{C}^2 \otimes H,$$

$E = S^3\mathbb{C}^2 = V(3/2\alpha)$ and $\bigwedge_0^2 E = V(2\alpha)$. In this case $i^{2,2} = 7 \neq 0$.

□

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